PERIODIC PARALLELOGRAM POLYOMINOES

ADRIEN BOUSSICAULT AND PATXI LABORDE-ZUBIETA

Abstract. A periodic parallelogram polyomino is a parallelogram polyomino such that we glue the first and the last column. In this work we extend a bijection between ordered trees and parallelogram polyominoes in order to compute the generating function of periodic parallelogram polyominoes with respect to the height, the width and the intrinsic thickness, a new statistic unrelated to the existing statistics on parallelogram polyominoes. Moreover we define a rotation over periodic parallelogram polyominoes, which induces a partitioning in equivalent classes called strips. We also compute the generating function of strips using the theory of Pólya.

1. Introduction

Convex polyominoes and parallelogram polyominoes are classical combinatorial objects which have been studied by Delest and Viennot in [4]. They are counted respectively by $f_{n+2} = (2n+11)4^n - 4(2n+1)\binom{2n}{n}$ and $c_n = \frac{1}{n+1}\binom{2n}{n}$.

In this article, we are interested in a new family of parallelogram polyominoes, the periodic parallelogram polyominoes (cf. Definition 2.2). They model infinite strips embedded in a cylinder. In order to study these objects, we adapt a bijection, between parallelogram polyominoes and forests of trees, introduced in [3]. The statistics that will be studied are the height, the width, the semi-perimeter and a new statistic called the intrinsic thickness. We were able to get a generating function with respect to these statistics. Our new statistic doesn't seem to be related to the area, which is another important statistic of parallelogram polyominoes (cf. article [2]). However, if we fix the intrinsic thickness, the number of parallelogram polyominoes of semi-perimeter equal to n is equal to the sum of the areas under all the dyck paths of semi-length equal to n. In this paper, we also give the generating function of periodic parallelogram polyominoes up to rotation of columns.

We start by giving (Section 2) the definition of a periodic parallelogram polyomino, we will denote them PPPs. Then, in a second section (Section 3), we will study the internal structure of a PPP, by constructing for each PPP a graph called *ordered cyclic forest*. In the section Section 4, we define a reversible operation over ordered cyclic forests called *pruning*, which will reduce the study of PPPs to trunk PPPs equipped with a marked ordered cyclic forest. We introduce in Section 5, the notion of strip by defining a rotation on PPPs, then, using trunk PPPs and ordered cyclic forests we decompose a strip as a cycle of 4-tuples. This bijective decomposition allow us to compute the generating function of strips (Section 6) and the generating function of PPPs (Section 7).

2. Periodic Parallelogram Polyominoes (PPPs)

We start by giving a definition of *parallelogram polyominoes*.

Definition 2.1 (PP)**.** *A* parallelogram polyomino *is a maximal set of cells of $\mathbb{Z}\times\mathbb{Z}$, defined up translation, contained in between two paths with North and East steps that intersect only at their starting and ending points.*

The Figure 1 shows an example of a parallelogram polyomino.

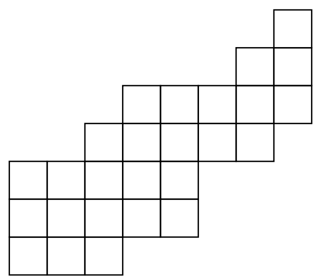

FIGURE 1. A parallelogram polyomino.

The *first* (resp. *last*) *column* corresponds to the leftmost (resp. rightmost) column.

Definition 2.2 (PPP)**.** *A* periodic parallelogram polyomino *is a PP with an admissible marked cell in the first column. A cell is called admissible if its height is less or equal to the size of the last column.*

This marking indicates the location where the first and the last column are glued in order to obtain a *periodic strip*. The Figure 2 gives an example of PPP and the beginning of the induced infinite strip. In the rest of the article, multiple rows joint together will count as one. For example, in the Figure 2 the second row and the last row, starting from the top, count as one. Following this convention, we define the height of a PPP as its number of rows, it is also the number of rows above the marked one. The number of column defines the width of the PPP and the semi-perimeter is equal to the sum of the width and the height. For example, the polyomino of Figure 2 is of height 5, width 8 and semi-perimeter 13.

From Section 3, the study does not hold for *degenerated* PPPs.

Definition 2.3. *A* degenerated periodic parallelogram polyomino *is a PPP of rectangular shape such that the marked cell is the topmost cell of the first column.*

For example, the left PPP of Figure 3 is degenerated while the right one is not.

The generating function of degenerated PPPs, is equal to $\frac{t}{1-t} \cdot \frac{y}{1-y}$, where t counts the intrinsic thickness and y width, it therefore remains to study the non degenerated case. In the rest of the article, a non degenerated PPP will be denoted by PPP*.

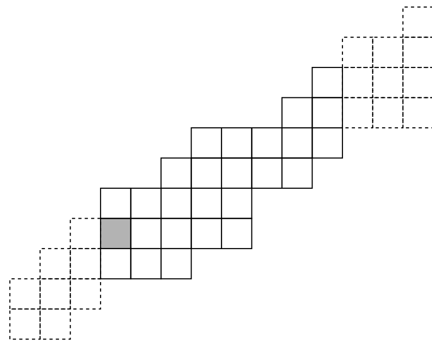

FIGURE 2. A periodic parallelogram polyomino.

Degenerated. Non degenerated

FIGURE 3. A degenerated PPP and a non degerated one.

3. Ordered cyclic forest of a PPP*

In this section we define the notion of the ordered cyclic forest of a PPP*. It is a graph which encodes the internal structure of a PPP* and gives a simpler description of it.

Let S be a finite set of elements and $father$ a map from S to S. Let G_{father} be the graph whose vertices are the elements of S and the adjacency map $sons := father^{-1}$. Each connected component of G_{father} contains exactly one cycle. Indeed, by the Euler formula, the cyclomatic number of each connected component is one since there are as much as edges as vertices. Moreover, if s is in a connected component C, the subgraph $(father^k(s))_{k\geq 0}$ contains a cycle, and so do C.

For each vertex s of G_{father}, we define a total order on $sons(s)$ denoted by $\overrightarrow{sons}(s)$. The graphs obtained by this construction will be called *ordered cyclic forests*. An example is given in the right part of Figure 4. In this figure, the sons are horizontally aligned and ordered from left to right (from the biggest to the smallest). In order to lighten the figure, the orientation of the edges is not explicitly written, since it is not so important in our study.

We will now define a map Φ from $PPPs^*$ to ordered cyclic forests. It is based on a bijection between parallelogram polyominoes and forests of ordered trees, introduced in [3]. Let P be a PPP^*, the ordered cyclic forest $\Phi(P)$ is obtained by defining $father$ and $\overrightarrow{sons}$ as follows (the fact that multiple joint rows count as one still holds) :

- the vertices are the topmost cells of columns and rightmost cells of rows;
- for all vertex s, if s is the topmost cell of a column (resp. rightmost cell of a row) then $father(s)$ is the rightmost cell (resp. topmost cell) of its row (resp. column);

- let s be a vertex and, f_1 and f_2 two of its sons. f_1 is bigger than f_2 if f_1 is at the right or above f_2 in P.

An example of Φ is given in Figure 4, the vertices correspond to the pointed cells.

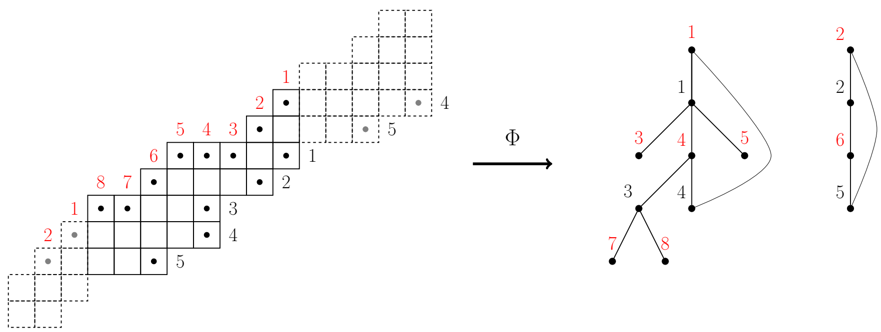

FIGURE 4. The map Φ.

4. Pruning and trunk PPP*

In the previous section, we introduced ordered cyclic forests. We will define now an operation consisting in deleting recursively the rows and the columns in P corresponding to leaves in $\Phi(P)$. We call this operation, pruning. The idea is to reduce the study of PPPs* to the ones whose ordered cyclic graph is a disjoint union of cycles.

The leaves in $\Phi(P)$ correspond to rows and columns of P which have only one pointed cell. Hence, deleting a leaf in $\Phi(P)$ is equivalent to deleting the corresponding row or column in P. If we know the position of the leaf among its brotherhood we can construct back the corresponding row or column, up to a rotation of column. For example, in Figure 5, if we want to reconstruct the leaf 7 we have no choice, but in the case of the leaf 8, we don't know if it was in the first position or in the last position. Therefore, pruning is reversible in the sens that we can reconstruct a PPP up to rotation of columns. In order to, recover P, we need to mark the vertex of $\Phi(P)$ corresponding to the first column of P.

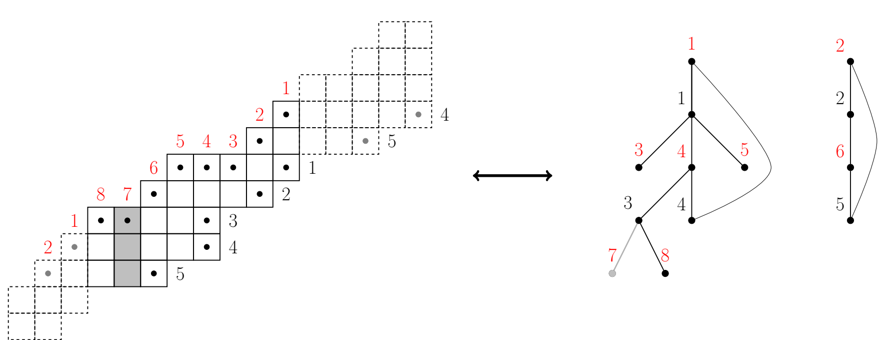

FIGURE 5. Deletion of the leaf 7 of Figure 4.

After doing the complete pruning to P, the graph becomes a disjoint union of cycles, we denote by $trunk(P)$ the PPP* we obtain. As the pruning is invertible, if we know $\Phi(P)$, $trunk(P)$ and the marked vertex in $\Phi(P)$, we can reconstruct P. Hence, we need to characterise the trunk PPPs*.

Proposition 4.1. *A PPPs* is a trunk PPP* if:*
- *the upper path is of the form $N^k(NE)^l$,*
- *the lower path is of the form $(EN)^l N^k$ and,*
- *the marked cell in the first column, is the topmost cell,*

with l and k two positive integers.

The Figure 6 shows the trunk PPP* of the PPP* in Figure 5. The corresponding integers k and l are respectively 2 and 4.

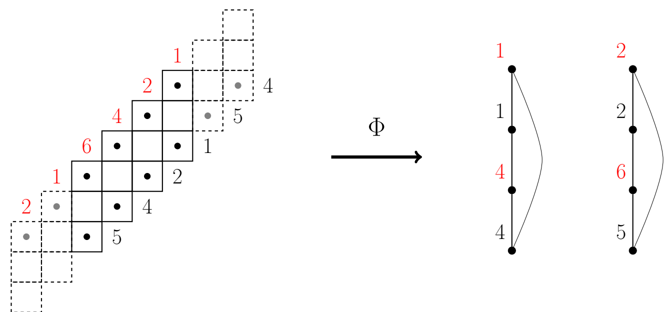

FIGURE 6. The trunk PPP* of the PPP* of Figure 5 and its ordered cyclic forest .

Proposition 4.2. *Let P be a PPP^*, then the cycles of the ordered cyclic forest $\Phi(P)$ are of the same size, and this size is even.*

Proof. The number of vertex of the cycles does not change when we prune the PPP^*, and the Proposition 4.2 holds for trunk $PPPs^*$. □

Remark 4.3. *It should be noted that multiple trunk PPPs* can give the same ordered cyclic forest. Indeed, $\Phi(P)$ contains $\frac{l}{\gcd(k,l)}$ disjoint cycles of size $2\gcd(k,l)$.*

Let P be a PPP* and k be the integer of $trunk(P)$ of Proposition 4.1. The integer k is called the *intrinsic thickness* of P.

5. Action of the rotation over the PPPs*: the strips

In this section, we will define strips by defining how we rotate a PPP*.

If we join recursively a copy of a PPP* in the first and in the last column, we obtain an infinite strip. Two different PPPs* can generate the same infinite strip, in particular, they will have the same ordered cyclic forest. In order to define more formally the notion of strip, we need to define the rotation of a PPP*. Let P be a PPP* of width at least 2. If we mark the second column at the level of the topmost cell of the first column, and we move the first column to the last position while respecting the initial marking. We obtain a PPP* that we note $r(P)$. In particular, $\Phi(p)$ and $\Phi(r(P))$ give the same ordered cyclic forest. This construction defines an automorphism r of PPPs* called *rotation*. Let us denote G the group generated by r.

Definition 5.1. *The rotation induces an action of the group G on the set of PPPs***. The orbits of this action will be called* strips*. Since the semi-perimeter and the intrinsic thickness are invariant by rotation, we extend the two statistics to strips.*

This notion of strip, model the embedding of an infinite strip in a cylinder whose circumference is give by the semi-perimeter. That is why, if two PPPs* of different semi-perimeter give graphically the same infinite strip by translation, their strips will be viewed as different.

6. Generating function of strips

In order to get the generating function of strips, we describe a bijection between strips and pairs (k, c) made of a positive integer k, that will correspond to the intrinsic thickness, and a cycle c of 4-tuples of ordered trees. Then, we will use the theory of Pólya to compute the generating function.

Proposition 6.1. *The strips are in bijection with pairs made of:*

- *a positive integer k*
- *a cycle of size l of 4-tuples of ordered trees.*

Proof (Sketched). Let P be a PPP*. $trunk(P)$ is uniquely define with the positive integers k and l of Proposition 4.2. Each column of $trunk(P)$ contains two consecutive vertices v_1 and v_2 of the same cycle of $\Phi(P)$. Hence, we associate to each column, the 4-tuple of ordered trees (two for each vertex) rooted in s_1 and s_2. Following the order of the columns in $trunk(P)$, we construct a list of 4-tuples. Instead of a list we will consider a cycle since the pruning is reversible up to rotation. We will denote by $\psi(P)$ the pair (k, c). □

The set of cycles composed of 4-tuples of ordered trees will be denoted by C. The intrinsic thickness and the semi-perimeter of a strip b are respectively denoted $it(b)$ and $sp(b)$.

We want to compute the generating function $B(z, t)$ of strips, with z counting the semi-perimeter and t the intrinsic thickness:

$$B(z,t) := \sum_{b \text{ strip}} t^{it(b)} z^{sp(b)} = \sum_{\substack{k \geq 1 \\ c \in C}} t^k z^{sp(\Psi^{-1}(k,c))} = \frac{t}{1-t} \cdot \sum_{c \in C} w(c).$$

with $w(c) = z^{sp(\Psi^{-1}((1,c)))}$.

In order to find the generating function of strips, we just need to count the cycles whose elements are colored with 4-tuples of ordered trees.

The theory of Pòlya (cf. [1, 5]) gives a formula to compute the generating function of cycles $c = (c_1, c_2, \ldots, c_n)$ with the weight $w(c) = \prod_i w(\text{ color}(c_i))$ and colored with colors taken in a set A:

$$Z(A) := \sum_{c \text{ cycle}} w(c) = -\sum_{i \geqslant 1} \frac{\varphi(i)}{i} \log\left(1 - \sum_{a \in A} w(a)^i\right),$$

with $\varphi(i) := |\{k < i \mid k \text{ and } i \text{ are relatively prime.}\}|$ is the Euler phi function. Let $w(a) = z^{|a|}$ and $\mathcal{G}(z)$ be the generating function $\sum_{a \in A} w(a)$ of A, then $\mathcal{G}(z^i)$ is equal to $\sum_{a\in A}(w(a))^i$. Hence, the generating function of cycles becomes

$$-\sum_{i\geqslant 1}\frac{\varphi(i)}{i}\log(1-\mathcal{G}(z^i)). \tag{1}$$

In our case, we want to compute the generating function of strips with respect to the semi-perimeter. Since the semi-perimeter of a strip b is equal to the number of vertices in its underlying graph $\Phi(P)$, hence we need to count the number of non-root vertices in $\Psi(P)$ plus 2 vertices for each 4-tuples in the cycle. That's why, the set A corresponds to the set of 4-tuples of ordered trees with the weight:

$$w((T_1,T_2,T_3,T_4)) = z^{|(T_1,T_2,T_3,T_4)|},$$

with $|(T_1,T_2,T_3,T_4)| = |T_1|+|T_3|+|T_3|+|T_4|+2$ and $|T|$ is the number of non-root vertices of T. All that remains is to compute the generating function $\mathcal{G}(z)$ of the set A, which is equal to $z^2\mathcal{A}(z)^4$, where $\mathcal{A}(z)$ is the generating function of ordered trees counted with respect to the number of non-root vertices. It satisfies the equation:

$$\mathcal{A}(z) = \frac{1}{1-z\mathcal{A}(z)},$$

which can be solved:

$$\mathcal{A}(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

Finally, the generating function of strips $\mathcal{B}(z,t)$ where z counts the semi-perimeter and t the intrinsic thickness, is equal to:

$$B(z,t) = -\frac{t}{1-t}\cdot\sum_{i\geqslant 1}\frac{\varphi(i)}{i}\log(1-\frac{(1-\sqrt{1-4z})^4}{16z^{2i}}). \tag{2}$$

This formula is quite surprising, because the intrinsic thickness is a statistic independent from other statistics. Hence, it is sufficient to study the family of strips with intrinsic thickness equal to 1 to characterise completely the combinatory of the entire family.

7. Generating function of PPPs

In this section we will give a non symmetric version of Proposition 6.1 in order to completely encode a PPP* with a pair (k,s) with k being the intrinsic thickness and s a sequence of 4-tuples of ordered trees such that the first 4-tuple has a marked vertex.

In order to extend the Proposition 6.1 to PPPs*, it is sufficient to mark the vertex corresponding the top cell of the first column. Hence, we color in black the vertices corresponding to the top cells of columns and in white the rightmost cells of rows. We obtain two types of bicolored trees, the ones with black vertices at odd height and the ones with black vertices at even height. In particular, each 4-tuple $(T_{i,1},T_{i,2},T_{i,3},T_{i,4})$ is composed of two black rooted trees, and two white rooted trees. Marking the top cell of a column correspond to marking a non-root black vertex or the two black roots of a same 4-tuple. If we put in first position the marked 4-tuple, we get a sequence instead of a cycle.

Proposition 7.1. *The PPPs* are in bijection with pairs composed of a positive integer k (the intrinsic thickness) and a non empty l-tuple of 4-tuples of bicolored ordered trees such that:*

– *each 4-tuple is composed of two black rooted trees and two white rooted trees,*
– *in the first 4-tuple, we mark a non-root black vertex or the two black roots.*

To compute the generating function of PPPs*, we need to find the generating functions of black rooted and white rooted ordered trees. We denote them respectively $\mathcal{A}_\bullet(z_\bullet, z_\circ)$ and $\mathcal{A}_\circ(z_\bullet, z_\circ)$, with $z_\bullet$ counting the black vertices and $z_\circ$ the white ones. Those two generating functions satisfies the following equations.

$$\mathcal{A}_\bullet(z_\bullet, z_\circ) = \frac{1}{1 - z_\circ \mathcal{A}_\circ} \text{ and } \mathcal{A}_\circ(z_\bullet, z_\circ) = \frac{1}{1 - z_\bullet \mathcal{A}_\bullet}.$$

After solving them, we obtain

$$\mathcal{A}_\bullet(z_\bullet, z_\circ) = \frac{z_\bullet - z_\circ + 1 - \sqrt{(z_\bullet - z_\circ + 1)^2 - 4z_\bullet}}{2z_\bullet} = \mathcal{A}_\circ(z_\circ, z_\bullet) \tag{3}$$

Hence, the generating function of a 4-tuple of ordered trees is

$$z_\bullet z_\circ \mathcal{A}_\bullet(z_\bullet, z_\circ)^2 \mathcal{A}_\circ(z_\bullet, z_\circ)^2. \tag{4}$$

More over, if we mark one of the black vertices, we get

$$z_\bullet \partial_{z_\bullet} z_\bullet z_\circ \mathcal{A}_\bullet(z_\bullet, z_\circ)^2 \mathcal{A}_\circ(z_\bullet, z_\circ)^2 \tag{5}$$

Finally, the generating function of the PPPs* with $z_\bullet$ counting the width, $z_\circ$ the height and t the intrinsic thickness is equal to

$$P(z_\bullet, z_\circ, t) = \frac{t}{1-t} \cdot \frac{z_\bullet \partial_{z_\bullet} z_\bullet z_\circ \mathcal{A}_\bullet(z_\bullet, z_\circ)^2 \mathcal{A}_\circ(z_\bullet, z_\circ)^2}{1 - z_\bullet z_\circ \mathcal{A}_\bullet(z_\bullet, z_\circ)^2 \mathcal{A}_\circ(z_\bullet, z_\circ)^2}. \tag{6}$$

As in the previous section, we notice that the intrinsic thickness is a statistic independent from the other statistics. Hence, in order to characterize combinatorially the PPPs*, it is sufficient to study the PPPs* of intrinsic thickness 1. The generating function of PPPs* of intrinsic height 1 is equal to:

$$P(z, z, t)[t] = 1 \cdot z + 6 \cdot z^2 + 29 \cdot z^3 + 130 \cdot z^4 + 562 \cdot z^5 + 2380 \cdot z^6 + \dots$$

The sequence of the previous coefficients appears in OEIS (cf. [7]), it corresponds to the sequence A008549. The n-th term is the sum of the areas under all the dyck paths of semi-length n.

Acknowledgement

The authors are grateful to Srečko Brlek for his attention, his advices and for the numerous discussions we had during our stays in the LaCIM (Laboratoire de Combinatoire et d'Informatique Mathématique) at the University of Quebec at Montreal.

We thank the LIRCO (Laboratoire International Franco-Québécois de Recherche en Combinatoire) for financing our respective stays at LaCIM.

This research was driven by computer exploration using the open-source software `Sage` [8] and its algebraic combinatorics features developed by the `Sage-Combinat` community [6].

References

[1] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial Species and Tree-like Structures*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1998.

[2] M. Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Math.*, 154(1-3):1–25, 1996.

[3] A. Boussicault, S. Rinaldi, and S. Socci. The number of directed k-convex polyominoes, 2015.

[4] M.-P. Delest and G Viennot. Algebraic languages and polyominoes enumeration. *Theoretical Computer Science*, 34(1):169 – 206, 1984.

[5] G. Pólya and R.C. Read. *Combinatorial enumeration of groups, graphs, and chemical compounds*. Springer-Verlag, New York, 1987.

[6] The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. `http://combinat.sagemath.org`.

[7] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. `http://oeis.org`.

[8] W. A. Stein et al. *Sage Mathematics Software (Version 7.2.beta1)*. The Sage Development Team, 2015. `http://www.sagemath.org`.

LaBRI - Université de Bordeaux, 351 cours de la Libération F-33405 Talence cedex
E-mail address: `plaborde@labri.fr`
E-mail address: `boussica@labri.fr`