

**COMPLETE DETERMINATION OF
THE ZETA FUNCTION OF THE HILBERT SCHEME OF n POINTS
ON A TWO-DIMENSIONAL TORUS**

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ABSTRACT. We compute the coefficients of the polynomials $C_n(q)$ defined by the equation

$$1 + \sum_{n \geq 1} \frac{C_n(q)}{q^n} t^n = \prod_{i \geq 1} \frac{(1-t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}}.$$

As an application we obtain an explicit formula for the zeta function of the Hilbert scheme of n points on a two-dimensional torus and show that this zeta function satisfies a remarkable functional equation. The polynomials $C_n(q)$ are divisible by $(q-1)^2$. We also compute the coefficients of the polynomials $P_n(q) = C_n(q)/(q-1)^2$: each coefficient counts the divisors of n in a certain interval; it is thus a non-negative integer. Finally we give arithmetical interpretations for the values of $C_n(q)$ and of $P_n(q)$ at $q = -1$ and at roots of unity of order 3, 4, 6.

1. INTRODUCTION

Let \mathbb{F}_q be a finite field of cardinality q and $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ be the algebra of Laurent polynomials in two variables with coefficients in \mathbb{F}_q . In [9, Th. 1.1] we gave the following formula for the number $C_n(q)$ of ideals of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$, where $n \geq 1$:

$$(1.1) \quad C_n(q) = \sum_{\lambda \vdash n} (q-1)^{2v(\lambda)} q^{n-\ell(\lambda)} \prod_{\substack{i=1, \dots, t \\ d_i \geq 1}} \frac{q^{2d_i} - 1}{q^2 - 1},$$

where the sum runs over all partitions λ of n (the notation $\ell(\lambda)$, $v(\lambda)$, d_i are defined in *loc. cit.*). This formula has the following immediate consequences.

- (i) $C_n(q)$ is a monic polynomial in the variable q with integer coefficients and of degree $2n$.
- (ii) The polynomial $C_n(q)$ is divisible by $(q-1)^2$.
- (iii) If we set

$$P_n(q) = \frac{C_n(q)}{(q-1)^2},$$

then $P_n(q)$ is a monic polynomial with integer coefficients and of degree $2n-2$.

- (iv) The value at $q = 1$ of $P_n(q)$ is given by

$$P_n(1) = \sigma(n) = \sum_{d|n; d \geq 1} d.$$

2010 *Mathematics Subject Classification.* (Primary) 05A17, 14C05, 14G10, 14N10 (Secondary) 05A30, 11P84, 14G15.

Key words and phrases. Infinite product, modular forms, zeta function, Hilbert scheme.

The aim of the present article is to compute the coefficients of the polynomials $C_n(q)$ and $P_n(q)$. To this end we use another consequence of (1.1), namely the following infinite product expression we obtained in *loc. cit.* for the generating function of the polynomials $C_n(q)$ (see [9, Cor. 1.3]):

$$(1.2) \quad 1 + \sum_{n \geq 1} \frac{C_n(q)}{q^n} t^n = \prod_{i \geq 1} \frac{(1-t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}}.$$

This infinite product is clearly invariant under the transformation $q \leftrightarrow q^{-1}$. Therefore,

$$\frac{C_n(q)}{q^n} = \frac{C_n(q^{-1})}{q^{-n}},$$

which implies that the polynomials $C_n(q)$ are palindromic. We can thus express them as follows:

$$(1.3) \quad C_n(q) = c_{n,0} q^n + \sum_{i=1}^n c_{n,i} (q^{n+i} + q^{n-i}),$$

where the coefficients $c_{n,i}$ ($0 \leq i \leq n$) are integers.

Our first main theorem is the following.

Theorem 1.1. *Let $c_{n,i}$ be the coefficients of the Laurent polynomial $C_n(q)$ as defined by (1.3).*

(a) *For the central coefficients we have*

$$c_{n,0} = \begin{cases} 2(-1)^k & \text{if } n = k(k+1)/2 \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *For the non-central coefficients ($i \geq 1$) we have*

$$c_{n,i} = \begin{cases} (-1)^k & \text{if } n = k(k+2i+1)/2 \text{ for some integer } k \geq 1, \\ (-1)^{k-1} & \text{if } n = k(k+2i-1)/2 \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the coefficients of $C_n(q)$ take only the values 0, ± 1 , and ± 2 , and that in Item (c) the first two conditions are mutually exclusive. A list of polynomials $C_n(q)$ ($1 \leq n \leq 12$) appears in Table 1.

In the next section we will apply Theorem 1.1 to give an explicit formula for the local zeta function of the Hilbert scheme of n points on a two-dimensional torus (see Theorem 2.1).

Since the polynomial $P_n(q)$ is the quotient of two palindromic polynomials, $P_n(q)$ is palindromic as well. We can thus expand it as follows:

$$P_n(q) = a_{n,0} q^{n-1} + \sum_{i=1}^{n-1} a_{n,i} (q^{n+i-1} + q^{n-i-1}),$$

where $a_{n,i}$ ($0 \leq i \leq n-1$) are integers.

We now state our second main theorem.

Theorem 1.2. *For $n \geq 1$ and $0 \leq i \leq n-1$, the integer $a_{n,i}$ is equal to the number of divisors d of n satisfying the inequalities*

$$\frac{i + \sqrt{2n + i^2}}{2} < d \leq i + \sqrt{2n + i^2}.$$

The most striking consequence of this theorem is that the coefficients of $P_n(q)$ are *non-negative* integers. Table 2 lists the polynomials $P_n(q)$ for $1 \leq n \leq 12$ (this table also displays their values at -1 , at a primitive third root of unity j and at $i = \sqrt{-1}$).

Remark 1.3. It follows from Theorem 1.2 that the central coefficient $a_{n,0}$ of $P_n(q)$ is equal to the number of “middle divisors” of n , i.e. of the divisors d in the half-open interval $(\sqrt{n/2}, \sqrt{2n}]$. The table¹ in [12, A067742] listing the central coefficients $a_{n,0}$ for $n \leq 10000$ suggests that the sequence $(a_{n,0})_n$ grows very slowly (see also [12, A128605]); Vatne recently proved that this sequence is unbounded (see [16]).

Remark 1.4. Since the degree of $P_n(q)$ is $2n - 2$, its coefficients are non-negative integers and their sum $P_n(1)$ is equal to the sum $\sigma(n)$ of divisors of n , necessarily at least one of the coefficients of $P_n(q)$ must be ≥ 2 when $\sigma(n) \geq 2n$, i.e. when n is a perfect number (such as 6 or 28) or an abundant number (such as 12, 18, 20, 24 or 30).

Our third main result is concerned with the values of $C_n(q)$ and $P_n(q)$ at certain roots of unity. Indeed, if ω is a root of unity of order $d = 2, 3, 4$, or 6 , then $\omega + \omega^{-1} \in \mathbb{Z}$, which together with (1.2) shows that $C_n(\omega)/\omega^n$ is an integer. It is easy to check that when we set $q = \omega$ in (1.2) the infinite product is expressible in terms of Dedekind’s eta function $\eta(z) = t^{1/24} \prod_{n \geq 1} (1 - t^n)$ where $t = e^{2\pi iz}$. More precisely,

$$(1.4) \quad 1 + \sum_{n \geq 1} \frac{C_n(\omega)}{\omega^n} t^n = \begin{cases} \frac{\eta(z)^4}{\eta(2z)^2} & \text{if } d = 2, \\ \frac{\eta(z)^3}{\eta(3z)} & \text{if } d = 3, \\ \frac{\eta(z)^2 \eta(2z)}{\eta(4z)} & \text{if } d = 4, \\ \frac{\eta(z) \eta(2z) \eta(3z)}{\eta(6z)} & \text{if } d = 6. \end{cases}$$

The eta-products appearing in the previous equality are *modular forms* of weight one and of level d (see [11]). We will determine the four sequences $(C_n(\omega))_{n \geq 1}$ explicitly (see Theorem 1.5). In view of their relationship with modular forms, it is not surprising that these sequences are related to well-known arithmetical sequences.

In order to state Theorem 1.5 we introduce some notation.

(i) Let $r(n)$ be the number of representations of n as a sum of two squares:

$$(1.5) \quad r(n) = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}.$$

The sequence $r(n)$ is Sequence A004018 of [12]. Its generating function is the theta series of the square lattice. Note that $r(n)$ is divisible by 4 for $n \geq 1$ due to the symmetries of the square lattice.

¹This table is due to R. Zumkeller

(ii) Similarly, we denote by $r'(n)$ the number of representations of n as a sum of a square and twice another square:

$$(1.6) \quad r'(n) = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + 2y^2 = n\}.$$

The sequence $r'(n)$ is Sequence A033715 of [12].

(iii) Let $\lambda(n)$ be the multiplicative function of n defined by on all prime powers by

$$(1.7) \quad \lambda(p^e) = \begin{cases} -2 & \text{if } p = 3 \text{ and } e \geq 1, \\ e + 1 & \text{if } p \equiv 1 \pmod{6}, \\ (1 + (-1)^e)/2 & \text{if } p \equiv 2, 5 \pmod{6}. \end{cases}$$

Recall that “multiplicative” means that $\lambda(mn) = \lambda(m)\lambda(n)$ whenever m and n are coprime. The function $\lambda(n)$ can also be expressed in terms of the excess function

$$(1.8) \quad E_1(n; 3) = \sum_{\substack{d|n \\ d \equiv 1 \pmod{3}}} 1 - \sum_{\substack{d|n \\ d \equiv 2 \pmod{3}}} 1.$$

Indeed, $\lambda(n) = E_1(n; 3) - 3E_1(n/3; 3)$, where we agree that $E_1(n/3; 3) = 0$ when n is not divisible by 3.

We can now state our results concerning the values of $C_n(q)$ and $P_n(q)$ at the roots of unity of order $d = 2, 3, 4, 6$. We express these values in terms of the notation we have just introduced.

Theorem 1.5. *Let n be an integer ≥ 1 .*

(a) *We have*

$$C_n(-1) = r(n) \quad \text{and} \quad P_n(-1) = \frac{r(n)}{4}.$$

(b) *Let $j = e^{2\pi i/3}$, a primitive third root of unity. Then*

$$C_n(j) = -3\lambda(n)j^n \quad \text{and} \quad P_n(j) = \lambda(n)j^{n-1}.$$

(c) *Let $\mathbf{i} = \sqrt{-1}$ be a square root of -1 . Then²*

$$C_n(\mathbf{i}) = (-1)^{\lfloor (n+1)/2 \rfloor} r'(n) \mathbf{i}^n \quad \text{and} \quad P_n(\mathbf{i}) = (-1)^{\lfloor (n-1)/2 \rfloor} \frac{r'(n)}{2} \mathbf{i}^{n-1}.$$

(d) *Let $-j = e^{\pi i/3}$, a primitive sixth root of unity. We have*

$$C_n(-j) = \begin{cases} r(n) & \text{if } n \equiv 0, \\ \frac{r(n)}{4} j & \text{if } n \equiv 1, \\ -\frac{r(n)}{2} j^2 & \text{if } n \equiv 2, \end{cases} \pmod{3}$$

and $P_n(-j) = C_n(-j)/j = C_n(-j) j^2$.

The present paper, which complements [9], is organized as follows. In Section 2 we compute the local zeta function of the Hilbert scheme of n points on a two-dimensional torus. Section 3 is devoted to the proofs of Theorems 1.1, 1.2 and 1.5. In Section 4 we collect additional results on the polynomials $C_n(q)$ and $P_n(q)$.

²If α is a real number, then $\lfloor \alpha \rfloor$ stands for the largest integer $\leq \alpha$.

TABLE 1. *The polynomials $C_n(q)$*

n	$C_n(q)$	$C_n(-1)$
1	$q^2 - 2q + 1$	4
2	$q^4 - q^3 - q + 1$	4
3	$q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 1$	0
4	$q^8 - q^7 - q + 1$	4
5	$q^{10} - q^9 - q^7 + q^6 + q^4 - q^3 - q + 1$	8
6	$q^{12} - q^{11} + q^7 - 2q^6 + q^5 - q + 1$	0
7	$q^{14} - q^{13} - q^{10} + q^9 + q^5 - q^4 - q + 1$	0
8	$q^{16} - q^{15} - q + 1$	4
9	$q^{18} - q^{17} - q^{13} + q^{12} + q^{11} - q^{10} - q^8 + q^7 + q^6 - q^5 - q + 1$	4
10	$q^{20} - q^{19} - q^{11} + 2q^{10} - q^9 - q + 1$	8
11	$q^{22} - q^{21} - q^{16} + q^{15} + q^7 - q^6 - q + 1$	0
12	$q^{24} - q^{23} + q^{15} - q^{14} - q^{10} + q^9 - q + 1$	0

TABLE 2. *The polynomials $P_n(q)$*

n	$P_n(q)$	$P_n(1)$	$P_n(-1)$	$ P_n(j) $	$ P_n(i) $	$a_{n,0}$
1	1	1	1	1	1	1
2	$q^2 + q + 1$	3	1	0	1	1
3	$q^4 + q^3 + q + 1$	4	0	2	2	0
4	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	7	1	1	1	1
5	$q^8 + q^7 + q^6 + q^2 + q + 1$	6	2	0	0	0
6	$q^{10} + q^9 + q^8 + q^7 + q^6 + 2q^5 + q^4 + q^3 + q^2 + q + 1$	12	0	0	2	2
7	$q^{12} + q^{11} + q^{10} + q^9 + q^3 + q^2 + q + 1$	8	0	2	0	0
8	$q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	15	1	0	1	1
9	$q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^9 + q^8 + q^7 + q^4 + q^3 + q^2 + q + 1$	13	1	2	3	1
10	$q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	18	2	0	0	0
11	$q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^5 + q^4 + q^3 + q^2 + q + 1$	12	0	0	2	0
12	$q^{22} + q^{21} + q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + 2q^{10} + 2q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	28	0	2	2	2

Finally, in Appendix A we give a proof of a result by Somos needed in Section 3.3 for the proof of Theorem 1.5.

2. THE LOCAL ZETA FUNCTION OF THE HILBERT SCHEME OF n POINTS ON A TWO-DIMENSIONAL TORUS

Let k be a field and n be a positive integer. The algebra $k[x, y, x^{-1}, y^{-1}]$ of Laurent polynomials with coefficients in k is the coordinate ring of the two-dimensional torus $(\mathbb{A}_k^1 \setminus \{0\}) \times (\mathbb{A}_k^1 \setminus \{0\})$. As is well known, the ideals of codimension n of the Laurent polynomial algebra $k[x, y, x^{-1}, y^{-1}]$ are in bijection with the k -points of the Hilbert scheme

$$H_k^n = \text{Hilb}^n((\mathbb{A}_k^1 \setminus \{0\}) \times (\mathbb{A}_k^1 \setminus \{0\}))$$

parametrizing finite subschemes of colength n of the two-dimensional torus. This scheme is a quasi-projective variety.

As an application of Theorem 1.1, we now give an explicit expression for the local zeta function of the Hilbert scheme $H_{\mathbb{F}_q}^n$. Since the scheme is quasi-projective, its local zeta function is by [5, 7] a rational function.

Recall that the *local zeta function* of an algebraic variety X defined over a finite field \mathbb{F}_q of cardinality q is the formal power series

$$(2.1) \quad Z_{X/\mathbb{F}_q}(t) = \exp\left(\sum_{m \geq 1} |X(\mathbb{F}_{q^m})| \frac{t^m}{m}\right),$$

where $|X(\mathbb{F}_{q^m})|$ is the number of points of X over the degree m extension \mathbb{F}_{q^m} of \mathbb{F}_q .

Theorem 2.1. *The local zeta function of the Hilbert scheme $H_{\mathbb{F}_q}^n$ is the rational function*

$$Z_{H_{\mathbb{F}_q}^n/\mathbb{F}_q}(t) = \frac{1}{(1 - q^n t)^{c_{n,0}}} \prod_{i=1}^n \frac{1}{[(1 - q^{n+i} t)(1 - q^{n-i} t)]^{c_{n,i}}},$$

where the exponents $c_{n,i}$ are the integers determined in Theorem 1.1.

Let us display a few examples of such rational functions. For $n = 3, 5, 6$, we have

$$Z_{H_{\mathbb{F}_q}^3/\mathbb{F}_q}(t) = \frac{(1 - qt)(1 - q^2t)(1 - q^4t)(1 - q^5t)}{(1 - t)(1 - q^3t)^2(1 - q^6t)},$$

$$Z_{H_{\mathbb{F}_q}^5/\mathbb{F}_q}(t) = \frac{(1 - qt)(1 - q^3t)(1 - q^7t)(1 - q^9t)}{(1 - t)(1 - q^4t)(1 - q^6t)(1 - q^{10}t)},$$

$$Z_{H_{\mathbb{F}_q}^6/\mathbb{F}_q}(t) = \frac{(1 - qt)(1 - q^6t)^2(1 - q^{11}t)}{(1 - t)(1 - q^5t)(1 - q^7t)(1 - q^{12}t)}.$$

Proof of Theorem 2.1. Let $Z(t)$ be the RHS of the equality in the theorem. Clearly, $Z(0) = 1$. By (2.1) it remains to check that

$$t \frac{Z'(t)}{Z(t)} = \sum_{m \geq 1} C_n(q^m) t^m.$$

Now

$$\begin{aligned}
t \frac{Z'(t)}{Z(t)} &= c_{n,0} \frac{q^n t}{1 - q^n t} + \sum_{i=1}^n c_{n,i} \left(\frac{q^{n+i} t}{1 - q^{n+i} t} + \frac{q^{n-i} t}{1 - q^{n-i} t} \right) \\
&= \sum_{m \geq 1} c_{n,0} q^{mn} t^m + \sum_{i=1}^n c_{n,i} \sum_{m \geq 1} (q^{m(n+i)} + q^{m(n-i)}) t^m \\
&= \sum_{m \geq 1} C_n(q^m) t^m.
\end{aligned}$$

□

As one immediately sees, the formula for $Z_{H_{\mathbb{F}_q}^n/\mathbb{F}_q}(t)$ has a striking symmetry, which we express as follows.

Corollary 2.2. *The local zeta function $Z_{H_{\mathbb{F}_q}^n/\mathbb{F}_q}(t)$ satisfies the functional equation*

$$Z_{H_{\mathbb{F}_q}^n/\mathbb{F}_q}\left(\frac{1}{q^{2n}t}\right) = Z_{H_{\mathbb{F}_q}^n/\mathbb{F}_q}(t).$$

Such functional equations exist for smooth projective schemes (see for instance [3, (2.6)]). Here we obtained one despite the fact that the Hilbert scheme H^n is not projective.

Proof. Using the expression for $Z_{H^n/\mathbb{F}_q}(t)$ in Theorem 2.1, we have

$$\begin{aligned}
Z_{H^n/\mathbb{F}_q}\left(\frac{1}{q^{2n}t}\right) &= \frac{1}{(1 - q^{-n}t^{-1})^{c_{n,0}}} \prod_{i=1}^n \frac{1}{[(1 - q^{-n+i}t^{-1})(1 - q^{-n-i}t^{-1})]^{c_{n,i}}} \\
&= \frac{(q^n t)^{c_{n,0}}}{(q^n t - 1)^{c_{n,0}}} \prod_{i=1}^n \frac{(q^{n-i}t)^{c_{n,i}} (q^{n+i}t)^{c_{n,i}}}{[(q^{n-i}t - 1)(q^{n+i}t - 1)]^{c_{n,i}}} \\
&= (-1)^{c_{n,0}} \frac{(q^n t)^{c_{n,0} + 2 \sum_{i=1}^n c_{n,i}}}{(1 - q^n t)^{c_{n,0}}} \prod_{i=1}^n \frac{1}{[(1 - q^{n-i}t)(1 - q^{n+i}t)]^{c_{n,i}}} \\
&= (-1)^{c_{n,0}} \frac{(q^n t)^{C_n(1)}}{(1 - q^n t)^{c_{n,0}}} \prod_{i=1}^n \frac{1}{[(1 - q^{n+i}t)(1 - q^{n-i}t)]^{c_{n,i}}}.
\end{aligned}$$

This latter is equal to $Z_{H^n/\mathbb{F}_q}(t)$ in view of the vanishing of $C_n(1)$ and of the fact that all integers $c_{n,0}$ are even (see Theorem 1.1 (a)). □

With Theorem 2.1 we can easily compute the *Hasse–Weil zeta function* of the above Hilbert scheme; this zeta function is defined by

$$(2.2) \quad \zeta_{H^n}(s) = \prod_{p \text{ prime}} Z_{H_{\mathbb{F}_p}^n/\mathbb{F}_p}(p^{-s}),$$

where the product is taken over all prime integers p . It follows from Theorem 2.1 that the Hasse–Weil zeta function of H^n is given by

$$\zeta_{H^n}(s) = \zeta(s - n)^{c_{n,0}} \prod_{i=1}^n [\zeta(s - n - i) \zeta(s - n + i)]^{c_{n,i}},$$

where $\zeta(s)$ is Riemann's zeta function. As a consequence of (2.2) and of Corollary 2.2, the Hasse–Weil zeta function $\zeta_{H^n}(s)$ satisfies the functional equation

$$\zeta_{H^n}(s) = \zeta_{H^n}(2n - s).$$

3. PROOFS

We now give the proofs of the results stated in the introduction. We start with the proof of Theorem 1.2.

3.1. Proof of Theorem 1.2. We first establish the positivity of the coefficients of $P_n(q)$, then we compute them.

(a) (*Positivity*) Replacing first $C_n(q)$ by $(q-1)^2 P_n(q)$, then the variables q by q^2 and t by t^2 in (1.2), we obtain

$$1 + (q^2 - 1)^2 \sum_{n \geq 1} \frac{P_n(q^2)}{q^{2n}} t^{2n} = \prod_{n \geq 1} \frac{(1 - t^{2n})^2}{(1 - q^2 t^{2n})(1 - q^{-2} t^{2n})}.$$

By [6, Eq. (10.1)],

$$\begin{aligned} \frac{2q}{1 - q^2} \prod_{n \geq 1} \frac{(1 - t^{2n})^2}{(1 - q^2 t^{2n})(1 - q^{-2} t^{2n})} \\ = \frac{2q}{1 - q^2} + \sum_{k, m \geq 1} ((-1)^m - (-1)^k) q^{k-m} t^{km}. \end{aligned}$$

Therefore,

$$1 + (q^2 - 1)^2 \sum_{n \geq 1} \frac{P_n(q^2)}{q^{2n}} t^{2n} = 1 + \frac{1 - q^2}{2q} \sum_{k, m \geq 1} ((-1)^m - (-1)^k) q^{k-m} t^{km}.$$

Equating the coefficients of t^{2n} for $n \geq 1$, we obtain

$$\frac{P_n(q^2)}{q^{2n}} = \frac{1}{2q(1 - q^2)} \sum_{\substack{k, m \geq 1 \\ km = 2n}} ((-1)^m - (-1)^k) q^{k-m}.$$

Now the integer $(-1)^m - (-1)^k$ vanishes if m, k are of the same parity, is equal to 2 if m is even and k is odd, and is equal to -2 if m is odd and k is even. In view of this remark, we have

$$\begin{aligned} P_n(q^2) &= \frac{q^{2n-1}}{2(1 - q^2)} \sum_{\substack{1 \leq k < m \\ km = 2n}} (((-1)^m - (-1)^k) q^{k-m} + ((-1)^k - (-1)^m) q^{m-k}) \\ &= \frac{1}{2(1 - q^2)} \sum_{\substack{1 \leq k < m \\ km = 2n}} ((-1)^m - (-1)^k) q^{2n-1} q^{k-m} (1 - q^{2m-2k}) \\ &= \sum_{\substack{1 \leq k < m \\ km = 2n}} \frac{(-1)^m - (-1)^k}{2} q^{2n-1+k-m} \frac{1 - q^{2m-2k}}{1 - q^2} \\ &= \sum_{\substack{1 \leq k < m \\ km = 2n}} \frac{(-1)^m - (-1)^k}{2} q^{2n-1+k-m} \left(\sum_{j=0}^{m-k-1} q^{2j} \right). \end{aligned}$$

We now restrict this sum to the pairs (k, m) of different parity, in which case $((-1)^m - (-1)^k)/2 = (-1)^{k-1}$. Using the notation $H(I) = \sum_{2k \in I} q^{2k}$ for any interval $I \subset \mathbb{N}$, we obtain

$$(3.1) \quad P_n(q^2) = \sum_{\substack{1 \leq k < m, km = 2n \\ k \not\equiv m \pmod{2}}} (-1)^{k-1} H([2n - 1 + k - m, 2n - 3 + m - k]).$$

Note that the intervals are all centered around 0 and their bounds are even.

Now the only contributions in the previous sum which may pose a positivity problem occur when k is even; so in this case write $k = 2^\alpha k'$ for some integer $\alpha \geq 1$ and some odd integer $k' \geq 1$. Setting $m' = 2^\alpha m$, we see that k' is odd, m' is even, $k'm' = km = 2n$ and $1 \leq k' < k < m < m'$. Moreover,

$$[2n - 1 + k - m, 2n - 3 + m - k] \subsetneq [2n - 1 + k' - m', 2n - 3 + m' - k'].$$

Observe that the function $k \mapsto k' = 2^{-\alpha}k$ is injective. It follows that the negative contributions in the sum (3.1) corresponding to even k are counterbalanced by the positive contributions corresponding to odd k' . This shows that $P_n(q^2)$, hence $P_n(q)$, has only non-negative coefficients.

(b) (*Computation of the coefficients of $P_n(q)$*) Using (3.1), we immediately derive

$$(3.2) \quad \frac{P_n(q)}{q^{n-1}} = \sum_{\substack{1 \leq k < m, km=2n \\ k \not\equiv m \pmod{2}}} (-1)^{k-1} \left[\left[-\frac{m-k-1}{2}, \frac{m-k-1}{2} \right] \right],$$

where $[[a, b]]$ stands for $q^a + q^{a+1} + \dots + q^b$ (for two integers $a \leq b$). Observe that $m-k-1$ is even so that the bounds in the RHS of (3.2) are integers. It follows from this expression of $P_n(q)$ that $q^{-(n-1)}P_n(q)$ is invariant under the map $q \mapsto q^{-1}$. This shows that the degree $2n-2$ polynomial $P_n(q)$ is palindromic (which we had already pointed out in the introduction as the infinite product in (1.2) is invariant under the transformation $q \mapsto q^{-1}$). Therefore,

$$(3.3) \quad \frac{P_n(q)}{q^{n-1}} = a_{n,0} + \sum_{i=1}^{n-1} a_{n,i} (q^i + q^{-i})$$

for some non-negative integers $a_{n,i}$, which we now determine. We assert the following.

Proposition 3.1. *For any integer $i \geq 0$, we have*

$$\sum_{n \geq 1} a_{n,i} t^n = \sum_{k \geq 1} (-1)^{k-1} \frac{t^{k(k+1)/2} t^{ki}}{1 - t^k}.$$

Proof. The coefficient $a_{n,i}$ is the coefficient of q^i in $q^{-(n-1)}P_n(q)$. It follows from (3.2) that

$$a_{n,i} = \sum_{A(n,i)} (-1)^{k-1},$$

where $A(n, i)$ is the set of integers k such that $km = 2n$, $1 \leq k < m$, $k \not\equiv m \pmod{2}$, and $(m-k-1)/2 \geq i$ (the latter condition is equivalent to $m-k \geq 2i+1$). Now the RHS in the proposition is equal to

$$\sum_{k \geq 1} (-1)^{k-1} t^{k(k+1)/2} t^{ki} \sum_{j \geq 0} t^{kj} = \sum_{k \geq 1} (-1)^{k-1} \sum_{j \geq 0} t^{k(k+2i+2j+1)/2}.$$

Thus the coefficient of q^i in the latter expression is equal to

$$\sum_{\substack{k \geq 1, j \geq 0 \\ k(k+2i+2j+1)=2n}} (-1)^{k-1}.$$

Setting $m = k + 2i + 2j + 1$, we see that the previous sum is the same as the one indexed by the set $A(n, i)$ above. \square

Note the following consequence of (3.3) and of Proposition 3.1.

Corollary 3.2. *We have*

$$\sum_{n \geq 1} \frac{P_n(q)}{q^{n-1}} t^n = \sum_{k \geq 1} (-1)^{k-1} t^{k(k+1)/2} \frac{1+t^k}{(1-qt^k)(1-q^{-1}t^k)}.$$

To compute the coefficients $a_{n,i}$, we adapt the proof in [1]. Using Proposition 3.1 and separating the even indices k from the odd ones, we have

$$\begin{aligned} \sum_{n \geq 1} a_{n,i} t^n &= \sum_{k \text{ odd} \geq 1} \frac{t^{k(k+2i+1)/2}}{1-t^k} - \sum_{k \geq 1} \frac{t^{k(2k+2i+1)}}{1-t^{2k}} \\ &= \sum_{k \text{ odd} \geq 1} \frac{t^{k(k+2i+1)/2}}{1-t^k} - \sum_{k \geq 1} \frac{t^{k(2k+2i+1)}(1+t^k)}{1-t^{2k}} + \sum_{k \geq 1} \frac{t^{k(2k+2i+2)}}{1-t^{2k}} \\ &= \sum_{k \text{ odd} \geq 1} \frac{t^{k(k+2i+1)/2}}{1-t^k} + \sum_{k \text{ even} \geq 1} \frac{t^{k(k+2i+2)/2}}{1-t^k} - \sum_{k \geq 1} \frac{t^{k(2k+2i+1)}}{1-t^k} \\ &= \sum_{k \geq 1} \frac{t^{k(\lfloor k/2 \rfloor + i + 1)}}{1-t^k} - \sum_{k \geq 1} \frac{t^{k(2k+2i+1)}}{1-t^k} \end{aligned}$$

since

$$\lfloor k/2 \rfloor + i + 1 = \begin{cases} (k+2i+1)/2 & \text{if } k \text{ is odd,} \\ (k+2i+2)/2 & \text{if } k \text{ is even.} \end{cases}$$

Using the inequality $\lfloor k/2 \rfloor + i + 1 < 2k + 2i + 1$ and the identity

$$\frac{t^{ka}}{1-t^k} - \frac{t^{kb}}{1-t^k} = \sum_{a \leq d < b} t^{kd} \quad (a < b)$$

in the special case $a = \lfloor k/2 \rfloor + i + 1$ and $b = 2k + 2i + 1$, we obtain

$$\sum_{n \geq 1} a_{n,i} t^n = \sum_{k \geq 1} \sum_{\lfloor k/2 \rfloor + i + 1 \leq d < 2k + 2i + 1} t^{kd}.$$

Therefore, $a_{n,i}$ is the number of divisors d of n such that

$$\left\lfloor \frac{n}{2d} \right\rfloor + i + 1 \leq d < \frac{2n}{d} + 2i + 1.$$

The leftmost inequality is equivalent to $n/(2d) + i < d$, which is equivalent to $d^2 - id - n/2 > 0$, which in turn is equivalent to $d > (i + \sqrt{2n + i^2})/2$. On the other hand, the rightmost inequality is equivalent to $d \leq 2n/d + 2i$, which is equivalent to $d^2 - 2id - 2n \leq 0$, which in turn is equivalent to $d \leq i + \sqrt{2n + i^2}$.

This completes the proof of Theorem 1.2.

3.2. Proof of Theorem 1.1. Let $c_{n,i}$ be the coefficients of $C_n(q)$ as defined by (1.3). We claim the following.

Proposition 3.3. *We have*

$$\sum_{n \geq 1} c_{n,0} t^n = 2 \sum_{k \geq 1} (-1)^k t^{k(k+1)/2},$$

and for any integer $i \geq 1$, we have

$$\sum_{n \geq 1} c_{n,i} t^n = \sum_{k \geq 1} (-1)^k \left(t^{k(k+2i+1)/2} - t^{k(k+2i-1)/2} \right).$$

Proof. The equality $C_n(q)/q^n = (q - 2 + q^{-1})P_n(q)/q^{n-1}$ implies that the coefficients $c_{n,i}$ of $C_n(q)$ are related to the coefficients $a_{n,i}$ of $P_n(q)$ by the relations

$$(3.4) \quad c_{n,0} = -2a_{n,0} + 2a_{n,1}$$

and

$$(3.5) \quad c_{n,i} = a_{n,i+1} - 2a_{n,i} + a_{n,i-1} \quad (i \geq 1).$$

From (3.4) and Proposition 3.1, we obtain

$$\sum_{n \geq 1} c_{n,0} t^n = 2 \sum_{k \geq 1} (-1)^{k-1} \frac{t^{k(k+1)/2} (t^k - 1)}{(1 - t^k)} = 2 \sum_{k \geq 1} (-1)^k t^{k(k+1)/2}.$$

If $i \geq 1$, then by (3.5) and Proposition 3.1,

$$\begin{aligned} \sum_{n \geq 1} c_{n,i} t^n &= \sum_{k \geq 1} (-1)^{k-1} \frac{t^{k(k+1)/2} (t^{k(i+1)} - 2t^{ki} + t^{k(i-1)})}{(1 - t^k)} \\ &= \sum_{k \geq 1} (-1)^{k-1} \frac{t^{k(k+1)/2} t^{k(i-1)} (1 - t^k)^2}{(1 - t^k)} \\ &= \sum_{k \geq 1} (-1)^{k-1} (1 - t^k) t^{k(k+2i-1)/2} \\ &= \sum_{k \geq 1} (-1)^k \left(t^{k(k+2i+1)/2} - t^{k(k+2i-1)/2} \right). \end{aligned}$$

□

It follows from the first equality in Proposition 3.3 that $c_{n,0} = 0$ unless n is *triangular*, i. e., of the form $n = k(k+1)/2$, in which case $c_{n,0} = 2(-1)^k$. This proves Theorem 1.1 (a).

For Part (b) of the theorem we introduce the following definition.

Definition 3.4. Fix an integer $i \geq 0$. An integer $n > 0$ is *i-trapezoidal* if there is an integer $k \geq 1$ such that

$$n = (i+1) + (i+2) + \cdots + (i+k) = \frac{k(k+2i+1)}{2}.$$

A 0-trapezoidal number is a triangular number.

Lemma 3.5. (1) An integer n is *i-trapezoidal* if and only if $8n + (2i+1)^2$ is a perfect square.

(2) If n is *i-trapezoidal* for some $i \geq 1$, then it is not $(i-1)$ -trapezoidal.

Proof. (1) An integer n is *i-trapezoidal* if and only if $x^2 + (2i+1)x - 2n = 0$ has an integral solution $k \geq 1$. This implies that the discriminant $8n + (2i+1)^2$ is a perfect square. Conversely, if the discriminant is the square of a positive integer δ , then δ must be odd, and $k = (\delta - (2i+1))/2$ is a positive integer.

(2) Consider the sequence $((2(i+j)+1)^2)_{j \geq 0}$ of odd perfect squares $\geq (2i+1)^2$. The distance between the square $(2(i+j)+1)^2$ and the subsequent one is equal to $8(i+j+1)$, which, if $j \geq 0$, is strictly bigger than the distance $8i$ between $(2i-1)^2$ and $(2i+1)^2$. Now suppose that n is both *i-trapezoidal* and $(i-1)$ -trapezoidal. By Part (1), both $8n + (2i+1)^2$ and $8n + (2i-1)^2$ are odd perfect squares, which are $> (2i-1)^2$ since $n \geq 1$. Hence they are $\geq (2i+1)^2$. Since their difference is $8i$, we obtain a contradiction. □

We now complete the proof of Theorem 1.1 (b). Since by Lemma 3.5 an integer n cannot be i -trapezoidal for two consecutive i , it follows from the second equality in Proposition 3.3 that

$$c_{n,i} = \begin{cases} (-1)^k & \text{if } n = k(k + 2i + 1)/2 \text{ for some integer } k, \\ (-1)^{k-1} & \text{if } n = k(k + 2i - 1)/2 \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.6. By Theorem 1.1 we have the inequalities $|c_{n,i}| \leq 2$. Together with (3.4) and (3.5), they imply that the variation of the coefficients $a_{n,i}$ of $P_n(q)$ is small. More precisely,

$$\left| a_{n,i} - \frac{a_{n,i-1} + a_{n,i+1}}{2} \right| \leq 1$$

for all $n \geq 1$ and all $i \geq 0$ (we use here the convention $a_{n,-1} = a_{n,1}$).

3.3. Proof of Theorem 1.5. Let ω be a complex root of unity of order $d = 2, 3, 4$, or 6 . For each such d we define the sequence $(a_d(n))_{n \geq 1}$ by

$$(3.6) \quad 1 + \sum_{n \geq 1} a_d(n) t^n = \prod_{i \geq 1} \frac{(1 - t^i)^2}{1 - (\omega + \omega^{-1})t^i + t^{2i}}.$$

As observed in the introduction each $a_d(n)$ is an integer. In view of (1.2) and of the definition of $P_n(q)$ we have

$$a_d(n) = \frac{C_n(\omega)}{\omega^n} = (\omega + \omega^{-1} - 2) \frac{P_n(\omega)}{\omega^{n-1}}.$$

In order to prove Theorem 1.5 we compute $a_2(n)$, $a_3(n)$, $a_4(n)$, and $a_6(n)$ successively. See Table 3 for the absolute values $|a_d(n)|$ with $1 \leq n \leq 18$.

3.3.1. The case $d = 2$. Setting $\omega = -1$ in (3.6), we obtain

$$(3.7) \quad 1 + \sum_{n \geq 1} a_2(n) t^n = \prod_{i \geq 1} \frac{(1 - t^i)^2}{1 + 2t^i + t^{2i}} = \left(\prod_{i \geq 1} \frac{1 - t^i}{1 + t^i} \right)^2.$$

Now recall the following identity of Gauss (see [6, (7.324)] or [8, 19.9 (i)]):

$$(3.8) \quad \prod_{i \geq 1} \frac{1 - t^i}{1 + t^i} = \sum_{k \in \mathbb{Z}} (-1)^k t^{k^2}.$$

It follows from this identity that the RHS of (3.7) is equal to

$$\left(\sum_{k \in \mathbb{Z}} (-1)^k t^{k^2} \right)^2.$$

Since k and k^2 are of the same parity, we obtain

$$1 + \sum_{n \geq 1} a_2(n) t^n = \left(\sum_{k \in \mathbb{Z}} (-t)^{k^2} \right)^2.$$

Now the latter is clearly equal to $\sum_{n \geq 0} r(n) (-t)^n$, where $r(n)$ is the positive integer defined by (1.5). Identifying the terms, we deduce $a_2(n) = (-1)^n r(n)$ for all $n \geq 1$. Since $a_2(n) = (-1)^n C_n(-1) = (-1)^n 4P_n(-1)$, we deduce the desired values for $C_n(-1)$ and $P_n(-1)$.

3.3.2. *The case $d = 3$.* Let j be a primitive third root of unity. Setting $\omega = j$ in (3.6), we obtain

$$1 + \sum_{n \geq 1} a_3(n) t^n = \prod_{i \geq 1} \frac{(1-t^i)^2}{1+t^i+t^{2i}} = \prod_{i \geq 1} \frac{(1-t^i)^3}{1-t^{3i}}.$$

By [6, § 32, p. 79],

$$a_3(n) = -3\lambda(n) = -3(E_1(n; 3) - 3E_1(n/3; 3))$$

where $\lambda(n)$ is the multiplicative function of n defined by (1.7) and $E_1(n; 3)$ is the excess function (1.8). Hence, the desired values for $C_n(j)$ and $P_n(j)$.

Remark 3.7. By [14], the series $\sum_{n \geq 0} 3\lambda(n)t^n$ is the theta series (with respect to a node) of the two-dimensional honeycomb (not a lattice) in which each node has three neighbors. The sequence $a_3(n)$ (resp. $\lambda(n)$) is Sequence A005928 (resp. A113063) of [12].

By [12, Seq. A005928], $a_3(3n+2) = 0$, and $-a_3(3n+1)$ is Sequence A005882 of [12]. More interestingly, the sequence $a_3(3n) = r''(3n)$, where

$$(3.9) \quad r''(n) = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + xy + y^2 = n\}.$$

The generating function of the sequence $r''(n)$ is the theta series of the hexagonal lattice in which each point has six neighbors (see Sequence A004016 of [12] or [2, Chap. 4, § 6.2]).

3.3.3. *The case $d = 4$.* Let $i = \sqrt{-1}$ be a square root of -1 . It suffices to compute $a_4(n)$. It follows from (3.6) that

$$(3.10) \quad 1 + \sum_{n \geq 1} a_4(n) t^n = \prod_{i \geq 1} \frac{(1-t^i)^2}{1+t^{2i}}.$$

Then $a_4(n)$ forms Sequence A082564 of [12] and we have

$$a_4(n) = (-1)^{\lfloor (n+1)/2 \rfloor} |a_4(n)|.$$

By Lemma A.1 of the appendix, for the absolute value of $a_4(n)$ we have

$$1 + \sum |a_4(n)| q^n = \varphi(q)\varphi(q^2) = \left(\sum_{n \in \mathbb{Z}} q^{n^2} \right) \left(\sum_{n \in \mathbb{Z}} q^{2n^2} \right),$$

which implies that $|a_4(n)| = r'(n)$, where $r'(n)$ is the positive integer defined by (1.6). Therefore, $|a_4(n)|$ forms Sequence A033715 of [12].

3.3.4. *The case $d = 6$.* Since $C_n(-j) = (-1)^n a_6(n) j^n$ and $P_n(-j) = (-1)^n a_6(n) j^{n-1}$, it suffices to compute $a_6(n)$. Setting $\omega = -j$ (which is a primitive sixth root of unity) in (3.6), we obtain

$$1 + \sum_{n \geq 1} a_6(n) t^n = \prod_{i \geq 1} \frac{(1-t^i)^2}{1-t^i+t^{2i}} = \prod_{i \geq 1} \frac{(1-t^i)(1-t^{2i})(1-t^{3i})}{1-t^{6i}}.$$

We computed the integers $a_6(n)$ in [10] (see Theorem 1.1 there); they form Sequence A258210 in [12]. This completes the proof of Theorem 1.5.

Remark 3.8. Our result for $a_6(n)$ shows that $a_6(n) = 0$ if and only if $a_2(n) = 0$, i.e. if and only if n is not the sum of two squares.

TABLE 3. *The absolute values of $a_d(n)$*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$ a_2(n) $	4	4	0	4	8	0	0	4	4	8	0	0	8	0	0	4	8	4
$ a_3(n) $	3	0	6	3	0	0	6	0	6	0	0	6	6	0	0	3	0	0
$ a_4(n) $	2	2	4	2	0	4	0	2	6	0	4	4	0	0	0	2	4	6
$ a_6(n) $	1	2	0	1	4	0	0	2	4	2	0	0	2	0	0	1	4	4

4. GROWTH AND SECTIONS

In this section we collect further results on the polynomials $C_n(q)$ and $P_n(q)$.

4.1. **Growth of $C_n(q)$.** (a) Fix an integer $n \geq 1$. Let us examine how $C_n(q)$ grows when q tends to infinity.

We know that $C_n(q)$ counts the number of \mathbb{F}_q -points of the Hilbert scheme $H_{\mathbb{F}_q}^n$ of n points on the two-dimensional torus. This scheme is an open subset of the Hilbert scheme $\text{Hilb}^n(\mathbb{A}_{\mathbb{F}_q}^2)$ of n points on the affine plane. Let $A_n(q)$ be the number of \mathbb{F}_q -points of $\text{Hilb}^n(\mathbb{A}_{\mathbb{F}_q}^2)$; by [9, Remark 4.7] $A_n(q)$ is a monic polynomial of degree $2n$ in q . Since $C_n(q)$ is also a monic polynomial of degree $2n$, we necessarily have

$$C_n(q) \sim A_n(q) \sim q^{2n}.$$

Moreover, the cardinality $A_n(q) - C_n(q)$ of the complement of $H_{\mathbb{F}_q}^n$ in $\text{Hilb}^n(\mathbb{A}_{\mathbb{F}_q}^2)$ must become small compared to $A_n(q)$ when q tends to ∞ . Indeed, using the expansions

$$A_n(q) = q^{2n} + q^{2n-1} + \text{terms of degree} \leq 2n - 2$$

and

$$\begin{aligned} C_n(q) &= (q-1)^2 (q^{2n-2} + q^{2n-3} + \text{terms of degree} \leq 2n-4) \\ &= q^{2n} - q^{2n-1} + \text{terms of degree} \leq 2n-2, \end{aligned}$$

we easily show that

$$\frac{A_n(q) - C_n(q)}{A_n(q)} = \frac{2}{q} + O\left(\frac{1}{q^2}\right).$$

(b) We now fix q and let n tend to infinity. Since by Theorem 1.1 we have $|c_{n,0}| \leq 2$ and $|c_{n,i}| \leq 1$ for $i \neq 0$, we obtain the inequality

$$|C_n(q)| \leq q^n + \sum_{i=0}^{2n} q^i = q^n + \frac{q^{2n+1} - 1}{q - 1},$$

which implies that $|C_n(q)|$ is bounded above by a function of n equivalent to

$$\frac{1}{q-1} q^{2n+1}.$$

Hence, $|P_n(q)|$ is bounded above by a function equivalent to $q^{2n+1}/(q-1)^3$.

4.2. Sections of the polynomials $P_n(q)$. Given an integer $k \geq 1$, we define the k -section $s_k(n)$ of $P_n(q)$ to be the sum of the (positive) coefficients of the monomials of $P_n(q)$ of the form q^{ki} ($i \geq 0$). We now compute $s_1(n)$, $s_2(n)$, $s_3(n)$, $s_4(n)$, and $s_6(n)$. We keep the notation introduced in the introduction.

(a) When $k = 1$, we clearly have $s_1(n) = P_n(1) = \sigma(n)$, where $\sigma(n)$ is the sum of positive divisors of n .

(b) When $k = 2$, then $s_2(n) = (P_n(1) + P_n(-1))/2$, which by Theorem 1.5 (a) implies

$$s_2(n) = \frac{\sigma(n) + r(n)/4}{2}.$$

By [13, Table 2, Symmetry *p4*], $s_2(n)$ is equal to the number of orbits of subgroups of index n of \mathbb{Z}^2 under the action of the cyclic group generated by the rotation of angle $\pi/2$.

(c) For $k = 3$, Theorem 1.5 (b) implies

$$s_3(n) = \frac{P_n(1) + P_n(j) + P_n(j^2)}{3} = \frac{\sigma(n) + \lambda(n)(j^{n-1} + j^{-(n-1)})}{3}.$$

By [13, Table 2, Symmetry *p6*],

$$s_3(n) = \frac{\sigma(n) + r''(n)/3}{3},$$

where $r''(n)$ is defined by (3.9), and $r''(n)/6$ counts those subgroups of index n of the lattice generated by 1 and j in \mathbb{C} which are fixed under the rotation of angle $\pi/3$. According to [4, § 51, p. 80, Exercise XXII.2], we have $r''(n) = 6 E_1(n; 3)$, where $E_1(n; 3)$ is the excess function (1.8). The integers $s_3(n)$ form Sequence A145394 of [12].

(d) Let $k = 4$. By Theorem 1.5 (c) we have

$$\begin{aligned} s_4(n) &= \frac{P_n(1) + P_n(i) + P_n(-1) + P_n(-i)}{4} \\ &= \frac{\sigma(n) + r(n)/4 + (-1)^{\lfloor (n-1)/2 \rfloor} r'(n)(i^{n-1} + i^{-(n-1)})/2}{4}. \end{aligned}$$

(e) For $k = 6$, by Theorem 1.5 (b) and (d) we have

$$\begin{aligned} s_6(n) &= \frac{P_n(1) + P_n(-j^2) + P_n(j) + P_n(-1) + P_n(j^2) + P_n(-j)}{6} \\ &= \begin{cases} \frac{\sigma(n) - 3r(n)/4 - \lambda(n)}{6} & \text{if } n \equiv 0, \\ \frac{\sigma(n) + 3r(n)/4 + 2\lambda(n)}{6} & \text{if } n \equiv 1, \\ \frac{\sigma(n) + 3r(n)/4 - \lambda(n)}{6} & \text{if } n \equiv 2. \end{cases} \pmod{3} \end{aligned}$$

The values of $s_2(n)$, $s_3(n)$, $s_4(n)$ and $s_6(n)$ for $n \leq 18$ are given in Table 4.

APPENDIX A. ON A RESULT BY MICHAEL SOMOS

This appendix is based on the use of multisections of power series, which is an idea due to Michael Somos (see [15]).

TABLE 4. Sections of the polynomials $P_n(q)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$s_2(n)$	1	2	2	4	4	6	4	8	7	10	6	14	7	12	12	16	10	20
$s_3(n)$	1	1	2	3	2	4	4	5	5	6	4	10	6	8	8	11	6	13
$s_4(n)$	1	1	2	2	2	3	2	4	5	5	4	7	4	6	6	8	6	10
$s_6(n)$	1	1	1	2	2	2	2	3	2	4	2	5	4	4	4	6	4	6

Replacing t by q in (3.10), we have

$$1 + \sum_{n \geq 1} a_4(n) q^n = \prod_{i \geq 1} \frac{(1 - q^i)^2}{1 + q^{2i}}.$$

We now express this generating function and the generating function of the absolute values $|a_4(n)|$ in terms of Ramanujan's φ -function

$$\varphi(q) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2 \sum_{n \geq 1} q^{n^2}.$$

The following lemma is due to Somos (see [12, A082564 and A033715]).

Lemma A.1. *We have*

$$1 + \sum_{n \geq 1} a_4(n) q^n = \varphi(-q)\varphi(-q^2) \quad \text{and} \quad 1 + \sum_{n \geq 1} |a_4(n)| q^n = \varphi(q)\varphi(q^2).$$

Proof. Replacing t by $-q$ in Gauss's identity (3.8), we deduce that $\varphi(q)$ can be expanded as the infinite products

$$(A.1) \quad \varphi(q) = \prod_{n \geq 1} \frac{1 - (-1)^n q^n}{1 + (-1)^n q^n} = \prod_{n \geq 1} \frac{(1 + q^{2n-1})(1 - q^{2n})}{(1 - q^{2n-1})(1 + q^{2n})}.$$

Using (A.1), we have

$$\begin{aligned} \varphi(-q)\varphi(-q^2) &= \prod_{n \geq 1} \frac{(1 - q^{2n-1})(1 - q^{2n})(1 - q^{4n-2})(1 - q^{4n})}{(1 + q^{2n-1})(1 + q^{2n})(1 + q^{4n-2})(1 + q^{4n})} \\ &= \prod_{n \geq 1} \frac{(1 - q^{2n})(1 - q^n)}{(1 + q^{2n})(1 + q^n)} \\ &= \prod_{n \geq 1} \frac{(1 - q^{2n})^2(1 - q^n)^2}{(1 + q^{2n})(1 + q^n)(1 - q^{2n})(1 - q^n)} \\ &= \prod_{n \geq 1} \frac{(1 - q^{2n})^2(1 - q^n)^2}{(1 - q^{4n})(1 - q^{2n})} \\ &= \prod_{n \geq 1} \frac{(1 - q^{2n})(1 - q^n)^2}{1 - q^{4n}} \\ &= \prod_{n \geq 1} \frac{(1 - q^n)^2}{1 + q^{2n}} = 1 + \sum_{n \geq 1} a_4(n) q^n. \end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi(q)\varphi(q^2) &= \prod_{n \geq 1} \frac{(1+q^{2n-1})(1-q^{2n})(1+q^{4n-2})(1-q^{4n})}{(1-q^{2n-1})(1+q^{2n})(1-q^{4n-2})(1+q^{4n})} \\
&= \prod_{n \geq 1} \frac{(1+q^n)(1-q^{2n})^2(1-q^{4n})^3}{(1-q^n)(1-q^{8n})^2} \\
&= \prod_{n \geq 1} \frac{(1-q^{2n})^3(1-q^{4n})^3}{(1-q^n)^2(1-q^{8n})^2}.
\end{aligned}$$

Introducing Ramanujan's ψ -function $\psi(q) = \sum_{n \geq 0} q^{n(n+1)/2}$, we obtain

$$\begin{aligned}
\varphi(q^4) + 2q\psi(q^8) &= \sum_{n \in \mathbb{Z}} q^{(2n)^2} + 2 \sum_{n \geq 0} q^{4n(n+1)+1} \\
&= \sum_{n \in \mathbb{Z}} q^{(2n)^2} + 2 \sum_{n \geq 0} q^{(2n+1)^2} = \varphi(q).
\end{aligned}$$

Similarly, $\varphi(q^4) - 2q\psi(q^8) = \varphi(-q)$. Therefore, $\varphi(\pm q^2) = \varphi(q^8) \pm 2q\psi(q^{16})$. Using the previous identities, we obtain

$$\begin{aligned}
\varphi(q)\varphi(q^2) &= \varphi(q^4)\varphi(q^8) + 2q\psi(q^8)\varphi(q^8) + 2q^2\psi(q^{16})\varphi(q^4) + 4q^3\psi(q^8)\varphi(q^{16})
\end{aligned}$$

and

$$\begin{aligned}
\varphi(-q)\varphi(-q^2) &= \varphi(q^4)\varphi(q^8) - 2q\psi(q^8)\varphi(q^8) - 2q^2\psi(q^{16})\varphi(q^4) + 4q^3\psi(q^8)\varphi(q^{16}).
\end{aligned}$$

Now, in view of the definitions of φ and ψ ,

$$\begin{aligned}
\varphi(q^4)\varphi(q^8) &= \sum_{n \geq 0} b_n q^{4n}, & \psi(q^8)\varphi(q^8) &= \sum_{n \geq 0} b'_n q^{4n}, \\
\psi(q^{16})\varphi(q^4) &= \sum_{n \geq 0} b''_n q^{4n}, & \psi(q^8)\varphi(q^{16}) &= \sum_{n \geq 0} b'''_n q^{4n},
\end{aligned}$$

where b_n, b'_n, b''_n, b'''_n are non-negative integers. Therefore,

$$\begin{aligned}
1 + \sum a_4(n) q^n &= \varphi(-q)\varphi(-q^2) \\
&= \sum_{n \geq 0} (b_n q^{4n} - 2b'_n q^{4n+1} - 2b''_n q^{4n+2} + 4b'''_n q^{4n+3}).
\end{aligned}$$

We deduce

$$\begin{aligned}
1 + \sum |a_4(n)| q^n &= \sum_{n \geq 0} (b_n q^{4n} + 2b'_n q^{4n+1} + 2b''_n q^{4n+2} + 4b'''_n q^{4n+3}) \\
&= \varphi(q)\varphi(q^2),
\end{aligned}$$

which completes the proof. \square

Remark A.2. Using the computations above, we can easily express the two generating functions considered in this appendix in terms of Dedekind's eta function $\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ where $q = e^{2\pi iz}$. Indeed,

$$1 + \sum_{n \geq 1} a_4(n) q^n = \frac{\eta(z)^2 \eta(2z)}{\eta(4z)} \quad \text{and} \quad 1 + \sum_{n \geq 1} |a_4(n)| q^n = \frac{\eta(2z)^3 \eta(4z)^3}{\eta(z)^2 \eta(8z)^2}.$$

ACKNOWLEDGEMENT

We are grateful to Giuseppe Ancona, Pierre Baumann, Olivier Benoist, François Bergeron, Mark Haiman, Günter Köhler, Emmanuel Letellier and Luca Migliorini for helpful discussions. We also thank Michael Somos for various comments and for the idea on which Appendix A is based.

The second-named author is grateful to the Université de Strasbourg for the invited professorship which allowed him to spend the month of June 2014 at IRMA; he was also supported by NSERC (Canada).

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