FLOW POLYTOPES AND THE SPACE OF DIAGONAL HARMONICS

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ABSTRACT. A result of Haglund implies that the (q,t)-bigraded Hilbert series of the space of diagonal harmonics is a (q,t)-Ehrhart function of the flow polytope of a complete graph with netflow vector $(-n,1,\dots,1)$. We study the (q,t)-Ehrhart functions of flow polytopes of threshold graphs with arbitrary netflow vectors. Our results generalize previously known specializations of the mentioned bigraded Hilbert series at $t=1$, 0, and q^{-1}. As a corollary to our results, we obtain a proof of a conjecture of Armstrong, Garsia, Haglund, Rhoades and Sagan about the (q,q^{-1})-Ehrhart function of the flow polytope of a complete graph with an arbitrary netflow vector.

1. Introduction

The *space of diagonal harmonics*

$$DH_n = \left\{ f \in \mathbf{C}[x_1,\dots,x_n,y_1,\dots,y_n] \;\middle|\; \sum_{i=1}^{n} \frac{\partial^h}{\partial x_i^h}\frac{\partial^k}{\partial y_i^k} f = 0 \text{ for all } h+k>0 \right\}$$

was introduced by Garsia and Haiman [8] in their study of Macdonald polynomials. Haiman [18] proved using algebro-geometric arguments that it has dimension $(n+1)^{n-1}$ as a vector space over $\mathbf{C}$. The space DH_n is naturally bigraded by the degree of the variables x_i and y_j. Thus, one can obtain a q,t-analogue of $(n+1)^{n-1}$ by considering the bigraded Hilbert series of DH_n, which we denote by $\mathrm{Hilb}_{q,t}(DH_n)$. This is a symmetric polynomial in q and t with nonnegative coefficients.

The number $(n+1)^{n-1}$ counts spanning trees of the complete graph on $n+1$ vertices or parking functions of size n. A combinatorial model for this bigraded Hilbert series in terms of these objects was conjectured by Haglund and Loehr [16] in 2002 and settled in 2015 by Carlsson and Mellit [4] in their proof of the more general Shuffle Conjecture [15]. Stated in terms of parking functions, the result is the following (see [16, Conj. 2]).

Theorem 1.1 (Carlsson–Mellit [4], Hilbert series conjecture of Haglund–Loehr [16])**.**

$$\mathrm{Hilb}_{q,t}(DH_n) = \sum_{p\in\mathcal{P}_n} q^{\mathrm{area}(p)} t^{\mathrm{dinv}(p)}, \tag{1.1}$$

where $\mathcal{P}_n$ denotes parking functions of size n.

For more background on DH_n and the Shuffle Conjecture see [2,12,13]. For the definition of area and dinv on parking functions see [16, §1, §2]. Special cases of this Hilbert series

Mészáros was partially supported by a National Science Foundation Grant (DMS 1501059). Morales was partially supported by an AMS-Simons travel grant.

when $t = 1, 0, q^{-1}$ are combinatorially appealing:

$$\text{Hilb}_{q,1}(DH_n) = \sum_{p \in \mathcal{P}_n} q^{\text{area}(p)}, \tag{1.2}$$

$$\text{Hilb}_{q,0}(DH_n) = \sum_{w \in S_n} q^{\text{inv}(w)} = [n]_q!, \tag{1.3}$$

$$q^{\binom{n}{2}} \text{Hilb}_{q,q^{-1}}(DH_n) = [n+1]_q^{n-1}, \tag{1.4}$$

where S_n is the symmetric group of size n, $\text{inv}(w)$ is the number of inversions of the permutation w, and $[k]_q = 1 + q + \cdots + q^{k-1}$. The right hand side of (1.1) evaluated at $(q, 1)$ is trivially the right hand side of (1.2). The fact that the right hand side of (1.1) evaluated at $(q, 0)$ yields the right hand side of (1.3) follows from [13, Theorem 5.3] together with the fact that the major index and number of inversions are equidistributed over S_n [29, §1.4]. Finally, Loehr [21] showed the case (q, q^{-1}) combinatorially. Showing directly that the Hilbert series has these evaluations is highly nontrivial and is due to Haiman [17].

Before the proof of Haglund and Loehr's Hilbert series conjecture in [4], Haglund [14] gave an expression for the Hilbert series as a weighted sum over certain upper triangular matrices called *Tesler matrices* [28, A008608]. In [25], Mészáros, Morales, and Rhoades noticed that these matrices can be easily reinterpreted as integer flows on the complete graph K_{n+1} with netflow vector $(-n, 1, \ldots, 1)$. With this interpretation, Haglund's result states that the Hilbert series equals a weighted sum over the lattice points of the polytope of flows on K_{n+1} with netflow vector $(-n, 1, \ldots, 1)$. Denoting this sum by $\text{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(-n, 1, \ldots, 1))$ (see Section 2 for the precise definition of flows and their weight), Haglund's result can be restated as follows.

Theorem 1.2 (Haglund [14])**.**

$$\text{Hilb}_{q,t}(DH_n) = \text{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(-n, 1, \ldots, 1)).$$

Combining Theorems 1.1 and 1.2, we obtain intriguing combinatorial identities between $\text{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(-n, 1, \ldots, 1))$ and (q, t)-analogues (for t, $t = 1$, $t = 0$, and $t = q^{-1}$) of the number of parking functions of size n:

$$\text{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(-n, 1, \ldots, 1)) = \sum_{p \in \mathcal{P}_n} q^{\text{area}(p)} t^{\text{dinv}(p)}. \tag{1.5}$$

There are many natural bijections between spanning trees of K_{n+1} and parking functions of size n. Correspondingly, there are various statistics $(\text{stat}_1, \text{stat}_2)$ on trees that can be used to rewrite the right hand side of (1.5) as a sum over spanning trees of K_{n+1}:

$$\text{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(-n, 1, \ldots, 1)) = \sum_{T} q^{\text{stat}_1(T)} t^{\text{stat}_2(T)}, \tag{1.6}$$

where the sum is over all spanning trees T of K_{n+1}: see for instance [16, §4] for one example. Equations (1.2), (1.3), and (1.4) can be rewritten as

$$\mathrm{Ehr}_{q,1}(\mathcal{F}_{K_{n+1}}(-n,1,\ldots,1)) = \sum_T q^{\mathrm{inv}(T)}, \tag{1.7}$$

$$\mathrm{Ehr}_{q,0}(\mathcal{F}_{K_{n+1}}(-n,1,\ldots,1)) = [n]_q!, \tag{1.8}$$

$$q^{\binom{n}{2}}\,\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_{K_{n+1}}(-n,1,\ldots,1)) = [n+1]_q^{n-1}, \tag{1.9}$$

where on the right hand side of (1.7), T ranges over all spanning trees of K_{n+1}, and $\mathrm{inv}(T)$ is the number of inversions of T (see Section 3 for the definition of inv statistic and the correspondence to the area statistic on parking functions). It is then natural to verify these identities directly. Doing so in the general case (q,t) would give an alternative proof of the now settled Haglund–Loehr conjecture. Progress in this direction started with Levande [20] who verified the cases $(q,0)$ using a sign-reversing involution. Armstrong et al. [1] verified the case $(q,1)$. We verify directly the (q,q^{-1}) case in this paper.

More generally, one could extend the identity (1.6) to flows with other netflow vectors (in [1], these are called *generalized Tesler matrices*) or to other graphs. The former was done in [30] for the $(q,0)$ case for binary netflows on K_{n+1} extending the involution approach of Levande. Formulas for the $(q,1)$ case for positive integral netflows were given in [1] and for integral flows in [30]. We generalize the known formulas for $\mathrm{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(-n,1,\ldots,1))$ for $t=1,0,q^{-1}$ (as in equations (1.7), (1.8), and (1.9)) to a family of graphs called *threshold graphs* [23] with arbitrary positive integral netflows. There are 2^n such graphs with $n+1$ vertices including the complete graph.

We now summarize our main results. First we state the case $t=1$ for netflow $(-n,1,\ldots,1)$, which implies (1.7) when G is the complete graph.

Theorem 3.8. *Let G be a threshold graph. Then*

$$\mathrm{Ehr}_{q,1}(\mathcal{F}_G(-n,1,\ldots,1)) = t_G(1,q) = \sum_T q^{\mathrm{inv}(T)},$$

where t_G is the Tutte polynomial of G, and T ranges over all spanning trees T of G.

This relationship between $\mathrm{Ehr}_{q,1}(\mathcal{F}_G(-n,1,\ldots,1))$ and the Tutte polynomial of G extends to general positive flows as follows.

Theorem 3.11. *For a connected threshold graph G and $\mathbf{a}\in\mathbf{Z}^n_{>0}$, let $\widetilde{G}$ be the multigraph obtained from G by replacing each edge (i,j) with $a_{\max\{i,j\}}$ parallel edges. Then*

$$\mathrm{Ehr}_{q,1}(\mathcal{F}_G(-\textstyle\sum_i a_i, a_1,\ldots,a_n)) = t_{\widetilde{G}}(1,q),$$

where $t_{\widetilde{G}}$ is the Tutte polynomial of $\widetilde{G}$.

We also state the case $t=0$, which implies (1.8) when G is the complete graph and $\mathbf{a}=(1,\ldots,1)$.

Theorem 4.2. *Let G be a threshold graph with degree sequence $(d_0,d_1,\ldots,d_n)$ and $\mathbf{a}\in\mathbf{Z}^n_{>0}$. Then*

$$\mathrm{Ehr}_{q,0}(\mathcal{F}_G(-\textstyle\sum_i a_i, a_1,\ldots,a_n)) = \prod_{i=1}^n q^{\bar{d}_i(a_i-1)}[\bar{d}_i]_q,$$

where $\bar{d}_i=\min\{d_i,i\}$ is the number of vertices $j<i$ adjacent to i.

Lastly, we state the case $t = q^{-1}$, which implies (1.9) when G is the complete graph and $\mathbf{a} = (1, \ldots, 1)$.

Theorem 5.2. *Let G be a threshold graph with degree sequence $(d_0, d_1, \ldots, d_n)$ and $\mathbf{a} \in \mathbf{Z}^n_{>0}$. Then*

$$\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_G(-\textstyle\sum_i a_i, a_1, \ldots, a_n)) = q^{-F} \displaystyle\prod_{i=1}^{n} b_i(q),$$

where $F = \sum_{i=1}^n \min\{d_i, i\} \cdot a_i - n$ and

$$b_i(q) = \begin{cases} [(i+1)a_i + \sum_{j=i+1}^{d_i} a_j]_q & \text{if } d_i > i, \\ [a_i]_{q^{i+1}} & \text{if } d_i = i, \\ [a_i]_{q^{d_i+1}}[d_i]_q & \text{if } d_i < i. \end{cases}$$

As a corollary, we prove a conjecture of Armstrong et al. [1] about the (q, q^{-1})-Ehrhart function of the flow polytope of a complete graph with an arbitrary netflow vector. The case $a_1 = \cdots = a_n = 1$ gives (1.9).

Corollary 5.4. *For positive integers $a_1, \ldots, a_n$ we have that*

$$\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_{K_{n+1}}(-\textstyle\sum_i a_i, a_1, \ldots, a_n)) = q^{n-\sum_{i=1}^n ia_i} \displaystyle\prod_{i=1}^{n-1} [(i+1)a_i + a_{i+1} + a_{i+2} + \cdots + a_n]_q.$$

Our proofs are self-contained and inductive on the netflow of the flow polytope without using machinery from symmetric functions. In the case $(q, 0)$ we do not use involutions like Levande in [20] and Wilson in [30].

The outline of this paper is as follows. In Section 2 we give the definitions of flow polytopes, (q,t)-Ehrhart functions, and threshold graphs. In Section 3 we calculate $\mathrm{Ehr}_{q,1}(\cdot)$ for flow polytopes of threshold graphs, while in Section 4 we do the same for the evaluation $(q, 0)$. In Section 5 we calculate the evaluation (q, q^{-1}) thereby also proving Conjecture 7.1 of Armstrong et al. in [1]. We conclude in Section 6 with positivity conjectures regarding the general (q,t) case of flow polytopes of threshold graphs.

2. Preliminaries

In this section, we give some background and preliminary results about flow polytopes and threshold graphs.

2.1. Flow polytopes and their (q,t)-Ehrhart functions.

We first discuss flow polytopes and define the (q,t)-Ehrhart functions.

Definition 2.1. Let $G = (V, E)$ be an acyclic directed graph on $V = \{0, 1, \ldots, n\}$, and let $\mathbf{a} \in \mathbf{Z}^{n+1}$. Let A_G be the $n \times |E|$ matrix with columns $e_i - e_j$ for each directed edge (i, j). Then the *flow polytope* $\mathcal{F}_G(\mathbf{a}) \subseteq \mathbf{R}^E_{\geq 0}$ is defined to be

$$\mathcal{F}_G(\mathbf{a}) = \{x \in \mathbf{R}^E_{\geq 0} \mid A_G \cdot x = \mathbf{a}\}.$$

In other words, the flow polytope is the set of all nonnegative flows that can be placed on the edges of G such that the net flow at each vertex is given by $\mathbf{a}$. By convention, we orient the edges from i to j if $i > j$. Since the sum of the entries of $\mathbf{a}$ must be 0 for the flow polytope to be nonempty, we will abuse notation and write, for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{Z}^n$, $\mathcal{F}_G(\mathbf{a}) = \mathcal{F}_G(-\sum a_i, a_1, \ldots, a_n)$. We also abbreviate $\mathcal{F}_G = \mathcal{F}_G(-n, 1, 1, \ldots, 1)$.

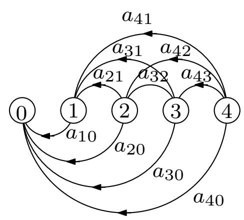
Remark 2.2. A *Tesler matrix* is an $n \times n$ upper triangular matrix $B = (b_{i,j})_{1 \le i \le j \le n}$ with nonnegative integer entries satisfying for $k = 1, \ldots, n$,

$$b_{k,k} + b_{k,k+1} + \cdots + b_{k,n} - (b_{1,k} + b_{2,k} + \cdots + b_{k-1,k}) = 1.$$

These matrices first appeared in Haglund's study of DH_n [14]. By an observation in [25], these matrices are in correspondence with integral flows on K_{n+1} with netflow $(-n, 1, 1, \ldots, 1)$. With the conventions on $\mathcal{F}_G$ in this paper, the correspondence is as follows: an integral flow $A = (a_{ij})_{0 \le j < i \le n}$ in $\mathcal{F}_{K_{n+1}}$ corresponds to the Tesler matrix $B = (b_{ij})_{1 \le i \le j \le n}$ where

$$b_{ij} = \begin{cases} a_{n+1-j,0} & \text{if } i = j, \\ a_{n+1-i,n+1-j} & \text{if } i < j. \end{cases}$$

For example, for $n = 4$ the correspondence is the following:



$$\leftrightarrow \quad \begin{bmatrix} a_{40} & a_{43} & a_{42} & a_{41} \\ & a_{30} & a_{32} & a_{31} \\ & & a_{20} & a_{21} \\ & & & a_{10} \end{bmatrix}$$

This correspondence can be extended to integral flows on subgraphs G of K_{n+1} by setting the entries corresponding to missing edges of G to zero.

For any nonnegative integer b, define the (q, t)-*weight*

$$wt_{q,t}(b) = \begin{cases} \frac{q^b - t^b}{q - t} & \text{if } b > 0, \\ 1 & \text{if } b = 0. \end{cases} \tag{2.1}$$

For a lattice point $A = (a_{ij}) \in \mathbf{R}^E_{\ge 0}$ with nonnegative entries, define

$$wt_{q,t}(A) = (-(1-t)(1-q))^{\#\{a_{ij} > 0\} - n} \cdot \prod_{i,j} wt_{q,t}(a_{ij}),$$

where $\#\{a_{ij} > 0\}$ denotes the number of nonzero entries of A. Finally, for an integer polytope $\mathcal{F}_G(\mathbf{a}) \subseteq \mathbf{R}^E_{\ge 0}$, define the (q, t)-*weighted Ehrhart function*

$$\operatorname{Ehr}_{q,t}(\mathcal{F}_G(\mathbf{a})) = \sum_{A \in \mathcal{F}_G(\mathbf{a}) \cap \mathbf{Z}^E} wt_{q,t}(A).$$

Note that if $\mathbf{a} \in \mathbf{Z}^n_{>0}$, then any $A \in \mathcal{F}_G(\mathbf{a})$ will have at least n nonzero entries, so $wt_{q,t}(A)$ and hence $\operatorname{Ehr}_{q,t}(\mathcal{F}_G(\mathbf{a}))$ will be *polynomials* in q and t. Moreover, this polynomial by construction is *symmetric* in q and t. There is no guarantee, however, that $\operatorname{Ehr}_{q,t}(\mathcal{F}_G(\mathbf{a}))$ will have nonnegative coefficients, and indeed it will not for general graphs G as illustrated in the next example.

Example 2.3. If $G = K_5 \setminus \{(3, 4)\}$, then there are 15 integer flows on G and one can check that

$$\operatorname{Ehr}_{q,t}(\mathcal{F}_G(-4,1,1,1,1)) = q^3t + 2q^2t^2 + qt^3 - 3q^3 - 5q^2t - 5qt^2 - 3t^3 - 5q^2 - 8qt - 5t^2 - 3q - 3t - 1.$$

2.2. **Threshold graphs.** We now define threshold graphs, a class of graphs of importance in computer science and optimization. For more information, see [23] and [29, Ex. 5.4].

Definition 2.4. A *threshold graph* G is a graph that can be constructed recursively starting from one vertex and no edges by repeatedly carrying out one of the following two steps:

- add a dominating vertex: a vertex that is connected to every other existing vertex;
- add an isolated vertex: a vertex that is not connected to any other existing vertex.

We say that a threshold graph G is labeled by *reverse degree sequence* if its vertices are labeled by $0, \ldots, n$ in such a way that $d_i \geq d_j$ for each pair of vertices $i < j$, where d_i is the degree of vertex i.

This family of graphs includes the complete graph and the star graph but excludes paths or cycles of 4 or more vertices. There are 2^{n-1} threshold graphs with n unlabeled vertices. The number $t(n)$ of threshold graphs with vertex set $[n]$ has exponential generating function $e^x(1-x)/(2-e^x)$, and $t(n) \sim n!(1-\log(2))/\log(2)^{n+1}$ (e.g. see [28, A005840]). A threshold graph is uniquely determined up to isomorphism by its degree sequence $d(G) = (d_0, d_1, \ldots, d_n)$. By convention, we will assume that all our threshold graphs are labeled by reverse degree sequence and that the edges are directed from i to j if $i > j$.

Alternatively, a graph G is a threshold graph if there exist real weights w_i for each vertex $i = 0, \ldots, n$ and a threshold value t such that i and j are adjacent if and only if $w_i + w_j > t$. If the vertices are labeled such that $w_0 > w_1 > \cdots > w_n$, then G is labeled by reverse degree sequence. Note that if i and j are adjacent in G, then so are i' and j' for any $i' \leq i$ and $j' \leq j$ (provided $i' \neq j'$).

Remark 2.5. A threshold graph with $n+1$ vertices can be encoded by a binary sequence $(\beta_0, \ldots, \beta_{n-1}) \in \{0,1\}^n$ where $\beta_i = 1$ or 0 depending on whether vertex i is a dominating or an isolated vertex with respect to vertices $i+1, \ldots, n$. In this labeling, if i and j are adjacent with $i < j$, then $d_i \geq d_j$ since all vertices at least i are adjacent to i and all vertices smaller than i are either adjacent to both j and i or to neither. Hence when we relabel the vertices by reverse degree sequence the orientation of the edges is preserved. Thus if $G = G(\beta)$ is a threshold graph with the labeling induced from β, and G' is the graph relabeled by reverse degree sequence, then

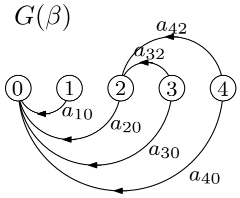
$$\mathrm{Ehr}_{q,t}(\mathcal{F}_{G(\beta)}(\mathbf{a})) \;=\; \mathrm{Ehr}_{q,t}(\mathcal{F}_{G'}(\mathbf{a}')),$$

where $\mathbf{a}'$ is obtained by permuting $\mathbf{a}$ according to the relabeling of the vertices. In the case when the graph is connected ($\beta_0 = 1$) and $\mathbf{a} = (-n, 1, 1, \ldots, 1)$, then $\mathbf{a} = \mathbf{a}'$ and the equation above becomes

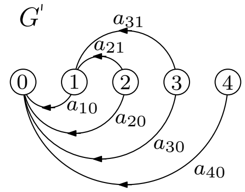
$$\mathrm{Ehr}_{q,t}(\mathcal{F}_{G(\beta)}) \;=\; \mathrm{Ehr}_{q,t}(\mathcal{F}_{G'}).$$

Using the correspondence between integral flows on graphs and Tesler matrices in Remark 2.2, the $n \times n$ matrices corresponding to the flows on threshold graph $G(\beta)$ have zero entries above the diagonal in column $i+1$ if $\beta_i = 0$.

Example 2.6. The threshold graph $G(1,0,1,0)$ corresponds to the graph G' with reverse degree sequence $(4,3,2,2,1)$. The map between integral flows on $G(1,0,1,0)$ and G' and Tesler matrices is the following:



$$\leftrightarrow \quad \begin{matrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ 1 & 0 & 1 & 0 \end{matrix}$$
$$\begin{bmatrix} a_{40} & 0 & a_{42} & 0 \\ & a_{30} & a_{32} & 0 \\ & & a_{20} & 0 \\ & & & a_{10} \end{bmatrix}$$



$$\leftrightarrow \begin{bmatrix} a_{40} & 0 & 0 & 0 \\ & a_{30} & 0 & a_{31} \\ & & a_{20} & a_{21} \\ & & & a_{10} \end{bmatrix}$$

3. Calculating the $(q, 1)$-Ehrhart function

In this section, we give a combinatorial formula for the weighted Ehrhart function of the flow polytope $\mathcal{F}_G(\mathbf{a})$ when G is a threshold graph and $t = 1$. We note that one such proof when G is the complete graph was given by Wilson [30, §6]. In particular, it will follow that when $q = t = 1$, the weighted Ehrhart function evaluates to the number of spanning trees of G, or equivalently, to the number of G-parking functions.

To begin, we will need some background about spanning trees, inversions, and parking functions, particularly in relation to threshold graphs.

3.1. **Spanning trees and inversions.** One important statistic on spanning trees is the number of inversions. The related notion of κ-inversions is due to [10]. We define both these notions below.

Definition 3.1. Let G be a graph on $0, 1, \ldots, n$, and let T be a spanning tree of G rooted at r. We say that v is a *descendant* of u if u lies on the unique path from r to v in T. We say u is the *parent* of a vertex v if v is a descendant of u in T, and u and v are adjacent in G.

Definition 3.2. An *inversion* of G is a pair of vertices (i, j) with $r \neq i > j$ such that j is a descendant of i. A κ*-inversion* of G is an inversion (i, j) such that j is adjacent to the parent of i in G. We denote the number of inversions of T by $\mathrm{inv}(T)$ and the number of κ-inversions by $\kappa(T)$.

We will assume our trees are rooted at $r = 0$ unless otherwise indicated.

We also briefly recall the definition of the Tutte polynomial of a graph.

Definition 3.3. Let $G = (V, E)$ be a multigraph. The *Tutte polynomial* of G is defined by

$$t_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-|V|},$$

where $k(A)$ denotes the number of connected components in the graph (V, A).

Define the *inversion enumerator* of G to be

$$I_G(q) = \sum_T q^{\kappa(T)},$$

where T ranges over all spanning trees of G. Gessel shows in [10] that $I_G(q)$ has the following properties.

Theorem 3.4. *[10] Let G be a graph on $0, 1, \ldots, n$.*

(a) The polynomial $I_G(q)$ does not depend on the labeling of G. In fact, $I_G(q) = t_G(1, q)$.

(b) *For any vertex* $i \neq 0$*, let* $\delta_{T,G}(i)$ *be the number of descendants of* i *in* T *(including* i *itself) that are adjacent in* G *to the parent of* i*. Then*

$$I_G(q) = \sum_{T:\ \kappa(T)=0} \prod_{i=1}^{n} [\delta_{T,G}(i)]_q,$$

where the sum ranges over all spanning trees T *for which* $\kappa(T) = 0$*.*

Here we use the standard notation $[k]_q = 1+q+q^2+\cdots+q^{k-1} = \frac{q^k-1}{q-1}$. In the case when G is a threshold graph, these results specialize as follows. Call a spanning tree of G *increasing* if it has no inversions.

Proposition 3.5. *Let* G *be a threshold graph (labeled by reverse degree sequence). Then*

$$I_G(q) = \sum_{T} q^{\mathrm{inv}(T)} = \sum_{T \text{ increasing}} \prod_{i=1}^{n} [\delta_T(i)]_q,$$

where $\delta_T(i)$ *is the number of descendants of* i *in* T *(including* i *itself).*

Proof. For a spanning tree of a threshold graph, any inversion (i, j) is a κ-inversion: $j < i$ implies that any vertex adjacent to i is also adjacent to j in G, particularly the parent of i (or see [26, Proposition 10]).

For the second equality, if j is a descendant of i, then the parent of j is a descendant of the parent of i. Thus in an increasing tree, the parent of i is at most the parent of j, so since j is adjacent to the latter, it must also be adjacent to the former in G. The result then follows from Theorem 3.4. □

3.2. **Parking functions.** The following notion of a G-parking function due to Postnikov and Shapiro [27] generalizes the usual notion of parking function (the latter corresponds to the complete graph). They are also called *superstable configurations* in the context of chip-firing.

Definition 3.6. Let $G = (V, E)$ be a graph on $V = \{0, 1, \ldots, n\}$. A *G-parking function* is a function $P\colon [n] \to \mathbf{Z}_{\geq 0}$ such that, for every nonempty set $S \subseteq [n]$, there exists $i \in S$ such that $P(i)$ is less than the number of vertices $j \notin S$ adjacent to i.

The *degree* of a parking function P is defined to be $\deg P = \sum_{i=1}^{n} P(i)$. The *codegree* of a parking function P is $\operatorname{codeg} P = g - \deg P$, where $g = |E| - |V| + 1$.

When G is the complete graph, P is a parking function if and only if, for $k = 1, 2, \ldots, n$, there are at least k vertices i such that $P(i) < k$. In the context of ordinary parking functions on the complete graph, the codeg statistic is usually referred to as area.

In general, G-parking functions are in bijection with the spanning trees of G. Merino [24] showed the following relationship (in the context of chip-firing) between parking functions and the Tutte polynomial of G.

Theorem 3.7. *[24] Let* G *be a graph. Then*

$$t_G(1, y) = \sum_{P} y^{\operatorname{codeg} P},$$

where the sum ranges over all G*-parking functions* P*.*

In light of Gessel's results on the inversion enumerator of G, it follows that the κ-inversion statistic on spanning trees of G has the same distribution as the codegree statistic on G-parking functions. (This was noted in the case of the complete graph by Kreweras [19].) The authors of [26] give an explicit bijection (called the *DFS-burning algorithm*) between spanning trees T and G-parking functions P that sends $\kappa(T)$ to $\operatorname{codeg}(P)$. If G is a threshold graph, then this bijection sends $\operatorname{inv}(T)$ to $\operatorname{codeg}(P)$.

3.3. **Relation to the Ehrhart function.** We are now ready to state the main result of this section.

Theorem 3.8. *Let G be a threshold graph. Then*

$$\operatorname{Ehr}_{q,1}(\mathcal{F}_G) = t_G(1,q) = I_G(q) = \sum_T q^{\operatorname{inv}(T)} = \sum_P q^{\operatorname{codeg}(P)},$$

where T ranges over all spanning trees of G, and P ranges over all G-parking functions.

Proof. Note that for any $A \in \mathcal{F}_G \cap \mathbf{Z}^E$, $wt_{q,1}(A) = 0$ unless A has exactly n nonzero entries. Hence to compute $\operatorname{Ehr}_{q,1}(\mathcal{F}_G)$, we need only sum $wt_{q,1}(A)$ over such A.

For any $A = (a_{ij}) \in \mathcal{F}_G \cap \mathbf{Z}^E$, the set of edges (i,j) for which $a_{ij} \neq 0$ forms a connected subgraph of G. Hence if A has exactly n nonzero entries, then these edges must form a spanning tree T of G. We claim that such A are in bijection with increasing spanning trees T of G. Indeed, if T were not increasing, then there is some vertex $i > 0$ that is smaller than its parent but larger than all of its descendants. But then i has no outgoing edges in T, so there cannot be a nonnegative flow supported on T with net flow 1 at i.

Given an increasing spanning tree T of G, there is a unique flow $A \in \mathcal{F}_G$ supported on the edges of T: we must have that the flow on the edge connecting i to its parent is $\delta_T(i)$. Hence

$$\operatorname{Ehr}_{q,1}(\mathcal{F}_G) = \sum_{T \text{ increasing}} \prod_{i=1}^{n} wt_{q,1}(\delta_T(i)) = \sum_{T \text{ increasing}} \prod_{i=1}^{n} [\delta_T(i)]_q = I_G(q)$$

by Proposition 3.5. □

As a corollary, we can specialize to the case when G is the complete graph K_{n+1}. This gives the $t = 1$ case of the Haglund-Loehr conjecture (Theorem 1.1) via Theorem 1.2.

Corollary 3.9. *We have*

$$\operatorname{Ehr}_{q,1}(\mathcal{F}_{K_{n+1}}) = t_{K_{n+1}}(1,q) = I_{K_{n+1}}(q) = \sum_T q^{\operatorname{inv}(T)} = \sum_P q^{\operatorname{area}(P)},$$

where T ranges over all spanning trees of K_{n+1}, and P ranges over all parking functions of length n.

3.4. **General flows.** We now give a combinatorial formula for $\operatorname{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a}))$ for arbitrary $\mathbf{a} \in \mathbf{Z}^n_{>0}$ as a weighted sum over spanning trees over G. This formula is analogous to a result by Armstrong et al. [1, Theorem 7.1] in the case of the complete graph.

Note that it is straightforward to give a combinatorial formula for $\operatorname{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a}))$ as a weighted sum over increasing spanning trees. For a similar result for the complete graph, see Wilson [30, §6].

Proposition 3.10. *Let G be a threshold graph. For any vertex $i > 0$, let $\delta_T^{\mathbf{a}}(i) = \sum_j a_j$, where j ranges over descendants of i (including i itself). Then*

$$\mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})) = \sum_{T \text{ increasing}} \prod_{i=1}^{n} [\delta_T^{\mathbf{a}}(i)]_q.$$

Proof. As in the proof of Theorem 3.8, the only nonzero terms in the sum for $\mathrm{Ehr}_{q,1}(\mathcal{F}_G)$ come from flows supported on increasing spanning trees of G. For any such tree, there is a unique flow in $\mathcal{F}_G(\mathbf{a})$ supported on it: the flow on the edge connecting i to its parent is $\delta_T^{\mathbf{a}}(i)$. The result follows easily. □

The following theorem converts this formula from a sum over increasing spanning trees of G to a sum over all spanning trees of G. For any spanning tree T, let $E(T)$ denote the edge set of T, $p_T(i)$ denote the parent of vertex i, and $\mathrm{Inv}(T)$ denote the set of inversions of T.

Theorem 3.11. *Let G be a threshold graph and $\mathbf{a} \in \mathbf{Z}_{>0}^n$.*

(a) Let $\widetilde{G}$ be the multigraph obtained from G by replacing each edge (i, j) with $a_{\max\{i,j\}}$ parallel edges. If G is connected, then $\mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})) = t_{\widetilde{G}}(1, q)$.

(b) For any spanning tree T of G, let

$$w(T) = \prod_{(i,j)\in E(T)} [a_{\max\{i,j\}}]_q \cdot \prod_{(i,j)\in \mathrm{Inv}(T)} q^{a_{\max\{p_T(i),j\}}}.$$

Then $\mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})) = \sum_T w(T)$, where T ranges over all spanning trees of G.

Note that if we set $a_1 = a_2 = \cdots = a_n = 1$, then $w(T) = q^{\mathrm{inv}(T)}$, so we recover Theorem 3.8. We will give two proofs of this result. The first is an adaptation of the proof of Theorem 3.4 above by Gessel in [10]. The second uses known properties of the Tutte polynomial.

Proof 1 of Theorem 3.11. For part (a), let $c_{\widetilde{G}}(q) = \sum_H q^{|E(H)|}$, where H ranges over connected sub-multigraphs of $\widetilde{G}$. For any such H and any fixed vertex r, $H\backslash\{r\}$ decomposes into connected components, yielding an unordered set partition $V_1, \ldots, V_k$ of $V\backslash\{r\}$. Let $a(r, V_j)$ denote the total number of edges in $\widetilde{G}$ from r to a vertex in V_j. Since H must have at least one edge from r to a vertex in V_j for each j, and the induced subgraphs $H[V_j]$ are all connected, we have

$$(*) \qquad c_{\widetilde{G}}(q) = \sum_{V_1,\ldots,V_k} \prod_{j=1}^{k} ((1+q)^{a(r,V_j)} - 1) \cdot c_{\widetilde{G}[V_j]}(q),$$

where the sum ranges over set partitions $V_1, \ldots, V_k$ of $V\backslash\{r\}$.

To prove that $\mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})) = (q-1)^{-n} c_{\widetilde{G}}(q-1) = t_{\widetilde{G}}(1, q)$, it suffices to show that, for $r = 0$, the weighted Ehrhart sum satisfies the appropriate recursion derived from $(*)$, namely

$$(**) \qquad \mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})) = \sum_{V_1,\ldots,V_k} \prod_{j=1}^{k} [a(r, V_j)]_q \cdot \mathrm{Ehr}_{q,1}(\mathcal{F}_{G[V_j]}(\mathbf{a}[V_j])),$$

where if $V_j = \{i_0, i_1, \ldots, i_s\}$ in order, then $\mathbf{a}[V_j] = (a_{i_1}, \ldots, a_{i_s})$. (Note that $G[V_j]$ is still a threshold graph for all V_j). If $r = 0$, then $a(0, V_j) = a_{i_0} + \cdots + a_{i_s} = \delta_T^{\mathbf{a}}(i_0)$, where T is any increasing spanning tree of G with a subtree supported on V_j. Part (a) now follows easily from Proposition 3.10. In particular, since $(*)$ is satisfied for all r, so must $(**)$ also be satisfied for all r.

We now show part (b) by induction on n—in fact, we will show that it holds for any choice of root r, not just $r = 0$. (Changing the root will usually change the second factor in $w(T)$.) To see this, let T be a spanning tree of G with subtrees $T_1, \dots, T_k$ on vertex sets $V_1, \dots, V_k$. If $v_j \in V_j$ is a child of the root r, then

$$w(T) = \prod_{j=1}^{k} w(T_j)[a_{\max\{r,v_j\}}]_q \prod_{\substack{i \in V_j \\ i < v_j}} q^{a_{\max\{r,i\}}}.$$

By induction, for a fixed v_j, $\sum_{T_j} w(T_j) = \mathrm{Ehr}_{q,1}(\mathcal{F}_{G[V_j]}(\mathbf{a}[V_j]))$, which does not depend on v_j. Moreover, as v_j ranges over vertices in V_j adjacent to r (noting that any $i < v_j$ is also adjacent to r since G is a threshold graph),

$$\sum_{v_j} [a_{\max\{r,v_j\}}]_q \prod_{\substack{i \in V_j \\ i < v_j}} q^{a_{\max\{r,i\}}} = [\textstyle\sum_{v_j} a_{\max\{r,v_j\}}]_q = [a(r, V_j)]_q.$$

Hence summing over all spanning trees T,

$$\begin{aligned}
\sum_T w(T) &= \sum_{V_1,\dots,V_k} \prod_{j=1}^{k} \sum_{v_j} \sum_{T_j} w(T_j)[a_{\max\{r,v_j\}}]_q \prod_{\substack{i \in V_j \\ i < v_j}} q^{a_{\max\{r,i\}}} \\
&= \sum_{V_1,\dots,V_k} \prod_{j=1}^{k} [a(r, V_j)]_q \, \mathrm{Ehr}_{q,1}(\mathcal{F}_{G[V_j]}(\mathbf{a}[V_j])) \\
&= \mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})).
\end{aligned}$$

□

For the second proof, we recall the following properties of the Tutte polynomial (see [3, Ch. X]). The first is that the Tutte polynomial satisfies the following *deletion-contraction* recurrence.

Proposition 3.12. *Let G be a multigraph and e an edge of G.*

(a) If e is not a bridge or loop of G, then $t_G(x, y) = t_{G-e}(x, y) + t_{G/e}(x, y)$, where $G - e$ and G/e are obtained from G by removing edge e and contracting edge e, respectively.
(b) If e is a bridge of G, then $t_G(x, y) = x t_{G/e}(x, y)$.
(c) If e is a loop of G, then $t_G(x, y) = y t_{G-e}(x, y)$.

The second is that the Tutte polynomial can be described in terms of internal and external activity as follows. Fix a total order $\prec$ on the edges of G. Given a spanning tree T, we call an edge $e \in T$ *internally active* if e is the smallest edge of G joining the two connected components of $T - e$. We call an edge $e \notin T$ *externally active* if e is the smallest edge in the unique cycle of $T \cup \{e\}$. Then the *internal* and *external* activities $ia(T)$ and $ea(T)$ are the total number of internally and externally active edges of T, respectively.

Proposition 3.13. *Let G be a multigraph. Then $t_G(x, y) = \sum_T x^{ia(T)} y^{ea(T)}$, where T ranges over spanning trees of G. (This does not depend on the choice of total order $\prec$.)*

We are now ready to give a second proof of Theorem 3.11. Although one can use the method of part (b) below to prove part (a) as well via Proposition 3.10, we present a proof using the deletion-contraction recurrence since a similar recurrence will appear in the proof of Theorem 5.2.

Proof 2 of Theorem 3.11. For (a), let m be the largest neighbor of vertex n in G. We will use the deletion-contraction recurrence on each of the a_n edges of $\widetilde{G}$ from n to m. At most one of these edges can be contracted, and all subsequent edges become loops. Let $\widetilde{G}'$ be the graph obtained by contracting any one of these edges and then removing all loops, and let $\widetilde{G}''$ be the graph obtained by deleting all of these edges. Then we get

$$t_{\widetilde{G}}(1,q) = \begin{cases} [a_n]_q \cdot t_{\widetilde{G}'}(1,q) + t_{\widetilde{G}''}(1,q) & \text{if } m > 0, \\ [a_n]_q \cdot t_{\widetilde{G}'}(1,q) & \text{if } m = 0. \end{cases}$$

(When $m = 0$, the last edge from n to m is a bridge so it cannot be deleted.) Note that for $j < m$, the number of edges in $\widetilde{G}'$ from m to j is $a_m + a_n$; hence $\widetilde{G}'$ comes from the threshold graph G' (G with vertex n removed) by multiplying edges according to the flow vector $\mathbf{a}' = (a_1, \ldots, a_{m-1}, a_m + a_n, a_{m+1}, \ldots, a_{n-1})$ if $m > 0$, and $\mathbf{a}' = (a_1, \ldots, a_{n-1})$ if $m = 0$. Likewise $\widetilde{G}''$ comes from the threshold graph G'' (G with edge (n, m) removed) with flow vector $\mathbf{a}$.

In fact, $\mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a}))$ satisfies the same recurrence, that is,

$$\mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})) = \begin{cases} [a_n]_q \cdot \mathrm{Ehr}_{q,1}(\mathcal{F}_{G'}(\mathbf{a}')) + \mathrm{Ehr}_{q,1}(\mathcal{F}_{G''}(\mathbf{a})) & \text{if } m > 0, \\ [a_n]_q \cdot \mathrm{Ehr}_{q,1}(\mathcal{F}_{G'}(\mathbf{a}')) & \text{if } m = 0. \end{cases}$$

Indeed, as in Proposition 3.10, we need only consider flows supported on spanning trees T of G. If $(n, m) \in T$, then it must support a flow of size a_n, which changes the net flow at m on the rest of T from a_m to $a_m + a_n$—this gives the first term in the sum. If $m > 0$ and $(n, m) \notin T$, then we get the second term in the sum. It follows that $\mathrm{Ehr}_{q,1}(\mathcal{F}_G(\mathbf{a})) = t_{\widetilde{G}}(1, q)$ by induction on the number of edges of G (the base case with one edge is trivial).

For (b), we again prove the claim for any root r by induction on n. We need to show that

$$(\dagger) \qquad \sum_T wt(T) = \sum_{\widetilde{T}} q^{ea(\widetilde{T})},$$

where T and $\widetilde{T}$ range over spanning trees of G and $\widetilde{G}$, respectively. (Recall that the right side does not depend on the choice of total order.) Fix a set partition $V_1, \ldots, V_k$ of $V \backslash \{r\}$ and vertices $v_j \in V_j$ adjacent to r. Then restrict both sides of (†) to trees T and $\widetilde{T}$ such that the v_j are the children of r, and the V_j are the vertex sets supporting the corresponding subtrees T_j and $\widetilde{T}_j$. The left hand side then becomes, by induction,

$$\prod_{j=1}^{k} \sum_{T_j} wt(T_j) \cdot [a_{\max\{r,v_j\}}]_q \prod_{\substack{i \in V_j \\ i < v_j}} q^{a_{\max\{r,i\}}} = \prod_{j=1}^{k} \sum_{\widetilde{T}_j} q^{ea(\widetilde{T}_j)} \cdot [a_{\max\{r,v_j\}}]_q \prod_{\substack{i \in V_j \\ i < v_j}} q^{a_{\max\{r,i\}}}.$$

We claim this is also what the right hand side of (†) becomes.

Choose any total order on the edges of $\widetilde{G}$ that starts with all edges between r and 0, then all edges between r and 1, and so forth. (The edges not containing r can be in any order after that.) No edges between distinct V_i and V_j are externally active, so the external activity of $\widetilde{T}$ is the sum of the external activities of its subtrees $\widetilde{T}_j$ plus the number of externally active edges containing r. Of the $a_{\max\{r,v_j\}}$ edges from r to v_j, any number from 0 to $a_{\max\{r,v_j\}} - 1$ are externally active depending on which parallel edge lies in $\widetilde{T}$. For $i \in V_j \backslash \{v_j\}$, all edges from r to i are externally active if $i < v_j$, otherwise none are. The result follows. $\square$

Remark 3.14. One special case worth noting is when $q = t = 1$. In this case, Theorem 3.11 implies that $\mathrm{Ehr}_{1,1}(\mathcal{F}_G(\mathbf{a}))$ is the total weight of all spanning trees T of G, where the weight of any edge (i,j) is $a_{\max\{i,j\}}$. Thus $\mathrm{Ehr}_{1,1}(\mathcal{F}_G(\mathbf{a}))$ can be expressed as a determinant using the Matrix-Tree Theorem. In fact, one can show that this determinant factors into linear factors. For instance, when $G = K_{n+1}$, $\mathrm{Ehr}_{1,1}(\mathcal{F}_{K_{n+1}}(\mathbf{a})) = \det M$, where

$$M = \begin{bmatrix} a_1 + a_2 + \cdots + a_n & -a_2 & -a_3 & \cdots & -a_n \\ -a_2 & 2a_2 + a_3 + \cdots + a_n & -a_3 & \cdots & -a_n \\ -a_3 & -a_3 & 3a_3 + a_4 + \cdots + a_n & \cdots & -a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_n & -a_n & \cdots & na_n \end{bmatrix}.$$

Multiplying the ith row by $i+1$ and adding all the lower rows to it for $i = 1, \ldots, n-1$ yields a lower triangular matrix, so one can easily recover the result of Armstrong et al. [1] that

$$\mathrm{Ehr}_{1,1}(\mathcal{F}_{K_{n+1}}(\mathbf{a})) = a_n \cdot \prod_{i=1}^{n-1} ((i+1)a_i + \textstyle\sum_{j=i+1}^{n} a_j).$$

We will see a generalization of this product formula for general threshold graphs G later in Section 5 when we compute $\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_G(\mathbf{a}))$.

4. Calculating the $(q,0)$-Ehrhart function

In this section, we give a product formula for the weighted Ehrhart function of the flow polytope $\mathcal{F}_G(\mathbf{a})$ when G is a threshold graph and $t = 0$. In particular, when $q = 1$ and $t = 0$, the weighted Ehrhart function evaluates to the number of increasing spanning trees of G, or equivalently the number of maximal G-parking functions.

For a threshold graph G, let $\bar{d}_i = \min\{d_i, i\}$ be the outdegree of vertex i, that is, the number of vertices j adjacent to i with $j < i$. It should be noted that these outdegrees are closely related to the number of increasing spanning trees of G.

Proposition 4.1. *Let G be a threshold graph. The number of increasing spanning trees of G is $\prod_{i=1}^{n} \bar{d}_i$.*

Proof. Each vertex $i > 0$ has a choice of $\bar{d}_i$ vertices to be its parent. $\square$

We now state the main result of this section. Observe that when $t = 0$, the weights specialize to

$$wt_{q,0}(A) = (q-1)^{\#\{a_{ij}>0\}-n} \cdot \prod_{n \geq i > j \geq 0} wt_{q,0}(a_{ij}), \quad \text{where} \quad wt_{q,0}(b) = \begin{cases} q^{b-1} & \text{if } b > 0, \\ 1 & \text{if } b = 0. \end{cases}$$

Theorem 4.2. *Let G be a threshold graph and $\mathbf{a} \in \mathbf{Z}^n_{>0}$. Then*

$$\mathrm{Ehr}_{q,0}(\mathcal{F}_G(\mathbf{a})) = \prod_{i=1}^{n} q^{\bar{d}_i(a_i-1)}[\bar{d}_i]_q.$$

Note that when $a_1 = a_2 = \cdots = a_n = 1$, this formula gives a q-analogue for the number of increasing spanning trees on G.

We will first need the following lemma.

Lemma 4.3. *For integers $c \geq 1$ and $k \geq 1$, let $\Delta = \Delta(k, c) = \{(b_0, \ldots, b_{k-1}) \mid \sum b_i = c\}$. For any $B \in \Delta \cap \mathbf{Z}^k$, define*

$$wt_{q,0}(B) = (q-1)^{\#\{b_i>0\}-1} \prod_i wt_{q,0}(b_i).$$

Then

$$\sum_{B \in \Delta \cap \mathbf{Z}^k} q^{b_1+2b_2+\cdots+(k-1)b_{k-1}} wt_{q,0}(B) = q^{k(c-1)}[k]_q.$$

Proof. We induct on k. When $k = 1$, Δ has a single point c, and both sides equal q^{c-1}. We therefore assume $k > 1$.

For $B = (b_0, \ldots, b_{k-1})$, write $B' = (b_0, \ldots, b_{k-2})$. Letting $b = b_{k-1}$, we have the decomposition

$$\Delta \cap \mathbf{Z}^k = \bigcup_{b=0}^{c} (\Delta(k-1, c-b) \times \{b\}) \cap \mathbf{Z}^k$$

(where $\Delta(k, 0)$ is the set containing the single point $0 \in \mathbf{Z}^k$). Since

$$wt_{q,0}(B) = \begin{cases} wt_{q,0}(B') & \text{if } b = 0, \\ wt_{q,0}(B')(q^b - q^{b-1}) & \text{if } 0 < b < c, \\ q^{c-1} & \text{if } b = c, \end{cases}$$

we have by the inductive hypothesis that, for fixed b,

$$\sum_{\substack{B \in \Delta \cap \mathbf{Z}^k \\ b_{k-1}=b}} q^{b_1+2b_2+\cdots+(k-1)b_{k-1}} wt_{q,0}(B) = \begin{cases} q^{(k-1)(c-1)}[k-1]_q & \text{if } b = 0, \\ q^{(k-1)(c-b-1)}[k-1]_q \cdot q^{(k-1)b}(q^b - q^{b-1}) & \text{if } 0 < b < c, \\ q^{(k-1)c} \cdot q^{c-1} & \text{if } b = c. \end{cases}$$

Summing over all b gives the telescoping sum

$$\begin{aligned} q^{(k-1)(c-1)}[k-1]_q + \sum_{b=1}^{c-1} (q^{(k-1)(c-1)}[k-1]_q)(q^b - q^{b-1}) + q^{ck-1} \\ = q^{(k-1)(c-1)}[k-1]_q \cdot q^{c-1} + q^{ck-1} \\ = q^{k(c-1)}([k-1]_q + q^{k-1}) \\ = q^{k(c-1)}[k]_q. \end{aligned}$$

□

We now proceed with the proof of the theorem.

Proof of Theorem 4.2. We may assume that G is connected for both sides to be nonzero. We induct on n. When $n = 1$, we must have $\bar{d}_1 = 1$, so both sides equal $wt_{q,0}(a_1) = q^{a_1-1}$.

Now assume $n > 1$. The vertex n is adjacent to vertices $0, 1, \ldots, \bar{d}_n - 1$. For any lattice point $A \in \mathcal{F}_G(\mathbf{a})$, write

$$B = (a_{n0}, a_{n1}, \ldots, a_{n,\bar{d}_n-1}) = (b_0, b_1, \ldots, b_{\bar{d}_n-1})$$

so that B ranges over all lattice points in $\Delta(\bar{d}_n, a_n)$. For fixed B, the remaining flow A' on the graph G' obtained from G by removing vertex n lies in $\mathcal{F}_{G'}(\mathbf{a}')$, where

$$\mathbf{a}' = (a_1, a_2, \ldots, a_{n-1}) + (b_1, \ldots, b_{\bar{d}_n-1}, 0, \ldots, 0) = (a'_1, \cdots, a'_{n-1}).$$

Note that the outdegree of a vertex $i < n$ in G' is still $\bar{d}_i$. Since $wt_{q,0}(A) = wt_{q,0}(B) \cdot wt_{q,0}(A')$, we have by induction and Lemma 4.3 that

$$\begin{aligned}
\sum_{A \in \mathcal{F}_G(\mathbf{a}) \cap \mathbf{Z}^E} wt_{q,0}(A) &= \sum_{B \in \Delta(\bar{d}_n, a_n) \cap \mathbf{Z}^{\bar{d}_n}} \left(wt_{q,0}(B) \cdot \sum_{A' \in \mathcal{F}_{G'}(\mathbf{a}')} wt_{q,0}(A') \right) \\
&= \sum_{B \in \Delta(\bar{d}_n, a_n) \cap \mathbf{Z}^{\bar{d}_n}} \left(wt_{q,0}(B) \cdot \prod_{i=1}^{n-1} q^{\bar{d}_i(a'_i - 1)} [\bar{d}_i]_q \right) \\
&= \sum_{B \in \Delta(\bar{d}_n, a_n) \cap \mathbf{Z}^{\bar{d}_n}} q^{b_1 + 2b_2 + \cdots + (\bar{d}_n - 1) b_{\bar{d}_n - 1}} wt_{q,0}(B) \cdot \prod_{i=1}^{n-1} q^{\bar{d}_i(a_i - 1)} [\bar{d}_i]_q \\
&= q^{\bar{d}_n(a_n - 1)} [\bar{d}_n]_q \cdot \prod_{i=1}^{n-1} q^{\bar{d}_i(a_i - 1)} [\bar{d}_i]_q \\
&= \prod_{i=1}^{n} q^{\bar{d}_i(a_i - 1)} [\bar{d}_i]_q.
\end{aligned}$$

□

Specializing to the case when $G = K_{n+1}$ gives the following corollary.

Corollary 4.4. *Let* $\mathbf{a} \in \mathbf{Z}^n_{>0}$*. Then*

$$\mathrm{Ehr}_{q,0}(\mathcal{F}_{K_{n+1}}(\mathbf{a})) = q^{a_1 + 2a_2 + \cdots + na_n - \binom{n+1}{2}} [n]_q!,$$

where $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$*. In particular,* $\mathrm{Ehr}_{q,0}(\mathcal{F}_{K_{n+1}}) = [n]_q!$.

Remark 4.5. The case $\mathrm{Ehr}_{q,0}(\mathcal{F}_{K_{n+1}}(-n, 1, \ldots, 1)) = [n]_q!$ was known by combining Theorem 1.2 with (1.3). There is an elegant proof of this result by Levande [20] who defined a function φ from integer flows on K_{n+1} with netflow $(-n, 1, \ldots, 1)$ to permutations in $\mathfrak{S}_n$ and used a sign-reversing involution to show that $\sum_{A \in \varphi^{-1}(w)} wt_{q,t}(A) = q^{\mathrm{inv}(w)}$. Wilson [30, §5] extended this involution to the case $\mathrm{Ehr}_{q,0}(\mathcal{F}_{K_{n+1}}(\mathbf{a}))$ where $a_i \in \{0, 1\}$. In contrast with these proofs, our proof is inductive and does not use involutions.

5. Calculating the (q, q^{-1})-Ehrhart function

In this section, we give a product formula for the weighted Ehrhart function of the flow polytope $\mathcal{F}_G(\mathbf{a})$ when G is a threshold graph and $t = q^{-1}$. When specialized to the case $G = K_{n+1}$, this proves a conjecture of Armstrong et al. [1].

From Theorem 3.8, we know that $\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_G(\mathbf{a}))$ should specialize to the number of spanning trees of G when $q=1$ and $\mathbf{a}=1$. In fact, for threshold graphs G, there is a simple product formula for the number of spanning trees. Let $c_i = \#\{j \mid d_j \geq i\}$. In other words, $(c_1, c_2, \ldots, c_n)$ is the conjugate partition to $d(G)$.

Proposition 5.1. *Let G be a threshold graph on $0, 1, \ldots, n$. The number of spanning trees of G is*

$$c_2 c_3 \cdots c_n = \prod_{i\colon 0<i<d_i} (d_i+1) \cdot \prod_{i\colon d_i<i} d_i.$$

This is a direct application of the Matrix-Tree Theorem; see also [5] for a combinatorial proof. Note that when G is the complete graph, we recover Cayley's formula $(n+1)^{n-1}$ for the number of spanning trees of K_{n+1}.

We now state the main result of this section. Observe that when $t = q^{-1}$, the weights specialize to

$$wt_{q,q^{-1}}(A) = (-(1-q)(1-q^{-1}))^{\#\{a_{ij}>0\}-n} \prod_{n\geq i>j\geq 0} wt_{q,q^{-1}}(a_{ij}),$$

where

$$wt_{q,q^{-1}}(b) = \begin{cases} \frac{q^b - q^{-b}}{q-q^{-1}} & \text{if } b>0, \\ 1 & \text{if } b=0. \end{cases}$$

Also recall that $\bar{d}_i = \min\{d_i, i\}$ is the outdegree of vertex i.

Theorem 5.2. *Let G be a threshold graph and $\mathbf{a} \in \mathbf{Z}^n_{>0}$. Then*

$$\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_G(\mathbf{a})) = q^{-F} \prod_{i=1}^{n} b_i(q),$$

where $F = \sum_{i=1}^n \bar{d}_i a_i - n$ and

$$b_i(q) = \begin{cases} [(i+1)a_i + \sum_{j=i+1}^{d_i} a_j]_q & \text{if } d_i > i, \\ [a_i]_{q^{i+1}} & \text{if } d_i = i, \\ [a_i]_{q^{d_i+1}} [d_i]_q & \text{if } d_i < i. \end{cases}$$

Before we get to the proof, note what happens when we specialize $a_1 = a_2 = \cdots = a_n = 1$. In this case,

$$b_i(q) = \begin{cases} [d_i+1]_q & \text{if } d_i > i, \\ 1 & \text{if } d_i = i, \\ [d_i]_q & \text{if } d_i < i, \end{cases}$$

so Theorem 5.2 gives a q-analogue of Proposition 5.1 in this case.

Proof of Theorem 5.2. We may assume G is connected and induct on n and d_n. When $n=1$, we have $d_1 = 1$, and $wt_{q,q^{-1}}(a_1) = q^{1-a_1}[a_1]_{q^2}$, so assume $n>1$.

If $d_n = 1$, let G' be the threshold graph obtained by removing vertex n. Then any flow $A \in \mathcal{F}_G(\mathbf{a})$ can be obtained from a flow in $\mathcal{F}_{G'}(a_1, \ldots, a_{n-1})$ by adding the vertex n and a single edge with flow a_n from n to 0. Hence

$$\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_G(\mathbf{a})) = \mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_{G'}(a_1, \ldots, a_{n-1})) \cdot wt_{q,q^{-1}}(a_n) = q^{-F'} q^{1-a_n} [a_n]_{q^2} \prod_{i=1}^{n-1} b'_i(q),$$

where b'_i and F' are the corresponding values of b_i and F for G'. But $b'_i(q) = b_i(q)$ for $i < n$, $b_n(q) = [a_n]_{q^2}[1]_q = [a_n]_{q^2}$, and $F = F' + a_n - 1$, so the right side is $q^{-F}\prod_{i=1}^n b_i(q)$, as desired.

Now suppose $d_n - 1 = m > 0$. Then vertex n is adjacent to $0, 1, \ldots, m$. Let G' be the threshold graph obtained by removing vertex n, and let G'' be the threshold graph obtained from G by removing only the edge from n to m. Choose any $A \in \mathcal{F}_G(\mathbf{a})$, and let $k = a_{n,m}$.

- If $k = 0$, then $A \in \mathcal{F}_{G''}(\mathbf{a})$.
- If $k = a_n$, then A can be obtained from a flow in $\mathcal{F}_{G'}(a_1, \ldots, a_m + a_n, \ldots, a_{n-1})$ by adding vertex n and flow a_n from n to m.
- If $0 < k < a_n$, then A can be obtained from a flow in $\mathcal{F}_{G''}(a_1, \ldots, a_m + k, \ldots, a_{n-1}, a_n - k)$ by adding flow k from n to m.

It follows that

$$\operatorname{Ehr}_{q,q^{-1}}(\mathcal{F}_G(\mathbf{a})) = \operatorname{Ehr}_{q,q^{-1}}(\mathcal{F}_{G''}(\mathbf{a})) + \operatorname{Ehr}_{q,q^{-1}}(\mathcal{F}_{G'}(a_1, \ldots, a_m + a_n, \ldots, a_{n-1})) \cdot wt_{q,q^{-1}}(a_n)$$
$$- \sum_{k=1}^{a_n - 1} (1-q)(1-q^{-1}) \operatorname{Ehr}_{q,q^{-1}}(\mathcal{F}_{G''}(a_1, \ldots, a_m + k, \ldots, a_{n-1}, a_n - k)) \cdot wt_{q,q^{-1}}(k).$$

By induction, we may expand each of the terms on the right side. First note that for any of the terms involving G'', the corresponding value of F is, for any $k = 0, \ldots, a_n - 1$,

$$F - a_n d_n + (a_n - k)(d_n - 1) + km = F - a_n,$$

while for the G' term, the corresponding value of F is $F - d_n a_n + a_n m + 1 = F - a_n + 1$.

Next observe that for $i \neq n, m$, the value of $b_i(q)$ is the same in all terms. Indeed, this is clear by the definition of $b_i(q)$ if $d_i \leq i$, so assume $d_i > i$. Then if $i < m$, vertex i is adjacent to n, so $b_i(q) = [(i+1)a_i + \sum_{j>i} a_j]_q$, which is the same in all terms. If instead $m < i < n$, then i is not adjacent to n, so $b_i(q) = [(i+1)a_i + \sum_{j=i+1}^{d_i} a_j]_q$ does not involve either a_n or a_m, so it is also unchanged. It follows that we need only compare $b_n(q)$ and $b_m(q)$ for each of the terms.

Therefore to prove the theorem, it suffices to show that, if $d_n < n$ (so $d_m = n > m + 1$),

$$[a_n]_{q^{d_n+1}}[d_n]_q[d_n a_m + \textstyle\sum_{j=m+1}^n a_j]_q =$$
$$q^{a_n}[a_n]_{q^{d_n}}[d_n - 1]_q[d_n a_m + \textstyle\sum_{j=m+1}^{n-1} a_j]_q + q^{a_n - 1} wt_{q,q^{-1}}(a_n)[d_n(a_m + a_n) + \sum_{j=m+1}^{n-1} a_j]_q$$
$$- q^{a_n}(1-q)(1-q^{-1}) \sum_{k=1}^{a_n - 1} wt_{q,q^{-1}}(k)[a_n - k]_{q^{d_n}}[d_n - 1]_q[d_n(a_m + k) + \textstyle\sum_{j=m+1}^{n-1} a_j]_q,$$

while if $d_n = n$,

$$[a_n]_{q^{n+1}}[n a_{n-1} + a_n]_q =$$
$$q^{a_n}[a_n]_{q^n}[n-1]_q[a_{n-1}]_{q^n} + q^{a_n - 1} wt_{q,q^{-1}}(a_n)[a_{n-1} + a_n]_{q^n}$$
$$- q^{a_n}(1-q)(1-q^{-1}) \sum_{k=1}^{a_n - 1} wt_{q,q^{-1}}(k)[a_n - k]_{q^n}[n-1]_q[a_{n-1} + k]_{q^n}.$$

Both of these follow from Lemma 5.3 below: the first follows by letting $a = a_n$, $d = d_n$, and $z = d_n a_m + \sum_{j=m+1}^{n-1} a_j$, while the second follows by letting $a = a_n$, $d = n$, and $z = n a_{n-1}$ and dividing both sides by $[n]_q$ (using the fact that $[nx]_q = [x]_{q^n}[n]_q$). $\square$

Lemma 5.3. *Let a, d, and z be positive integers. Then*

$$[a]_{q^{d+1}}[d]_q[z+a]_q = q^a[a]_{q^d}[d-1]_q[z]_q + q^{a-1}wt_{q,q^{-1}}(a)[z+da]_q$$
$$- q^a(1-q)(1-q^{-1})\sum_{k=1}^{a-1} wt_{q,q^{-1}}(k)[a-k]_{q^d}[d-1]_q[z+dk]_q.$$

Proof. We compute

$$\begin{aligned}
f(x) &= 1 - q(1-q)(1-q^{-1})\sum_{k\geq 1} q^k[k]_{q^d}[d-1]_q x^k \\
&= 1 + (1-q)(1-q^{d-1})\sum_{k\geq 1}[k]_{q^d}(qx)^k \\
&= 1 + \frac{(1-q)(1-q^{d-1})qx}{(1-qx)(1-q^{d+1}x)} \\
&= \frac{(1-q^2x)(1-q^dx)}{(1-qx)(1-q^{d+1}x)}
\end{aligned}$$

and

$$\begin{aligned}
g(x) &= \frac{-[z]_q}{q(1-q)(1-q^{-1})} + \sum_{k\geq 1} q^{k-1}wt_{q,q^{-1}}(k)[z+dk]_q x^k \\
&= \frac{1-q^z}{(1-q)^3} + \sum_{k\geq 1}[k]_{q^2}[z+dk]_q x^k \\
&= \bar{g}(x) - q^z\bar{g}(q^dx),
\end{aligned}$$

where

$$\begin{aligned}
\bar{g}(x) &= \frac{1}{(1-q)^3} + \frac{1}{1-q}\cdot\sum_{k\geq 1}[k]_{q^2}x^k \\
&= \frac{1}{(1-q)^3} + \frac{x}{(1-q)(1-x)(1-q^2x)} \\
&= \frac{(1-qx)^2}{(1-q)^3(1-x)(1-q^2x)}.
\end{aligned}$$

For $a \geq 1$, the desired right hand side is the coefficient of x^a in $f(x)g(x) = f(x)\bar{g}(x) - q^z f(x)\bar{g}(q^dx)$. Since

$$\sum_{a\geq 1}[a]_{q^{d+1}}[d]_q[z+a]_q x^a = h(x) - q^z h(qx),$$

where

$$h(x) = \sum_{a\geq 1}\frac{[a]_{q^{d+1}}[d]_q x^a}{1-q} = \frac{(1-q^d)x}{(1-q)^2(1-x)(1-q^{d+1}x)},$$

it suffices to check that the difference between the two sides,

$$(f(x)\bar{g}(x) - h(x)) - q^z(f(x)\bar{g}(q^dx) - h(qx)),$$

is independent of x. Indeed, we will show that

$$f(x)\bar{g}(x) - h(x) = f(x)\bar{g}(q^d x) - h(qx) = \frac{1}{(1-q)^3}.$$

This is straightforward:

$$\begin{aligned} f(x)\bar{g}(x) - h(x) &= \frac{(1-qx)(1-q^d x)}{(1-q)^3(1-x)(1-q^{d+1}x)} - \frac{(1-q^d)x}{(1-q)^2(1-x)(1-q^{d+1}x)} \\ &= \frac{1}{(1-q)^3}, \end{aligned}$$

and

$$f(x)\bar{g}(q^d x) = \frac{(1-q^2x)(1-q^{d+1}x)}{(1-q)^3(1-qx)(1-q^{d+2}x)} = f(qx)\bar{g}(qx),$$

so $f(x)\bar{g}(q^d x) - h(qx) = f(qx)\bar{g}(qx) - h(qx) = \frac{1}{(1-q)^3}$ as well. $\square$

If we specialize to the case $G = K_{n+1}$, we arrive at the following corollary, conjectured by Armstrong et al. in [1, Conjecture 7.1].

Corollary 5.4. *Let* $\mathbf{a} \in \mathbf{Z}^n_{>0}$. *Then*

$$\mathrm{Ehr}_{q,q^{-1}}(\mathcal{F}_{K_{n+1}}(\mathbf{a})) = q^{-F}[a_n]_{q^{n+1}} \prod_{i=1}^{n-1} [(i+1)a_i + \textstyle\sum_{j=i+1}^{n} a_j]_q,$$

where $F = \sum_{i=1}^{n} i a_i - n$.

6. About the (q,t)-Ehrhart function

In this section we look at the weighted Ehrhart series of the flow polytope $\mathcal{F}_G(-n,1,\ldots,1)$ when G is a threshold graph with $n+1$ vertices.

6.1. **Conjectured q,t-positivity.** By Haglund's result [14], the weighted Ehrhart series $\mathrm{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(-n,1,\ldots,1))$ is the bigraded Hilbert series of the space of diagonal harmonics, so it must lie in $\mathbf{N}[q,t]$. By Example 2.3, the polynomial $\mathrm{Ehr}_{q,t}(\mathcal{F}_G)$ for other graphs G sometimes has negative coefficients. However, experimentation suggests some positivity properties of the polynomials $\mathrm{Ehr}_{q,t}(\mathcal{F}_G)$ for threshold graphs G and netflow $(-n,1,\ldots,1)$.

Conjecture 6.1. Let G be a threshold graph with $n+1$ vertices. Then

$$\mathrm{Ehr}_{q,t}\big(\mathcal{F}_G(-n,1,\ldots,1)\big) \in \mathbf{N}[q,t].$$

This conjecture has been verified up to $n = 9$.

In [7], it was conjectured that $\mathrm{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}(\mathbf{a})) \in \mathbf{N}[q,t]$ for integral netflows $\mathbf{a}$ satisfying $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. The analogous q,t-positivity statement for threshold graphs does not hold even though Theorem 5.2 gives product formulas when $t = q^{-1}$.

Example 6.2. For the threshold graph G with degree sequence $(3,3,2,2)$ and $\mathbf{a} = (-9,3,3,3)$, there are 16 integral flows, and we have that

$$\mathrm{Ehr}_{q,t}(\mathcal{F}_G(\mathbf{a})) = q^{12} + q^{11}t + q^{10}t^2 + \cdots + 2q^4t^3 + 2q^3t^4 - q^3t^3 \notin \mathbf{N}[q,t].$$

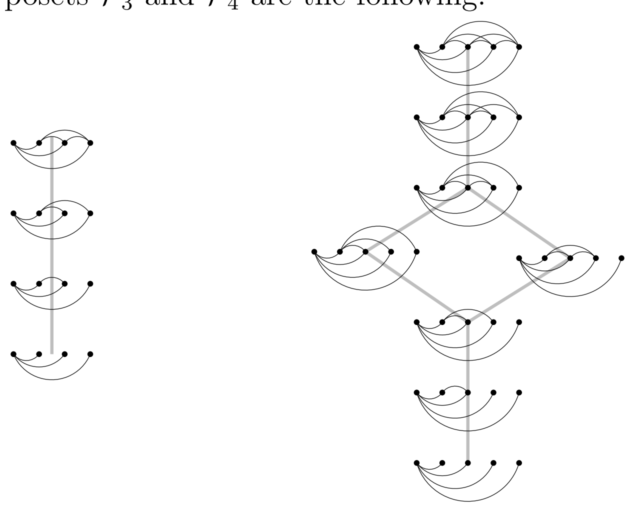
Along with Theorem 3.8, Conjecture 6.1 suggests that there may be some statistic stat$(\cdot)$ on spanning trees T of G or on G-parking functions such that $\sum_T q^{\mathrm{inv}(T)} t^{\mathrm{stat}(T)}$ equals $\mathrm{Ehr}_{q,t}(\mathcal{F}_G)$. We have so far been unable to find such a statistic (see Section 6.3).

A spanning tree T of a connected threshold graph G with $n+1$ vertices is also a spanning tree of the complete graph. A stronger positivity result would be that each monomial $q^{\mathrm{inv}(T)} t^{\mathrm{stat}(T)}$ in $\mathrm{Ehr}_{q,t}(\mathcal{F}_G)$ appeared also in $\mathrm{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}})$. Calculations up to $n = 9$ suggest that this is also the case.

Conjecture 6.3. Let G be a threshold graph with $n+1$ vertices. Then

$$\mathrm{Ehr}_{q,t}(\mathcal{F}_{K_{n+1}}) - \mathrm{Ehr}_{q,t}(\mathcal{F}_G) \in \mathbf{N}[q,t].$$

The above computations suggest to check positivity of differences of (q,t)-Ehrhart functions between a threshold graph and a subgraph that is also a threshold graph. Let $\mathcal{P}_n$ be the poset of connected threshold graphs with vertices $0, 1, \ldots, n$, where $H \preceq G$ if H is a subgraph of G. This poset is isomorphic to the poset of shifted Young diagrams (or partitions with distinct parts) contained in $(n-1, n-2, \ldots, 0)$, ordered by inclusion. For example the Hasse diagrams of the posets $\mathcal{P}_3$ and $\mathcal{P}_4$ are the following:



Calculations up to $n = 9$ suggest positivity of differences of (q,t)-Ehrhart functions along the cover relations of this poset.

Conjecture 6.4. For threshold graphs $G \succeq H$ in $\mathcal{P}_n$,

$$\mathrm{Ehr}_{q,t}(\mathcal{F}_G) - \mathrm{Ehr}_{q,t}(\mathcal{F}_H) \in \mathbf{N}[q,t].$$

Note that Conjecture 6.4 implies Conjecture 6.3.

Lastly, one might try to use the poset structure of $\mathcal{P}_n$ to refine further each (q,t)-Ehrhart series in the following way. For a threshold graph G in $\mathcal{P}_n$, let

$$S_{q,t}(G) = \sum_{H \preceq G} \mu(H,G)\, \mathrm{Ehr}_{q,t}(\mathcal{F}_H),$$

where $\mu(\cdot,\cdot)$ is the Möbius function of $\mathcal{P}_n$. By Möbius inversion we then have that

$$\mathrm{Ehr}_{q,t}(\mathcal{F}_G) = \sum_{H \preceq G} S_{q,t}(H).$$

One might hope that $S_{q,t}(G)$ is q,t-positive in general, but this is not the case.

Example 6.5. Let G be the threshold graph G with degree sequence $(6,6,6,6,5,5,4)$. Then

$$S_{q,t}(G) = q^{13} + q^{12}t + q^{11}t^2 + \cdots + q^3t^2 + q^2t^3 - q^2t^2 \notin \mathbf{N}[q,t].$$

This shows that if the statistic $\mathrm{stat}(T)$ exists such that $\mathrm{Ehr}_{q,t}(\mathcal{F}_G) = \sum_T q^{\mathrm{inv}(T)} t^{\mathrm{stat}(T)}$, then it must depend on the underlying threshold graph G.

6.2. **Positivity with Gorsky–Negut weight.** One could explore generalizations to other positivity results for Tesler matrices. The *alternant* DH_n^{ε} is a certain subspace of DH_n of dimension $\frac{1}{n+1}\binom{2n}{n}$ [9], the nth Catalan number. The bigraded Hilbert series of this subspace is the q,t-Catalan number $C_n(q,t)$. Gorsky and Negut [11] expressed $C_n(q,t)$ as a different weighted sum over the integral flows of $\mathcal{F}_{K_{n+1}}(-n,1,\ldots,1)$ (see Remark 2.2 for translating from integral flows to Tesler matrices).

Theorem 6.6 (Gorsky–Negut [11])**.**

$$C_n(q,t) = \sum_{A \in \mathcal{F}_{K_{n+1}} \cap \mathbf{Z}^E} wt'_{q,t}(A),$$

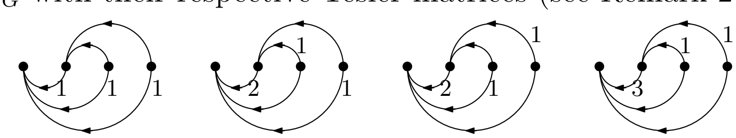
where

$$wt'_{q,t}(A) = \prod_{\substack{i>1 \\ a_{i,i-1}>0}} \left(wt_{q,t}(a_{i,i-1}+1) - wt_{q,t}(a_{i,i-1})\right) \prod_{\substack{i-1>j>0 \\ a_{i,j}>0}} \left(-(1-t)(1-q)wt_{q,t}(a_{i,j})\right), \tag{6.1}$$

for $wt_{q,t}(b)$ as defined in (2.1)*.*

In contrast with the evidence for Conjecture 6.1, this weighted sum does not necessarily give a polynomial in $\mathbf{N}[q,t]$ for threshold graphs.

Example 6.7. For the threshold graph G with degree sequence $(3,3,2,2)$, there are four integral flows in $\mathcal{F}_G$ with their respective Tesler matrices (see Remark 2.2):



$$\begin{bmatrix} 1 & \underline{0} & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \underline{0} & 0 \\ & 0 & 1 \\ & & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & \underline{0} & 1 \\ & 1 & 0 \\ & & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & \underline{0} & 1 \\ & 0 & 1 \\ & & 3 \end{bmatrix}$$

The weighted sum $wt'_{q,t}(A)$ of these flows gives

$$1 + (q+t-1) - (1-q)(1-t) - (q+t-1)(1-q)(1-t) = q^2 + 2qt + t^2 - q^2t - qt^2 \notin \mathbf{N}[q,t].$$

The Gorsky–Negut weight (6.1) is not as natural for threshold graphs G as it is for the complete graph since G might not contain the edge $(i,i-1)$. Instead, one could consider a weight

$$wt''_{q,t}(A) = \prod_{\substack{i>1 \\ a_{i,\bar{d}_i-1}>0}} \left(wt_{q,t}(a_{i,\bar{d}_i-1}+1) - wt_{q,t}(a_{i,\bar{d}_i-1})\right) \prod_{\substack{\bar{d}_i-1>j>0 \\ a_{i,j}>0}} \left(-(1-t)(1-q)wt_{q,t}(a_{i,j})\right),$$

where $\bar{d}_i - 1$ is the largest neighbor of i less than i. Still, summing over this weight does not necessarily yield a polynomial in $\mathbf{N}[q,t]$.

Example 6.8. For the threshold graph G with degree sequence $(5,5,5,3,3,3)$, we have $\bar{d}_2 = 2$, and $\bar{d}_3 = \bar{d}_4 = \bar{d}_5 = 3$. There are 81 integral flows in $\mathcal{F}_G$. The weighted sum of these flows gives

$$\sum_{A \in \mathcal{F}_G \cap \mathbf{Z}^E} wt''_{q,t}(A) = q^7 + q^6 t + q^5 t^2 + \cdots + 3q^3t^2 + 3q^2t^3 - q^2t^2 \notin \mathbf{N}[q,t].$$

Garsia and Haglund [6] gave a weight over certain integral flows in $\mathcal{F}_{K_{n+1}}$ that yields the *Frobenius series* of the space DH_n, a certain symmetric function that in the Schur basis has coefficients in $\mathbf{N}[q,t]$ (for more details see [13, Ch. 2, Ch. 6]). Using the same weight on flows in $\mathcal{F}_G$ for threshold graphs G does not give symmetric functions with a Schur expansion with coefficients in $\mathbf{N}[q,t]$.

6.3. **A note on the statistic pmaj.** Loehr and Remmel [22] defined a statistic pmaj on parking functions and showed that (dinv, area) and (area, pmaj) are equidistributed. Hence Theorem 1.1 implies that

$$\mathrm{Hilb}_{q,t}(DH_n) = \sum_P q^{\mathrm{area}(P)} t^{\mathrm{pmaj}(P)}.$$

One way to define pmaj is as follows. Parking functions have a natural partial order: $P \leq Q$ if $P(i) \leq Q(i)$ for all i. For the complete graph, the maximal parking functions are those with area 0, namely the bijections $Q\colon [n] \to \{0,1,\ldots,n-1\}$. For a maximal parking function Q,

$$\mathrm{pmaj}(Q) = \sum_{i\colon Q(i)<Q(i+1)} (n-i),$$

while for any other parking function P, $\mathrm{pmaj}(P) = \min_{Q>P} \mathrm{pmaj}(Q)$. Note that on the maximal parking functions,

$$\sum_{Q \text{ maximal}} t^{\mathrm{pmaj}(Q)} = \mathrm{Hilb}_{0,t}(DH_n) = [n]_t!$$

(and on maximal parking functions, pmaj is easily seen to be equidistributed with major index on permutations).

The area and codeg statistics coincide on parking functions, which suggests that, in accordance with Conjecture 6.1, if there exists a statistic stat on G-parking functions such that

$$\mathrm{Ehr}_{q,t}(\mathcal{F}_G) = \sum_P q^{\mathrm{codeg}(P)} t^{\mathrm{stat}(P)},$$

then we should be able to construct stat to be analogous to pmaj. Then we might expect to be able to define stat on G-parking functions such that

$$\sum_{Q \text{ maximal}} t^{\mathrm{stat}(Q)} = \mathrm{Ehr}_{0,t}(\mathcal{F}_G) = [\bar{d}_i]_t!$$

by Theorem 4.2, while for any other parking function P, $\mathrm{stat}(P) = \min_{Q>P} \mathrm{stat}(Q)$. This could simplify the task of defining stat since it would only need to be defined on the maximal G-parking functions. Initial computations suggest that it is possible to find such a statistic for small graphs, though we have not yet found a suitable statistic that works for all threshold graphs.

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