Walks on Graphs and Their Connections with Tensor Invariants and Centralizer Algebras

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Abstract

The number of walks of k steps from the node 0 to the node λ on the representation graph (McKay quiver) determined by a finite group G and a G-module V is the multiplicity of the irreducible G-module G_λ in the tensor power $\mathsf{V}^{\otimes k}$, and it is also the dimension of the irreducible module labeled by λ for the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k})$. This paper explores ways to effectively calculate that number using the character theory of G. We determine the corresponding Poincaré series. The special case $\lambda = 0$ gives the Poincaré series for the tensor invariants $\mathsf{T}(\mathsf{V})^\mathsf{G} = \bigoplus_{k=0}^{\infty}(\mathsf{V}^{\otimes k})^\mathsf{G}$. When G is abelian, we show that the exponential generating function for the number of walks is a product of generalized hyperbolic functions. Many graphs (such as circulant graphs) can be viewed as representation graphs, and the methods presented here provide efficient ways to compute the number of walks on them.

1 Introduction

Let G be a finite group, and assume that the elements λ of $\Lambda(\mathsf{G})$ index the irreducible complex representations of G, hence also the conjugacy classes of G. Let G_λ denote the irreducible G-module indexed by λ, and let χ_λ be its character. The module G_0 denotes the trivial one-dimensional G-module with $\chi_0(g) = 1$ for all $g \in \mathsf{G}$.

The *representation graph* $\mathcal{R}_\mathsf{V}(\mathsf{G})$ (also known as the *McKay quiver*) associated to a finite-dimensional G-module V over the complex field $\mathbb{C}$ has nodes corresponding to the irreducible G-modules $\{\mathsf{G}_\lambda \mid \lambda \in \Lambda(\mathsf{G})\}$. For $\nu \in \Lambda(\mathsf{G})$, there are $a_{\nu,\lambda}$ edges from ν to λ in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ if

$$\mathsf{G}_\nu \otimes \mathsf{V} = \bigoplus_{\lambda \in \Lambda(\mathsf{G})} a_{\nu,\lambda} \mathsf{G}_\lambda. \tag{1.1}$$

If $a_{\nu,\lambda} = a_{\lambda,\nu}$, then we draw $a_{\nu,\lambda}$ edges without arrows between ν and λ. The number of edges $a_{\nu,\lambda}$ from ν to λ in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is the multiplicity of G_λ as a summand of $\mathsf{G}_\nu \otimes \mathsf{V}$. Since each step on the graph is achieved by tensoring with V,

$$\begin{aligned} \mathsf{m}_k^\lambda :&= \text{number of walks of } k \text{ steps from } 0 \text{ to } \lambda \\ &= \text{multiplicity of } \mathsf{G}_\lambda \text{ in } \ \mathsf{G}_0 \otimes \mathsf{V}^{\otimes k} \cong \mathsf{V}^{\otimes k}. \end{aligned} \tag{1.2}$$

For a faithful G-module V, any irreducible G-module G_λ occurs in $\mathsf{V}^{\otimes \ell}$ for some ℓ by Burnside's theorem (in fact, for some ℓ such that $0 \le \ell \le |\mathsf{G}|$ by Brauer's strengthening of that result [CR, Thm. 9.34]). This implies that there is a directed path with ℓ steps from G_0 to G_λ in $\mathcal{R}_\mathsf{V}(\mathsf{G})$.

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The *centralizer algebra*,

$$\mathsf{Z}_k(\mathsf{G}) = \{z \in \mathsf{End}(\mathsf{V}^{\otimes k}) \mid z(g.w) = g.z(w) \ \ \forall\, g \in \mathsf{G}, w \in \mathsf{V}^{\otimes k}\}, \tag{1.3}$$

plays a critical role in studying $\mathsf{V}^{\otimes k}$, as it contains the projection maps onto the irreducible summands of $\mathsf{V}^{\otimes k}$.

Let $\Lambda_k(\mathsf{G})$ denote the subset of $\Lambda(\mathsf{G})$ corresponding to the irreducible G-modules that occur in $\mathsf{V}^{\otimes k}$ with multiplicity at least one. *Schur-Weyl duality* establishes essential connections between the representation theories of G and $\mathsf{Z}_k(\mathsf{G})$:

- $\mathsf{Z}_k(\mathsf{G})$ is a semisimple associative $\mathbb{C}$-algebra whose irreducible modules $\mathsf{Z}_k^\lambda(\mathsf{G})$ are in bijection with the elements λ of $\Lambda_k(\mathsf{G})$.

- $\dim \mathsf{Z}_k^\lambda(\mathsf{G}) = \mathsf{m}_k^\lambda$, the number of walks of k steps from the trivial G-module G_0 to G_λ on $\mathcal{R}_\mathsf{V}(\mathsf{G})$.

- If $d_\lambda = \dim \mathsf{G}_\lambda$, then the tensor space $\mathsf{V}^{\otimes k}$ has the following decompositions:

$$\begin{aligned} \mathsf{V}^{\otimes k} &\cong \bigoplus_{\lambda\in\Lambda_k(\mathsf{G})} \mathsf{m}_k^\lambda\, \mathsf{G}_\lambda && \text{as a G-module,} \\ &\cong \bigoplus_{\lambda\in\Lambda_k(\mathsf{G})} d_\lambda\, \mathsf{Z}_k^\lambda(\mathsf{G}) && \text{as a } \mathsf{Z}_k(\mathsf{G})\text{-module,} \\ &\cong \bigoplus_{\lambda\in\Lambda_k(\mathsf{G})} \Big(\mathsf{G}_\lambda \otimes \mathsf{Z}_k^\lambda(\mathsf{G})\Big) && \text{as a } (\mathsf{G}, \mathsf{Z}_k(\mathsf{G}))\text{-bimodule.} \end{aligned} \tag{1.4}$$

- $\dim \mathsf{Z}_k(\mathsf{G}) = \dim \mathsf{Z}_{2k}^0(\mathsf{G}) = \mathsf{m}_{2k}^0$ if V is isomorphic to its dual G-module.

Thus, the following numbers are the same, and our aim in this paper is to demonstrate various ways to compute these values effectively:

(1) the number of walks of k steps from 0 to $\lambda \in \Lambda(\mathsf{G})$ on $\mathcal{R}_\mathsf{V}(\mathsf{G})$,

(2) the $(0,\lambda)$-entry $(\mathsf{A}^k)_{0,\lambda}$ of A^k, where $\mathsf{A} = (a_{\nu,\lambda})$ is the adjacency matrix of $\mathcal{R}_\mathsf{V}(\mathsf{G})$,

(3) the multiplicity m_k^λ of the irreducible G-module G_λ in $\mathsf{V}^{\otimes k}$,

(4) the dimension of the irreducible module $\mathsf{Z}_k^\lambda(\mathsf{G})$ labeled by $\lambda \in \Lambda_k(\mathsf{G})$ for the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k})$,

(5) the number of paths from 0 at level 0 to λ at level k on the Bratteli diagram $\mathcal{B}_\mathsf{V}(\mathsf{G})$ (see Section 4.3 for the definition).

(∗) Moreover, when $\lambda = 0$, these values are all equal to the dimension $\dim (\mathsf{V}^{\otimes k})^\mathsf{G}$ of the space of G-invariants $(\mathsf{V}^{\otimes k})^\mathsf{G} = \{w \in \mathsf{V}^{\otimes k} \mid g.w = w \ \ \forall g \in \mathsf{G}\}$ in $\mathsf{V}^{\otimes k}$.

Many graphs can be viewed as representation graphs $\mathcal{R}_\mathsf{V}(\mathsf{G})$ for some choice of G and V, and the methods described here provide an efficient approach to computing walks on them. This is true, for example, of circulant graphs, as illustrated in Section 3.2.

We fix a set $\{\mathsf{c}_\mu\}_{\mu\in\Lambda(\mathsf{G})}$ of conjugacy class representatives of G, and let $\mathcal{C}_\mu$ denote the conjugacy class of c_μ. Then c_0 is the identity element, $|\mathcal{C}_0| = 1$, and the following result holds:

Theorem 1.5. (Theorem 2.3) *Assume* V *is a finite-dimensional module over* $\mathbb{C}$ *for the finite group* G. *The number of walks of* k*-steps from node* ν *to node* λ *on the representation graph* $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ *is*

$$(\mathsf{A}^k)_{\nu,\lambda} = |\mathsf{G}|^{-1} \sum_{\mu\in\Lambda(\mathsf{G})} |\mathcal{C}_\mu|\; \chi_\nu(\mathsf{c}_\mu)\; \chi_{\mathsf{V}}(\mathsf{c}_\mu)^k\; \overline{\chi_\lambda(\mathsf{c}_\mu)} = |\mathsf{G}|^{-1} \sum_{g\in\mathsf{G}} \chi_\nu(g)\; \chi_{\mathsf{V}}(g)^k\; \overline{\chi_\lambda(g)}. \tag{1.6}$$

Therefore, the Poincaré series for the number of walks from 0 on λ on $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ (hence also for the multiplicities of the G-module G_λ in the tensor powers $\mathsf{V}^{\otimes k}$ and for the dimensions of the centralizer algebra modules $\dim \mathsf{Z}_k^\lambda(\mathsf{G})$) is given by

$$\mathsf{P}^\lambda(t) = \sum_{k=0}^{\infty} (\mathsf{A}^k)_{0,\lambda}\, t^k = |\mathsf{G}|^{-1} \sum_{\mu\in\Lambda(\mathsf{G})} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(\mathsf{c}_\mu)}}{1-\chi_{\mathsf{V}}(\mathsf{c}_\mu)t} = |\mathsf{G}|^{-1} \sum_{g\in\mathsf{G}} \frac{\overline{\chi_\lambda(g)}}{1-\chi_{\mathsf{V}}(g)t}. \tag{1.7}$$

Since the space $\mathsf{T}(\mathsf{V})^{\mathsf{G}} = \bigoplus_{k=0}^{\infty} (\mathsf{V}^{\otimes k})^{\mathsf{G}}$ of G-invariants in $\mathsf{T}(\mathsf{V}) = \bigoplus_{k=0}^{\infty} \mathsf{V}^{\otimes k}$ is the sum of the trivial G-summands G_0 in $\mathsf{T}(\mathsf{V})$, it follows that the Poincaré series for the tensor invariants is given by

$$\mathsf{P}^0(t) = |\mathsf{G}|^{-1} \sum_{\mu\in\Lambda(\mathsf{G})} |\mathcal{C}_\mu|\; \frac{1}{1-\chi_{\mathsf{V}}(\mathsf{c}_\mu)t} = |\mathsf{G}|^{-1} \sum_{g\in\mathsf{G}} \frac{1}{1-\chi_{\mathsf{V}}(g)t}. \tag{1.8}$$

(An alternate derivation of (1.8) can be found in [DF].) The results in (1.7) and (1.8) are tensor analogues of Molien's 1897 formulas for polynomials that have played a prominent role in combinatorics, coding theory, commutative algebra, and physics (see, for example, Stanley [S1], Sloane [Sl], Murai [Mu], and Forger [Fo]). Let $\{z_1, \dots, z_n\}$ be a basis for V, and let $\mathsf{S}(\mathsf{V}) = \mathbb{C}[z_1, \dots, z_n]$ be the symmetric algebra of polynomials in the z_i. Assume $\mathsf{S}_k(\mathsf{V})$ is the space of polynomials in $\mathsf{S}(\mathsf{V})$ of total degree k, and let $\mathsf{S}_k^\lambda(\mathsf{V})$ be the sum of all the copies of G_λ in $\mathsf{S}_k(\mathsf{V})$ (the λ-isotypic component). According to [Mo], the Poincaré series are given by

$$\mathsf{P}_{\mathsf{S}}^\lambda(t) = \sum_{k=0}^{\infty} \dim \mathsf{S}_k^\lambda(\mathsf{V})\, t^k = |\mathsf{G}|^{-1} \sum_{\mu\in\Lambda(\mathsf{G})} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(\mathsf{c}_\mu)}}{\mathsf{det}_{\mathsf{V}}(\mathrm{I}-t\mathsf{c}_\mu)} = |\mathsf{G}|^{-1} \sum_{g\in\mathsf{G}} \frac{\overline{\chi_\lambda(g)}}{\mathsf{det}_{\mathsf{V}}(\mathrm{I}-tg)}, \tag{1.9}$$

$$\mathsf{P}_{\mathsf{S}}^0(t) = |\mathsf{G}|^{-1} \sum_{\mu\in\Lambda(\mathsf{G})} |\mathcal{C}_\mu| \frac{1}{\mathsf{det}_{\mathsf{V}}(\mathrm{I}-t\mathsf{c}_\mu)} = |\mathsf{G}|^{-1} \sum_{g\in\mathsf{G}} \frac{1}{\mathsf{det}_{\mathsf{V}}(\mathrm{I}-tg)}. \tag{1.10}$$

From (1.1) we see that

$$\sum_{\lambda\in\Lambda(\mathsf{G})} a_{\nu,\lambda}\, \chi_\lambda(\mathsf{c}_\mu) = \chi_{\mathsf{V}}(\mathsf{c}_\mu)\chi_\nu(\mathsf{c}_\mu), \tag{1.11}$$

which implies that the eigenvalues of the adjacency matrix A of $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ are the character values $\chi_{\mathsf{V}}(\mathsf{c}_\mu)$ as μ ranges over the elements of $\Lambda(\mathsf{G})$, and the eigenvector corresponding to $\chi_{\mathsf{V}}(\mathsf{c}_\mu)$ is the column vector with entries $\chi_\lambda(\mathsf{c}_\mu)$ for $\lambda \in \Lambda(\mathsf{G})$. The matrix of these eigenvectors is exactly the character table of G. (Compare [St, Sec. 1] which considers the matrix $\mathsf{dI} - \mathsf{A}$, where $\mathsf{d} = \chi_{\mathsf{V}}(\mathsf{c}_0) = \dim \mathsf{V}$.)

Theorem 2.1 of [B2] shows that the Poincaré series $\mathsf{P}^\lambda(t)$ can be expressed as a quotient of two determinants under the assumption that the module V is isomorphic to its dual G-module. But that assumption is unnecessary if the matrix A is replaced by its transpose in computing the determinant in the numerator, as in the statement below. A proof of this result can be deduced from the proposition in Appendix I, which holds for walks on arbitrary finite directed graphs. In considering the rows and columns of the adjacency matrix A in the next theorem, we assume that the elements of $\Lambda(\mathsf{G})$ have been ordered in some fashion and that 0 is always the first element relative to that ordering.

Theorem 1.12. *Let G be a finite group with irreducible modules G_λ, $\lambda \in \Lambda(\mathsf{G})$, over $\mathbb{C}$, and let V be a finite-dimensional G-module. Let $\mathsf{A} = \left(a_{\nu,\lambda}\right)$ be the adjacency matrix of the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$, and let M^λ be the matrix $\mathrm{I} - t\mathsf{A}^\mathsf{T}$ with the column indexed by λ replaced by $\delta_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then*

$$\mathsf{P}^\lambda(t) = \frac{\det(\mathsf{M}^\lambda)}{\det(\mathrm{I} - t\mathsf{A})} = \frac{\det(\mathsf{M}^\lambda)}{\prod_{\mu \in \Lambda(\mathsf{G})} \left(1 - \chi_\mathsf{V}(\mathsf{c}_\mu)t\right)}. \tag{1.13}$$

In [Mc], John McKay described a remarkable correspondence between the finite subgroups G of the special unitary group SU_2 and the simply laced affine Dynkin diagrams. Almost a century earlier, Felix Klein had determined that a finite subgroup of SU_2 must be isomorphic to one of the following: (a) a cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of order n, (b) a binary dihedral group $\mathbf{D}_n$ of order $4n$, or (c) one of the 3 exceptional groups: the binary tetrahedral group $\mathbf{T}$ of order 24, the binary octahedral group $\mathbf{O}$ of order 48, or the binary icosahedral group $\mathbf{I}$ of order 120. McKay's observation was that the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ for $\mathsf{G} = \mathbb{Z}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$ relative to its defining representation $\mathsf{V} = \mathbb{C}^2$ corresponds exactly to the affine Dynkin diagram $\hat{\mathsf{A}}_{n-1}$, $\hat{\mathsf{D}}_{n+2}$, $\hat{\mathsf{E}}_6$, $\hat{\mathsf{E}}_7$, $\hat{\mathsf{E}}_8$, respectively, where the node labeled by 0 corresponding to the trivial G-module is the affine node. The matrix $\mathsf{C} = 2\,\mathrm{I} - \mathsf{A}$, where A is adjacency matrix of $\mathcal{R}_\mathsf{V}(\mathsf{G})$, is the associated affine Cartan matrix. In this case, the Poincaré series for the tensor invariants in Theorem 1.12 specializes to the following:

Theorem 1.14. [B2, Thm. 3.1] *Let G be a finite subgroup of SU_2 and $\mathsf{V} = \mathbb{C}^2$. Then the Poincaré series for the G-invariants $\mathsf{T}(\mathsf{V})^\mathsf{G}$ in $\mathsf{T}(\mathsf{V}) = \bigoplus_{k=0}^\infty \mathsf{V}^{\otimes k}$ is*

$$\mathsf{P}^0(t) = \frac{\det\left(\mathrm{I} - t\mathring{\mathsf{A}}\right)}{\det\left(\mathrm{I} - t\mathsf{A}\right)} = \frac{\det\left(\mathrm{I} - t\mathring{\mathsf{A}}\right)}{\prod_{\mu \in \Lambda(\mathsf{G})} \left(1 - \chi_\mathsf{V}(\mathsf{c}_\mu)t\right)}, \tag{1.15}$$

where A is the adjacency matrix of the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ (i.e. the affine Dynkin diagram corresponding to G), and $\mathring{\mathsf{A}}$ is the adjacency matrix of the finite Dynkin diagram obtained by removing the affine node.

As shown in [B2, Sec. 3], the eigenvalues of $\mathring{\mathsf{A}}$ and A are related to the exponents of the finite and affine root systems respectively, and the determinants in this formula can be expressed as Chebyshev polynomials of the second kind. Results in a similar vein for the doubly laced root systems can be found in [B1].

We illustrate the results in our paper by computing many examples, as described below for various choices of G and V. When G is abelian, the conjugacy classes consist of a single element of G, so we will always identify $\Lambda(\mathsf{G})$ with G when G is abelian.

1. $\mathsf{G} = \mathbb{Z}_r$ *(a cyclic group of order r) and* $\mathsf{V} = \mathsf{G}_1 \oplus \mathsf{G}_{r-1}$:
   In Section 3.1, we obtain a formula for the number of walks of k steps on a circular graph with r nodes.

2. $\mathsf{G} = \mathbb{Z}_{13}$ *and* $\mathsf{V} = \bigoplus_j \mathsf{G}_j$, *where* $j = 1, 3, 4, 9, 10, 12$*; or* $\mathsf{G} = \mathbb{Z}_{2m}$ *and* $\mathsf{V} = \bigoplus_j \mathsf{G}_j$, *where* $j = 1, m, 2m-1$:
   As shown in Section 3.2, the first example leads to an expression for the number of walks on the Paley graph $\mathcal{P}_{13}$ of order 13. Paley graphs arise in studying quadratic residues in finite fields, and the key fact germane to the results here is that Paley graphs are circulant graphs (their adjacency matrices are circulant matrices). The same method used for $\mathcal{P}_{13}$ can be applied to compute walks on any circulant graph. We demonstrate this further with the second example which yields a formula for the number of walks on the Möbius ladder graph of order $2m$.

In Section 3.3, we adopt a different approach and determine closed-form formulas for the number of walks of k steps from 0 to any node on a Paley (di)graph $\mathcal{P}_p$ of order p for an arbitrary odd prime p using Theorem 2.3 and number-theoretic properties of Gauss sums. When $p \equiv 1 \bmod 4$, $\mathcal{P}_p$ is an undirected graph, and when $p \equiv 3 \bmod 4$, $\mathcal{P}_p$ is a directed graph (digraph).

3. $\mathsf{G} = \mathrm{S}_n$, *the symmetric group on* n *letters, and* V *is its* n*-dimensional permutation module*:
Our results here lead to a proof of the relation

$$\operatorname{dim} \mathsf{Z}_k(\mathrm{S}_n) = (n\,!)^{-1} \sum_{\sigma \in \mathrm{S}_n} \mathsf{F}(\sigma)^{2k} = \sum_{\ell=0}^{n} \left\{ {2k \atop \ell} \right\} \tag{1.16}$$

between the number of fixed points $\mathsf{F}(\sigma)$ of permutations σ, and the Stirling numbers $\left\{ {2k \atop \ell} \right\}$ of the second kind, which count the number of ways to partition a set of $2k$ objects into ℓ nonempty disjoint parts. (Note that $\left\{ {0 \atop \ell} \right\} = 0$ unless $\ell = 0$, in which case it is 1.) The relation in (1.16) was proven by Farina and Halverson in [FaH] under the additional assumption that $n \geq 2k$ using the characters of the partition algebra $\mathsf{P}_k(n)$, which is the centralizer algebra $\mathsf{Z}_k(\mathrm{S}_n) = \mathsf{End}_{\mathrm{S}_n}(\mathsf{V}^{\otimes k})$ when $n \geq 2k$.

The partitions λ of n index the irreducible S_n-modules. Using [BHH, Thm. 5.5(a)], we determine that

$$\operatorname{dim} \mathsf{Z}_k^{\lambda}(\mathrm{S}_n) = (n!)^{-1} \sum_{\sigma \in \mathrm{S}_n} \mathsf{F}(\sigma)^k \, \overline{\chi_\lambda(\sigma)} = \sum_{\ell=0}^{n} \left\{ {k \atop \ell} \right\} \mathsf{K}_{\lambda,(n-\ell,1^\ell)}, \tag{1.17}$$

where $\mathsf{K}_{\lambda,(n-\ell,1^\ell)}$ is the *Kostka number*, and $(n-\ell, 1^\ell)$ is the partition of n with one part of size $n-\ell$ and ℓ parts of size 1. Equation (1.16) is a special case of (1.17), since $\operatorname{dim} \mathsf{Z}_k(\mathrm{S}_n) = \operatorname{dim} \mathsf{Z}_{2k}^{\mathsf{0}}(\mathrm{S}_n)$, and the relevant Kostka numbers are all 1 in this case. It follows from (1.17) with $\lambda = \mathsf{0}$ that the dimension of the S_n-invariants in $\mathsf{V}^{\otimes k}$ is given by

$$\operatorname{dim} (\mathsf{V}^{\otimes k})^{\mathrm{S}_n} = (n!)^{-1} \sum_{\sigma \in \mathrm{S}_n} \mathsf{F}(\sigma)^k = \sum_{\ell=0}^{n} \left\{ {k \atop \ell} \right\}, \tag{1.18}$$

and the Poincaré series for the tensor invariants is given by

$$\mathsf{P}^{\mathsf{0}}(t) = \sum_{k=0}^{\infty} \operatorname{dim} (\mathsf{V}^{\otimes k})^{\mathrm{S}_n} t^k = (n!)^{-1} \sum_{\sigma \in \mathrm{S}_n} \frac{1}{1 - \mathsf{F}(\sigma) t}. \tag{1.19}$$

It would be nice to have a bijective combinatorial proof of the identity in (1.17).

4. $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n$ *(the wreath product) and* V *is its* n*-dimensional module over* $\mathbb{C}$ *on which* G *acts by* $n \times n$ *monomial matrices with entries of the form* ω^j *for* $j = 0, 1, \ldots, r-1$*, where* ω *is a primitive* r*th root of unity for* $r \geq 2$:
In Theorem 4.9, we show that

$$\operatorname{dim} (\mathsf{V}^{\otimes k})^{\mathsf{G}} = \frac{1}{r^n n\,!} \sum_{m=1}^{n} r^m \mathsf{F}_n(m)^k \left(\sum_{\ell_1, \ell_2, \ldots, \ell_m} \binom{k}{\ell_1, \ell_2, \ldots, \ell_m} \right),$$

where the inner sum of multinomial coefficients is over all $0 \leq \ell_1, \ell_2, \ldots, \ell_m \leq k$ such that $\ell_1 + \ell_2 + \cdots + \ell_m = k$ and $\ell_1 \equiv \ell_2 \equiv \cdots \equiv \ell_m \equiv 0 \bmod r$, and $\mathsf{F}_n(m) = \dfrac{n!}{m!} \displaystyle\sum_{j=0}^{n-m} \frac{(-1)^j}{j\,!}$ is the number of permutations in S_n with exactly m fixed points. Equation (4.18) gives a second expression for the dimension of the invariants using the fact that the irreducible modules for $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n$ are indexed by r-tuples $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r)})$ of partitions $\alpha^{(i)}$ with $\sum_{i=1}^{r} |\alpha^{(i)}| = n$:

$$\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \sum_{\underline{\alpha} \in \Lambda(\mathsf{G})} \frac{\left(\sum_{i=1}^{r} \mathsf{F}(\alpha^{(i)})\,\omega^{i-1}\right)^{k}}{r^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{n} j^{\mathsf{p}_j(\underline{\alpha})} \left(\prod_{i=1}^{r} \mathsf{p}_j(\alpha^{(i)})!\right)} \quad \text{for } \mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n. \tag{1.20}$$

In this formula $\mathsf{p}_j(\alpha^{(i)})$ is the number of parts of $\alpha^{(i)}$ of size j; $\mathsf{p}_j(\underline{\alpha}) = \sum_{i=1}^{r} \mathsf{p}_j(\alpha^{(i)})$; $\mathsf{p}(\underline{\alpha}) = \sum_{j=1}^{n} \mathsf{p}_j(\underline{\alpha})$; and $\mathsf{F}(\alpha^{(i)}) = \mathsf{p}_1(\alpha^{(i)})$, the number of parts of $\alpha^{(i)}$ of size 1, (the number of fixed points of a permutation with cycle type $\alpha^{(i)}$). It is desirable to have a direct proof of the equivalence of these two formulas for $\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}}$. When $r = 2$, the group $\mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_n$ is the Weyl group corresponding to the root systems B_n and C_n, and the dimension of the tensor invariants can be obtained by specializations of these formulas (see (4.19)). Some particular cases are worked out explicitly in Sections 4.7 and 4.8.

5. G *is the general linear group* $\mathrm{GL}_2(\mathbb{F}_q)$ *of invertible* 2×2 *matrices over a finite field* $\mathbb{F}_q$ *of q elements, where* q *is odd, or* G *is the special linear subgroup* $\mathrm{SL}_2(\mathbb{F}_q)$ *of matrices of determinant 1. The* G*-module* V *is the* $(q+1)$*-dimensional module over* $\mathbb{C}$ *obtained by inducing the trivial module for the Borel subgroup* B *of upper-triangular matrices in* G:
The module V decomposes as a G-module, $\mathsf{V} = \mathsf{G}_0 \oplus \mathsf{V}_q$, where G_0 is the trivial G-module and V_q is the q-dimensional irreducible Steinberg module. In Theorems 5.3 and 5.11, we derive formulas for the dimension of the spaces $(\mathsf{V}^{\otimes k})^{\mathsf{G}}$ and $(\mathsf{V}_q^{\otimes k})^{\mathsf{G}}$ of G-invariants and determine the Poincaré series for the tensor invariants $\mathsf{T}(\mathsf{V})^{\mathsf{G}}$ and $\mathsf{T}(\mathsf{V}_q)^{\mathsf{G}}$.

6. G *is an arbitrary finite abelian group, say* $\mathsf{G} = \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_n}$, *and* $\mathsf{V} = \mathsf{G}_{\varepsilon_1} \oplus \cdots \oplus \mathsf{G}_{\varepsilon_n}$, *where* ε_j *is the element of* G *with 1 as its* j*th component and* 0 *as its other components*:
In Section 6, we show that the *exponential* generating function for the number of walks on the representation graph (equivalently, for the multiplicities of the irreducible G-modules in $\mathsf{V}^{\otimes k}$; also, for the dimensions of the irreducible modules $\mathsf{Z}_k^{\lambda}(\mathsf{G})$ for the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$), is a product of generalized hyperbolic functions. We deduce that the number of walks can be expressed as a sum of multinomial coefficients. When $r_1 = r_2 = \cdots = r_n = 2$, we obtain a formula for the number of walks on a hypercube of dimension n and the expression for the exponential generating function for the number of walks as a product of hyperbolic sines and cosines that was given in [BM, Cor. 4.29]. In Sections 6.2 and 6.3, we exhibit a basis for $\mathsf{Z}_k(\mathsf{G})$ and view $\mathsf{Z}_k(\mathsf{G})$ as a diagram algebra by giving a diagrammatic realization of the basis elements.

2 Walks and Poincaré series

2.1 Expressions for counting walks, multiplicities, and centralizer algebra dimensions

There is a Hermitian inner product on the class functions of a finite group G defined by

$$\langle \phi, \psi \rangle = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \phi(g)\overline{\psi(g)} = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu|\, \phi(\mathsf{c}_\mu)\overline{\psi(\mathsf{c}_\mu)},$$

where "−" denotes the complex conjugate. The irreducible characters χ_λ for $\lambda \in \Lambda(\mathsf{G})$ satisfy the well-known orthogonality relations relative to this inner product (see for example, [FuH, (2.10) and Ex. 2.21]):

$$\langle \chi_\nu, \chi_\lambda \rangle = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \chi_\nu(g)\overline{\chi_\lambda(g)} = \delta_{\nu,\lambda}, \tag{2.1}$$

$$|\mathsf{G}|^{-1} \sum_{\lambda \in \Lambda(\mathsf{G})} \chi_\lambda(c_\mu)\chi_\lambda(c_\nu) = \begin{cases} |\mathcal{C}_\mu| & \text{if } \mu = \nu, \\ 0 & \text{if } \mu \neq \nu. \end{cases} \tag{2.2}$$

Therefore, if U is a G-module over $\mathbb{C}$ with character χ_{U}, then (2.1) implies that

$$\langle \chi_{\mathsf{U}}, \chi_\lambda \rangle = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \chi_{\mathsf{U}}(g)\overline{\chi_\lambda(g)} = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu|\, \chi_{\mathsf{U}}(\mathsf{c}_\mu)\overline{\chi_\lambda(\mathsf{c}_\mu)}$$

is the multiplicity of G_λ as a summand of U. Applying this to the G-module $\mathsf{G}_\nu \otimes \mathsf{V}^{\otimes k}$, which has character $\chi_\nu \chi_{\mathsf{V}}^k$, gives the following result.

Theorem 2.3. *Assume* V *is finite-dimensional module for the finite group* G. *The number of walks of* k*-steps from node* ν *to node* λ *on the representation graph* $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ *(equivalently, the multiplicity of* G_λ *in* $\mathsf{G}_\nu \otimes \mathsf{V}^{\otimes k}$*) is equal to*

$$(\mathsf{A}^k)_{\nu,\lambda} = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu|\, \chi_\nu(\mathsf{c}_\mu)\, \chi_{\mathsf{V}}(\mathsf{c}_\mu)^k\, \overline{\chi_\lambda(\mathsf{c}_\mu)}. \tag{2.4}$$

Corollary 2.5. *Under the hypotheses of Theorem 2.3, the dimension of the irreducible module* $\mathsf{Z}_k^\lambda(\mathsf{G})$ *for the centralizer algebra* $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$ *is given by*

$$\dim \mathsf{Z}_k^\lambda(\mathsf{G}) = (\mathsf{A}^k)_{0,\lambda} = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu|\, \chi_{\mathsf{V}}(\mathsf{c}_\mu)^k = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \chi_{\mathsf{V}}(g)^k\, \overline{\chi_\lambda(g)}, \tag{2.6}$$

and when V *is a self-dual* G*-module,*

$$\dim \mathsf{Z}_k(\mathsf{G}) = \dim \mathsf{Z}_{2k}^0(\mathsf{G}) = (\mathsf{A}^{2k})_{0,0} = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu|\, \chi_{\mathsf{V}}(\mathsf{c}_\mu)^{2k} = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \chi_{\mathsf{V}}(g)^{2k}. \tag{2.7}$$

2.2 Poincaré series

It is a consequence of the results in (2.6) and (2.7) that the Poincaré series

$$\mathsf{P}^\lambda(t) := \sum_{k=0}^{\infty} (\mathsf{A}^k)_{0,\lambda}\, t^k = \sum_{k=0}^{\infty} \mathsf{m}_k^\lambda\, t^k = \sum_{k=0}^{\infty} \dim \mathsf{Z}_k^\lambda(\mathsf{G})\, t^k \tag{2.8}$$

has the following expression

$$\mathsf{P}^\lambda(t) = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(\mathsf{c}_\mu)}}{1 - \chi_{\mathsf{V}}(\mathsf{c}_\mu)t} = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \frac{\overline{\chi_\lambda(g)}}{1 - \chi_{\mathsf{V}}(g)t} \tag{2.9}$$

$$= \frac{\det(\mathsf{M}^\lambda)}{\det(\mathrm{I} - t\mathsf{A})} = \frac{\det(\mathsf{M}^\lambda)}{\prod_{\mu \in \Lambda(\mathsf{G})} (1 - \chi_{\mathsf{V}}(\mathsf{c}_\mu)t)}, \tag{2.10}$$

where M^λ is the matrix $\mathrm{I} - t\mathsf{A}^{\mathsf{T}}$ with the column indexed by λ replaced by $\delta_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ as in Theorem 1.12. Then a special case of this formula is the Poincaré series for the tensor invariants $\mathsf{T}(\mathsf{V})^{\mathsf{G}}$ in $\mathsf{T}(\mathsf{V}) = \bigoplus_{k=0}^{\infty} \mathsf{V}^{\otimes k}$:

$$\begin{aligned} \mathsf{P}^0(t) &= |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu| \frac{1}{1 - \chi_{\mathsf{V}}(\mathsf{c}_\mu) t} = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \frac{1}{1 - \chi_{\mathsf{V}}(g) t} \\ &= \frac{\det(\mathsf{M}^0)}{\det(\mathrm{I} - t\mathsf{A})} = \frac{\det(\mathsf{M}^0)}{\prod_{\mu \in \Lambda(\mathsf{G})} (1 - \chi_{\mathsf{V}}(\mathsf{c}_\mu) t)}. \end{aligned} \tag{2.11}$$

These are analogs of Molien's formulas

$$\mathsf{P}_{\mathsf{S}}^\lambda(t) = \sum_{k=0}^{\infty} \dim \mathsf{S}_k^\lambda(\mathsf{V})\, t^k = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu| \frac{\overline{\chi_\lambda(\mathsf{c}_\mu)}}{\det_{\mathsf{V}}(\mathrm{I} - t\mathsf{c}_\mu)}, \tag{2.12}$$

$$\mathsf{P}_{\mathsf{S}}^0(t) = |\mathsf{G}|^{-1} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu| \frac{1}{\det_{\mathsf{V}}(\mathrm{I} - t\mathsf{c}_\mu)} = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \frac{1}{\det_{\mathsf{V}}(\mathrm{I} - tg)}. \tag{2.13}$$

for multiplicities of G-modules and invariants in polynomials, as described in the Introduction.

3 Cyclic examples

3.1 $\mathsf{G} = \mathbb{Z}_r$

When $\mathsf{G} = \mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$, we identify the elements of $\Lambda(\mathsf{G})$ with the elements $\{0, 1, \dots, r-1\}$ of $\mathbb{Z}_r$. Then for $a \in \mathsf{G}$, the character χ_a of G_a is given by $\chi_a(b) = \omega^{ab}$ for $a, b \in \mathsf{G}$, where $\omega = \mathrm{e}^{2\pi i/r}$. We assume $\mathsf{V} = \mathsf{G}_1 \oplus \mathsf{G}_{r-1}$. The representation graph $\mathcal{R}_{\mathsf{V}}(\mathbb{Z}_r)$ is a circular graph with r nodes, and a step from a node on the graph amounts to moving one step to the left or to the right. Then for $b \in \mathsf{G}$, we have $\chi_{\mathsf{V}}(b) = \chi_1(b) + \chi_{r-1}(b) = \omega^b + \omega^{-b} = 2\cos(2\pi i b/r)$. Therefore

$$\chi_{\mathsf{V}^{\otimes k}}(b) = \chi_{\mathsf{V}}(b)^k = (\omega^b + \omega^{-b})^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^{(k-\ell)b} \omega^{-\ell b} = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^{(k-2\ell)b}.$$

Now using the fact that

$$\sum_{b=0}^{r-1} \omega^{mb} = \begin{cases} r & \text{if } m \equiv 0 \bmod r, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

and Theorem 2.3, we have the following expression for the number of walks of k steps from a to c on $\mathcal{R}_{\mathsf{V}}(\mathbb{Z}_r)$:

$$\begin{aligned} (\mathsf{A}^k)_{a,c} &= r^{-1} \sum_{b \in \mathbb{Z}_r} \chi_a(b) \chi_{\mathsf{V}}(b)^k \overline{\chi_c(b)} = r^{-1} \sum_{b=0}^{r-1} \omega^{(a-c)b} \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^{(k-2\ell)b} \\ &= r^{-1} \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{b=0}^{r-1} \omega^{(k-2\ell+a-c)b} = \sum_{\substack{0 \le \ell \le k \\ k-2\ell \equiv c-a \bmod r}} \binom{k}{\ell}. \end{aligned} \tag{3.2}$$

Therefore, the dimension of the irreducible module $\mathsf{Z}_k^c(\mathbb{Z}_r)$ for the centralizer algebra $\mathsf{Z}_k(\mathbb{Z}_r) = \mathsf{End}_{\mathbb{Z}_r}(\mathsf{V}^{\otimes k})$ is

$$\dim \mathsf{Z}_k^c(\mathbb{Z}_r) = (\mathsf{A}^k)_{0,c} = \sum_{\substack{0\le \ell\le k \\ k-2\ell \,\equiv\, c \bmod r}} \binom{k}{\ell}.$$

In particular, in order for the irreducible $\mathbb{Z}_r$-module labeled by c to occur in $\mathsf{V}^{\otimes k}$ with multiplicity at least one, equivalently, in order for $\dim \mathsf{Z}_k^c(\mathbb{Z}_r)$ to be nonzero, it must be that $k - c \equiv 2\ell \bmod r$ for some ℓ. Let ℓ_c be the least nonnegative integer with that property. Then

$$\dim \mathsf{Z}_k^c(\mathbb{Z}_r) = \sum_{\substack{0\le \ell\le k \\ \ell \,\equiv\, \ell_c \bmod \tilde{r}}} \binom{k}{\ell},$$

where $\tilde{r} = r$ if r is odd, and $\tilde{r} = r/2$ if r is even. Since the module V is self dual,

$$\dim \mathsf{Z}_k(\mathbb{Z}_r) = \dim \mathsf{Z}_{2k}^0(\mathbb{Z}_r) = \sum_{\substack{0\le \ell\le k \\ k-\ell\equiv 0 \bmod \tilde{r}}} \binom{2k}{\ell}.$$

(Compare [BBH, Thm. 2.17(i) and Thm. 2.8(d)].) These formulas can be interpreted as computing Pascal's triangle on a cylinder of diameter $\tilde{r}$. (See [BBH, Sec. 4.2] for more details.)

Here is a specific example to demonstrate the above results.

Example 3.3. When $k = 6$ and $r = 10$,

$$\begin{aligned}\dim \mathsf{Z}_6(\mathbb{Z}_{10}) &= \sum_{\substack{0\le \ell\le 12 \\ 6-\ell \,\equiv\, 0 \bmod 5}} \binom{12}{\ell} \\ &= \binom{12}{1} + \binom{12}{6} + \binom{12}{11} = 12 + 924 + 12 = 948.\end{aligned}$$

This can be seen from the Bratteli diagram for the cyclic group of order 10 (which can be found in the Appendix of this paper and in [BBH, Sec. 4.2]). The right-hand column there displays the dimension of the centralizer algebra. Since the dimension of the irreducible module $\mathsf{Z}_6^8(\mathbb{Z}_{10})$ is the number of walks of 6 steps from 0 to 8 on the representation graph for $\mathsf{G} = \mathbb{Z}_{10}$ and $\mathsf{V} = \mathsf{G}_1 \oplus \mathsf{G}_9$, we have from (3.2),

$$\dim \mathsf{Z}_6^8(\mathbb{Z}_{10}) = \sum_{\substack{0\le \ell\le k \\ 6-2\ell \,\equiv\, 8 \bmod 5}} \binom{6}{\ell} = \binom{6}{4} = 15.$$

This is the subscript on the node labeled 8 on level 6 of the Bratteli diagram for the cyclic group of order 10.

3.2 Circulant graphs

The Paley graphs are a family of graphs constructed from quadratic residues in finite fields. The Paley graph $\mathcal{P}_{13}$ of order 13 is pictured below. Every Paley graph is a circulant graph, which is equivalent to saying its adjacency matrix is a circulant matrix. There are many different characterizations of circulant graphs and circulant matrices. (The article by Kra and Simanca [KS] nicely summarizes many of them.) Most relevant here is the fact that a graph is circulant if and only if its automorphism group contains a cyclic group acting transitively on its nodes. For $\mathcal{P}_{13}$ this group is $\mathbb{Z}_{13}$. In the notation of the previous example, we can take the module V so that $\chi_\mathsf{V} = \sum_j \chi_j$, where the sum is over $j = 1, 3, 4, 9, 10, 12$. Then a step on $\mathcal{P}_{13}$ corresponds

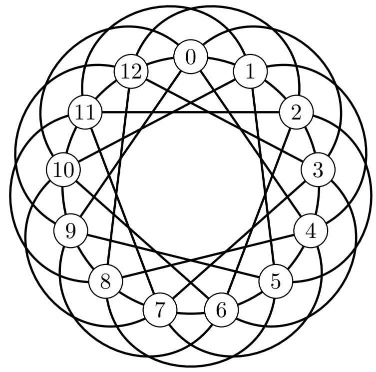


Figure 1: Paley graph $\mathcal{P}_{13}$

to tensoring with this particular choice of $\mathbb{Z}_{13}$-module V. Using that fact and Theorem 2.3, we have the following (where ω is a primitive 13th root of 1):

Corollary 3.4. *The number of walks of k steps from $\mathsf{0}$ to $c \in \{0, 1, \ldots, 12\}$ on the Paley graph $\mathcal{P}_{13}$ is*

$$\begin{aligned}\left(\mathsf{A}^k\right)_{\mathsf{0},c} &= (13)^{-1} \sum_{\substack{0 \leq \ell_1, \ell_2, \ldots, \ell_6 \leq k \\ \ell_1 + \cdots + \ell_6 = k}} \binom{k}{\ell_1, \ell_2, \ldots, \ell_6} \left(\sum_{b=0}^{12} \omega^{(\ell_1 + 3\ell_2 + 4\ell_3 + 9\ell_4 + 10\ell_5 + 12\ell_6 - c)b} \right) \\ &= \sum_{\substack{0 \leq \ell_1, \ell_2, \ldots \ell_6 \leq k,\ \ell_1 + \cdots + \ell_6 = k \\ \ell_1 + 3\ell_2 + \cdots + 12\ell_6 \,\equiv\, c \bmod 13}} \binom{k}{\ell_1, \ell_2, \ldots, \ell_6}.\end{aligned}$$

Walks on any circulant graph can be enumerated by exactly the same type of argument.

To illustrate this point with one further family of graphs, we consider the Möbius ladder graph M_{2m} with $2m$ nodes, which is obtained from a prism graph of order $2m$ by applying a twist, as pictured below for M_{16}. These are toroidal graphs that embed without crossings on a torus or projective plane. Since these graphs are known to be circulant, we can take $\mathsf{G} = \mathbb{Z}_{2m}$ and assume the G-module V is chosen so that $\chi_{\mathsf{V}} = \chi_1 + \chi_m + \chi_{2m-1}$. The next corollary follows readily from Theorem 2.3 and (3.1) with $\omega = \mathrm{e}^{2\pi i/2m}$.

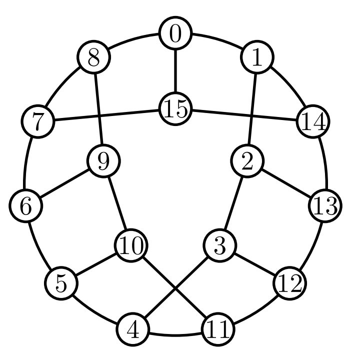


Figure 2: Möbius ladder graph M_{16}

Corollary 3.5. *The number of walks of k steps from $\mathsf{0}$ to $c \in \{0, 1, \ldots, 2m-1\}$ on the Möbius ladder* *graph* M_{2m} *is*

$$\left(\mathsf{A}^k\right)_{\mathsf{0},c} = (2m)^{-1} \sum_{\substack{0\le \ell_1,\ell_2,\ell_3\le k\\ \ell_1+\ell_2+\ell_3=k}} \binom{k}{\ell_1,\ell_2,\ell_3} \sum_{b=0}^{2m-1} \omega^{(\ell_1+m\ell_2+(2m-1)\ell_3-c)b} = \sum_{\substack{0\le \ell_1,\ell_2,\ell_3\le k,\ \ell_1+\ell_2+\ell_3=k\\ \ell_1+m\ell_2+(2m-1)\ell_3 \,\equiv\, c \,\mathsf{mod}\, 2m}} \binom{k}{\ell_1,\ell_2,\ell_3}.$$

3.3 Paley (di)graphs $\mathcal{P}_p$ of order p an odd prime

Suppose p is an odd prime and $\omega = \mathrm{e}^{2\pi i/p}$. The nodes in the Paley (di)graph $\mathcal{P}_p$ are labeled by the elements in $\{0, 1, \ldots, p-1\}$, and the ones connected to $\mathsf{0}$ are labeled by the distinct square values x^2 in $\mathbb{Z}_p^\times = \{1, 2, \ldots, p-1\}$ (the quadratic residues modulo p). For $p = 13$, these are the values $x^2 = 1, 3, 4, 9, 10, 12$. When $p \equiv 1 \,\mathsf{mod}\, 4$, $\mathcal{P}_p$ is an undirected graph, and for $p \equiv 3 \,\mathsf{mod}\, 4$ it is a digraph, as illustrated below for $p = 7$.

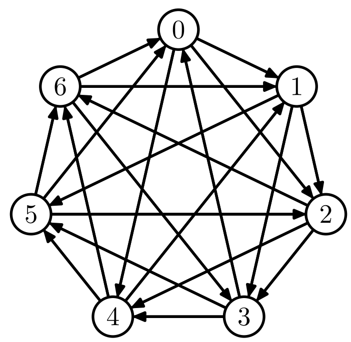


Figure 3: Paley digraph $\mathcal{P}_7$

We take V so that $\mathcal{R}_{\mathsf{V}}(\mathbb{Z}_p)$ is $\mathcal{P}_p$. Then

$$\chi_{\mathsf{V}}(b) = f(b) := \sum_{x^2 \in \mathbb{Z}_p^\times} \omega^{bx^2},$$

and we know from (2.6) that the number of walks of k steps from $\mathsf{0}$ to c on the graph $\mathcal{P}_p$ is given by

$$(\mathsf{A}^k)_{\mathsf{0},c} = \frac{1}{p} \sum_{b\in\mathbb{Z}_p} \chi_{\mathsf{V}}(b)^k \overline{\chi_c(b)} = \frac{1}{p} \sum_{b=0}^{p-1} f(b)^k \omega^{-cb} \tag{3.6}$$

We evaluate this expression using well-known facts about Gauss sums, which can be found for example in [IR, Chap. 8]. Suppose

$$\xi = \begin{cases} 1 & \text{if } \ p \equiv 1 \ \mathsf{mod}\, 4, \\ i = \sqrt{-1} & \text{if } \ p \equiv 3 \ \mathsf{mod}\, 4. \end{cases} \tag{3.7}$$

The Gauss sum $g(b) = \sum_{x=0}^{p-1} \omega^{bx^2}$ equals p when $b = 0$, and for $b \in \mathbb{Z}_p^\times$

$$g(b) = \left(\frac{b}{p}\right) g(1) = \begin{cases} \ \xi\sqrt{p} & \text{if } b \text{ is a quadratic residue modulo } p, \\ -\xi\sqrt{p} & \text{if } b \text{ is a quadratic nonresidue modulo } p, \end{cases}$$

where $\left(\frac{b}{p}\right)$ is the Legendre symbol, which is 1 if b is a quadratic residue and -1 otherwise. Since the number of quadratic residues equals the number of quadratic nonresidues, it follows that

$$f(b) = \frac{1}{2}\left(g(b) - 1\right) = \begin{cases} \ \frac{1}{2}(\xi\sqrt{p} - 1) & \text{if } b \text{ is a nonzero quadratic residue modulo } p, \\ -\frac{1}{2}(\xi\sqrt{p} + 1) & \text{if } b \text{ is a quadratic nonresidue modulo } p, \\ \ \frac{1}{2}(p-1) & \text{if } b = \mathsf{0}. \end{cases}$$

Our aim in this section is to prove

Theorem 3.8. *Assume $\mathcal{P}_p$ is the Paley (di)graph of order p a prime and ξ is as in* (3.7). *Then the number of walks of k steps from 0 to c on $\mathcal{P}_p$ is given by one of the following:*

(i) *If c is a nonzero quadratic residue, then*

$$(\mathsf{A}^k)_{0,c} = \begin{cases} \frac{1}{2^{k+1}p}\left(2\,(p-1)^k + \left(\sqrt{p}-1\right)^{k+1} + (-1)^{k+1}\left(\sqrt{p}+1\right)^{k+1}\right) & \text{if } p \equiv 1 \bmod 4, \\ \frac{1}{2^{k+1}p}\left(2\,(p-1)^k + (p+1)\left(i\sqrt{p}-1\right)^{k-1} + (-1)^k(p+1)\left(i\sqrt{p}+1\right)^{k-1}\right) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(ii) *If c is a quadratic nonresidue, then*

$$(\mathsf{A}^k)_{0,c} = \begin{cases} \frac{p-1}{2^{k+1}p}\left(2\,(p-1)^{k-1} + \left(\sqrt{p}-1\right)^{k-1} + (-1)^k\left(\sqrt{p}+1\right)^{k-1}\right) & \text{if } p \equiv 1 \bmod 4 \\ \frac{1}{2^{k+1}p}\left(2\,(p-1)^k - \left(i\sqrt{p}+1\right)^{k+1} + (-1)^{k+1}\left(i\sqrt{p}+1\right)^{k+1}\right) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(iii) *If $c = 0$, then*

$$(\mathsf{A}^k)_{0,0} = \frac{p-1}{2^{k+1}p}\left(2\,(p-1)^{k-1} + (\xi\sqrt{p}-1)^k + (-1)^k\,(\xi\sqrt{p}+1)^k\right).$$

Proof. Since the quadratic nonresidues modulo p are all of the form ax^2 for some fixed quadratic nonresidue a, we have from (3.6)

$$\begin{aligned} (\mathsf{A}^k)_{0,c} &= \frac{1}{p}\left(\left(\frac{p-1}{2}\right)^k + \sum_{x^2\in\mathbb{Z}_p^\times}\left(\frac{\xi\sqrt{p}-1}{2}\right)^k \omega^{-x^2c} + \sum_{x^2\in\mathbb{Z}_p^\times}(-1)^k\left(\frac{\xi\sqrt{p}+1}{2}\right)^k \omega^{-ax^2c}\right) \\ &= \frac{1}{p}\left(\left(\frac{p-1}{2}\right)^k + \left(\frac{\xi\sqrt{p}-1}{2}\right)^k \sum_{x^2\in\mathbb{Z}_p^\times}\omega^{-x^2c} + (-1)^k\left(\frac{\xi\sqrt{p}+1}{2}\right)^k \sum_{x^2\in\mathbb{Z}_p^\times}\omega^{-ax^2c}\right). \end{aligned} \tag{3.9}$$

Now if $c \neq 0$, then

$$g(-c) = \left(\frac{-c}{p}\right)g(1) = \left(\frac{c}{p}\right)\left(\frac{-1}{p}\right)g(1) = \begin{cases} g(c) & \text{if } p \equiv 1 \bmod 4 \\ -g(c) & \text{if } p \equiv 3 \bmod 4, \end{cases}$$

so that

$$f(-c) = \begin{cases} f(c) & \text{if } p \equiv 1 \bmod 4, \\ -(f(c)+1) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Therefore when $c \neq 0$,

$$(\mathsf{A}^k)_{0,c} = \begin{cases} \frac{1}{p}\left(\left(\frac{p-1}{2}\right)^k + \left(\frac{\sqrt{p}-1}{2}\right)^k f(c) + (-1)^k\left(\frac{\sqrt{p}+1}{2}\right)^k f(ac)\right) & \text{if } p \equiv 1 \bmod 4 \\ \frac{1}{p}\left(\left(\frac{p-1}{2}\right)^k - \left(\frac{i\sqrt{p}-1}{2}\right)^k (f(c)+1) + (-1)^{k+1}\left(\frac{i\sqrt{p}+1}{2}\right)^k (f(ac)+1)\right) & \text{if } p \equiv 3 \bmod 4. \end{cases} \tag{3.10}$$

We examine the expression in (3.10) for the scenarios in (i) and (ii) of Theorem 3.8.

(i) When $c \in \mathbb{Z}_p^\times$ is a quadratic residue modulo p, then

$$(\mathsf{A}^k)_{0,c} = \begin{cases} \frac{1}{2^{k+1}p}\left(2\,(p-1)^k + \left(\sqrt{p}-1\right)^{k+1} + (-1)^{k+1}\left(\sqrt{p}+1\right)^{k+1}\right) & \text{if } p \equiv 1 \bmod 4, \\ \frac{1}{2^{k+1}p}\left(2\,(p-1)^k - \left(i\sqrt{p}-1\right)^k (i\sqrt{p}+1) + (-1)^{k+1}\left(i\sqrt{p}+1\right)^k \left(i\sqrt{p}-1\right)\right) & \\ = \frac{1}{2^{k+1}p}\left(2\,(p-1)^k + (p+1)\left(i\sqrt{p}-1\right)^{k-1} + (-1)^k (p+1)\left(i\sqrt{p}+1\right)^{k-1}\right) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(ii) When c is a quadratic nonresidue modulo p,

$$(\mathsf{A}^k)_{0,c} = \begin{cases} \frac{p-1}{2^{k+1}p}\left(2\,(p-1)^{k-1} + \left(\sqrt{p}-1\right)^{k-1} + (-1)^k\left(\sqrt{p}+1\right)^{k-1}\right) & \text{if } p \equiv 1 \bmod 4, \\ \frac{1}{2^{k+1}p}\left(2\,(p-1)^k - \left(i\sqrt{p}+1\right)^{k+1} + (-1)^{k+1}\left(i\sqrt{p}+1\right)^{k+1}\right) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(iii) Finally, when $c = 0$, then (3.9) implies

$$\begin{aligned} (\mathsf{A}^k)_{0,0} &= \frac{1}{2^{k+1}p}\left(2\,(p-1)^k + (\xi\sqrt{p}-1)^k\,(p-1) + (-1)^k\,(\xi\sqrt{p}+1)^k\,(p-1)\right) \\ &= \frac{p-1}{2^{k+1}p}\left(2\,(p-1)^{k-1} + (\xi\sqrt{p}-1)^k + (-1)^k\,(\xi\sqrt{p}+1)^k\right), \end{aligned}$$

to give the assertion in part (iii). □

4 The groups S_n and $\mathbb{Z}_r \wr \mathrm{S}_n$

4.1 The symmetric group S_n

The irreducible modules for the symmetric group S_n are in one-to-one correspondence with the partitions $\lambda \vdash n$, and the conjugacy classes are determined by the cycle decomposition of the permutations, hence they also are indexed by the partitions of n. If V is taken to be the n-dimensional permutation module on which S_n acts by permuting the basis elements, then for all $\sigma \in \mathrm{S}_n$,

$$\chi_{\mathsf{V}}(\sigma) = \mathsf{tr}_{\mathsf{V}}(\sigma) = \mathsf{F}(\sigma), \tag{4.1}$$

where $\mathsf{F}(\sigma)$ is the number of fixed points of σ. As a result, we know from (2.11) that the Poincaré series for the tensor invariants $\mathsf{T}(\mathsf{V})^{\mathrm{S}_n}$ is given by

$$\begin{aligned} \mathsf{P}^0(t) &= (n\,!)^{-1}\sum_{\mu\vdash n}|\mathcal{C}_\mu|\frac{1}{1-\mathsf{F}(\mathsf{c}_\mu)t} = (n\,!)^{-1}\sum_{\sigma\in\mathrm{S}_n}\frac{1}{1-\mathsf{F}(\sigma)t} \\ &= \frac{\mathsf{det}(\mathsf{M}^0)}{\mathsf{det}(\mathrm{I}-t\mathsf{A})} = \frac{\mathsf{det}(\mathsf{M}^0)}{\prod_{\mu\vdash n}(1-\mathsf{F}(c_\mu)t)} \end{aligned} \tag{4.2}$$

where M^0 and A are as in Theorem 1.12. For the centralizer algebra $\mathsf{Z}_k(\mathrm{S}_n) = \mathsf{End}_{\mathrm{S}_n}(\mathsf{V}^{\otimes k})$ and its irreducible module $\mathsf{Z}_k^\lambda(\mathrm{S}_n)$,

$$\begin{aligned} \dim \mathsf{Z}_k^\lambda(\mathrm{S}_n) &= (n\,!)^{-1}\sum_{\sigma\in\mathrm{S}_n}\mathsf{F}(\sigma)^k\,\overline{\chi_\lambda(\sigma)}, \\ \dim \mathsf{Z}_k(\mathrm{S}_n) &= (n\,!)^{-1}\sum_{\mu\vdash n}|\mathcal{C}_\mu|\;\mathsf{F}(\mathsf{c}_\mu)^{2k} = (n\,!)^{-1}\sum_{\sigma\in\mathrm{S}_n}\mathsf{F}(\sigma)^{2k}. \end{aligned} \tag{4.3}$$

The centralizer algebra $\mathsf{Z}_k(\mathrm{S}_n)$ for the S_n-action on the k-fold tensor power of its permutation module V is a homomorphic image of the partition algebra $\mathsf{P}_k(n) \twoheadrightarrow \mathsf{Z}_k(\mathrm{S}_n) = \mathsf{End}_{\mathrm{S}_n}(\mathsf{V}^{\otimes k})$, and $\mathsf{Z}_k(\mathrm{S}_n)$ is isomorphic to $\mathsf{P}_k(n)$ when $n \geq 2k$ (see for example [HR] for basic facts about partition algebras). Parts (a) and (c) of [BHH, Thm. 5.5] give expressions for the dimension of $\mathsf{Z}_k^\lambda(\mathrm{S}_n)$ and $\mathsf{Z}_k(\mathrm{S}_n)$ respectively in terms of Stirling numbers of the second kind, and these expressions combine with the ones above to show that

$$\begin{aligned}(n!)^{-1} \sum_{\sigma \in \mathrm{S}_n} \mathsf{F}(\sigma)^k \overline{\chi_\lambda(\sigma)} &= \dim \mathsf{Z}_k^\lambda(\mathrm{S}_n) = \sum_{\ell=0}^{n} \mathsf{K}_{\lambda,(n-\ell,1^\ell)} \left\{ {k \atop \ell} \right\}, \\ (n!)^{-1} \sum_{\sigma \in \mathrm{S}_n} \mathsf{F}(\sigma)^{2k} &= \dim \mathsf{Z}_k(\mathrm{S}_n) = \sum_{\ell=0}^{n} \left\{ {2k \atop \ell} \right\}.\end{aligned} \tag{4.4}$$

The *Kostka number* $\mathsf{K}_{\lambda,(n-\ell,1^\ell)}$ counts the number of semistandard tableaux of shape λ with $n-\ell$ entries equal to 0 and one entry equal to each of the numbers $1, 2, \dots, \ell$ such that the entries weakly increase across the rows and strictly increase down the columns of the Young diagram of λ (more details on Kostka numbers can be found in [Sa, Sec. 2.11] or [S2, Sec. 7.10]). The first relation in (4.4) was proven by Farina and Halverson in [FaH] under the additional assumption that $n \geq 2k$. In that case, $\mathsf{Z}_k(\mathrm{S}_n) \cong \mathsf{P}_k(n)$, and the right-hand side $\sum_{\ell=0}^{n} \left\{ {2k \atop \ell} \right\} = \sum_{\ell=0}^{2k} \left\{ {2k \atop \ell} \right\}$ equals the Bell number $\mathsf{B}(2k)$. The relations in (4.4) hold for all $n, k \in \mathbb{Z}_{\geq 1}$.

Next we examine the particular case of the symmetric group S_4 to illustrate the above results.

4.2 The special case of the symmetric group S_4

The irreducible modules and conjugacy classes for the symmetric group S_4 are indexed by the partitions $\lambda \vdash 4$, where $\lambda \in \{(4), (3,1), (2^2), (2,1^2), (1^4)\}$. The trivial module corresponds to the partition (4) with just one part, and the 4-dimensional permutation module for S_4 is given by $\mathsf{V} = (\mathrm{S}_4)_{(4)} \oplus (\mathrm{S}_4)_{(3,1)}$. The corresponding representation graph $\mathcal{R}_\mathsf{V}(\mathrm{S}_4)$ is pictured in Figure 4. Hence, by (2.4), the dimensions of the

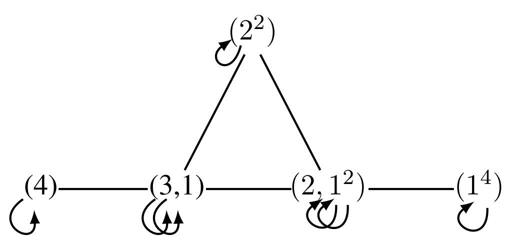


Figure 4: Representation graph $\mathcal{R}_\mathsf{V}(\mathrm{S}_4)$ for $\mathsf{V} = (\mathrm{S}_4)_{(4)} \oplus (\mathrm{S}_4)_{(3,1)}$

irreducible modules $\mathsf{Z}_k^\lambda(\mathrm{S}_4)$ for the centralizer algebra $\mathsf{Z}_k(\mathrm{S}_4) = \mathsf{End}_{\mathrm{S}_4}(\mathsf{V}^{\otimes k})$ are given by

$$\dim \mathsf{Z}_k^\lambda(\mathrm{S}_4) = (\mathsf{A}^k)_{(4),\lambda} = (24)^{-1} \sum_{\mu \vdash 4} |\mathcal{C}_\mu| \, \chi_\mathsf{V}(\mathsf{c}_\mu)^k \overline{\chi_\lambda(\mathsf{c}_\mu)}.$$

The necessary information to evaluate this expression is displayed in the table below and can be gotten from the character table for S_4 (see for example [FuH, Sec. 2.3]).

(4.5)

$\lambda \setminus \mu$	(1^4)	$(2,1^2)$	(2^2)	$(3,1)$	(4)
$\lvert\mathcal{C}_\mu\rvert$	1	6	3	8	6
$\chi_{(4)}(\mathsf{c}_\mu)$	1	1	1	1	1
$\chi_{(3,1)}(\mathsf{c}_\mu)$	3	1	-1	0	-1
$\chi_{(2^2)}(\mathsf{c}_\mu)$	2	0	2	-1	0
$\chi_{(2,1^2)}(\mathsf{c}_\mu)$	3	-1	-1	0	1
$\chi_{(1^4)}(\mathsf{c}_\mu)$	1	-1	1	1	-1
$\chi_{\mathsf{V}}^k(\mathsf{c}_\mu)$	4^k	2^k	0	1	0

From this we determine that for $k \geq 1$,

$$
\begin{aligned}
\dim \mathsf{Z}_k^{(4)}(\mathrm{S}_4) &= \frac{1}{24}\left(4^k + 6\cdot 2^k + 8\right) \quad \left(= \sum_{\ell=1}^{4} \left\{ {k \atop \ell} \right\}\right)\\
\dim \mathsf{Z}_k^{(3,1)}(\mathrm{S}_4) &= \frac{1}{24}\left(3\cdot 4^k + 6\cdot 2^k\right) \quad \left(= \left\{ {k \atop 1} \right\} + 2\left\{ {k \atop 2} \right\} + 3\left\{ {k \atop 3} \right\} + 3\left\{ {k \atop 4} \right\}\right)\\
\dim \mathsf{Z}_k^{(2^2)}(\mathrm{S}_4) &= \frac{1}{24}\left(2\cdot 4^k - 8\right) \quad \left(= \left\{ {k \atop 2} \right\} + 2\left\{ {k \atop 3} \right\} + 2\left\{ {k \atop 4} \right\}\right)\\
\dim \mathsf{Z}_k^{(2,1^2)}(\mathrm{S}_4) &= \frac{1}{24}\left(3\cdot 4^k - 6\cdot 2^k\right) \quad \left(= \left\{ {k \atop 2} \right\} + 3\left\{ {k \atop 3} \right\} + 3\left\{ {k \atop 4} \right\}\right)\\
\dim \mathsf{Z}_k^{(1^4)}(\mathrm{S}_4) &= \frac{1}{24}\left(4^k - 6\cdot 2^k + 8\right) \quad \left(= \left\{ {k \atop 3} \right\} + \left\{ {k \atop 4} \right\}\right)\\
\dim \mathsf{Z}_k(\mathrm{S}_4) = \dim \mathsf{Z}_{2k}^{(4)}(\mathrm{S}_4) &= \frac{1}{24}\left(4^{2k} + 6\cdot 2^{2k} + 8\right) \quad \left(= \sum_{\ell=1}^{4} \left\{ {2k \atop \ell} \right\}\right).
\end{aligned}
\tag{4.6}
$$

On the right-hand side above, we have given expressions for the dimensions in terms of Stirling numbers of the second kind, which were derived using the following closed-form formula:

$$
\left\{ {k \atop \ell} \right\} = \frac{1}{\ell\,!}\sum_{j=0}^{\ell} (-1)^{\ell-j}\binom{\ell}{j} j^k. \tag{4.7}
$$

The coefficients of the Stirling numbers $\left\{ {k \atop \ell} \right\}$ are the Kostka numbers $\mathsf{K}_{\lambda,(n-\ell,1^\ell)}$ for $n = 4$, and they enumerate the semistandard tableaux of shape λ and type $(4-\ell, 1^\ell)$ as pictured below for $\lambda = (2^2)$:

$$
\underset{\ell=2}{\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 2 \\ \hline\end{array}}
\qquad
\underset{\ell\,=\,3}{\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 & 3 \\ \hline\end{array}}
\qquad
\underset{\ell\,=\,4}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}}\ .
$$

4.3 Bratteli diagram

The *Bratteli diagram* $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ is an infinite graph with vertices labeled by the elements of $\Lambda_k(\mathsf{G})$ on level k. A walk of k steps on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ from $\mathsf{0}$ to λ is a sequence

$\big(\lambda^{(0)} = 0, \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)} = \lambda\big)$ starting at $\lambda^{(0)} = 0$, such that $\lambda^{(j)} \in \Lambda_j(\mathsf{G})$ for each $1 \le j \le k$, and $\lambda^{(j-1)}$ is connected to $\lambda^{(j)}$ by an edge in $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$. Such a walk is equivalent to a unique path of length k on the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ from 0 at the top to $\lambda \in \Lambda_k(\mathsf{G})$ on level k. The subscript on vertex $\lambda \in \Lambda_k(\mathsf{G})$ in $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ indicates the number m_k^λ of paths from 0 on the top to λ at level k. This can be easily computed by summing, in a Pascal triangle fashion, the subscripts of the vertices at level $k-1$ that are connected to λ. This is dimension of the irreducible $\mathsf{Z}_k(\mathsf{G})$-module $\mathsf{Z}_k^\lambda(\mathsf{G})$, which is also the multiplicity of G_λ in $\mathsf{V}^{\otimes k}$. The sum of the squares of those dimensions at level k is the number on the right, which is the dimension of the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$ by Wedderburn theory.

The top levels of the Bratteli diagram for the group $\mathsf{G} = \mathrm{S}_4$ and its 4-dimensional permutation module V are exhibited in Figure 5.

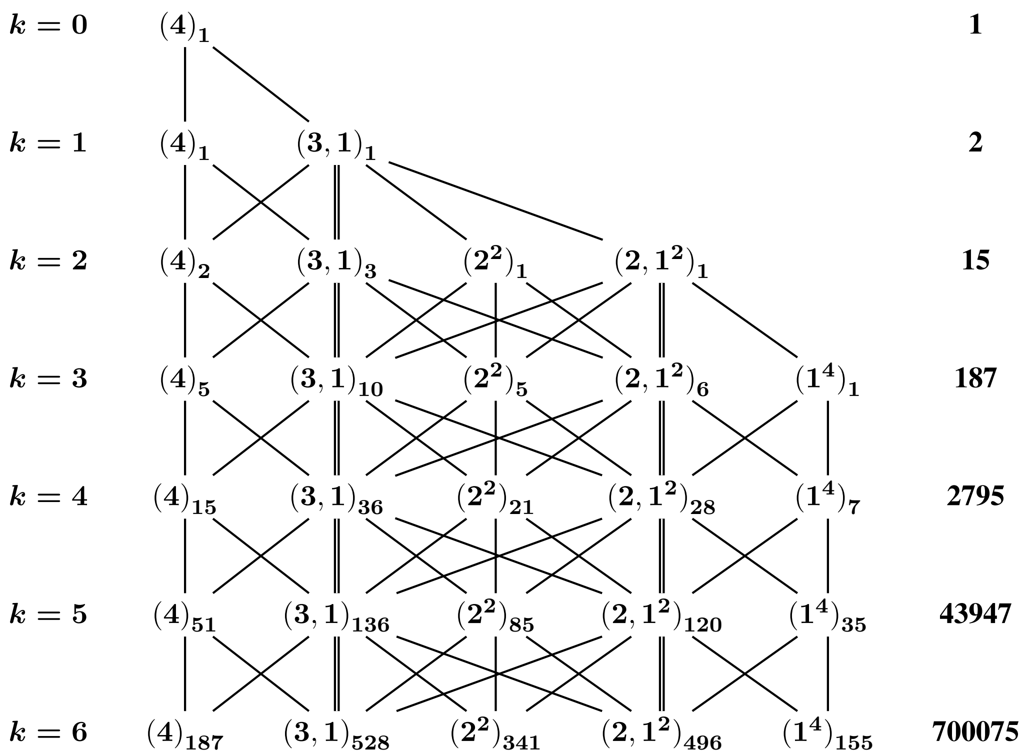


Figure 5: Levels $k = 0, 1, \dots, 6$ of the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathrm{S}_4)$ for S_4 and its permutation module V

4.4 The group $\mathbb{Z}_r \wr \mathrm{S}_n$

In this section, G is the wreath product $\mathbb{Z}_r \wr \mathrm{S}_n$ viewed as $n \times n$ monomial matrices with entries of the form ω^j for $j = 0, 1, \dots, r-1$, where $\omega = \mathrm{e}^{2\pi i/r}$, a primitive rth root of unity for $r \ge 2$. The module V is the space of $n \times 1$ column vectors with complex entries on which G acts by matrix multiplication. We present a formula for the dimension of the G-invariants $(\mathsf{V}^{\otimes k})^{\mathsf{G}}$ in $\mathsf{V}^{\otimes k}$, equivalently, for the dimension $\dim \mathsf{Z}_k^0(\mathsf{G}) = |\mathsf{G}|^{-1} \sum_{g \in \mathsf{G}} \chi_{\mathsf{V}}(g)^k$ of the irreducible module labeled by 0 for the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$. Our formula will depend on the number of entries on the main diagonal of a monomial matrix in G (the number of fixed points of the underlying permutation in S_n), and so for $m = 1, 2, \dots, n$, we set $\mathsf{F}_n(m) := |\{\sigma \in \mathrm{S}_n \mid \mathsf{F}(\sigma) = m\}|$. This number, which is sometimes referred to as a *rencontres* *number*, counts the number of "partial derangements" of n with m fixed points. It equals $\binom{n}{m}\mathsf{D}_{n-m}$, where D_{n-m} is the number of *derangements* of $n-m$ (permutations in S_{n-m} with no fixed points). From known expressions for the derangement numbers, we have

$$\mathsf{F}_n(m) = \binom{n}{m}\mathsf{D}_{n-m} = \binom{n}{m}(n-m)!\sum_{j=0}^{n-m}\frac{(-1)^j}{j\,!} = \frac{n!}{m!}\sum_{j=0}^{n-m}\frac{(-1)^j}{j\,!}. \tag{4.8}$$

Theorem 4.9. *For $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n$ and V the n-dimensional G-module on which G acts by monomial matrices, the dimension of the space of G-invariants in $\mathsf{V}^{\otimes k}$ (equivalently, $\dim \mathsf{Z}_k^0(\mathsf{G})$) is given by*

$$\dim(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \frac{1}{r^n n\,!}\sum_{m=1}^{n} r^m \mathsf{F}_n(m)^k \left(\sum_{\ell_1,\ell_2,\ldots,\ell_m}\binom{k}{\ell_1,\ell_2,\ldots,\ell_m}\right), \tag{4.10}$$

where the sum is over all $0 \le \ell_1,\ell_2,\ldots,\ell_m \le k$ such that $\ell_1+\ell_2+\cdots+\ell_m = k$ and $\ell_1 \equiv \ell_2 \equiv \cdots \equiv \ell_m \equiv 0 \bmod r$, and $\mathsf{F}_n(m) = \dfrac{n!}{m!}\displaystyle\sum_{j=0}^{n-m}\frac{(-1)^j}{j\,!}$. In particular, the space $(\mathsf{V}^{\otimes k})^{\mathsf{G}}$ of invariants is 0 unless $k \equiv 0 \bmod r$.

Proof. We know from Theorem 2.3 that $\dim(\mathsf{V}^{\otimes k})^{\mathsf{G}} = (\mathsf{A}^k)_{0,0} = |\mathsf{G}|^{-1}\sum_{g\in\mathsf{G}}\chi_{\mathsf{V}}(g)^k$, from which we have

$$\begin{aligned}
\dim(\mathsf{V}^{\otimes k})^{\mathsf{G}} &= \frac{1}{r^n\,n\,!}\sum_{m=1}^{n}\mathsf{F}_n(m)^k \sum_{b_1,b_2,\ldots,b_m\in\{0,1,\ldots,r-1\}}\left(\omega^{b_1}+\omega^{b_2}+\cdots+\omega^{b_m}\right)^k \\
&= \frac{1}{r^n\,n\,!}\sum_{m=1}^{n}\mathsf{F}_n(m)^k\left(\sum_{\ell_1+\ell_2+\cdots+\ell_m=k}\binom{k}{\ell_1,\ell_2,\ldots,\ell_m}\left(\sum_{b_1=0}^{r-1}\omega^{\ell_1 b_1}\right)\left(\sum_{b_2=0}^{r-1}\omega^{\ell_2 b_2}\right)\cdots\left(\sum_{b_m=0}^{r-1}\omega^{\ell_m b_m}\right)\right) \\
&= \frac{1}{r^n\,n\,!}\sum_{m=1}^{n}\mathsf{F}_n(m)^k r^m\left(\sum_{\substack{\ell_1+\ell_2+\cdots+\ell_m=k\\ \ell_1\equiv\ell_2\equiv\cdots\equiv\ell_m\equiv 0 \bmod r}}\binom{k}{\ell_1,\ell_2,\ldots,\ell_m}\right) \qquad \text{by (3.1).}
\end{aligned}$$

□

Remark 4.11. It is a consequence of (4.10) that for $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n$,

$$\dim(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \frac{1}{r^n n\,!}\sum_{m=1}^{n} r^m\mathsf{F}_n(m)^k\left(\sum_{(q_1+q_2+\cdots+q_m)r=k}\binom{k}{q_1r,q_2r,\ldots,q_mr}\right). \tag{4.12}$$

Therefore, the exponential generating function for the invariants is given by

$$\begin{aligned}
\mathrm{g}^0(t) = \sum_{k=0}^{\infty}\dim(\mathsf{V}^{\otimes k})^{\mathsf{G}}\frac{t^k}{k!} &= \frac{1}{r^n n\,!}\sum_{m=1}^{n} r^m\sum_{k=0}^{\infty}\mathsf{F}_n(m)^k\left(\sum_{(q_1+q_2+\cdots+q_m)r=k}\binom{k}{q_1r,q_2r,\ldots,q_mr}\frac{t^k}{k!}\right) \\
&= \frac{1}{r^n n\,!}\sum_{m=1}^{n} r^{2m}\left(r^{-1}\sum_{q_1=0}^{\infty}\frac{(\mathsf{F}_n(m)t)^{q_1r}}{(q_1r)!}\right)\left(r^{-1}\sum_{q_2=0}^{\infty}\frac{(\mathsf{F}_n(m)t)^{q_2r}}{(q_2r)!}\right)\cdots\left(r^{-1}\sum_{q_m=0}^{\infty}\frac{(\mathsf{F}_n(m)t)^{q_mr}}{(q_mr)!}\right) \\
&= \frac{1}{r^n n\,!}\sum_{m=1}^{n} r^{2m}\,\mathsf{h}_1\left(\mathsf{F}_n(m)t, r\right)^m,
\end{aligned} \tag{4.13}$$

where h_1 is a generalized hyperbolic function (see (6.10) and (6.14) below for more details.)

4.5 $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n$ for some special choices of r and n

Assume $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_2$ and $\mathsf{V} = \mathbb{C}^2$. Then since $\mathsf{F}_2(1) = \binom{2}{1}\mathsf{D}_1 = 0$, and $\mathsf{F}_2(2) = \binom{2}{2}\mathsf{D}_0 = 1$, we have

$$\dim(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^0(\mathsf{G}) = \frac{1}{2} \sum_{\substack{\ell_1+\ell_2=k \\ \ell_1 \equiv \ell_2 \equiv 0 \bmod r}} \binom{k}{\ell_1, \ell_2}. \tag{4.14}$$

So, for example, when $r = 2$,

$$\dim(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \begin{cases} \frac{1}{2} \sum_{\ell=0}^{\frac{1}{2}k} \binom{k}{2\ell} = \frac{1}{2} 2^{k-1} = 2^{k-2} & \text{if } k \text{ is even and } k \geq 2, \\ 0 & \text{if } k \text{ is odd and } k \geq 1. \end{cases} \tag{4.15}$$

$$\mathsf{P}^0(t) = \sum_{k=0}^{\infty} \dim(\mathsf{V}^{\otimes k})^{\mathsf{G}}\, t^k = 1 + t^2 \sum_{j=0}^{\infty} (4t^2)^j = 1 + \frac{t^2}{1-4t^2} = \frac{1-3t^2}{1-4t^2}.$$

4.6 The group $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n$ – a different approach

The irreducible modules $\mathsf{G}_{\underline{\alpha}}$ for $\mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n$, hence also the G-conjugacy classes $\mathcal{C}_{\underline{\alpha}}$, are labeled by r-tuples of partitions $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r)})$ such that $n = \sum_{i=1}^r |\alpha^{(i)}|$ (see for example [AK, Sec. 2]).

For $x \in \mathbb{C}$, let $\mathsf{J}_\ell(x)$ be the $\ell \times \ell$ Jordan block matrix given by

$$\mathsf{J}_\ell(x) = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & & & \ddots & & \\ \vdots & & & & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ x & 0 & & & 0 & 0 \end{pmatrix}$$

Then a conjugacy class representative of G corresponding to $\underline{\alpha}$ is

$$\mathsf{c}_{\underline{\alpha}} = \bigoplus_{i=1}^{r} \bigoplus_{p} \mathsf{J}_{\alpha_p^{(i)}}(\omega^{i-1}),$$

where $\omega = \mathrm{e}^{2\pi i/r}$, the parts $\alpha_p^{(i)}$ of the ith partition $\alpha^{(i)}$ are $\alpha_1^{(i)} \geq \alpha_2^{(i)} \geq \ldots$, and this sum represents the $n \times n$ matrix with blocks down the main diagonal starting with $\mathsf{J}_{\alpha_1^{(1)}}(\omega^0)$, then $\mathsf{J}_{\alpha_2^{(1)}}(\omega^0)$, …, and continuing down to $\mathsf{J}_{\alpha_\ell^{(r)}}(\omega^{r-1})$ corresponding to the last part $\alpha_\ell^{(r)}$ of the last partition $\alpha^{(r)}$.

For a partition λ, assume $\mathsf{p}_j(\lambda)$ is the number of parts of λ equal to j. Set

$$\mathsf{z}_\lambda = \prod_{j=1}^{n} j^{\mathsf{p}_j(\lambda)}\, \mathsf{p}_j(\lambda)!.$$

This is the order of the centralizer of an element of $\mathrm{S}_{|\lambda|}$ with cycle structure given by the partition λ. Now for $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r)})$, we define

$$\mathsf{p}_j(\underline{\alpha}) = \sum_{i=1}^{r} \mathsf{p}_j(\alpha^{(i)}) \qquad \text{and} \qquad \mathsf{p}(\underline{\alpha}) = \sum_{j=1}^{n} \mathsf{p}_j(\underline{\alpha}). \tag{4.16}$$

Thus, $\mathsf{p}_j(\underline{\alpha})$ is the total number of parts equal to j in the partitions comprising $\underline{\alpha}$, and $\mathsf{p}(\underline{\alpha})$ is the total number of nonzero parts in the partitions of $\underline{\alpha}$. Then according to [AK, Sec. 2], the size of the centralizer of $\mathsf{c}_{\underline{\alpha}}$ in G is given by

$$\mathsf{z}_{\underline{\alpha}} = \prod_{i,j} (rj)^{\mathsf{p}_j(\alpha^{(i)})}\, \mathsf{p}_j(\alpha^{(i)})! = r^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{n} j^{\mathsf{p}_j(\underline{\alpha})} \left(\prod_{i=1}^{r} \mathsf{p}_j(\alpha^{(i)})! \right) = r^{\mathsf{p}(\underline{\alpha})} \prod_{i=1}^{r} \mathsf{z}_{\alpha^{(i)}}. \tag{4.17}$$

Hence, the size of the conjugacy class $\mathcal{C}_{\underline{\alpha}}$ corresponding to the element $\mathsf{c}_{\underline{\alpha}}$ is given by

$$|\mathcal{C}_{\underline{\alpha}}| = \frac{|\mathsf{G}|}{\mathsf{z}_{\underline{\alpha}}} = \frac{|\mathsf{G}|}{r^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{n} j^{\mathsf{p}_j(\underline{\alpha})} \left(\prod_{i=1}^{r} \mathsf{p}_j(\alpha^{(i)})! \right)}.$$

Thus, we know that

$$\dim (\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^0(\mathsf{G}) = |\mathsf{G}|^{-1} \sum_{\underline{\alpha} \in \Lambda(\mathsf{G})} |\mathcal{C}_{\underline{\alpha}}|\, \chi_{\mathsf{V}}(\mathsf{c}_{\underline{\alpha}})^k = \sum_{\underline{\alpha} \in \Lambda(\mathsf{G})} \frac{\chi_{\mathsf{V}}(\mathsf{c}_{\underline{\alpha}})^k}{r^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{n} j^{\mathsf{p}_j(\underline{\alpha})} \left(\prod_{i=1}^{r} \mathsf{p}_j(\alpha^{(i)})! \right)}.$$

Observe that

$$\chi_{\mathsf{V}}(\mathsf{c}_{\underline{\alpha}}) = \mathsf{tr}_{\mathsf{V}}\left(\mathsf{c}_{\underline{\alpha}}\right) = \sum_{i=1}^{r} \mathsf{p}_1(\alpha^{(i)})\, \omega^{i-1} = \sum_{i=1}^{r} \mathsf{F}(\alpha^{(i)})\, \omega^{i-1}$$

where $\mathsf{p}_1(\alpha^{(i)})$ is the number of parts equal to 1 in $\alpha^{(i)}$, as the only contributions to the trace come from the matrix blocks of size one in $\mathsf{c}_{\underline{\alpha}}$. Since that is the number of fixed points of a permutation of cycle type $\alpha^{(i)}$, we write $\mathsf{F}(\alpha^{(i)})$ by a slight abuse of notation. Therefore, we obtain a second expression for the dimension of the G-invariants in $\mathsf{V}^{\otimes k}$ using the definitions in (4.16):

$$\dim (\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^0(\mathsf{G}) = \sum_{\underline{\alpha} \in \Lambda(\mathsf{G})} \frac{\left(\sum_{i=1}^{r} \mathsf{F}(\alpha^{(i)})\, \omega^{i-1} \right)^k}{r^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{n} j^{\mathsf{p}_j(\underline{\alpha})} \left(\prod_{i=1}^{r} \mathsf{p}_j(\alpha^{(i)})! \right)} \quad \text{for } \mathsf{G} = \mathbb{Z}_r \wr \mathrm{S}_n, \tag{4.18}$$

The group $\mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_n$ is the Weyl group for a root system of type B_n or C_n. The irreducible G-modules are labeled by pairs $\underline{\alpha} = (\alpha^{(1)}, \alpha^{(2)})$ of partitions such that $|\alpha^{(1)}| + |\alpha^{(2)}| = n$. Since $\omega = -1$ in this case, we have the following formula for the dimension of the space of G-invariants in $\mathsf{V}^{\otimes k}$:

$$\dim (\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^0(\mathsf{G}) = \sum_{\underline{\alpha} \in \Lambda(\mathsf{G})} \frac{\left(\mathsf{F}(\alpha^{(1)}) - \mathsf{F}(\alpha^{(2)}) \right)^k}{2^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{n} j^{\mathsf{p}_j(\underline{\alpha})} \left(\mathsf{p}_j(\alpha^{(1)})! \cdot \mathsf{p}_j(\alpha^{(2)})! \right)} \quad \text{for } \mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_n, \tag{4.19}$$

where $\mathsf{p}_j(\alpha^{(i)})$ and $\mathsf{p}(\underline{\alpha})$ are as in (4.16).

Remark 4.20. In [T], Tanabe investigated the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$, where G is a complex reflection group $\mathsf{G}(m, p, n)$ viewed as $n \times n$ matrices acting on $\mathsf{V} = \mathbb{C}^n$. The group $\mathsf{G}(r, 1, n)$ is the wreath product $\mathbb{Z}_r \wr \mathrm{S}_n$. Using results from [T], we showed in [BM] for $\mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_n$ that

$$\dim \mathsf{Z}_k(\mathsf{G}) = \sum_{s=1}^{n} \mathrm{T}(k, s),$$

where $\mathrm{T}(k, s)$ is the number of set partitions of a set of size $2k$ into s nonempty disjoint parts of *even* size. The numbers $\mathrm{T}(k, s)$ correspond to sequence A156289 in the Online Encyclopedia of Integer Sequences [OEIS] and have many different interpretations. They are known to satisfy

$$\mathrm{T}(k, s) = \frac{1}{s!\, 2^{s-1}} \sum_{j=1}^{s} (-1)^{s-j} \binom{2s}{s-j} j^{2k} = \sum_{\lambda} \frac{1}{\prod_{j \geq 1} \mathsf{p}_j(\lambda)} \binom{2k}{2\lambda_1, 2\lambda_2, \ldots, 2\lambda_s},$$

where the last sum is over all partitions $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0\}$ of k into s nonzero parts λ_i (see [BM, Sec. 4.2] for details). In particular, since V is self-dual, we see that

$$\dim\,(\mathsf{V}^{\otimes 2k})^{\mathsf{G}} = \dim \mathsf{Z}_k(\mathsf{G}) = \sum_{s=1}^{n} \mathrm{T}(k,s), \qquad \text{for } \mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_n. \tag{4.21}$$

It would be interesting to show the equivalence of the formulas in Theorem 4.9 and (4.19) and then relate them (with $2k$ in place of k) to (4.21).

Next we derive a few special instances of the formula in (4.19).

4.7 The $\mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_2$ case revisited

It is convenient to display the information needed to compute $\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^0(\mathsf{G})$ using (4.19) in the following table. Since the partitions in $\underline{\alpha}$ are small, we won't bother using parentheses in listing them.

$\underline{\alpha}$	$\mathsf{F}(\alpha^{(1)})$	$\mathsf{F}(\alpha^{(2)})$	$\mathsf{tr}_{\mathsf{V}}(\mathsf{c}_{\underline{\alpha}}) = \mathsf{F}(\alpha^{(1)}) - \mathsf{F}(\alpha^{(2)})$	$\mathsf{p}(\underline{\alpha})$	$2^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{2} j^{\mathsf{p}_j(\underline{\alpha})} \left(\mathsf{p}_j(\alpha^{(1)})! \cdot \mathsf{p}_j(\alpha^{(2)})!\right)$
$(2,\emptyset)$	0	0	0	1	4
$(1^2,\emptyset)$	2	0	2	2	8
$(1,1)$	1	1	0	2	8
$(\emptyset,1^2)$	0	2	-2	2	8
$(\emptyset,2)$	0	0	0	1	4

(4.22)

Therefore, we have

$$\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^0(\mathsf{G}) = \sum_{\underline{\alpha}\in\Lambda(\mathsf{G})} \frac{\left(\mathsf{F}(\alpha^{(1)}) - \mathsf{F}(\alpha^{(2)})\right)^k}{2^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{2} j^{\mathsf{p}_j(\underline{\alpha})} \left(\prod_{i=1}^{2} \mathsf{p}_j(\alpha^{(i)})!\right)} = \frac{2^k + (-2)^k}{8} \quad \text{for } \mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_2$$
$$= \begin{cases} 2^{k-2} & \text{if } k \text{ is even and } k \geq 2, \\ 0 & \text{if } k \text{ is odd and } k \geq 1, \end{cases} \tag{4.23}$$

in agreement with (4.15).

4.8 $\mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_3$

The relevant information for applying (4.19) is given in the table below.

$\underline{\alpha}$	$\mathsf{F}(\alpha^{(1)})$	$\mathsf{F}(\alpha^{(2)})$	$\mathsf{tr}_{\mathsf{V}}(\mathsf{c}_{\underline{\alpha}}) = \mathsf{F}(\alpha^{(1)}) - \mathsf{F}(\alpha^{(2)})$	$\mathsf{p}(\underline{\alpha})$	$2^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{3} j^{\mathsf{p}_j(\underline{\alpha})} \left(\mathsf{p}_j(\alpha^{(1)})! \cdot \mathsf{p}_j(\alpha^{(2)})!\right)$
$(3, \emptyset)$	0	0	0	1	6
$((2,1), \emptyset)$	1	0	1	2	8
$(1^3, \emptyset)$	3	0	3	3	48
$(2, 1)$	0	1	-1	2	8
$(1^2, 1)$	2	1	1	3	16
$(1, 2)$	1	0	1	2	8
$(1, 1^2)$	1	2	-1	3	16
$(\emptyset, 1^3)$	0	3	-3	3	48
$(\emptyset, (2,1))$	0	1	-1	2	8
$(\emptyset, 3)$	0	0	0	1	6

(4.24)

$$\begin{aligned}
\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^0(\mathsf{G}) &= \sum_{\underline{\alpha} \in \Lambda(\mathsf{G})} \left(\frac{\left(\mathsf{F}(\alpha^{(1)}) - \mathsf{F}(\alpha^{(2)})\right)^k}{2^{\mathsf{p}(\underline{\alpha})} \prod_{j=1}^{3} j^{\mathsf{p}_j(\underline{\alpha})} \left(\mathsf{p}_j(\alpha^{(1)})! \cdot \mathsf{p}_j(\alpha^{(2)})!\right)} \right) \\
&= \frac{15(1^k + (-1)^k) + (3^k + (-3)^k)}{48} \qquad \text{for } \mathsf{G} = \mathbb{Z}_2 \wr \mathrm{S}_3 \\
&= \begin{cases} \dfrac{3^{k-1}+5}{8} & \text{if } k \text{ is even and } \geq 2, \\ 0 & \text{if } k \text{ is odd and } \geq 1. \end{cases}
\end{aligned} \tag{4.25}$$

$$\mathsf{P}^0(t) = \sum_{k=0}^{\infty} \dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}}\, t^k = 1 + \frac{1}{8} \sum_{j=1}^{\infty} (3^{2j-1} + 5)\, t^{2j} = \frac{1 - 9t^2 + 3t^4}{(1-t^2)(1-9t^2)}.$$

5 $\mathsf{G} = \mathrm{GL}_2(\mathbb{F}_q)$ and $\mathsf{G} = \mathrm{SL}_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field of q elements. Then $q = p^\ell$ for some prime p and some $\ell \geq 1$, and we assume p is odd to simplify considerations. In this section, G is the general linear group $\mathrm{GL}_2(\mathbb{F}_q)$ of 2×2 invertible matrices over $\mathbb{F}_q$ or the special linear subgroup $\mathrm{SL}_2(\mathbb{F}_q)$ of matrices with determinant equal to 1. We assume $\mathsf{V} = \mathsf{Ind}_{\mathsf{B}}^{\mathsf{G}} \mathsf{B}_0$, the G-module induced from the trivial module B_0 for the subgroup B of upper triangular matrices in G, and V_q is its q-dimensional irreducible summand, which is Steinberg module. (Here we write V_q rather than the customary St, to emphasize its analogy to V in previous sections.) Our aim in this section is to develop a formula for $\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}}$ and for $\dim\,(\mathsf{V}_q^{\otimes k})^{\mathsf{G}}$ and to determine the corresponding Poincaré series for the tensor invariants.

5.1 $\mathsf{G} = \mathrm{GL}_2(\mathbb{F}_q)$

Let $\mathsf{B} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \,\middle|\, x, z \in \mathbb{F}_q^{\times},\, y \in \mathbb{F}_q \right\}$ be the Borel subgroup of upper-triangular matrices in $\mathsf{G} = \mathrm{GL}_2(\mathbb{F}_q)$ and V be the induced G-module $\mathsf{V} = \mathsf{Ind}_{\mathsf{B}}^{\mathsf{G}} \mathsf{B}_0 = \mathbb{C}[\mathsf{G}] \otimes_{\mathbb{C}[\mathsf{B}]} \mathsf{B}_0$. Since the order of G is

$q(q+1)(q-1)^2$ and the order of B is $q(q-1)^2$, we have $\dim \mathsf{V} = q+1$. The module V decomposes into a sum $\mathsf{V} = \mathsf{G}_0 \oplus \mathsf{V}_q$ of a copy of the trivial G-module G_0 and a copy of a q-dimensional irreducible G-module V_q (the Steinberg module).

Let ε be a non-square in $\mathbb{F}_q^\times$, and define the following elements of G,

$$\mathsf{a}_x = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad (x \in \mathbb{F}_q^\times) \qquad \mathsf{b}_x = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \quad (x \in \mathbb{F}_q^\times) \qquad \mathsf{c}_{x,y} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad (x, y \in \mathbb{F}_q^\times, x \neq y) \qquad \mathsf{d}_{x,y} = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} \quad (y \in \mathbb{F}_q^\times) \tag{5.1}$$

We will use the information in the table below, which can be derived from [FuH, Sec. 5.2]. As before, c_μ, $\mu \in \Lambda(\mathsf{G})$, is a representative of the conjugacy class $\mathcal{C}_\mu$ of G.

(5.2)

c_μ	a_x	b_x	$\mathsf{c}_{x,y}$	$\mathsf{d}_{x,y}$
no. of such classes	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	$\frac{1}{2}q(q-1)$
$\lvert\mathcal{C}_\mu\rvert$	1	q^2-1	q^2+q	q^2-q
$\chi_{\mathsf{V}}(\mathsf{c}_\mu)$	$q+1$	1	2	0
$\chi_{\mathsf{V}_q}(\mathsf{c}_\mu)$	q	0	1	-1

Therefore, we have the following consequence of Theorem 2.3.

Theorem 5.3. *Assume* $\mathsf{G} = \mathrm{GL}_2(\mathbb{F}_q)$ *where* q *is odd.*

(a) *For the* G*-module* $\mathsf{V} = \mathsf{Ind}_{\mathsf{B}}^{\mathsf{G}}\mathsf{B}_0 = \mathsf{G}_0 \oplus \mathsf{V}_q$ *induced from the trivial module* B_0 *for the Borel subgroup* B *of upper-triangular matrices in* G,

$$\dim (\mathsf{V}^{\otimes k})^{\mathsf{G}} = \begin{cases} 1 & \text{when } k = 0, \\ \dfrac{1}{q(q-1)}\Big((q+1)^{k-1} + q(q-2)\cdot 2^{k-1} + q - 1\Big) & \text{when } k \geq 1. \end{cases} \tag{5.4}$$

The Poincaré series for the G*-invariants* $\mathsf{T}(\mathsf{V})^{\mathsf{G}}$ *in* $\mathsf{T}(\mathsf{V}) = \bigoplus_{k=0}^\infty \mathsf{V}^{\otimes k}$ *is*

$$\mathsf{P}^0(t) = \sum_{k=0}^{\infty} \dim (\mathsf{V}^{\otimes k})^{\mathsf{G}}\, t^k = \frac{1-(q+3)t+(2q+3)t^2-qt^3}{(1-t)\,(1-2t)\,(1-(1+q)t)}. \tag{5.5}$$

(b) *For the Steinberg module* V_q, $\dim (\mathsf{V}_q^{\otimes k})^{\mathsf{G}} = 1$ *when* $k = 0$*, and*

$$\dim (\mathsf{V}_q^{\otimes k})^{\mathsf{G}} = \frac{1}{2(q^2-1)}\Big(2q^{k-1} - q(q-1)(-1)^{k-1} + (q+1)(q-2)\Big) \quad \text{when } k \geq 1, \tag{5.6}$$

$$= \begin{cases} \dfrac{q^{2\ell}-1}{q^2-1} = \displaystyle\sum_{j=0}^{\ell-1} q^{2j} & \text{if} \quad k = 2\ell+1 \geq 1, \\ 1 + q\dfrac{q^{2\ell-2}-1}{q^2-1} = 1 + \displaystyle\sum_{j=0}^{\ell-2} q^{2j+1} & \text{if} \quad k = 2\ell \geq 2. \end{cases} \tag{5.7}$$

The Poincaré series $\mathsf{P}_q^0(t)$ *for the* G*-invariants* $\mathsf{T}(\mathsf{V}_q)^{\mathsf{G}}$ *in* $\mathsf{T}(\mathsf{V}_q) = \bigoplus_{k=0}^\infty \mathsf{V}_q^{\otimes k}$ *is*

$$\mathsf{P}_q^0(t) = \sum_{k=0}^{\infty} \dim (\mathsf{V}_q^{\otimes k})^{\mathsf{G}}\, t^k = \frac{1-qt+t^3}{(1-t)\,(1+t)\,(1-qt)}. \tag{5.8}$$

Proof. (a) From Theorem 2.3 and Table 5.2 we know that

$$\dim (\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k(\mathsf{G}) = \frac{1}{|\mathsf{G}|} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu| \chi_{\mathsf{V}}(\mathsf{c}_\mu)^k$$
$$= \frac{1}{(q-1)^2 q(q+1)} \Big((q-1)(q+1)^k + (q-1)(q^2-1)\, 1^k + \frac{1}{2} q(q+1)(q-1)(q-2)\, 2^k + \frac{1}{2} q^2 (q-1)^2\, 0^k \Big)$$
$$= \frac{1}{q(q-1)} \Big((q+1)^{k-1} + q(q-2)\cdot 2^{k-1} + q - 1 \Big) \quad \text{when } k \geq 1.$$

Therefore,

$$\begin{aligned}
\mathsf{P}^0(t) &= \sum_{k=0}^{\infty} \dim (\mathsf{V}^{\otimes k})^{\mathsf{G}}\, t^k = 1 + \frac{1}{q(q-1)} \left(\sum_{k=1}^{\infty} (q+1)^{k-1} + q(q-2)\cdot 2^{k-1} + (q-1) \right) t^k \\
&= 1 + \frac{1}{q(q-1)} \left(t \sum_{k=1}^{\infty} (q+1)^{k-1} t^{k-1} + q(q-2) t \sum_{k=1}^{\infty} 2^{k-1} t^{k-1} + (q-1) t \sum_{k=1}^{\infty} t^{k-1} \right) \\
&= 1 + \frac{1}{q(q-1)} \left(\frac{t}{1-(q+1)t} + \frac{q(q-2)t}{1-2t} + \frac{(q-1)t}{1-t} \right) \\
&= \frac{1-(q+3)t + (2q+3)t^2 - qt^3}{(1-t)\,(1-2t)\,(1-(q+1)t)}.
\end{aligned}$$

(b) Now for V_q and $k \geq 1$, we have

$$\dim (\mathsf{V}_q^{\otimes k})^{\mathsf{G}} = \frac{1}{|\mathsf{G}|} \sum_{\mu \in \Lambda(\mathsf{G})} |\mathcal{C}_\mu|\, \chi_{\mathsf{V}_q}(\mathsf{c}_\mu)^k$$
$$= \frac{1}{(q-1)^2 q(q+1)} \Big((q-1) q^k + (q-1)(q^2-1)\, 0^k + \frac{1}{2} q(q+1)(q-1)(q-2)\, 1^k + \frac{1}{2} q^2 (q-1)^2\, (-1)^k \Big)$$
$$= \frac{1}{2(q^2-1)} \Big(2q^{k-1} + q(q-1)(-1)^k + (q+1)(q-2) \Big)$$
$$= \begin{cases} \dfrac{q^{2\ell}-1}{q^2-1} = \displaystyle\sum_{j=0}^{\ell-1} q^{2j} & \text{if} \quad k = 2\ell+1 \geq 1, \\ 1 + q\dfrac{q^{2\ell-2}-1}{q^2-1} = 1 + q \displaystyle\sum_{j=0}^{\ell-2} q^{2j} & \text{if} \quad k = 2\ell \geq 2. \end{cases}$$

□

5.2 $\mathsf{G} = \mathrm{SL}_2(\mathbb{F}_q)$

For the group $\mathsf{G} = \mathrm{SL}_2(\mathbb{F}_q)$ (q odd), we introduce the following elements of G:

$$\mathsf{u}_x = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \ (x \neq 0), \quad \mathsf{v}_y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \quad \mathsf{w}_{x,y} = \begin{pmatrix} x & y \\ y\varepsilon & x \end{pmatrix} \ (x^2 - \varepsilon y^2 = 1). \tag{5.9}$$

We will use the information in the following table, which can be derived from [Mur, Chap. 3] or [FuH, Sec. 5.2]. As before, c_μ, $\mu \in \Lambda(\mathsf{G})$, is a representative of the conjugacy class $\mathcal{C}_\mu$ of G.

(5.10)

c_μ	$\pm\mathrm{I}$	$\mathsf{u}_x,\ x \neq \pm 1$	$\mathsf{v}_y,\ y = 1, \varepsilon$	$-\mathsf{v}_y,\ y = -1, -\varepsilon$	$\mathsf{w}_{x,y},\ x \neq \pm 1$
no. of such classes	2	$\frac{1}{2}(q-3)$	2	2	$\frac{1}{2}(q-1)$
$\lvert\mathcal{C}_\mu\rvert$	1	$q(q+1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q-1)$
$\chi_{\mathsf{V}}(\mathsf{c}_\mu)$	$q+1$	2	1	1	0
$\chi_{\mathsf{V}_q}(\mathsf{c}_\mu)$	q	1	0	0	-1

The order of $\mathsf{G} = \mathrm{SL}_2(\mathbb{F}_q)$ is $q(q-1)(q+1)$ and the order of its Borel subgroup B of upper triangular matrices is $q(q-1)$. Therefore, the induced G-module $\mathsf{V} = \mathsf{Ind}_{\mathsf{B}}^{\mathsf{G}}\mathsf{B}_0$ has dimension $q+1$, and $\mathsf{V} = \mathsf{G}_0 \oplus \mathsf{V}_q$, where V_q is the q-dimensional irreducible Steinberg module for G. Using this Table 5.10 and Theorem 2.3, we have the next result.

Theorem 5.11. *Assume* $\mathsf{G} = \mathrm{SL}_2(\mathbb{F}_q)$*, where* q *is odd.*

(a) *For* $\mathsf{V} = \mathsf{Ind}_{\mathsf{B}}^{\mathsf{G}}\mathsf{B}_0 = \mathsf{G}_0 \oplus \mathsf{V}_q$*, the* G*-module over* $\mathbb{C}$ *induced from the trivial module* B_0 *for the Borel subgroup* B *of upper-triangular matrices in* G*, we have*

$$\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \begin{cases} 1 & \text{when } k = 0 \\ \dfrac{1}{q(q-1)}\Big(2(q+1)^{k-1} + q(q-3)\cdot 2^{k-1} + 2(q-1)\Big) & \text{when } k \geq 1. \end{cases} \tag{5.12}$$

The Poincaré series for the G*-invariants* $\mathsf{T}(\mathsf{V})^{\mathsf{G}}$ *in* $\mathsf{T}(\mathsf{V}) = \bigoplus_{k=0}^{\infty} \mathsf{V}^{\otimes k}$ *is*

$$\mathsf{P}^0(t) = \sum_{k=0}^{\infty} \dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}}\, t^k = \frac{1-(q+3)t+(2q+3)t^2-(q-1)t^3}{(1-t)\,(1-2t)\,(1-(q+1)t)}. \tag{5.13}$$

(b) *For the Steinberg module* V_q*,* $\dim\,(\mathsf{V}_q^{\otimes k})^{\mathsf{G}} = 1$ *when* $k = 0$*, and*

$$\begin{aligned} \dim\,(\mathsf{V}_q^{\otimes k})^{\mathsf{G}} &= \frac{1}{2(q^2-1)}\Big(4q^{k-1} + (q-1)^2(-1)^k + (q-3)(q+1)\Big) \quad \text{when } k \geq 1, \\ &= \begin{cases} \dfrac{2(q^{2\ell}-1)}{q^2-1} = 2\displaystyle\sum_{j=0}^{\ell-1} q^{2j} & \text{if} \quad k = 2\ell+1 \geq 1, \\ 1 + 2q\dfrac{(q^{2\ell-2}-1)}{q^2-1} = 1 + 2\displaystyle\sum_{j=0}^{\ell-2} q^{2j+1} & \text{if} \quad k = 2\ell \geq 2. \end{cases} \end{aligned} \tag{5.14}$$

(b) *The Poincaré series* $\mathsf{P}_q^0(t)$ *for the* G*-invariants* $\mathsf{T}(\mathsf{V}_q)^{\mathsf{G}}$ *in* $\mathsf{T}(\mathsf{V}_q) = \bigoplus_{k=0}^{\infty} \mathsf{V}_q^{\otimes k}$ *is*

$$\mathsf{P}_q^0(t) = \sum_{k=0}^{\infty} \dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}}\, t^k = \frac{1-qt+2t^3}{(1+t)\,(1-t)\,(1-qt)}. \tag{5.15}$$

Proof. The proofs are analogous to those for Theorem 5.3 and are left to the reader. □

6 The case G is abelian and exponential generating functions

It is convenient to regard an arbitrary finite abelian group $(\mathsf{G},+)$ as a multiplicative group and write e^{a} for $\mathsf{a} \in \mathsf{G}$, so that the group operation is given by $\mathsf{e}^{\mathsf{a}}\mathsf{e}^{\mathsf{b}} = \mathsf{e}^{\mathsf{a}+\mathsf{b}}$, $\mathsf{a},\mathsf{b} \in \mathsf{G}$, where the sum $\mathsf{a}+\mathsf{b}$ is addition in G. The identity element is e^{0}. Since G is abelian, the irreducible G-modules are all one-dimensional, and we label them and the conjugacy classes with the elements of G. Thus, for $\mathsf{a} \in \mathsf{G}$, let $\mathsf{G}_{\mathsf{a}} = \mathbb{C}x_{\mathsf{a}}$, where $\mathsf{e}^{\mathsf{b}}x_{\mathsf{a}} = \chi_{\mathsf{a}}(\mathsf{b})x_{\mathsf{a}}$, and let χ_{a} denote the corresponding character. The characters satisfy

$$\chi_{\mathsf{a}}(\mathsf{b}+\mathsf{b}') = \chi_{\mathsf{a}}(\mathsf{b})\chi_{\mathsf{a}}(\mathsf{b}') \qquad \text{for all } \mathsf{a},\mathsf{b},\mathsf{b}' \in \mathsf{G}, \text{ and} \tag{6.1}$$

$$\chi_{\mathsf{a}+\mathsf{a}'}(\mathsf{b}) = \chi_{\mathsf{a}}(\mathsf{b})\chi_{\mathsf{a}'}(\mathsf{b}) \qquad \text{for all } \mathsf{a},\mathsf{a}',\mathsf{b} \in \mathsf{G}, \tag{6.2}$$

as $\mathsf{G}_{\mathsf{a}} \otimes \mathsf{G}_{\mathsf{a}'} \cong \mathsf{G}_{\mathsf{a}+\mathsf{a}'}$. for all $\mathsf{a},\mathsf{a}' \in \mathsf{G}$. Since $\chi_{\mathsf{a}}(\mathsf{b})\chi_{-\mathsf{a}}(\mathsf{b}) = \chi_{\mathsf{a}-\mathsf{a}'}(\mathsf{b}) = \chi_0(\mathsf{b}) = 1$ and $\chi_{\mathsf{a}}(0) = 1$ for all $\mathsf{a},\mathsf{b} \in \mathsf{G}$, the following hold:

$$\begin{aligned} \chi_{-\mathsf{a}}(\mathsf{b}) &= \chi_{\mathsf{a}}(\mathsf{b})^{-1} = \overline{\chi_{\mathsf{a}}(\mathsf{b})} \\ \chi_{\mathsf{a}}(-\mathsf{b}) &= \chi_{\mathsf{a}}(\mathsf{b})^{-1} = \overline{\chi_{\mathsf{a}}(\mathsf{b})}. \end{aligned} \tag{6.3}$$

By the fundamental theorem of finite abelian groups, we may suppose that $\mathsf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ where the r_j are powers of not necessarily distinct primes. The elements of G have the form e^{b}, where $\mathsf{b} = (b_1, b_2, \ldots, b_n)$ and $b_j \in \mathbb{Z}_{r_j}$ for each j. Set $\omega_j = \mathrm{e}^{2\pi i/r_j}$. Then $\mathsf{G}_{\mathsf{a}} = \mathbb{C}x_{\mathsf{a}}$, where

$$\mathsf{e}^{\mathsf{b}}x_{\mathsf{a}} = \chi_{\mathsf{a}}(\mathsf{b})x_{\mathsf{a}} \quad \text{and} \quad \chi_{\mathsf{a}}(\mathsf{b}) = \omega_1^{a_1b_1}\omega_2^{a_2b_2}\cdots\omega_n^{a_nb_n}. \tag{6.4}$$

Let ε_j be the n-tuple with 1 in position j and 0 for all its other components. Here we suppose that $\mathsf{V} = \mathsf{G}_{\varepsilon_1} \oplus \cdots \oplus \mathsf{G}_{\varepsilon_n}$, so for $\mathsf{b} = (b_1, b_2, \ldots, b_n) \in \mathsf{G}$, the character values are given by

$$\chi_{\mathsf{V}}(\mathsf{b}) = \sum_{j=1}^{n} \chi_{\varepsilon_j}(\mathsf{b}) = \sum_{j=1}^{n} \omega_j^{b_j} \qquad \chi_{\mathsf{V}^{\otimes k}}(\mathsf{b}) = \chi_{\mathsf{V}}(\mathsf{b})^k = \left(\sum_{j=1}^{n} \omega_j^{b_j}\right)^k. \tag{6.5}$$

We have the following corollary to Theorem 2.3:

Corollary 6.6. *The number of walks of k-steps from node a to node c on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ for $\mathsf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ and $\mathsf{V} = \mathsf{G}_{\varepsilon_1} \oplus \cdots \oplus \mathsf{G}_{\varepsilon_n}$ is*

$$(\mathsf{A}^k)_{\mathsf{a},\mathsf{c}} = \sum_{0 \le \ell_1, \ell_2, \ldots, \ell_n \le k} \binom{k}{\ell_1, \ell_2, \ldots, \ell_n} \tag{6.7}$$

where the sum is over all $\ell_1, \ell_2, \ldots, \ell_n$ such that $\ell_1 + \ell_2 + \cdots + \ell_n = k$ and $c_i - a_i \equiv \ell_i \bmod r_i$ for all $i \in [1,n] = \{1, 2, \ldots, n\}$.

Proof. Now

$$
\begin{aligned}
(\mathsf{A}^k)_{\mathsf{a},\mathsf{c}} &= \sum_{0\le \ell_1,\dots,\ell_n\le k} |\mathsf{G}|^{-1}\sum_{\mathsf{b}\in\mathsf{G}} \chi_{\mathsf{a}}(\mathsf{b})\chi_{\mathsf{V}}^k(\mathsf{b})\overline{\chi_{\mathsf{c}}(\mathsf{b})} = |\mathsf{G}|^{-1}\sum_{\mathsf{b}\in\mathsf{G}}\chi_{\mathsf{a}-\mathsf{c}}(\mathsf{b})\chi_{\mathsf{V}}^k(\mathsf{b})\\
&= |\mathsf{G}|^{-1}\sum_{\mathsf{b}\in\mathsf{G}} \omega_1^{(a_1-c_1)b_1}\cdots\omega_n^{(a_n-c_n)b_n}\left(\sum_{j=1}^n \omega_j^{b_j}\right)^k\\
&= |\mathsf{G}|^{-1}\sum_{\mathsf{b}\in\mathsf{G}} \omega_1^{(a_1-c_1)b_1}\cdots\omega_n^{(a_n-c_n)b_n}\left(\sum_{0\le \ell_1,\dots,\ell_n\le k}\binom{k}{\ell_1,\ell_2,\dots,\ell_n}\omega_1^{\ell_1 b_1}\cdots\omega_n^{\ell_n b_n}\right)\\
&= |\mathsf{G}|^{-1}\sum_{0\le \ell_1,\dots,\ell_n\le k}\binom{k}{\ell_1,\ell_2,\dots,\ell_n}\left(\left(\sum_{b_1=0}^{r_1-1}\omega_1^{(a_1-c_1+\ell_1)b_1}\right)\cdots\left(\sum_{b_n=0}^{r_n-1}\omega_1^{(a_n-c_n+\ell_n)b_n}\right)\right)\\
&= \sum_{0\le \ell_1,\dots,\ell_n\le k}\binom{k}{\ell_1,\ell_2,\dots,\ell_n}
\end{aligned}
\tag{6.8}
$$

by applying (3.1) repeatedly, where the sum is over all $\ell_1,\ell_2,\dots,\ell_n$ such that $\ell_1+\ell_2+\cdots+\ell_n=k$ and $\ell_i\equiv c_i-a_i \bmod r_i$ for all $i\in[1,n]$. □

6.1 Exponential generating functions

For $\mathsf{c}\in\mathsf{G}=\mathbb{Z}_{r_1}\times\mathbb{Z}_{r_2}\times\cdots\times\mathbb{Z}_{r_n}$ and $\mathsf{V}=\mathsf{G}_{\varepsilon_1}\oplus\cdots\oplus\mathsf{G}_{\varepsilon_n}$, let

$$\mathrm{g}^{\mathsf{c}}(t) := \sum_{k=0}^{\infty}(\mathsf{A}^k)_{0,\mathsf{c}}\frac{t^k}{k!}$$

denote the exponential generating function for walks of k steps from 0 to c on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ (and also for the multiplicity of G_{c} in $\mathsf{V}^{\otimes k}$ and for dimension of the irreducible module $\mathsf{Z}_k^{\mathsf{c}}(\mathsf{G})$ for the centralizer algebra). We determine an expression for $\mathrm{g}^{\mathsf{c}}(t)$ in terms of generalized hyperbolic functions.

The *generalized hyperbolic function* $\mathsf{h}_j(t,r)$ for $j\in\mathbb{Z}$ is defined by

$$\mathsf{h}_j(t,r) := r^{-1}\sum_{m=0}^{r-1}\omega^{(1-j)m}\mathrm{e}^{\omega^m t}, \tag{6.9}$$

where $\omega=\mathrm{e}^{2\pi i/r}$. In particular,

$$\mathsf{h}_1(t,r) = r^{-1}\sum_{m=0}^{r-1}\mathrm{e}^{\omega^m t}, \tag{6.10}$$

so that $\mathsf{h}_1(t,1)=\mathrm{e}^t$ and $\mathsf{h}_1(t,2)=\cosh t$. Because

$$\mathsf{h}_{j+r}(t,r)=\mathsf{h}_j(t,r)\qquad \text{for } j\in\mathbb{Z},$$

there are r distinct generalized hyperbolic functions $\mathsf{h}_j(t,r)$ for a fixed value of r.

Theorem 6.11. *For $\mathsf{G}=\mathbb{Z}_{r_1}\times\mathbb{Z}_{r_2}\times\cdots\times\mathbb{Z}_{r_n}$ and $\mathsf{c}=(c_1,c_2,\dots,c_n)\in\mathsf{G}$, the exponential generating function for the number of walks of k steps from 0 to c on $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ is*

$$\mathrm{g}^{\mathsf{c}}(t)=\sum_{k=0}^{\infty}(\mathsf{A}^k)_{0,\mathsf{c}}\frac{t^k}{k!}=\mathsf{h}_{1+c_1}(t,r_1)\mathsf{h}_{1+c_2}(t,r_2)\,\cdots\,\mathsf{h}_{1+c_n}(t,r_n).$$

Before giving the proof, we note the following immediate consequences.

Corollary 6.12. *For* $\mathsf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ *and* $\mathsf{V} = \mathsf{G}_{\varepsilon_1} \oplus \cdots \oplus \mathsf{G}_{\varepsilon_n}$,

(a) $\mathrm{g}^{\mathsf{0}}(t) = \sum_{k=0}^{\infty} (\mathsf{A}^k)_{\mathsf{0},\mathsf{0}} \frac{t^k}{k!} = \mathsf{h}_1(t,r_1)\mathsf{h}_1(t,r_2)\,\cdots\,\mathsf{h}_1(t,r_n).$

(b) *When* $\mathsf{G} = \mathbb{Z}_r^n$, *then* $\mathrm{g}^{\mathsf{0}}(t) = \mathsf{h}_1(t,r)^n$.

Remark 6.13. Part (b) of this corollary generalizes [BM, Cor. 4.29], which says that the generating function for the number of walks on a hypercube of order n is given by $\mathrm{g}^{\mathsf{0}}(t) = (\cosh t)^n = \mathsf{h}_1(t,2)^n$. Theorem 4.25 of [BM] shows that for $\mathbb{Z}_2^n$,

$$\mathrm{g}^{\mathsf{c}}(t) = (\cosh t)^{r-\mathfrak{h}(\mathsf{c})}(\sinh t)^{\mathfrak{h}(\mathsf{c})},$$

where $\mathfrak{h}(\mathsf{c})$ is the Hamming weight of c (the number of ones in c). This follows directly from Theorem 6.11, since each component of c equal to 1 contributes a factor $\mathsf{h}_2(t,2) = \sinh t$, and each component of c equal to 0 gives a factor $\mathsf{h}_1(t,2) = \cosh t$.

Proof of Theorem 6.11. Observe that by (6.5) and Corollary 6.6,

$$\begin{aligned}
\mathrm{g}^{\mathsf{c}}(t) &= \sum_{k=0}^{\infty} (\mathsf{A})^k_{\mathsf{0},\mathsf{c}} \frac{t^k}{k!} \\
&= |\mathsf{G}|^{-1} \sum_{k=0}^{\infty} \sum_{\mathsf{b}=(b_1,\dots,b_n)\in\mathsf{G}} \omega_1^{-b_1c_1}\,\cdots\,\omega_n^{-b_nc_n} \left(\sum_{j=1}^{n} \omega_j^{b_j}\right)^k \frac{t^k}{k!} \\
&= r_1^{-1}\cdots r_n^{-1} \sum_{k=0}^{\infty}\sum_{\mathsf{b}\in\mathsf{G}} \omega_1^{-b_1c_1}\cdots\omega_n^{-b_nc_n} \left(\sum_{\ell_1+\cdots+\ell_n=k} \frac{\omega_1^{b_1\ell_1}t^{\ell_1}}{\ell_1\,!}\,\cdots\,\frac{\omega_n^{b_n\ell_n}t^{\ell_n}}{\ell_n\,!}\right) \\
&= \left(r_1^{-1}\sum_{b_1=0}^{r_1-1}\sum_{\ell_1=0}^{\infty} \omega^{-b_1c_1}\frac{\omega_1^{b_1\ell_1}t^{\ell_1}}{\ell_1\,!}\right) \times\cdots\times \left(r_n^{-1}\sum_{b_n=0}^{r_n-1}\sum_{\ell_n=0}^{\infty} \omega_n^{-b_nc_n}\frac{\omega_n^{b_n\ell_n}t^{\ell_n}}{\ell_n\,!}\right) \\
&= \left(r_1^{-1}\sum_{b_1=0}^{r_1-1} \omega_1^{-b_1c_1}\mathrm{e}^{\omega_1^{b_1}t}\right) \times\cdots\times \left(r_n^{-1}\sum_{b_n=0}^{r_n-1} \omega^{-b_nc_n}\mathrm{e}^{\omega_n^{b_n}t}\right) \\
&= \mathsf{h}_{1+c_1}(t,r_1)\,\mathsf{h}_{1+c_2}(t,r_2)\,\cdots\,\mathsf{h}_{1+c_n}(t,r_n).
\end{aligned}$$

□

Using (3.1) and the definition of the generalized hyperbolic function $\mathsf{h}_j(t,r)$, one sees that the Taylor series expansion of $\mathsf{h}_j(t,r)$ is given by

$$\mathsf{h}_j(t,r) = \sum_{m=0}^{\infty} \frac{t^{mr+j-1}}{(mr+j-1)!} \tag{6.14}$$

Suppose $\mathsf{c} = (c_1,c_2,\dots,c_n) \in \mathsf{G} = \mathbb{Z}_{r_1}\times\mathbb{Z}_{r_2}\times\cdots\times\mathbb{Z}_{r_n}$, where $0 \le c_j < r_j$ for all j, and let $|\mathsf{c}| = \sum_{j=1}^{n} c_j$. We have shown in Theorem 6.11 that the exponential generating function $\mathrm{g}^{\mathsf{c}}(t)$ is given by

$$\mathrm{g}^{\mathsf{c}}(t) = \sum_{k=0}^{\infty} (\mathsf{A}^k)_{\mathsf{0},\mathsf{c}} \frac{t^k}{k!} = \mathsf{h}_{1+c_1}(t,r_1)\mathsf{h}_{1+c_2}(t,r_2)\,\cdots\,\mathsf{h}_{1+c_n}(t,r_n).$$

Combining that with the expressions coming from (6.14), we have

$$
\begin{aligned}
\mathsf{g}^{\mathsf{c}}(t) &= \mathsf{h}_{1+c_1}(t,r_1)\mathsf{h}_{1+c_2}(t,r_2)\ \cdots\ \mathsf{h}_{1+c_n}(t,r_n) \\
&= \left(\sum_{q_1=0}^{\infty} \frac{t^{q_1r_1+c_1}}{(q_1r_1+c_1)!}\right)\left(\sum_{q_2=0}^{\infty} \frac{t^{q_2r_2+c_2}}{(q_2r_2+c_2)!}\right)\cdots\left(\sum_{\ell_n=0}^{\infty} \frac{t^{q_nr_n+c_n}}{(q_nr_n+c_n)!}\right) \\
&= \sum_{k=0}^{\infty} \sum_{q_1r_1+\cdots+q_nr_n+|\mathsf{c}|=k} \frac{k!}{(q_1r_1+c_1)!(q_2r_2+c_2)!\cdots(q_nr_n+c_n)!}\ \frac{t^k}{k!}
\end{aligned}
$$

Setting $q_ir_i+c_i=\ell_i$ for $i=1,\ldots,n$ gives the result in Corollary 6.6 with $\mathsf{a}=0$, which provides a formula for the dimension of the irreducible module $\mathsf{Z}_k^{\mathsf{c}}(\mathsf{G})$ for the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$:

$$
\dim \mathsf{Z}_k^{\mathsf{c}}(\mathsf{G}) = (\mathsf{A}^k)_{\mathsf{0},\mathsf{c}} = \sum_{0\le \ell_1,\ell_2,\ldots,\ell_n\le k} \binom{k}{\ell_1,\ell_2,\ldots,\ell_n}. \tag{6.15}
$$

The sum is over all $0\le \ell_1,\ell_2,\ldots,\ell_n\le k$ such that $\ell_1+\cdots+\ell_n=k$ and $\ell_i\equiv c_i \mod r_i$ for all $i\in[1,n]$. In particular, when $\mathsf{G}=\mathbb{Z}_{r_1}\times\mathbb{Z}_{r_2}\times\cdots\times\mathbb{Z}_{r_n}$ and $\mathsf{c}=\mathsf{0}$, then

$$
\dim\,(\mathsf{V}^{\otimes k})^{\mathsf{G}} = \dim \mathsf{Z}_k^{\mathsf{0}}(\mathsf{G}) = \sum_{0\le \ell_1,\ell_2,\ldots,\ell_n\le k} \binom{k}{\ell_1,\ \ell_2,\ \ldots,\ \ell_n}, \tag{6.16}
$$

where $\ell_1+\ell_2+\cdots+\ell_n=k$ and $\ell_i\equiv 0 \bmod r_i$ for all $i\in[1,n]$.

An alternate approach to the result in (6.15) is via characters. For $\mathsf{G}=\mathbb{Z}_{r_1}\times\cdots\times\mathbb{Z}_{r_n}$ and $\mathsf{V}=\mathsf{G}_{\varepsilon_1}\oplus\cdots\oplus\mathsf{G}_{\varepsilon_n}$, where $\mathsf{G}_{\varepsilon_j}=\mathbb{C}x_{\varepsilon_j}$ for all j, the character of the kth tensor power of V is given by

$$
\begin{aligned}
\chi_{\mathsf{V}^{\otimes k}} &= \chi_{\mathsf{V}}^k = (\chi_{\varepsilon_1}+\cdots+\chi_{\varepsilon_n})^k \\
&= \sum_{\substack{0\le \ell_1,\ell_2,\ldots,\ell_n\le k\\ \ell_1+\ell_2+\cdots+\ell_n=k}} \binom{k}{\ell_1,\ell_2,\ \ldots,\ \ell_n}\chi_{\varepsilon_1}^{\ell_1}\cdots\chi_{\varepsilon_n}^{\ell_n} \\
&= \sum_{\substack{0\le \ell_1,\ell_2,\ldots,\ell_n\le k\\ \ell_1+\ell_2+\cdots+\ell_n=k}} \binom{k}{\ell_1,\ell_2,\ \ldots,\ \ell_n}\chi_{\ell_1\varepsilon_1+\ell_2\varepsilon_2+\cdots+\ell_n\varepsilon_n}.
\end{aligned}
$$

Now for $\mathsf{c}=(c_1,c_2,\ldots,c_n)$ with $0\le c_i<r_i$ for all $i\in[1,n]$, the multiplicity of the character χ_{c} in this expression is exactly the number of n-tuples $(\ell_1,\ell_2,\ldots,\ell_n)$ such that $\ell_i\equiv c_i \bmod r_i$ for all $i\in[1,n]$, as in (6.15).

Example 6.17. *Consider* $\mathsf{G}=\mathbb{Z}_4\times\mathbb{Z}_2$ *and the tensor power* $\mathsf{V}^{\otimes 6}$ *for* $\mathsf{V}=\mathsf{G}_{\varepsilon_1}\oplus\mathsf{G}_{\varepsilon_2}$. *Then*

$$
\begin{aligned}
(\chi_{\varepsilon_1}+\chi_{\varepsilon_2})^6 &= \chi_{6\varepsilon_1}+6\chi_{5\varepsilon_1+\varepsilon_2}+15\chi_{4\varepsilon_1+2\varepsilon_2}+20\chi_{3\varepsilon_1+3\varepsilon_2} \\
&\qquad +15\chi_{2\varepsilon_1+4\varepsilon_2}+6\chi_{\varepsilon_1+5\varepsilon_2}+\chi_{6\varepsilon_2} \\
&= 16\chi_{2\varepsilon_1}+12\chi_{\varepsilon_1+\varepsilon_2}+16\chi_{\mathsf{0}}+20\chi_{3\varepsilon_1+\varepsilon_2}.
\end{aligned}
$$

Thus, $\dim \mathsf{Z}_6^{(2,0)}(\mathsf{G})=16$, $\dim \mathsf{Z}_6^{(1,1)}(\mathsf{G})=12$, $\dim \mathsf{Z}_6^{(0,0)}(\mathsf{G})=16$, *and* $\dim \mathsf{Z}_6^{(3,1)}(\mathsf{G})=20$.

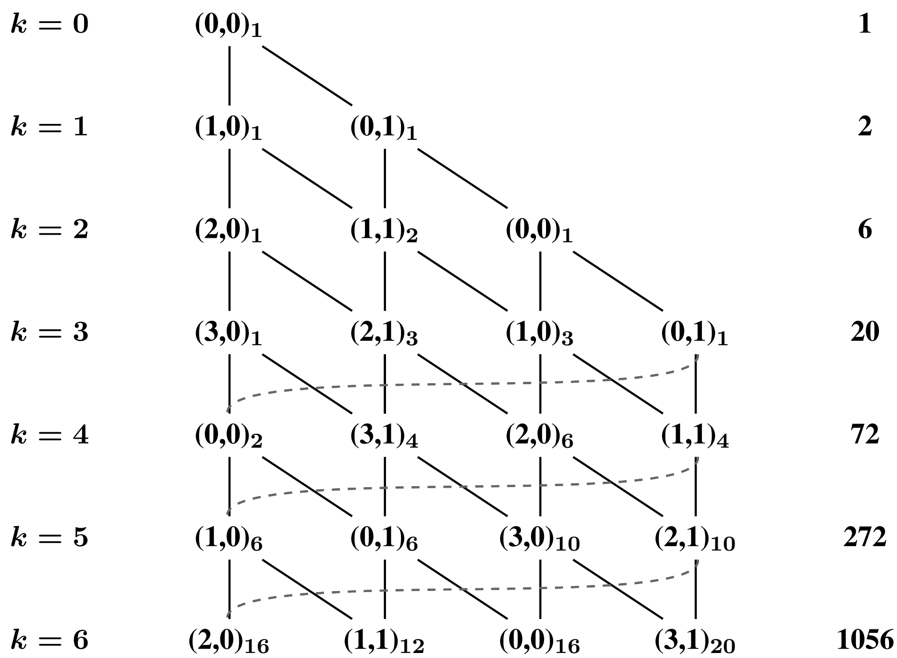


Figure 6: Levels $k = 0, 1, \ldots, 6$ of the Bratteli diagram for $\mathbb{Z}_4 \times \mathbb{Z}_2$

6.2 The Bratteli diagram and a basis for $\mathsf{Z}_k(\mathsf{G})$ when $\mathsf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ and $\mathsf{V} = \mathsf{G}_{\varepsilon_1} \oplus \cdots \oplus \mathsf{G}_{\varepsilon_n}$

A walk of k steps on the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ from 0 to c corresponds to a path $\left(\mathsf{c}^{(0)}, \mathsf{c}^{(1)}, \ldots, \mathsf{c}^{(k)}\right)$ on the Bratteli diagram $\mathcal{B}_\mathsf{V}(\mathsf{G})$ starting at $\mathsf{c}^{(0)} = 0 = (0, \ldots, 0)$ at level 0 and ending at $\mathsf{c} = \mathsf{c}^{(k)}$ at level k such that $\mathsf{c}^{(i)} \in \mathsf{G}$ for each $1 \leq i \leq k$, and $\mathsf{c}^{(i)} = \mathsf{c}^{(i-1)} + \varepsilon_{\gamma_i}$ for some $\gamma_i \in [1, n]$, where $\mathsf{c}^{(i)}$ is connected to $\mathsf{c}^{(i-1)}$ by the edge corresponding to γ_i in $\mathcal{R}_\mathsf{V}(\mathsf{G})$. The subscript on node c at level k in $\mathcal{B}_\mathsf{V}(\mathsf{G})$ indicates the number of such paths, which is the multiplicity of the irreducible G-module G_c in $\mathsf{V}^{\otimes k}$ and also equal to the dimension of the irreducible $\mathsf{Z}_k(\mathsf{G})$-module $\mathsf{Z}_k^\mathsf{c}(\mathsf{G})$. The sum of the squares of those dimensions at level k is the number on the right, which is the dimension of the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$. Levels 0,1,…,6 of the Bratteli diagram for $\mathbb{Z}_4 \times \mathbb{Z}_2$ are displayed in Figure 6. The nodes of the diagram correspond to elements $\mathsf{c} = (c_1, c_2) \in \mathbb{Z}_4 \times \mathbb{Z}_2$ and have $c_1 \in \{0, 1, 2, 3\}$ and $c_2 \in \{0, 1\}$.

Remark 6.18. The subscripts in the last row of the Bratteli diagram in Figure 6, exactly match with the dimensions determined in Example 6.17. The sequence of numbers in the right-hand column of Figure 6 (i.e. the dimension $d(k)$ of the centralizer algebra $\mathsf{Z}_k(\mathbb{Z}_4 \times \mathbb{Z}_2)$) satisfies $d(k) = a(k-1)$ in sequence [OEIS, A063376], where $a(-1) = 1$ and $a(k-1) = 2^{k-1} + 4^{k-1}$ for $k \geq 1$. Among the objects that $a(k-1)$ counts is the number of closed walks of length $2k$ at a vertex of the circular graph on 8 nodes, which is the same as $\dim \mathsf{Z}_k(\mathsf{G})$ for $\mathsf{G} = \mathbb{Z}_8$ and $\mathsf{V} = \mathsf{G}_1 \oplus \mathsf{G}_7$ (see Section 3.1).

Much of the next result is evident from the above considerations.

Theorem 6.19. *Assume* $\mathsf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$ *and* $\mathsf{V} = \mathsf{G}_{\varepsilon_1} \oplus \cdots \oplus \mathsf{G}_{\varepsilon_n}$. *Then the following hold:*

(i) *For* $\mathsf{c} = (c_1, \ldots, c_n) \in \mathsf{G}$, *a basis for the irreducible* $\mathsf{Z}_k(\mathsf{G})$*-module* $\mathsf{Z}_k^\mathsf{c}(\mathsf{G}) \subseteq \mathsf{V}^{\otimes k}$ *is*

$$\left\{ x(\gamma) := x_{\varepsilon_{\gamma_1}} \otimes \cdots \otimes x_{\varepsilon_{\gamma_k}} \;\middle|\; \gamma_i \in [1, n] \text{ for all } i \in [1, k], \text{ and } \textstyle\sum_{i=1}^k \varepsilon_{\gamma_i} = \mathsf{c} \right\}.$$

(ii) $\mathsf{e}^{\mathsf{a}}x(\gamma) = \chi_{\mathsf{c}}(\mathsf{a})x(\gamma)$ *for all* $\mathsf{a} \in \mathsf{G}$ *and all* $x(\gamma)$ *in* (i), *where* $\chi_{\mathsf{c}}(\mathsf{a}) = \prod_{j=1}^{n} \omega_j^{a_j c_j}$ *and* $\omega_j = \mathrm{e}^{2\pi i/r_j}$ *for all* $j \in [1, n]$, *so that* $\mathsf{Z}_k^{\mathsf{c}}(\mathsf{G})$ *is also a* G*-submodule of* $\mathsf{V}^{\otimes k}$*; it is the sum of all the copies of the irreducible* G*-module* G_{c} *in* $\mathsf{V}^{\otimes k}$.

(iii) *For* $\gamma = (\gamma_1, \dots, \gamma_k), \beta = (\beta_1, \dots, \beta_k) \in [1, n]^k$ *with* $\sum_{i=1}^{k} \varepsilon_{\gamma_i} = \sum_{i=1}^{k} \varepsilon_{\beta_i}$, *let* $\mathsf{E}_\gamma^\beta \in \mathsf{End}(\mathsf{V}^{\otimes k})$ *be defined by* $\mathsf{E}_\gamma^\beta x(\alpha) = \delta_{\alpha,\gamma} x(\beta)$ *for* $\alpha \in [1, n]^k$. *Then* $\mathsf{E}_\vartheta^\eta \mathsf{E}_\gamma^\beta = \delta_{\beta,\vartheta} \mathsf{E}_\eta^\beta$ *for all such* ϑ, η, *and the* E_γ^β *determine a basis for* $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$.

Proof. From the calculation below it is easy to see that the transformations E_γ^β for $\gamma, \beta \in [1, n]^k$ as in (iii) of Theorem 6.19 commute with the action of G on $\mathsf{V}^{\otimes k}$, hence belong to $\mathsf{Z}_k(\mathsf{G})$. Indeed, suppose $\alpha \in [1, n]^k$ with $\sum_{i=1}^{k} \varepsilon_{\alpha_i} = \mathsf{c}' \in \mathsf{G}$, and assume $\mathsf{a} \in \mathsf{G}$. Then

$$\begin{aligned} \mathsf{e}^{\mathsf{a}} \mathsf{E}_\gamma^\beta\big(x(\alpha)\big) &= \delta_{\alpha,\gamma} \mathsf{e}^{\mathsf{a}} x(\beta) = \delta_{\alpha,\gamma} \chi_{\mathsf{c}}(\mathsf{a}) x(\beta) \\ \mathsf{E}_\gamma^\beta \mathsf{e}^{\mathsf{a}}\big(x(\alpha)\big) &= \chi_{\mathsf{c}'}(\mathsf{a}) \delta_{\alpha,\gamma} x(\beta). \end{aligned}$$

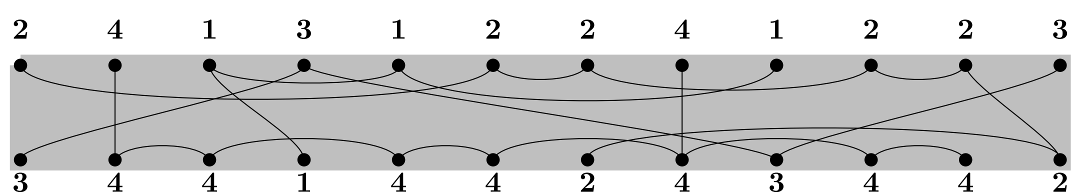
Both expressions are 0 when $\alpha \neq \gamma$, and when $\alpha = \gamma$, then $\mathsf{c}' = \mathsf{c}$, and the two expressions are identical. The transformations E_γ^β are clearly linearly independent. The number of $\gamma = (\gamma_1, \dots, \gamma_k)$ such that $\sum_{i=1}^{k} \varepsilon_{\gamma_i} = \mathsf{c}$ is the number of paths from 0 at level 0 to c at level k of the Bratteli diagram $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$, which is $\dim \mathsf{Z}_k^{\mathsf{c}}(\mathsf{G})$. Therefore, the number of E_γ^β in (iii) equals $\big(\dim \mathsf{Z}_k^{\mathsf{c}}(\mathsf{G})\big)^2$, and since $\dim \mathsf{Z}_k(\mathsf{G}) = \sum_{\mathsf{c} \in \mathsf{G}} \big(\dim \mathsf{Z}_k^{\mathsf{c}}(\mathsf{G})\big)^2$, taking the union of the sets of transformations E_γ^β as c ranges over all the elements of G will give a basis for $\mathsf{Z}_k(\mathsf{G})$. □

Remark 6.20. The condition $\sum_{i=1}^{k} \varepsilon_{\gamma_i} = \sum_{i=1}^{k} \varepsilon_{\beta_i}$ in Theorem 6.19 is equivalent to saying $(\#\gamma_i = j) \equiv (\#\beta_i = j) \bmod r_j$ for all $j = 1, \dots, n$. That interpretation leads to the diagrammatic point of view that we describe next.

6.3 A diagram basis for $\mathsf{Z}_k(\mathsf{G})$ for $\mathsf{G} = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_n}$

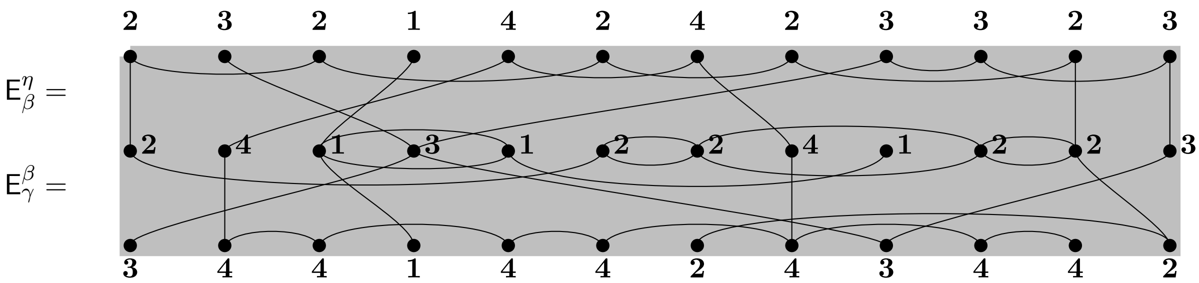
In this section, we present a realization $\mathsf{Z}_k(\mathsf{G})$ as a diagram algebra. We identify the basis element E_γ^β with a diagram having two rows of k nodes. The components of $\gamma = (\gamma_1, \dots, \gamma_k)$, which lie in $[1, n]$, label the nodes on the bottom row, and those of $\beta = (\beta_1, \dots, \beta_k)$ the top row. Nodes having the same labels are connected, but the way the edges are drawn is immaterial. What matters is that nodes with identical labels are all connected somehow, and those with different labels are not. Thus, for $\gamma = (3, 4, 4, 1, 4, 4, 2, 4, 3, 4, 4, 2)$ and $\beta = (2, 4, 1, 3, 1, 2, 2, 4, 1, 2, 2, 3)$ in $[1, 4]^{12}$, the basis element E_γ^β is identified with the diagram

$\mathsf{E}_\gamma^\beta =$



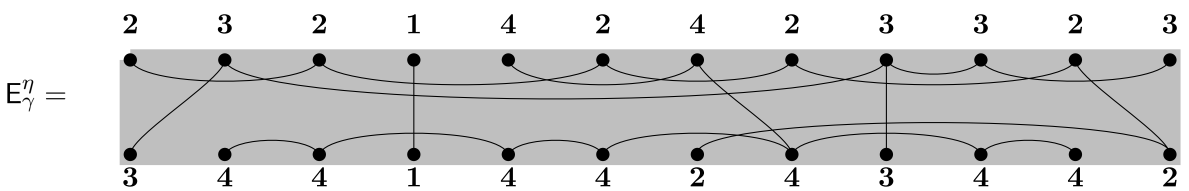
. (6.21)

Observe that in this example $(\#\gamma_i = j) \equiv (\#\beta_i = j) \bmod r_j$ for $r_1 = 2, r_2 = 3, r_3 = 2, r_4 = 5$. Thus, E_γ^β is a legitimate basis element for $\mathsf{Z}_{12}(\mathsf{G})$, where $\mathsf{G} = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Since $\mathsf{E}_\vartheta^\eta \mathsf{E}_\gamma^\beta = \delta_{\beta,\vartheta} \mathsf{E}_\gamma^\eta$, the top row of E_γ^β must exactly match the bottom row of $\mathsf{E}_\vartheta^\eta$ to achieve a nonzero product. Thus for E_β^η with $\eta = (2, 3, 2, 1, 4, 2, 4, 2, 3, 3, 2, 3)$, we place the diagram for E_β^η on top of the diagram for E_γ^β and concatenate the two diagrams, as pictured below.



(6.22)

The result is $\mathsf{E}^{\eta}_{\beta}\mathsf{E}^{\beta}_{\gamma} = \mathsf{E}^{\eta}_{\gamma}$ where



(6.23)

7 Appendix I

Let $\mathcal{G}$ be a directed graph with finite vertex set Γ and adjacency matrix $\mathsf{A} = (a_{\alpha,\gamma})_{\alpha,\gamma\in\Gamma}$. Then $a_{\alpha,\gamma}$ is the number of edges (arrows) from α to γ in $\mathcal{G}$, and $(\mathsf{A}^k)_{\alpha,\gamma}$ is the number of walks of k steps from α to γ on $\mathcal{G}$. We consider the corresponding generating function for the number of walks from α to γ,

$$\mathsf{w}_{\alpha,\gamma}(t) = \sum_{k=0}^{\infty} (\mathsf{A}^k)_{\alpha,\gamma}\, t^k,$$

where $\mathsf{A}^0 = \mathrm{I}$, the identity matrix.

Proposition 7.1. *Let δ_α be the $|\Gamma| \times 1$ matrix with 1 in row α and zeros elsewhere so that entry γ of δ_α is the Kronecker delta $\delta_{\alpha,\gamma}$, and assume $\mathsf{M}^{\gamma}_{\alpha}$ is the matrix $\mathrm{I} - t\mathsf{A}^{\mathtt{T}}$ with column γ replaced by δ_α (here $\mathtt{T}$ denotes the transpose). Then*

$$\mathsf{w}_{\alpha,\gamma}(t) = \frac{\det(\mathsf{M}^{\gamma}_{\alpha})}{\det(\mathrm{I} - t\mathsf{A})}.$$

Proof. First a simple observation: $(\mathsf{A}^{k+1})_{\alpha,\gamma} = \sum_{\beta\in\Gamma}(\mathsf{A}^k)_{\alpha,\beta}\, a_{\beta,\gamma}$, for all $k \geq 0$. Then

$$
\begin{aligned}
\mathsf{w}_{\alpha,\gamma}(t) &= \sum_{k=0}^{\infty} (\mathsf{A}^k)_{\alpha,\gamma}\, t^k \\
&= \delta_{\alpha,\gamma} + t \sum_{k\geq 1} (\mathsf{A}^k)_{\alpha,\gamma}\, t^{k-1} \\
&= \delta_{\alpha,\gamma} + t \sum_{k\geq 0} (\mathsf{A}^{k+1})_{\alpha,\gamma}\, t^k \\
&= \delta_{\alpha,\gamma} + t \sum_{k\geq 0} \left(\sum_{\beta\in\Gamma} (\mathsf{A}^k)_{\alpha,\beta}\, a_{\beta,\gamma} \right) t^k \\
&= \delta_{\alpha.\gamma} + t \sum_{\beta\in\Gamma} a_{\beta,\gamma} \left(\sum_{k\geq 0} (\mathsf{A}^k)_{\alpha,\beta}\, t^k \right) \\
&= \delta_{\alpha,\gamma} + t \sum_{\beta\in\Gamma} a_{\beta,\gamma}\, \mathsf{w}_{\alpha,\beta}(t).
\end{aligned}
$$

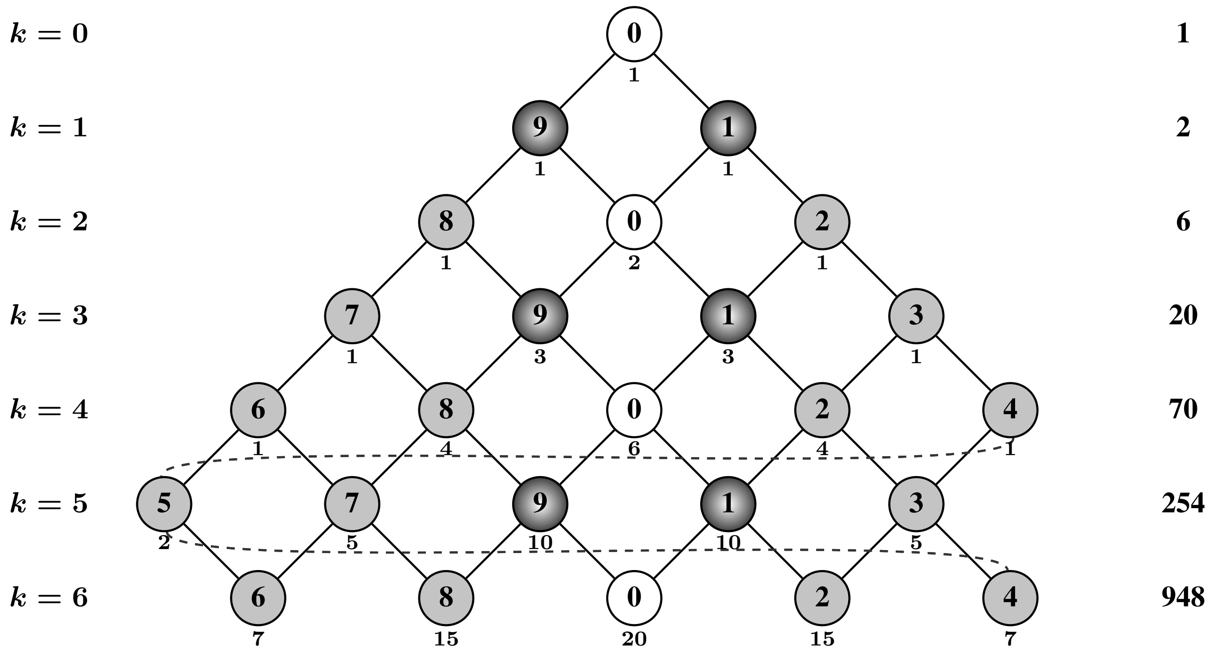
Letting w_α be the $|\Gamma| \times 1$ matrix with $\mathsf{w}_{\alpha,\gamma}(t)$ in row γ, we see from the above calculation that the matrix equation $\mathsf{w}_\alpha^{\mathsf{T}}\,(\mathrm{I} - t\mathsf{A}) = \delta_\alpha^{\mathsf{T}}$, or equivalently, $\left(\mathrm{I} - t\mathsf{A}^{\mathsf{T}}\right) \mathsf{w}_\alpha = \delta_\alpha$ holds. It follows then from Cramer's rule that

$$
\mathsf{w}_{\alpha,\gamma}(t) = \frac{\mathsf{det}(\mathsf{M}_\alpha^\gamma)}{\mathsf{det}(\mathrm{I} - t\mathsf{A}^{\mathsf{T}})} = \frac{\mathsf{det}(\mathsf{M}_\alpha^\gamma)}{\mathsf{det}(\mathrm{I} - t\mathsf{A})}.
$$

□

8 Appendix II

Levels 0-6 of the Bratteli diagram for the cyclic group $\mathsf{G} = \mathbb{Z}_{10}$ and its module $\mathsf{V} = \mathsf{G}_1 \oplus \mathsf{G}_9$ are pictured below. The label inside the node is the index of the irreducible G-module. The trivial module is indicated in white, and the module V in black. The subscript on node λ on level k indicates the number of paths from 0 at the top to λ at level k (equivalently, the number of walks from 0 to λ of k steps on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$; also the multiplicity of G_λ in $\mathsf{V}^{\otimes k}$; also the dimension of the irreducible module $\mathsf{Z}_k^\lambda(\mathsf{G})$ for the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$).



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