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Connected chord diagrams and bridgeless maps

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Abstract. We present a surprisingly new connection between two well-studied combinatorial classes: rooted connected chord diagrams on one hand, and rooted bridgeless combinatorial maps on the other hand. We describe a bijection between these two classes, which naturally extends to indecomposable diagrams and general rooted maps. As an application, this bijection provides a simplifying framework for some technical quantum field theory work realized by some of the authors. Most notably, an important but technical parameter naturally translates to vertices at the level of maps. Finally, we give a new interpretation of a famous equation due to Arquès and Béraud, which characterizes the generating function of rooted maps, in terms of indecomposable chord diagrams. This interpretation can be specialized to connected diagrams, and refined to incorporate the number of crossings.

Keywords: bijection; connected chord diagrams; combinatorial maps; combinatorial Dyson-Schwinger equations

1 Introduction

Connected chord diagrams are well-studied combinatorial objects that appear in numerous mathematical areas such as knot theory, graph sampling, analysis of data structures, and bioinformatics (see the references for example in [5, 4]). Their counting sequence (Sloane's A000699) has been known since Touchard's early work published in 1952 [12]. In this document we present a bijection with an arguably even more basic class of combinatorial objects: bridgeless combinatorial maps. Despite the ubiquity of both families of objects in the literature, this bijection is, to our knowledge, new. Furthermore, it is fruitful in the sense that it generalizes and restricts well, and useful parameters carry through it.

1.1 Definitions

Rooted chord diagrams (or simply *chord diagrams* or *diagrams* as there will be no confusion) are graphical representations of fixed point free involutions or matchings on the set $\{1, \ldots, 2n\}$. A chord diagram is obtained by arranging $2n$ dots on a line, and drawing n

chords above the line, linking the dots in disjoint pairs. A diagram is said to be *connected* or *irreducible* if the intersection graph (the graph where the vertices are chords and the edges link two crossing chords) is connected. The *size* of a diagram is the total number of chords. Small connected diagrams are depicted in the first row of Table 1.

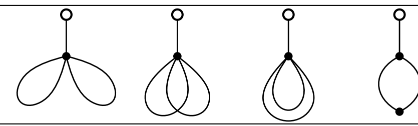
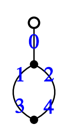
Objects	Size 1	Size 2	Size 3
Connected diagrams			
Bridgeless maps			

Table 1: Small connected diagrams and bridgeless maps

Combinatorial maps are a purely algebraic way of representing an embedding of a graph into any compact oriented surface [7]. Formally, a combinatorial map is given by a triple (H, σ, α), where H is a set and σ and α are respectively a permutation and a fixed point free involution that together act transitively on H. Intuitively, this data determines a connected graph equipped with a cyclic ordering of the half-edges around each vertex: H represents the set of half-edges, the cycles of σ form the vertices, and the matching α describes the gluing together of half-edges to form edges. Typically, combinatorial maps are rooted by distinguishing a half-edge, but it is essentially equivalent (and we will find it more convenient) to take the root to be a unique fixed point of the involution α. This fixed point may be thought of as a "*dangling*" half-edge, with only one end attached to a vertex. The *size* of a rooted combinatorial map is here the total number of edges including the half-edge where the map is rooted. Some examples of rooted combinatorial maps are given in the second row of Table 1, where we indicate the unattached end of the root half-edge by a white vertex. For example, if we label the half-edges of the map of size 3 at the far right end of Table 1 like so,



the corresponding permutations are $\sigma = (0\,1\,2)(3\,4)$ and $\alpha = (0)(1\,3)(2\,4)$.

We distinguish the root from the root edge: the *root edge* is the edge which follows the root in the cyclic order given by σ. The *root vertex* is the unique vertex which is incident to both the root and the root edge. A *corner* is the angular section between two adjacent half-edges. Half-edges are in obvious bijection with corners, but it is often more convenient to work with the corners: for example, pointing out two corners is a clear way to show how to insert an edge in a map. The *root corner* is the corner between the root and the root edge.

The examples in Table 1 have the special property of being *bridgeless*, in the sense that they do not contain any edge whose deletion disconnects the underlying graph. We show in this paper that rooted bridgeless combinatorial maps and rooted connected diagrams are equinumerous.

Theorem 1. *Rooted connected diagrams of size n are in bijection with rooted bridgeless combinatorial maps of size n.*

We should hasten to point out that it is already known that general (i.e., not necessarily bridgeless) rooted combinatorial maps are equinumerous with *indecomposable diagrams*, where a diagram is said to be indecomposable if it cannot be expressed as the concatenation of two diagrams. Note that connected implies indecomposable, but the converse is not true (cf. Table 2). According to Ossona de Mendez and Rosenstiehl [11], in 2006 the fact that rooted maps and indecomposable diagrams are equinumerous was "*known for years, in particular in quantum physics, [... but] no bijective proof of this numerical equivalence was known*". Their article [11] gives a bijective account (as a specialization of a more general bijection they described in [10] between *rooted hypermaps* and *indecomposable permutations*), and this bijection was later revisited and analyzed by Cori [3]. However, what may be surprising in light of our Theorem 1 is that the Ossona de Mendez and Rosenstiehl bijection does *not* restrict to a bijection between bridgeless maps and connected diagrams (a point we will return to in Section 4). In other words, the bijection we will describe here appears to be essentially new.

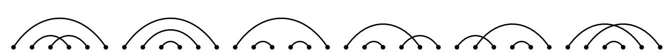
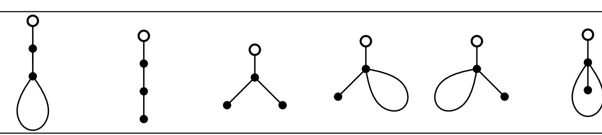
Objects	Size 2	Size 3
Indecomposable disconnected diagrams		
Maps with at least one bridge		

Table 2: Small indecomposable diagrams and maps that are not displayed in Table 1

1.2 Structure of the document

Section 2 describes the above mentioned bijection between connected diagrams and bridgeless maps, thus proving Theorem 1. It also shows how to generalize this bijection to indecomposable diagrams and bridgeless maps, recovering the result of Ossona de Mendez and Rosenstiehl [11]. It finally puts into correspondence the set of planar maps with a subset of diagrams, defined by a forbidden configuration.

Section 3 shows how this bijection can be used to simplify existing results in quantum field theory. In fact, it puts in a new light some technical parameters occurring in solutions of a family of Dyson-Schwinger equations.

Finally, Section 4 establishes a new combinatorial decomposition of indecomposable and connected chord diagrams. It extends a formula found by Arquès and Béraud [1] for rooted maps.

2 Description of the main bijection

2.1 Connected diagrams and bridgeless maps

In this section, we establish a recursive correspondence between connected diagrams and bridgeless maps, which implies Theorem 1. The first step is to prove that the cardinalities of both classes are the same. To do so, we combinatorially show the following recurrence, which characterizes the sequence `A000699` in the OEIS.

Proposition 2. *The number of rooted connected diagrams of size n and the number of rooted bridgeless maps of size n both satisfy $c_1 = 1$ and*

$$c_n = \sum_{k=1}^{n-1} (2k-1)\, c_k\, c_{n-k}.$$

Proof. The recurrence relation translates the fact that it is possible to combine two objects, one of which is weighted by twice its size (minus 1), to bijectively give a bigger object of cumulated size. We only need to describe how to do so for our two classes.

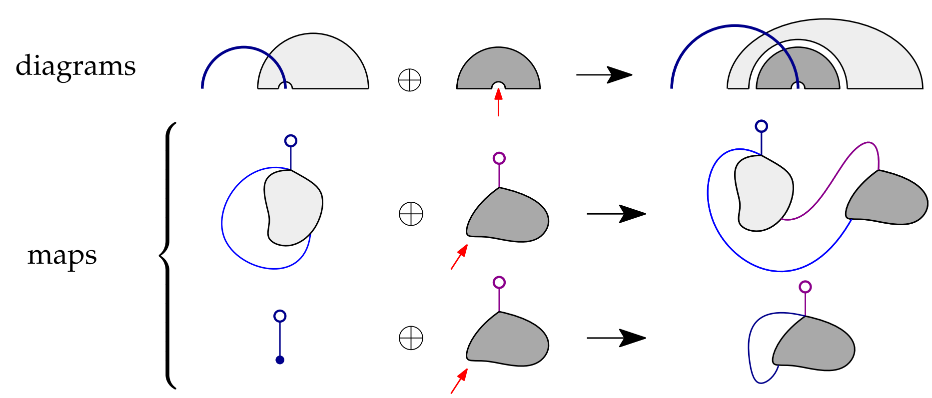


Figure 1: Schematic decomposition of connected diagrams and bridgeless maps.

Connected diagrams. For connected diagrams, $2k-1$ counts the number of intervals delimited by k chords. In other words, it means there are $2k-1$ ways to insert a new root chord in a diagram of size k. We can find in the literature numerous ways to combine a diagram C_1 with another diagram C_2 with a marked interval [9]. The one we choose comes from [4] and is illustrated in Figure 1. The idea is to insert C_2 into C_1, just after the root chord of C_1. Then, we move the right endpoint of the root chord of C_1 to the marked interval of C_2. We obtain thus our final combined diagram.

To recover C_1 and C_2, we mark the interval just after the root chord. Then, we pull the right endpoint of the diagram to the left until the diagram disconnects into two connected components. The first component is C_1, the second one C_2.

Bridgeless maps. In maps of size k, the number $2k-1$ refers to the number of corners. Given two maps M_1 and M_2 where M_2 has a marked corner, we construct a larger map as follows (this is also illustrated in Figure 1).

If M_1 has size 1, we insert a new edge in M_2 which links the root corner of M_2 to its marked corner. If M_1 has size > 1 then it has a root edge. Let us unstick the second endpoint of the root edge and insert it in the marked corner of M_2. Then, we take the root of M_2 and insert it where the second endpoint of the root chord of M_1 was. We thus obtain our final map. Note that no bridge has been created in the process.

To recover M_1 and M_2, we start by marking the corner after the second endpoint of the root chord of the new map. Then, grab this endpoint and slide it up, towards the root. When a bridge appears, we stop the process and cut the bridge, marking it as a root. The two resulting diagrams are M_1 are M_2. If we reach the root vertex with this process without creating any bridge, then it means that M_1 was the trivial map with one half-edge. In that case, we obtain M_2 by just removing the root chord. □

The previous proposition yields an implicit correspondence between connected diagrams and bridgeless maps, but it does not say which bridgeless map a given connected diagram must be sent to. In other words, no bijection is explicitly defined.

A bijection via generating trees. To solve this problem, we use the generating trees of both classes, and put them into bijection. The additional information we need is how the marking goes from one object to the other, which means that we want to find a correspondence between the intervals of a diagram and the corners of a map. There are several ways to proceed, but none is really canonical. The one we choose is based on induction, and will give the most significant results (see Section 3).

We will next describe the map from bridgeless maps to connected diagrams, see Figure 2 for an example. First, we recursively decompose the map as described in the proof of Proposition 2. We obtain a recursion tree. Then, we label every corner of every map in this recursion tree, using the following rules: Label the unique corner on each leaf of the recursion tree. Propagate from the leaves to the root in the recursion tree; whenever a corner is split in half by a new edge, create a new label for the right half.

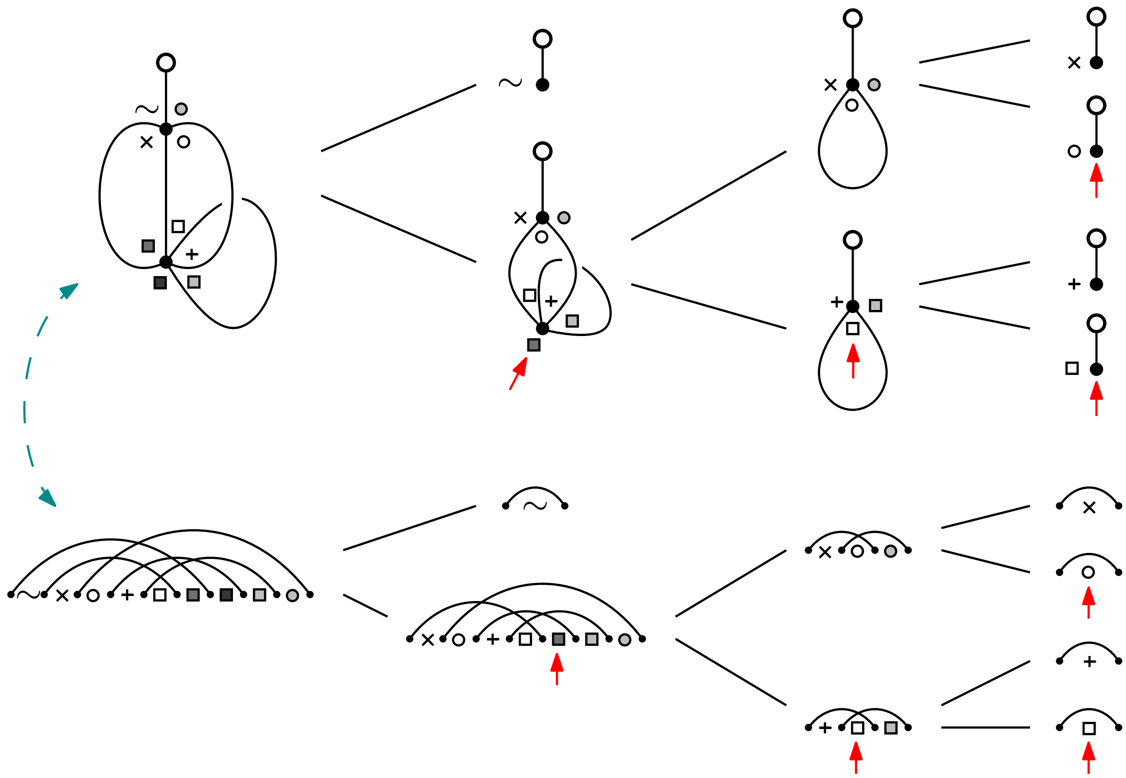


Figure 2: An example of bijection between a bridgeless map and a connected diagram. The symbols (crosses, tildes, squares, ...) are used to label the corners and the intervals of the diagrams.

We now want to carry these labels over to the intervals between the end points of the chord diagrams. The base cases are clear since there is only one interval. When a corner is split in half, it means in the world of diagrams, that an interval is also split in half. We then transfer the labels in such a way that the counterclockwise order in the maps corresponds to the left-right order in diagrams.

2.2 Extension to the indecomposable diagrams and general maps

As said in the introduction, it was already known that rooted maps are in bijection with indecomposable diagrams [11, 3]. However, these known bijections do not restrict to our bijection of bridgeless maps to connected diagrams, so we will now give one which does.

Proposition 3. *There exists a bijection between the rooted maps and the indecomposable diagrams in which the bridgeless maps are sent to the connected diagrams.*

Proof. (Idea) Here again, it is sufficient to remark that both objects have equivalent decompositions. On one hand, a rooted map is a bridgeless map where we have attached on each of its corners (sequences of) maps via bridges. On the other hand, an indecomposable diagram is a connected diagram where we have inserted (sequences of) indecomposable maps in each of its intervals. The above bijection for bridgeless maps thus transfers to general maps (see Table 2 for small examples of that correspondence). □

Other descriptions of this bijection exist, notably in terms of root edge/root chord insertion in a Tutte style recursion.

2.3 Planar maps as diagrams with forbidden pattern

We here characterize the image of planar maps under the previous bijection.

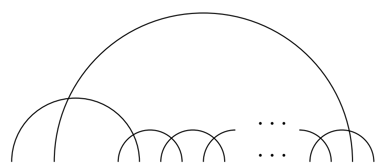


Figure 3: Forbidden configuration for diagrams corresponding to planar maps.

Proposition 4. *Planar rooted maps with n edges are in bijection with indecomposable diagrams with n chords which do not contain the configuration of Figure 3 as a subdiagram. Additionally, under this bijection, a planar map is bridgeless if and only if the corresponding diagram is connected.*

Proof. (Idea) The core idea is to characterize by induction the *covered intervals*, i.e., the intervals which correspond to the corners which break the planarity if we insert a root chord inside. They are precisely those intervals which are (i) under the root chord but not the first interval under the root chord or (ii) are already covered in the diagram obtained by removing the root chord. □

3 New perspectives on chord diagram expansions in quantum field theory

3.1 Context

Interestingly, by the work of some of the authors [8, 6, 4], rooted chord diagrams appear in quantum field theory where they give series solutions to certain Dyson-Schwinger equations. We are going to see that the bijection of Section 2 will simplify some formulas in this theory.

To proceed, we need two non-standard definitions concerning rooted connected chord diagrams. First we define the *intersection order* of the diagram as follows. The root chord is the smallest chord. Remove the root chord and let $C_1, C_2, \ldots$ be the connected components. Next sort all the chords of C_1 in this recursive order, then all the chords of C_2 and so on. This intersection order is not in general the same as the order by first endpoint — see [4] for an example. Then we say a chord is *terminal* if it does not cross any chords larger than it in this order. Equivalently, a chord c is terminal if every chord intersecting c is to the left of c.

The main result of [6], generalizing [8], consists of solving a family of Dyson-Schwinger equations in terms of a sum over weighted connected chord diagrams:

Theorem 5 (Hihn, Yeats [6])**.** *For a positive integer s, we define the Dyson-Schwinger equation*

$$G(x,L) = 1 - \sum_{k\geq 1} x^k G(x, \partial_{-\rho})^{1-sk}(e^{-L\rho}-1)F_k(\rho)_{\rho=0}$$

where $F_k(\rho) = \sum_{i\geq 0} a_{k,i}\rho^{i-1}$*. This equation is solved by*

$$G(x,L) = 1 - \sum_{i\geq 1}\frac{(-L)^i}{i!}\sum_{\substack{C\\ b(C)\geq i}} x^{\|C\|}w(C)a_{b(C),b(C)-i}\prod_{\substack{c \text{ not}\\ \text{terminal}}} a_{d(c),0}\prod_{j=2}^{\ell} a_{d(t_j),t_j-t_{j-1}}, \tag{3.1}$$

where the second sum runs over all connected diagrams C, carrying a positive integer weight $d(c)$ on each of its chords c, and such that the position of the first terminal chord is $b(C) \geq i$. As for the other parameters, $|C|$ denotes the number of chords; $\|C\|$ is the sum of the chord weights; $t_1 = b(C) < t_2 < \cdots < t_\ell = |C|$ lists the positions of all the terminal chords; and $w(C) = \prod_{\ell=1}^{|C|}\binom{d(\ell)s+\nu(\ell)-2}{\nu(\ell)}$. For the last quantity, we need another parameter $\nu(c)$ which will be subject to further discussion.

This theorem was shown by checking that the Dyson-Schwinger equation and the eventual solution both satisfy the same recurrences with the same initial conditions. Parts of the proof went through some opaque equations coming from some recursion trees. With the bijection to bridgeless maps, we can better understand the underlying logic.

3.2 Old and new definitions of the ν parameter

The first definition of the ν parameter given in [6] was quite involved as it could not be directly interpreted at the level of the diagrams. It was based on some recursion trees of the objects and the lengths of leftwards paths up these trees.

We give here a new definition of the ν parameter. It is not equivalent to the previous definition in the sense the two parameters can have different values for a given object, but both are consistent with Equation (3.1) overall. This generalizes some observations made by Hihn and will be proved in future work.

Parameters in connected chord diagrams	Parameters in bridgeless maps
chords	edges
terminal chords	vertices; edges in the spanning tree induced by the rightmost DFS
position $b(C)$ of the first terminal chord	number of ingoing edges (for the rightmost DFS) incident to the root vertex
position t_j of the jth terminal chord	number of ingoing edges (for the rightmost DFS) incident to the vertex which has been visited at position j in the rightmost DFS

Table 3: How parameters intervening in (3.1) *transfer from diagrams to maps.*

Let us consider the chords of a connected diagram C in intersection order: $c_1, c_2, \ldots$ For every k in increasing order, we label each interval below c_k by the number k. If some number was already on an interval, we rewrite k onto it. At the end of this procedure, $\nu(k)$ is defined as the number of remaining k-labels.

3.3 Interpretations in terms of maps

It turns out that the parameters occurring in Theorem 5 become more natural when viewed at the level of maps. In fact, the bijection of Section 2 brings out, maybe surprisingly, the Depth First Search (DFS) of the map where we favor the rightmost edges (call this a *rightmost DFS*). This traversal induces an orientation of the edges on a map, including the root half-edge. This orientation characterizes most of the aforementioned parameters.

Proposition 6. *Under the bijection of Section 2, the parameters of Theorem 5 are transferred as indicated by Table 3.*

This proposition can be shown by a simple induction.

Remarkably, the terminal chords correspond naturally to the vertices of the map. Consequently, all the asymptotic results of [4] translate over to asymptotics about vertices of bridgeless maps[1].

With some more thought, we can describe the analogue of the ν vector for maps, but this will be detailed in future work. Let us mention that this analogue enables us to interpret some formula which was at the time obscure but crucial for the proof of Theorem 5 (precisely [6, Proposition 6.10]).

[1] For example, it proves that the number of vertices in a random bridgeless map asymptotically obeys to a Gaussian law of mean $\sim \ln n$.

4 New interpretation of Arquès and Béraud's equation

4.1 Statement of the equation and implications

In this section we revisit the equation

$$B(z,u) = u + zB(z,u)B(z,u+1) \tag{4.1}$$

derived by Arquès and Béraud [1], who proved that it is satisfied by the generating function counting rooted maps with respect to edges and vertices. Cori [3] gave a shorter proof of this equation after analyzing Ossona de Mendez and Rosenstiehl's bijection between maps and indecomposable involutions [11], which sends vertices of a map to *left-to-right maxima* of the corresponding involution. Speaking in terms of diagrams, left-to-right maxima correspond to *top chords*: that is, chords which are not below another chord. For example, the number of top chords in the diagrams of size 3 of Tables 1 and 2 are respectively 3, 3, 2, and 2 for the connected diagrams, and 1, 1, 1, 2, 2, and 2 for the disconnected diagrams. Notice, however, that the association between maps and chord diagrams described in the columns (coming from the bijection of Section 2) does *not* in general pair a map with k vertices to a diagram with k top chords.

The question naturally arises of whether Equation (4.1) can be justified directly via a bijective interpretation, either in the case of rooted maps or of indecomposable chord diagrams. This question has been previously considered by Arquès and Micheli [2], who derived Equation (4.1) by introducing certain topological operations on maps of arbitrary genus[2]. Here we give a new bijective interpretation directly on indecomposable diagrams, with the crucial property that it restricts naturally to connected diagrams. (We also take the opportunity to refine the equation to keep track of the number of crossings.)

Theorem 7. *Let $B(z,u,v)$ be the ordinary generating function of indecomposable diagrams counted with respect to the number of chords minus one (z), the number of top chords (u) and the number of crossings (v). Similarly, let $C(z,u,v)$ be the generating function for connected diagrams with the same interpretation of the parameters. The following holds:*

$$B(z,u,v) = u + zB(z,1+uv,v)\, B(z,u,v), \tag{4.2}$$

$$C(z,u,v) = u + z(C(z,1+uv,v) - C(z,1,v))\, C(z,u,v). \tag{4.3}$$

By Theorem 1, we know that $C(z,1,1)$ is also the generating function for bridgeless maps counted by number of edges. However, the question of the interpretation of the parameters u and v for bridgeless maps remains open. In particular, an easy inspection reveals that $C(z,u,1) = u + zu^2 + z^2(2u^2 + 2u^3) + z^3(10u^2 + 12u^3 + 5u^4) + \dots$ does *not* count bridgeless maps by edges and vertices.

[2]Thanks to Maciej Dołęga for pointing us to this reference.

4.2 Combinatorial interpretation

For the sake of simplicity, let us forget the variable v counting the number of crossings (it can be incorporated quite simply afterwards), leaving just the equation $B(z,u) = u + zB(z,1+u)B(z,u)$. From a combinatorial point of view, this equation says that every indecomposable diagram with a least two chords can be seen as the product of two indecomposable diagrams, one of which has a marked subset of top chords.

We are going to describe the combination part, building a diagram from two smaller ones. The decomposition part can be deduced quite simply and will be described in a longer version of this paper. Figure 4 gives an example of such a combination.

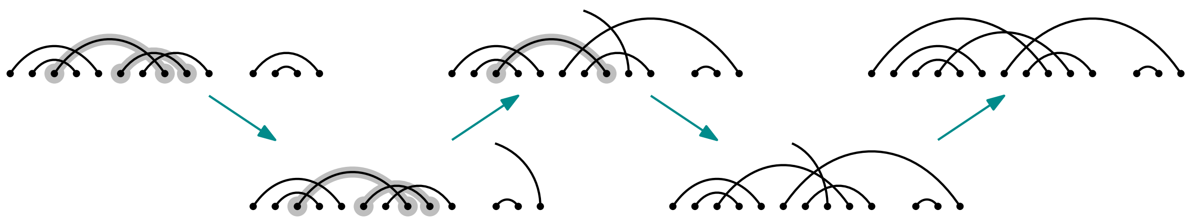


Figure 4: An example of how to combine an indecomposable diagram with another indecomposable diagram in which a subset of top chords is marked. The first diagram has 4 top chords, 2 of which are marked. The second diagram has only one top chord. The combination of both induces 3 top chords, as expected.

Let us thus consider two indecomposable diagrams D_1 and D_2, where some top chords of D_1 are marked. We run the following algorithm:

1. Put D_2 on the right of D_1.
2. Open the left endpoint of the root chord D_2.
3. Consider the rightmost marked top chord. (The top chords are sorted from left to right without ambiguity.) If there is no more marked top chord, go to 5.
4. Forget the marking of that chord. Then, open its left endpoint, and replace it by the left endpoint of the other open arc. Go to 3.
5. Close the open arc at the left of D_1.

The composition of two diagrams is thus defined. The non-marked top chords are now below a chord, so they are not top chords anymore.

Note that a new connected component is created by this process if and only if no top chord is marked. Indeed, the only way to form a new component is to close the root chord of D_2 directly at the left of D_1, which can be done by jumping Item 4. So if we want to enumerate connected diagrams, we have to force diagrams D_1 to have at least one marked top edge. Such diagrams are counted by $C(z,1+u) - C(z,1)$. We recover Equation (4.3).

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