Some elementary observations on Narayana polynomials and related topics

Johann Cigler

Fakultät für Mathematik

Universität Wien

johann.cigler@univie.ac.at

Abstract.
We give an elementary account of generalized Fibonacci and Lucas polynomials whose moments are Narayana polynomials of type A and type B.

Introduction

Consider the Fibonacci polynomials $F_n(x)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j\binom{n-j}{j}x^{n-2j}$ and the corresponding Lucas polynomials $L_n(x)=\sum_{j=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}x^{n-2j}$ and let L be the linear functional defined by $L\left(F_n(x)\right)=[n=0]$ and M be the linear functional defined by $M\left(L_n(x)\right)=[n=0]$. Then the moments $L\left(x^{2n}\right)=C_n$ are Catalan numbers and the moments $M\left(x^{2n}\right)=M_n=\binom{2n}{n}$ are central binomial coefficients. An analogous situation holds by replacing the Catalan numbers C_n by the Narayana polynomials $C_n(t)=\sum_{k\geq 0}\binom{n-1}{k}\binom{n}{k}\frac{1}{k+1}t^k$ and the central binomial coefficients M_n by the polynomials $M_n(t)=\sum_{j=0}^{n}\binom{n}{j}^2 t^j$, which are sometimes called Narayana polynomials of type B.

In this survey article I give an elementary and self-contained account of the corresponding polynomials and the associated Catalan-Stieltjes matrices. I want to thank Dennis Stanton and Jiang Zeng for helpful remarks and references to the literature.

1. 1. Background material on Fibonacci polynomials and Catalan numbers

The basic facts about Fibonacci and Lucas polynomials are very old and well known (cf. e.g. [5]).

The *Fibonacci polynomials* $f_n(x,s)=\sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1-k}{k}x^{n-1-2k}s^k$ satisfy the recursion

$f_n(x,s) = xf_{n-1}(x,s) + sf_{n-2}(x,s)$ with initial values $f_0(x,s) = 0$ and $f_1(x,s) = 1.$

We will consider the *special Fibonacci polynomials* $F_n(x) = f_{n+1}(x,-1).$ If $U_n(x)$ denotes a *Chebyshev polynomial of the second kind* then we can equivalently write $F_n(x) = U_n\left(\frac{x}{2}\right).$

The first terms of the sequence $\left(F_n(x)\right)_{n\geq 0}$ are

$1,\ x,\ -1+x^2,\ -2x+x^3,\ 1-3x^2+x^4,\ 3x-4x^3+x^5,\ \ldots$

Remark

Let me recall some well-known facts about orthogonal polynomials (cf. [4], [13],[17]). These are polynomials $\left(p_n(x)\right)_{n\geq -1}$ satisfying a recursion of the form $p_n(x) = (x - s_{n-1})p_{n-1}(x) - t_{n-2}p_{n-2}(x)$ with initial values $p_{-1}(x) = 0$ and $p_0(x) = 1.$ The corresponding *Catalan-Stieltjes matrix* $\left(a(n,k)\right)$ (cf. [13]) consists of the uniquely determined numbers $a(n,k)$ which satisfy $x^n = \sum_{k=0}^{n} a(n,k)p_k(x).$

It satisfies

$$a(n,k) = a(n-1,k-1) + s_k a(n-1,k) + t_k a(n-1,k+1) \tag{1.1}$$

with $a(0,k) = [k=0]$ and $a(n,-1) = 0$ because

$$\sum_{k=0}^{n} a(n,k)p_k(x) = x \cdot x^{n-1} = \sum_{k=0}^{n} a(n-1,k)xp_k(x) = \sum_{k=0}^{n} a(n-1,k)\left(p_{k+1}(x) + s_k p_k(x) + t_{k-1}p_{k-1}(x)\right)$$
$$= \sum a(n-1,k-1)p_k(x) + \sum s_k a(n-1,k)p_k(x) + \sum t_k a(n-1,k+1)p_k(x).$$

The numbers s_k and t_k uniquely determine both the polynomials $p_n(x)$ and the corresponding Catalan-Stieltjes matrix.

Let L be the linear functional defined by $L\left(p_n\right) = [n=0].$ Here we use Iverson's convention $[P] = 1$ if property P is true and $[P] = 0$ else. The polynomials satisfy moreover $L(p_n p_m) = 0$ for $m \neq n,$ i.e. they are orthogonal with respect to $L.$ But we shall not use this property.

The numbers $L\left(x^n\right)$ are called moments of the sequence $\left(p_n(x)\right).$

If all $s_k = 0$ then $P_n(x) = p_{2n}\left(\sqrt{x}\right)$ satisfies

$P_1(x) = x - t_0$ and $P_n(x) = \left(x - t_{2n-1} - t_{2n}\right)P_{n-1}(x) - t_{2n}t_{2n+1}P_{n-2}(x)$

and $Q_n(x) = \dfrac{p_{2n+1}\left(\sqrt{x}\right)}{\sqrt{x}}$ satisfies $Q_n(x) = \left(x - t_{2n} - t_{2n+1}\right)Q_{n-1}(x) - t_{2n+1}t_{2n+2}Q_{n-2}(x).$

This splitting is equivalent with the odd-even trick in [6].

For the Fibonacci polynomials $F_n(x)$ the numbers $a(n,k)$ satisfy

$$a(n,k)=a(n-1,k-1)+a(n-1,k+1) \tag{1.2}$$

with $a(0,k)=[k=0]$.

Thus $a(n,k)$ can be interpreted as the number of elements of the set of $n-$letter words $w_1w_2\cdots w_n$ in the alphabet $\{-1,1\}$ that add up to k, and all whose partial sums are non-negative because for $w_n=1$ the word $w_1w_2\cdots w_{n-1}$ adds up to $k-1$ and for $w_n=-1$ to $k+1$.

These so-called *ballot numbers* are well known and satisfy

$$a(2n+k,k)=\binom{2n+k}{n}-\binom{2n+k}{n-1}. \tag{1.3}$$

or equivalently

$$x^n=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\left(\binom{n}{k}-\binom{n}{k-1}\right)F_{n-2k}(x). \tag{1.4}$$

Let L be the linear functional defined by $L\left(F_n\right)=[n=0]$. Here $[P]=1$ if property P is true and $[P]=0$ else. Then (1.4) implies

$$L\left(x^{2n}\right)=\binom{2n}{n}-\binom{2n}{n-1}=C_n=\binom{2n}{n}\frac{1}{n+1} \tag{1.5}$$

is a *Catalan number* and $L\left(x^{2n+1}\right)=0$.

The first terms of the sequence $\left(C_n\right)_{n\ge 0}$ are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...

Let us compute the generating functions $f_k(z)=\sum_{n\ge 0}a(n,k)z^n$. Then (1.2) translates into

$$f_k(z)=z\left(f_{k-1}(z)+f_{k+1}(z)\right) \tag{1.6}$$

and

$$f_0(z)=1+zf_1(z). \tag{1.7}$$

The uniquely determined solution of these equations is $f_k(z)=z^k f(z)^{k+1}$ if we set $f(z)=f_0(z)$.

This can easily be verified by comparing coefficients.

By (1.7) $f(z)$ satisfies $f(z)=1+z^2 f(z)^2$ which implies the well-known result

$$f(z)=\sum_{n\geq 0} C_n z^{2n} = \frac{1-\sqrt{1-4z^2}}{2z^2}. \tag{1.8}$$

Let us also consider the polynomials

$$P_n(x)=F_{2n}\left(\sqrt{x}\right)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{2k}x^k \tag{1.9}$$

and

$$Q_n(x)=\frac{F_{2n+1}\left(\sqrt{x}\right)}{\sqrt{x}}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k+1}{2k+1}x^k. \tag{1.10}$$

By (1.4) we get

$$x^n = \sum_{k=0}^{n}\left(\binom{2n}{k}-\binom{2n}{k-1}\right)P_{n-k}(x). \tag{1.11}$$

Let L_0 denote the linear functional defined by $L_0\left(P_n\right)=[n=0]$.

Then we get for the moments

$$L_0\left(x^n\right)=C_n. \tag{1.12}$$

Analogously we get

$$x^n = \sum_{k=0}^{n}\left(\binom{2n+1}{k}-\binom{2n+1}{k-1}\right)Q_{n-k}(x). \tag{1.13}$$

Let L_1 denote the linear functional defined by $L_1\left(Q_n\right)=[n=0]$.

Then we get for the moments

$$L_1\left(x^n\right)=\binom{2n+1}{n}-\binom{2n+1}{n-1}=C_{n+1}. \tag{1.14}$$

1.2. Narayana polynomials as moments

The Catalan numbers are special cases for $t=1$ of the *Narayana polynomials*

$$C_n(t)=\sum_{k\geq 0}\binom{n-1}{k}\binom{n}{k}\frac{1}{k+1}t^k \tag{1.15}$$

for $n>0$ and $C_0(t)=1$. (cf. [14]).

The first terms of $\left(C_n(t)\right)_{n\geq 0}$ are

1, 1, 1 + t, 1 + 3 t + t², 1 + 6 t + 6 t² + t³, 1 + 10 t + 20 t² + 10 t³ + t⁴, ...

For $t=2$ they reduce to the *little Schroeder numbers* $\left(C_n(2)\right)_{n\geq 0}=\left(1,1,3,11,45,197,\cdots\right)$, OEIS [12], A001003.

Let $\tau_{2n}\left(t\right)=1$ and $\tau_{2n+1}\left(t\right)=t.$ Define polynomials $F_n(x,t)$ by the recursion

$$F_n(x,t)=xF_{n-1}(x,t)-\tau_{n-2}(t)F_{n-2}(x,t) \tag{1.16}$$

with initial values $F_0(x,t)=1$ and $F_1(x,t)=x.$

The first terms of the sequence $\left(F_n(x,t)\right)_{n\geq 0}$ are

$1,\ x,\ -1+x^2,\ -x-t\,x+x^3,\ 1-2\,x^2-t\,x^2+x^4,\ x+t\,x+t^2\,x-2\,x^3-2\,t\,x^3+x^5,\ \ldots$

Their generating function is

$$\sum_{n\geq 0}F_n(x,t)z^n=\frac{1+xz+tz^2}{1-\left(x^2-1-t\right)z^2+tz^4}. \tag{1.17}$$

Then we get

Theorem 1 ([1],[3], [11], [13], [16],[17])

Let L *be the linear functional defined by* $L\left(F_n(x,t)\right)=[n=0].$ *Then the moments satisfy*

$$\begin{aligned} L\left(x^{2n}\right)&=C_n(t),\\ L\left(x^{2n+1}\right)&=0. \end{aligned} \tag{1.18}$$

Remark

By starting with $C_n(t)$ it is easy to guess (1.16) in the same manner as I have done in [4].

In order to guess explicit formulae for $F_n(x,t)$ it is convenient to consider the polynomials with odd and even degrees separately. To this end we consider the polynomials

$$P_n(x,t)=F_{2n}\left(\sqrt{x},t\right) \text{ and } Q_n(x,t)=\frac{F_{2n+1}\left(\sqrt{x},t\right)}{\sqrt{x}}.$$

Then (1.32) and (1.21) can be summarized to give the formula

$$F_n(x,t)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\sum_{j=0}^{k}\binom{\left\lfloor\frac{n}{2}\right\rfloor-j}{k-j}\binom{\left\lfloor\frac{n-1}{2}\right\rfloor-k+j}{j}t^j x^{n-2k}. \tag{1.19}$$

1.2.1. The polynomials $Q_n(x,t)$.

The polynomials $Q_n(x,t)$ satisfy the recurrence

$$Q_n(x,t) = (x-1-t)Q_{n-1}(x,t) - tQ_{n-2}(x,t) \tag{1.20}$$

with initial values $Q_0(x,t) = 1$ and $Q_1(x,t) = x-1-t.$

Thus $Q_n(x,t) = f_{n+1}(x-1-t,-t).$ Binet's formula gives $Q_n(x,t) = \dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$

with $\alpha = \alpha(x,t) = \dfrac{x-1-t+\sqrt{(x-1-t)^2-4t}}{2}$ and $\beta = \beta(x,t) = \dfrac{x-1-t-\sqrt{(x-1-t)^2-4t}}{2}.$

A more general class of polynomials has been considered in [1].

By induction we get $Q_n(x,t) = \sum_{k=0}^{n} (-1)^{n-k} q_{n,k}(t) x^k$ with

$$q_{n,k}(t) = \sum_{j=0}^{n-k} \binom{n-j}{k}\binom{k+j}{j} t^j. \tag{1.21}$$

From (1.10) we see that $q_{n,k}(1) = \binom{n+k+1}{2k+1}.$

The first terms of $q_{n,k}(t)$ are

1					
$1+t$	1				
$1+t+t^2$	$2+2t$	1			
$1+t+t^2+t^3$	$3+4t+3t^2$	$3+3t$	1		
$1+t+t^2+t^3+t^4$	$4+6t+6t^2+4t^3$	$6+9t+6t^2$	$4+4t$	1	
$1+t+t^2+t^3+t^4+t^5$	$5+8t+9t^2+8t^3+5t^4$	$10+18t+18t^2+10t^3$	$10+16t+10t^2$	$5+5t$	1

Note that the polynomials $q_{n,k}(t)$ are palindromic.

Let $B_{n,k}(t)$ be the uniquely determined polynomials such that

$$x^n = \sum_{k=0}^{n} B_{n,k}(t) Q_k(x,t). \tag{1.22}$$

The recursion of $Q_n(x,t)$ implies that

$$B_{n,k}(t) = B_{n-1,k-1}(t) + (1+t)B_{n-1,k}(t) + tB_{n-1,k+1}(t) \tag{1.23}$$

with $B_{0,k}(t) = [k=0]$ and $B_{n,-1}(t) = 0.$

The first terms of the sequence $\left(B_{n,0}(t), B_{n,1}(t), \cdots, B_{n,n}(t)\right)_{n\geq 0}$ are

1				
$1+t$	1			
$1+3t+t^2$	$2+2t$	1		
$1+6t+6t^2+t^3$	$3+8t+3t^2$	$3+3t$	1	
$1+10t+20t^2+10t^3+t^4$	$4+20t+20t^2+4t^3$	$6+15t+6t^2$	$4+4t$	1

By induction we can verify that

$$B_{n,k}(t)=\sum_{j=0}^{n}\binom{n+1}{k+1+j}\binom{n+1}{j}\frac{k+1}{n+1}t^j=\sum_j\left(\binom{n}{j}\binom{n+1}{k+j+1}-\binom{n+1}{j}\binom{n}{k+j+1}\right)t^j. \tag{1.24}$$

For $k=0$ we get

$$B_{n,0}(t)=C_{n+1}(t). \tag{1.25}$$

From (1.13) we see that $B_{n,k}(1)=\binom{2n+1}{n-k}-\binom{2n+1}{n-k-1}=\frac{2k+2}{n+k+2}\binom{2n+1}{n-k}.$

This gives the Catalan triangle OEIS[12], A039598

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 \\ 42 & 48 & 27 & 8 & 1 \end{pmatrix}$$

For the little Schroeder numbers the corresponding triangle is OEIS [12], A110440,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 11 & 6 & 1 & 0 & 0 \\ 45 & 31 & 9 & 1 & 0 \\ 197 & 156 & 60 & 12 & 1 \end{pmatrix}$$

There is a nice interpretation in terms of weighted NSEW-paths. A *NSEW-path* is a path consisting of North, South, East and West steps of length 1. (Cf. [9] and [10]). We consider only NSEW- paths which start at $(0,0)$ and end on height $k\ge 0$ and never cross the $x-$axis.

$B_{n,k}(t)$ is the weight of all those NSEW-paths with n steps which end on height k, if the weight is defined by $w(N)=w(E)=1$ and $w(S)=w(W)=t.$ This follows immediately from (1.23) because there are 4 possibilities to reach a point of height $k.$ For $k=0$ this reduces to

$B_{n,0}(t)=(1+t)B_{n-1,0}(t)+tB_{n-1,1}(t).$

For example for $n=2$ and $k=0$ we get $w\left(EE\right)=1, w\left(NS+EW+WE\right)=3t, w(WW)=t^2.$

For $k=1$ we get $w(NE)+w(EN)=2$ and $w(NW)+w(WN)=2t.$

Let $y\geq 0$ and let $w_n(x,y)$ be the number of NSEW-paths from $(0,0)$ to (x,y) which do not cross the $x-$ axis. It has been shown in [9] that

$$w_n(-n+k+2j,k)=\binom{n}{j}\binom{n}{k+j}-\binom{n}{j-1}\binom{n}{k+j+1}=\binom{n+1}{k+1+j}\binom{n+1}{j}\frac{k+1}{n+1}.$$

A purely combinatorial proof has been given in [10] and can be considered as another proof of (1.24).

All these polynomials are palindromic and *gamma-nonnegative*, i.e. they have a representation of the form $\sum \gamma_{n,j}t^j(1+t)^{n-2j}$ where $\gamma_{n,j}$ are non-negative integers. (Cf. [14] for this notion).

More precisely we have

$$B_{n,k}(t)=\sum_{i=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor}\binom{k+2i}{i}\frac{k+1}{i+k+1}\binom{n}{2i+k}t^i(1+t)^{n-k-2i}, \tag{1.26}$$

which for $k=0$ reduces to

$$C_{n+1}(t)=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}C_i\binom{n}{2i}t^i(1+t)^{n-2i}. \tag{1.27}$$

In order to prove this we modify a method developed in [15]. Let $f(N)=1,\ f(S)=-1,\ f(E)=f(W)=0.$

To each non-negative NSEW- path $u_1\cdots u_n$ with $u_i\in\{N,S,E,W\}$ whose endpoint is on height k we associate the $n-$ letter word $f(u_1)f(u_2)\cdots f(u_n)$ in the alphabet $\{-1,1,0\}$ that adds up to $k,$ and all whose partial sums are non-negative.

For each such sequence there are i terms $f(u_j)=-1$ and $i+k$ terms $f(u_j)=1$ for some $i.$

On the other hand we can choose $2i+k$ places where $u_j=N$ or $u_j=S,$ i.e. $f(u_j)=\pm 1$ in $\binom{n}{2i+k}$ ways. By (1.3) we can order the signs in such a way that the corresponding path is non-negative in $\binom{k+2i}{i}-\binom{k+2i}{i-1}=\binom{k+2i}{i}\frac{k+1}{i+k+1}$ ways. In the remaining $n-2i-k$ places we can arbitrarily put W or $E.$ The weight of all such paths is therefore $\binom{n}{2i+k}\binom{k+2i}{i}\frac{k+1}{i+k+1}t^i(1+t)^{n-k-2i}.$

If we define the linear functional L_1 by $L_1\left(Q_n(x,t)\right)=[n=0]$ we get from (1.27) that

$$L_1\left(x^n\right)=C_{n+1}(t). \tag{1.28}$$

Let us compute the generating functions $f_k\left(z,t\right)=\sum_{n\geq 0} B_{n,k}(t)z^n.$ As above we see that they satisfy

$f_k(z,t)=z\left(f_{k-1}(z,t)+(1+t)f_k(z,t)+tf_{k+1}(z,t)\right)$ with $f_0(z,t)=1+(1+t)zf_0(z,t)+tzf_1(z,t).$

The unique solution is

$f_k(z,t)=z^k f(z,t)^{k+1}$ where $f(z,t)$ satisfies $1-(1-(1+t)z)f(z,t)+tz^2 f(z,t)^2=0.$

This implies

$$f(z,t)=\sum_{n\geq 0} C_{n+1}(t)z^n=\frac{1-(1+t)z-\sqrt{(1-(1+t)z)^2-4tz^2}}{2tz^2}. \tag{1.29}$$

Since $1-(1-(1+t)z)f(z,t)+tz^2 f(z,t)^2=0$ we get

$$\sum_k B_{n,k}(t)\frac{t^{k+1}-1}{t-1}z^n=\frac{1}{t-1}\left(\sum_k z^k f(z,t)^{k+1}t^{k+1}-\sum_k z^k f(z,t)^{k+1}\right)=\frac{f(z,t)}{t-1}\left(\frac{t}{1-tzf(z,t)}-\frac{1}{1-zf(z,t)}\right)$$
$$=\frac{f(z,t)}{t-1}\frac{(t-1)}{\left(1-zf(z,t)\right)\left(1-tzf(z,t)\right)}=\frac{f(z,t)}{1-(1+t)zf(z,t)+tz^2 f(z,t)^2}=\frac{f(z,t)}{f(z,t)-2(1+t)zf(z,t)}=\frac{1}{1-2(1+t)z}.$$

This implies

$$\sum_{k=0}^{n} B_{n,k}(t)\left(1+t+\cdots+t^k\right)=(2t+2)^n. \tag{1.30}$$

A combinatorial proof of (1.30) has been given in [2], proof of identity 1, in a somewhat different context which we will translate into our terminology.

The right-hand side of (1.30) is the weight of all NSWE-paths of length $n.$

Let $\mathbf{B}_{n,k}$ be the set of all non-negative NSWE-paths of length n which end on height $k.$

For $p\in\mathbf{B}_{n,k}$ we define $k+1$ different paths $\varphi_i(p),\ 0\leq i\leq k,$ of length n such that $w\left(\varphi_i(p)\right)=t^i w(p).$

To this end define the last ascent to height i of p to be the last step N from height $i-1$ to $i.$ Let $\varphi_i(p)$ denote the path obtained by changing each of the last ascents to heights $1,2,\cdots,i$ to downsteps $S.$ For $i=0$ let $\varphi_0(p)=p.$ Then all $\varphi_i(p)$ are different and for $i>0$ not non-negative. The height of $\varphi_i(p)$ is $k-2i$ and the weight is $w\left(\varphi_i(p)\right)=t^i w(p).$

Let on the other hand q be a path with height j, which crosses the $x-$axis. Then it has a set of premier descents below the $x-$axis, i.e. the first (from left to right) down steps S from height m to $m-1$ for $m=0,-1,\cdots$. Suppose q has i premier descents below the $x-$ axis. Then changing each of these S to upsteps N gives a new path p which is non-negative and ends on height $j+2i$. It is clear that $\varphi_i(p)=q$ and $w(\varphi_i(p))=t^i w(p)$.

For example

$\mathbf{B}(2,0)=\{EE,EW,WE,NS,WW\}$,

$\mathbf{B}(2,1)=\{NE,NW,EN,WN\}$, $\varphi_1(\mathbf{B}(2,1))=\{SE,SW,ES,WS\}$,

$\mathbf{B}(2,2)=\{NN\}$, $\varphi_1(\mathbf{B}(2,2))=\{SN\}$, $\varphi_2(\mathbf{B}(2,2))=\{SS\}$.

1.2.2. The polynomials $P_n(x,t)$.

The polynomials $P_n(x,t)$ satisfy the recurrence

$P_n(x,t)=(x-\sigma_{n-1}(t))P_{n-1}(x,t)-tP_{n-2}(x,t)$

with initial values $P_0(x,t)=1$ and $P_1(x,t)=x-1$,

where $\sigma_0(t)=1$ and $\sigma_n(t)=1+t$ for $n>0$.

We have for $n>0$

$$P_n(x,t)=Q_n(x,t)+tQ_{n-1}(x,t). \qquad (1.31)$$

For (1.31) holds for $n=1$ and $n=2$ and for $n\geq 3$ both sides satisfy the same recursion.

Let us set $P_n(x,t)=\sum_{k=0}^{n}(-1)^{n-k}p_{n,k}(t)x^k$.

Then we get

$$p_{n,k}(t)=\sum_{j=0}^{n-k}\binom{n-j}{k}\binom{k-1+j}{j}t^j. \qquad (1.32)$$

The first terms of the sequence $\left(p_{n,0}(t)=q_{n,0}(t)-q_{n-1,0}(t), p_{n,1}(t)=q_{n,1}(t)-q_{n-1,1}(t),\cdots, p_{n,n}(t)=q_{n,n}(t)-q_{n-1,n}(t)\right)_{n\geq 0}$ are

1					
1	1				
1	2 + t	1			
1	3 + 2 t + t²	3 + 2 t	1		
1	4 + 3 t + 2 t² + t³	6 + 6 t + 3 t²	4 + 3 t	1	
1	5 + 4 t + 3 t² + 2 t³ + t⁴	10 + 12 t + 9 t² + 4 t³	10 + 12 t + 6 t²	5 + 4 t	1

Let $A_{n,k}(t)$ be the uniquely determined polynomials satisfying

$$x^n = \sum_{k=0}^{n} A_{n,k}(t) P_k(x,t). \tag{1.33}$$

Then

$$A_{n,k}(t) = A_{n-1,k-1}(t) + \sigma_k(t) A_{n-1,k}(t) + t A_{n-1,k+1}(t) \tag{1.34}$$

with $A_{0,k}(t) = [k=0]$ and $A_{n,-1}(t) = 0.$

This means that $A_{n,k}(t)$ can be interpreted as the weight of all NSEW - paths of length n which end on height k and which have no W-step on height 0.

For example let $n=3.$ For $k=0$ we have $w(EEE)=1,$ $w(NSE+ENS+NES)=3t$ and $w(NWS)=t^2.$ For $k=2$ we have $w(NNE+ENN+NEN)=3$ and $w(NNW+NWN)=2t.$

The first terms of the sequence $\left(A_{n,0}(t), A_{n,1}(t), \cdots, A_{n,n}(t)\right)_{n\geq 0}$ are

1				
1	1			
$1+t$	$2+t$	1		
$1+3t+t^2$	$3+5t+t^2$	$3+2t$	1	
$1+6t+6t^2+t^3$	$4+14t+9t^2+t^3$	$6+11t+3t^2$	$4+3t$	1

From (1.31) we get $A_{n,k} + tA_{n,k+1} = B_{n,k}.$

In general we get for $n>0$

$$\begin{aligned} A_{n,k}(t) &= \sum_{j=0}^{n-k} \binom{n-1}{j}\binom{n}{k+j} \frac{kn+n-j}{(n-j)(k+1+j)} t^j \\ &= \sum_{j=0}^{n-k} \left(\binom{n-1}{j}\binom{n+1}{k+j+1} - \binom{n}{j}\binom{n}{k+j+1} \right) t^j. \end{aligned} \tag{1.35}$$

For $k=0$ this reduces to

$$A_{n,0}(t) = C_n(t). \tag{1.36}$$

For $t=1$ we get the triangle OEIS [12], A039599,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 5 & 9 & 5 & 1 & 0 \\ 14 & 28 & 20 & 7 & 1 \end{pmatrix}$$

For $t=2$ we get OEIS [12], 172094,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 11 & 17 & 7 & 1 & 0 \\ 45 & 76 & 40 & 10 & 1 \end{pmatrix}$$

From (1.33) we get

$$\sum_{k=0}^{n} A_{n,k}(t) F_{2k}(x,t) = x^{2n}. \tag{1.37}$$

Applying the linear functional L gives

$$L\left(x^{2n}\right) = A_{n,0}(t) = C_n(t). \tag{1.38}$$

By (1.22) we get $x^{2n+1} = \sum_{k=0}^{n} B_{n,k}(t) F_{2k+1}(x,t)$ which implies $L\left(x^{2n+1}\right)=0$ and thus proves Theorem 1.

If we define the linear functional L_0 by $L_0\left(P_n(x,t)\right)=[n=0]$ then we get

$$L_0\left(x^n\right) = C_n(t). \tag{1.39}$$

Let us also compute the generating functions $f_k(z,t) = \sum_{n\geq 0} A_{n,k}(t) z^n$. They satisfy

$$\begin{aligned} f_k(z,t) &= z\left(f_{k-1}(z,t) + (1+t) f_k(z,t) + t f_{k+1}(z,t)\right), \\ f_0(z,t) &= 1 + z\left(f_0(z,t) + t f_1(z,t)\right). \end{aligned} \tag{1.40}$$

Let $f(z,t)$ satisfy $f(z,t) = 1 + (1+t) z f(z,t) + t z^2 f(z,t)^2$. Then $f_k(z,t) = z^k f_0(z,t) f(z,t)^k$ satisfies the first equation in (1.40). From the second equation and (1.29) we get the well-known formula (cf. e.g. [14])

$$f_0(z) = C(t,z) = \sum_{n\geq 0} C_n(t) z^n = \frac{1 + z(t-1) - \sqrt{1 - 2z(t+1) + z^2(t-1)^2}}{2tz}. \tag{1.41}$$

Remarks

In terms of $C(t,z)$ we get

$$\sum_{n\geq 0} A_{n,k}(t)z^n = C(t,z)\left(C(t,z)-1\right)^k, \qquad (1.42)$$
$$\sum_{n\geq 0} B_{n,k}(t)z^n = \frac{\left(C(t,z)-1\right)^{k+1}}{z}.$$

For $t=1$ it is well known that $\left(F_n(1,1)\right)=\left(1,1,0,-1,-1,0,1,1,0,-1,-1,0,\cdots\right)$ is periodic with period 6 because $\alpha(1,1)=\frac{-1+\sqrt{-3}}{2}$ and $\beta(1,1)=\frac{-1-\sqrt{-3}}{2}$ satisfy $\alpha(1,1)^3=\beta(1,1)^3=1.$

For $t=2$ and $t=3$ an analogous situation obtains: $\alpha(1,2)=-1+i$ and $\beta(1,2)=-1-i$ satisfy $\alpha(1,2)^8=\beta(1,2)^8=2^4$ and $\alpha(1,3)=\frac{-3+\sqrt{-3}}{2}$ and $\beta(1,3)=\frac{-3-\sqrt{-3}}{2}$ satisfy $\alpha(1,3)^{12}=\beta(1,3)^{12}=3^6.$ This implies that the sequence $\left(\frac{F_n(1,2)}{4^{\left\lfloor \frac{n}{8}\right\rfloor}}\right)_{n\geq 0}$ is periodic with period 16 and the sequence $\left(\frac{F_n(1,3)}{27^{\left\lfloor \frac{n}{12}\right\rfloor}}\right)_{n\geq 0}$ is periodic with period $24.$

We get $\left(\frac{F_n(1,2)}{4^{\left\lfloor \frac{n}{8}\right\rfloor}}\right)_{n\geq 0}=\left(1,1,0,-2,-2,2,4,0,-1,-1,0,2,2,-2,-4,0,\cdots\right)$

and

$$\left(\frac{F_n(1,3)}{27^{\left\lfloor \frac{n}{12}\right\rfloor}}\right)_{n\geq 0}=\left(1,1,0,-3,-3,6,9,-9,-18,9,27,0,-1,-1,0,3,3,-6,-9,9,18,-9,-27,0,\cdots\right).$$

2.1. Background material on Lucas polynomials and central binomial coefficients

The *Lucas polynomials* $l_n(x,s)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{n}{n-k}\binom{n-k}{k}s^k x^{n-2k}$ satisfy the recurrence relation $l_n(x,s)=xl_{n-1}(x)+sl_{n-2}(x)$ with initial values $l_0(x,s)=2$ and $l_1(x,s)=x.$

Let us consider the *special Lucas polynomials* $L_n(x)$ defined by $L_n(x) = l_n(x,-1)$ for $n > 0$ and $L_0(x) = 1.$

Then $L_n(x)$ satisfies the recursion

$$L_n(x) = xL_{n-1}(x) - \tau_{n-2}L_{n-2}(x) \tag{2.1}$$

with $\tau_0 = 2$ and $\tau_n = 1$ for $n > 0.$

The first terms of $\left(L_n(x)\right)_{n\geq 0}$ are

$1,\ x,\ -2+x^2,\ -3x+x^3,\ 2-4x^2+x^4,\ 5x-5x^3+x^5,\ \ldots$

Note that $L_n(x) = 2T_n\left(\frac{x}{2}\right)$ for $n > 0$ if $T_n(x)$ is a *Chebyshev polynomial of the first kind*.

Let $\left(a(n,k)\right)$ be the corresponding Catalan-Stieltjes matrix.

Then we get

$a(n,k) = a(n-1,k-1) + a(n-1,k+1)$ for $k > 0$ and $a(n,0) = 2a(n-1,1).$

Thus $a(n,k)$ is the weight of all non-negative NSEW-paths of length n whose endpoints are on height k where all weights $w(E) = w(N) = w(W) = w(S) = 1$ except that $w(S) = 2$ if the endpoint of S is on the $x-$ axis.

The first terms are OEIS [12], A 108044,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 10 & 0 & 5 & 0 & 1 & 0 \\ 20 & 0 & 15 & 0 & 6 & 0 & 1 \end{pmatrix}$$

This gives $a(2n,2k) = \binom{2n}{n-k}$ and $a(2n+1,2k+1) = \binom{2n+1}{n-k}$ and all other terms vanish.

With other words we get the identities

$$\sum_{k=0}^{n}\binom{2n}{n-k}L_{2k}(x)=x^{2n},$$
$$\sum_{k=0}^{n}\binom{2n+1}{n-k}L_{2k+1}(x)=x^{2n+1}. \tag{2.2}$$

Let M be the linear functional defined by $M\left(L_n\right)=[n=0].$ Then

$$M\left(x^{2n}\right)=\binom{2n}{n} \tag{2.3}$$

is a central binomial coefficient and $M\left(x^{2n+1}\right)=0.$

Let now $f_k(z)=\sum_{n\geq 0}a(n,k)z^n.$ Then we have $f_k(z)=f_{k-1}(z)+f_{k+1}(z)$ for $k>0$ and $f_0(z)=1+2zf_1(z).$ Then we get $f_k(z)=z^k f_0(z)f(z)^k$ with $f(z)=\sum_{n\geq 0}C_nz^n=\dfrac{1-\sqrt{1-4z}}{2z}$ by (1.8). This gives $f_0(z)=1+2zf_0(z)f(z)$ or

$$f_0(z)=M(z)=\sum_{n\geq 0}\binom{2n}{n}z^n=\frac{1}{\sqrt{1-4z}}. \tag{2.4}$$

Let us also consider the polynomials

$$R_n(x)=L_{2n}\left(\sqrt{x}\right)=\sum_{k=0}^{n}(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{2k}x^k \tag{2.5}$$

and

$$S_n(x)=\frac{L_{2n+1}\left(\sqrt{x}\right)}{\sqrt{x}}=\sum_{k=0}^{n}(-1)^{n-k}\frac{2n+1}{n+k+1}\binom{n+k+1}{2k+1}x^k. \tag{2.6}$$

Let M_0 be the linear functional defined by $M_0\left(R_n\right)=[n=0].$ Then (2.2) gives

$$M_0\left(x^n\right)=\binom{2n}{n}=M_n. \tag{2.7}$$

If M_1 is the linear functional defined by $M_1\left(S_n\right)=[n=0]$ then we get

$$M_1\left(x^n\right)=\binom{2n+1}{n}=\frac{1}{2}\binom{2n+2}{n+1}=\frac{M_{n+1}}{2}. \tag{2.8}$$

2.2. The Narayana polynomials of type B as moments

The *central binomial coefficients* are the special case for $t=1$ of the *Narayana polynomials* $M_n(t)=\sum_{k=0}^{n}\binom{n}{k}^2 t^k$ *of type B.*

For $t=2$ we get the *central Delannoy numbers* $(M_n(2))_{n\geq 0}=(1,3,13,63,321,1683,\cdots)$. Here $M_n(2)=d_n=\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{2k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}$.

Let

$$\begin{aligned}
&\tau_0(t)=1+t,\\
&\tau_{2n}(t)=\frac{1+t^{n+1}}{1+t^n} \ \textit{for} \ n>0,\\
&\tau_{2n+1}(t)=\frac{t\left(1+t^n\right)}{1+t^{n+1}}.
\end{aligned} \tag{2.9}$$

Thus the sequence $\tau_n(t)$ satisfies $\tau_{2n}(t)=1+t-\tau_{2n-1}(t)$ and $\tau_{2n+1}(t)=\frac{t}{\tau_{2n}(t)}$ with initial values $\tau_0(t)=1+t$ and $\tau_1(t)=\frac{2t}{1+t}$.

Define polynomials $L_n(x,t)$ by the recurrence

$$L_n(x,t)=xL_{n-1}(x,t)-\tau_{n-2}(t)L_{n-2}(x,t) \tag{2.10}$$

with initial values $L_0(x,t)=1$ and $L_1(x,t)=x$.

The first terms of the sequence $(L_n(x,t))_{n\geq 0}$ are

$$1,\ x,\ -1-t+x^2,\ -\frac{x\left(1+4t+t^2-x^2-tx^2\right)}{1+t},\ 1+t^2-2x^2-2tx^2+x^4,\ \ldots$$

It is clear that $L_n(x,1)=L_n(x)$.

Let now

$$R_n(x,t)=L_{2n}\left(\sqrt{x},t\right). \tag{2.11}$$

These polynomials satisfy

$$R_n(x,t)=\left(x-1-t\right)R_{n-1}(x,t)-T_{n-2}(t)R_{n-2}(x,t) \tag{2.12}$$

with

$$T_n(t) = t \ for \ \ n > 0, \\ T_0(t) = 2t. \tag{2.13}$$

Then we get

$$R_n(x,t) = Q_n(x,t) - tQ_{n-2}(x,t) \tag{2.14}$$

for $n \geq 2$ and $R_0(x,t) = 1$ and $R_1(x,t) = x - 1 - t.$

For $n > 0$ we get

$$R_n(x,t) = (-1)^n \left(1 + t^n\right) + \sum_{\ell=1}^{n} (-1)^{n-\ell} \binom{n}{\ell} x^\ell \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} \frac{\binom{\ell + j - 1}{j}}{\binom{n-1}{j}} t^j. \tag{2.15}$$

We also have $R_n(x,t) = \alpha^n + \beta^n$ for $n > 0.$ This means that $R_n(x,t)$ are the Lucas polynomials corresponding to $Q_n(x,t).$

If we set $R_0(x,t) = 2$ then the sequence $\left(R_n(1,1)\right)_{n\geq 0} = \left(2, -1, -1, \cdots\right)$ is periodic with period $3,$ the sequence $\left(\frac{R_n(1,2)}{\left(2^4\right)^{\left\lfloor \frac{n}{8} \right\rfloor}}\right)_{n\geq 0} = \left(2, -2, 0, 4, -8, 8, 0, -16, \cdots\right)$ is periodic with period $8,$ and the sequence $\left(\frac{R_n(1,3)}{\left(3^6\right)^{\left\lfloor \frac{n}{12} \right\rfloor}}\right)_{n\geq 0} = \left(2, -3, 3, 0, -9, 27, -54, 81, -81, 0, 243, -729, \cdots\right)$ is periodic with period 12.

Let $D_{n,k}(t)$ be the uniquely determined polynomials such that

$$x^n = \sum_{k=0}^{n} D_{n,k}(t) R_k(x,t). \tag{2.16}$$

They satisfy

$$D_{n,k}(t) = D_{n-1,k-1}(t) + (1+t) D_{n-1,k}(t) + T_k(t) D_{n-1,k+1}(t) \tag{2.17}$$

with $D_{0,k}(t) = [k = 0]$ and $D_{n,-1}(t) = 0.$

This implies that

$$D_{n,k}(t) = \left[x^{n-k}\right]\left(1 + (1+t)x + tx^2\right)^n. \tag{2.18}$$

Let $a(n,k)=\left[x^{n-k}\right]\left(1+(1+t)x+tx^2\right)^n$. Since $\left(1+\frac{1+t}{\sqrt{t}}x+x^2\right)^n$ is palindromic we have

$\left[x^{2n-j}\right]\left(1+(1+t)x+tx^2\right)^n=t^{n-j}\left[x^j\right]\left(1+(1+t)x+tx^2\right)^n$ and thus

$\left[x^n\right]\left(1+(1+t)x+tx^2\right)^{n-1}=t\left[x^{n-2}\right]\left(1+(1+t)x+tx^2\right)^{n-1}$.

For $k\geq 1$ we have

$$a(n,k)=\left[x^{n-k}\right]\left(1+(1+t)x+tx^2\right)^n=\left[x^{n-k}\right]\left(1+(1+t)x+tx^2\right)\left(1+(1+t)x+tx^2\right)^{n-1}$$
$$=\left[x^{n-1-(k-1)}\right]\left(1+(1+t)x+tx^2\right)^{n-1}+(1+t)\left[x^{n-1-k}\right]\left(1+(1+t)x+tx^2\right)^{n-1}+t\left[x^{n-1-(k+1)}\right]\left(1+(1+t)x+tx^2\right)^{n-1}$$
$$=a(n-1,k-1)+(1+t)a(n-1,k)+ta(n-1,k+1).$$

For $k=0$ we get

$$a(n,0)=\left[x^n\right]\left(1+(1+t)x+tx^2\right)^n$$
$$=\left[x^n\right]\left(1+(1+t)x+tx^2\right)^{n-1}+(1+t)\left[x^{n-1-0}\right]\left(1+(1+t)x+tx^2\right)^{n-1}+2t\left[x^{n-1-(1)}\right]\left(1+(1+t)x+tx^2\right)^{n-1}$$
$$=ta(n-1,k-1)+(1+t)a(n-1,0)+ta(n-1,1)=(1+t)a(n-1,0)+2ta(n-1,1).$$

Another formula for $n>0$ is

$$D_{n,k}(t)=\sum_{j=0}^{n}\binom{n}{j}\binom{n}{k+j}t^j. \tag{2.19}$$

This follows from $\left(1+x+tx(1+x)\right)^n=\sum_{j=0}^{n}\binom{n}{j}t^jx^j(1+x)^{n-j}$ by considering the coefficient of x^{n-k}.

By (2.17) the polynomials $D_{n,k}(t)$ can also been interpreted as the weight of all NSEW-paths of length n and whose endpoint is on height k with weights $w(E)=w(N)=1$, $w(W)=t$, $w(S)=2t$ if the endpoint of S is on the $x-$axis and $w(S)=t$ else.

Let for example $n=2$ and $k=0$. Then we have $w(EE)=1$, $w(WW)=t^2$, $w\left(NS\right)=2t$, $w(EW)=w(WE)=t$. For $n=2$ and $k=1$ we get $w(NE)=w(EN)=1$ and $w(WN)=w(NW)=t$.

The first terms of the sequence $\left(D_{n,0}(t),D_{n,1}(t),\cdots,D_{n,n}(t)\right)_{n\geq 0}$ are

1				
$1+t$	1			
$1+4t+t^2$	$2+2t$	1		
$1+9t+9t^2+t^3$	$3+9t+3t^2$	$3+3t$	1	
$1+16t+36t^2+16t^3+t^4$	$4+24t+24t^2+4t^3$	$6+16t+6t^2$	$4+4t$	1

For $t=1$ $D_{n,k}(t)$ reduces to $D_{n,k}(1)=\binom{2n}{n-k}$ and we get the triangle OEIS [12], A094527,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 \\ 20 & 15 & 6 & 1 & 0 \\ 70 & 56 & 28 & 8 & 1 \end{pmatrix}$$

For $t=2$ we get OEIS [12], A118384,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 13 & 6 & 1 & 0 & 0 \\ 63 & 33 & 9 & 1 & 0 \\ 321 & 180 & 62 & 12 & 1 \end{pmatrix}$$

The polynomials $D_{n,k}(t)$ are gamma -nonnegative. More precisely we have

$$D_{n,k}(t)=\sum_{j=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor}\binom{2j+k}{j}\binom{n}{2j+k}t^{j}(1+t)^{n-k-2j}. \tag{2.20}$$

The proof is analogous to the corresponding proof of (1.26).

For each non-negative NSEW- path $u_1\cdots u_n$ with $u_i\in\{N,S,E,W\}$ whose endpoint is on height k there are i terms $f(u_j)$ negative and $i+k$ terms $f(u_j)=1$ for some i. We can choose $2i+k$ places where $u_j=N$ or $u_j=S$ in $\binom{n}{2i+k}$ ways. By (2.2) for $t=1$ the weight of all non-negative paths is $\binom{k+2i}{i}$. The remaining $n-2i-k$ places can arbitrarily be filled with W or E. Therefore for arbitrary t the weight of all such paths is $\binom{n}{2i+k}\binom{k+2i}{i}t^{i}\left(1+t\right)^{n-k-2i}$.

Let M_0 be the linear functional defined by $M_0\left(R_n(x,t)\right)=[n=0]$. Then (2.16) and (2.19) imply

$$M_0\left(x^n\right)=M_n(t). \tag{2.21}$$

This result can be found in [1] and [13] and is implicitly contained in [17].

Formula (2.16) implies $x^{2n}=\sum_{k=0}^{n}D_{n,k}(t)L_{2k}(x,t)$ and therefore

$$M\left(x^{2n}\right)=D_{n,0}(t)=M_n(t). \tag{2.22}$$

In the same way there are rational functions $E_{n,k}(t)$ such that $x^{2n+1}=\sum_{k=0}^{n}E_{n,k}(t)L_{2k+1}(x,t)$ which implies $M\left(x^{2n+1}\right)=0.$ This gives

Theorem 2 ([1], [13], [17])

Let M *be the linear functional defined by* $M\left(L_n(x,t)\right)=[n=0].$ *Then the moments satisfy*

$$\begin{aligned} M\left(x^{2n}\right)&=M_n(t),\\ M\left(x^{2n+1}\right)&=0. \end{aligned} \tag{2.23}$$

Let us now compute the generating functions $f_k(z,t)=\sum_{n\geq 0}D_{n,k}(t)z^n.$

We get $f_k(z,t)=z\left(f_{k-1}(z,t)+(1+t)f_k(z,t)+tf_{k+1}(z,t)\right)$ for $k>0$ and $f_0(z,t)=1+(1+t)zf_0(z,t)+2tzf_1(z,t).$

This gives $f_k(z,t)=z^k f_0(z,t)f(z,t)^k$ with

$f(z,t)=\frac{1-(1+t)z-\sqrt{(1-(1+t)z)^2-4tz^2}}{2tz^2}=\frac{C(t,z)-1}{z}$ by (1.29) . Thus

$f_0(z,t)=\frac{1}{1-(1+t)z-2tz^2f(z,t)}=\frac{1}{\sqrt{(1-(1+t)z)^2-4tz^2}}.$

This gives

$$M(t,z)=\sum_{n\geq 0}M_n(t)z^n=\frac{1}{\sqrt{(1-(1+t)z)^2-4tz^2}} \tag{2.24}$$

and

$$\sum_{n\geq 0}D_{n,k}(t)z^n=M(t,z)\left(C(t,z)-1\right)^k. \tag{2.25}$$

Corollary

Let

$$c_n(m,t)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+m}{k+m}\frac{m}{n+m}t^k$$

with $c_0(m,t)=1$ *be the* $m-$*fold convolution of* $C_n(t)$ *with itself (cf. (3.2)).*

Then for $m\geq 1$

$$\frac{1}{\prod_{j=0}^{m-1}(n-j)}\sum_{k=0}^{n}\left(\frac{\partial^m}{\partial t^m}D_{n,k}(t)\right)R_k(x,t)=\sum_{j=0}^{n-m}c_{n-m-j}(m,t)x^j. \tag{2.26}$$

Proof

By (3.4) we have

$$\frac{\partial^m}{\partial t^m}\sum_{n\geq 0}\frac{D_{n+m,k}(t)}{(n+m)\cdots(n+1)}z^n=C(t,z)^m\sum_{n\geq 0}D_{n,k}(t)z^n.$$

Therefore the left-hand side of (2.26) is the coefficient of z^{n-m} of the power series

$$C(t,z)^m\sum_{n\geq 0}\sum_{k=0}^{n}D_{n,k}(t)R_k(x,t)z^n=C(t,z)^m\sum_{n\geq 0}x^nz^n=\sum_{i\geq 0}c_i(m,t)z^i\sum_{\ell\geq 0}x^\ell z^\ell$$

and the coefficient of z^{n-m} is $\sum_{j=0}^{n-m}c_{n-m-j}(m,t)x^j.$

Since $\frac{\partial^m}{\partial t^m}D_{n,k}(t)\Big|_{t=1}=\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k+j}\binom{j}{m}=\binom{n}{m}\sum_{k=0}^{n}\binom{n}{k+j}\binom{n-m}{n-j}=\binom{n}{m}\binom{2n-m}{k+n}$

(2.26) for $t=1$ implies

$$\sum_{k=0}^{n-m}\binom{2n-m}{n+k}L_{2k}(x)=\sum_{j=0}^{n-m}c_j(m,1)x^{2(n-m-j)}=\sum_{j=0}^{n-m}\frac{m}{m+2j}\binom{m+2j}{j}x^{2(n-m-j)}.$$

For $m=1$ this reduces to

$$\sum_{k=0}^{n-1}\binom{2n-1}{n+k}L_{2k}(x)=\sum_{j=0}^{n-1}\frac{1}{1+2j}\binom{1+2j}{j}x^{2(n-1-j)}=\sum_{j=0}^{n-1}C_jx^{2(n-1-j)}.$$

It seems that there are also similar extensions of (1.22) and (1.33).

Conjecture 1

$$\sum_{k=0}^{n}\left(\frac{\partial^m}{\partial t^m}A_{n,k}(t)\right)P_k(x,t)=\prod_{j=1}^{m-1}(n-j)\sum_{j=0}^{n-m-1}(j+1)x^{j+1}c_{n-m-j-1}(m,t), \tag{2.27}$$

$$\sum_{k=0}^{n}\left(\frac{\partial^m}{\partial t^m}B_{n,k}(t)\right)Q_k(x,t)=\prod_{j=1}^{m-1}(n+1-j)\sum_{j=0}^{n-m}(j+1)x^{j}c_{n-m-j}(m,t). \tag{2.28}$$

Let me only mention one special case for $m=1$.

Since $\frac{\partial B_n(k,t)}{\partial t}\Big|_{t=1}=(k+1)\binom{2n+1}{n-k-1}$ we get

$$\sum_{k=0}^{n}(k+1)\binom{2n+1}{n-k-1}F_{2k+1}(x)=\sum_{j=0}^{n-1}(j+1)C_{n-1-j}x^{2j+1}.$$

2.3. The polynomials $S_n(x,t)=\frac{L_{2n+1}\left(\sqrt{x},t\right)}{\sqrt{x}}$.

Let $\sigma_0(t)=\frac{1+4t+t^2}{1+t}$ and $\sigma_n(t)=\frac{1+t^{n+1}}{1+t^n}+t\frac{1+t^n}{1+t^{n+1}}$.

The polynomials

$$S_n(x,t)=\frac{L_{2n+1}\left(\sqrt{x},t\right)}{\sqrt{x}} \tag{2.29}$$

satisfy the recursion

$$S_n(x,t)=\left(x-\sigma(n-1,t)\right)S_{n-1}(x,t)-\frac{t\left(1+t^{n-2}\right)\left(1+t^n\right)}{\left(1+t^{n-1}\right)^2}S_{n-2}(x,t)$$

with initial values $S_0(x,t)=1$ and $S_1(x,t)=x-\frac{1+4t+t^2}{1+t}$.

Theorem 3

The polynomials $S_n(x,t)$ are explicitly given by

$$S_n(x,t)=\frac{1}{1+t^n}\sum_{k=0}^{n}(-1)^{n-k}G_{n,k}(t)x^k \tag{2.30}$$

with

$$G_{n,k}(t)=\sum_{j=0}^{n-k}\frac{\binom{j+k}{k}\binom{n-j-1}{k-1}\left(n(k+1)-j\right)}{k(k+1)}\left(t^{j}+t^{2n-k-j}\right). \tag{2.31}$$

for $k>0$ *and*

$$G_{n,0}(t)=(2n+1)t^{n}+\sum_{j=0}^{2n}t^{j}. \tag{2.32}$$

The first terms of the sequence $\left(G_{n,0}(t),G_{n,1}(t),\cdots,G_{n,n}(t)\right)_{n\geq 0}$ are

2			
$1+4t+t^2$	$1+t$		
$1+t+6t^2+t^3+t^4$	$2+3t+3t^2+2t^3$	$1+t^2$	
$1+t+t^2+8t^3+t^4+t^5+t^6$	$3+5t+6t^2+6t^3+5t^4+3t^5$	$3+4t+4t^3+3t^4$	$1+t^3$

To prove this observe that by (2.10) we get

$xS_n(x,t)=R_{n+1}(x,t)+\tau(2n,t)R_n(x,t).$

This is equivalent with

$\left[x^{k+1}\right]\left(\left(1+t^{n}\right)R_{n+1}(x,t)+\left(1+t^{n+1}\right)R_n(x,t)\right)=(-1)^{n-k}G_{n,k}(t).$

Let us first consider the coefficient of t^j with $j<n.$

Comparing coefficients gives the easily verified identity

$$-\binom{n}{k+1}\binom{n-k-1}{j}\frac{\binom{k+j}{j}}{\binom{n-1}{j}}+\binom{n+1}{k+1}\binom{n-k}{j}\frac{\binom{k+j}{j}}{\binom{n}{j}}=$$

$$\frac{\binom{n-j-1}{k-1}\binom{k+j}{j}\left((k+1)n-j\right)}{k(k+1)}.$$

Now let us consider the coefficient of t^{2n-k-j}. Here we have to show that

$$(-1)^{n-k}[t^{n-k-j}x^{k+1}]\left(R_{n+1}(x,t)+tR_n(x,t)\right)=\frac{\binom{j+k}{k}\binom{n-j-1}{k-1}\left(n(k+1)-j\right)}{k(k+1)}.$$

The left-hand side is

$$\binom{n+1}{k+1}\binom{n-k}{j}\binom{n-j}{k}\frac{1}{\binom{n}{k+j}}-\binom{n}{k+1}\binom{n-k-1}{j-1}\binom{n-j}{k}\frac{1}{\binom{n-1}{k+j-1}}$$

which can be simplified to give the right-hand side.

The coefficients of $G_{n,k}(t)$ are related to the numbers $g(n,j,k)$ in OEIS [12] A051340, A141419, A185874, A185875, A185876.

Theorem 4

The functions $E_{n,k}(t)$ *which satisfy*

$$\sum_{k=0}^{n} E_{n,k}(t)S_k(x,t)=x^n \tag{2.33}$$

are

$$E_{n,k}(t)=\frac{\sum_{j=0}^{n-k}\binom{n}{k+j}\binom{n+1}{j}\left(t^j+t^{n+1-j}\right)}{1+t^{k+1}} \tag{2.34}$$

for $n\geq k$ *and* $E_{n,k}(t)=0$ *else.*

As special case note that

$$E_{n,0}(t)=\frac{\sum_{j=0}^{n}\binom{n}{j}\binom{n+1}{j}\left(t^j+t^{n+1-j}\right)}{1+t}=\frac{\sum_{j=0}^{n+1}\binom{n+1}{j}^2 t^j}{1+t}=\frac{M_{n+1}(t)}{1+t}. \tag{2.35}$$

Proof

By (1.1) this follows from

$$E_{n,k}(t)=D_{n,k}(t)+\tau(2k+1)D_{n,k+1}(t)=\sum_{j=0}^{n-k}\binom{n}{j}\binom{n}{k+j}t^j+\frac{t\left(1+t^k\right)}{1+t^{k+1}}\sum_{j=0}^{n}\binom{n}{j}\binom{n}{k+j+1}t^j$$

$$=\frac{1}{1+t^{k+1}}\left(\sum_{j=0}^{n-k}\binom{n+1}{j}\binom{n}{k+j}t^j+\sum_{j=0}^{n-k}\binom{n}{j}\binom{n+1}{k+j+1}t^{j+k+1}\right)$$

$$=\frac{1}{1+t^{k+1}}\left(\sum_{j=0}^{n-k}\binom{n}{j}\binom{n}{k+j}t^j+\sum_{j=0}^{n-k}\binom{n}{k+j}\binom{n+1}{j}t^{n-j+1}\right).$$

Thus the linear functional M_1 defined by $M_1\left(S_n(x,t)\right)=[n=0]$ has the moments

$$M_1\left(x^n\right)=\frac{M_{n+1}(t)}{1+t}. \tag{2.36}$$

The first terms of the triangle $\left((1+t)E_{n,0}(t),\left(1+t^2\right)E_{n,1}(t),\cdots,\left(1+t^{n+1}\right)E_{n,n}(t)\right)_{n\geq 0}$ are

$1+t$			
$1+4t+t^2$	$1+t^2$		
$1+9t+9t^2+t^3$	$2+3t+3t^2+2t^3$	$1+t^3$	
$1+16t+36t^2+16t^3+t^4$	$3+12t+12t^2+12t^3+3t^4$	$3+4t+4t^3+3t^4$	$1+t^4$

The first terms of the triangle $\left(E_{n,0}(2),E_{n,1}(2),\cdots,E_{n,n}(2)\right)_{n\geq 0}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{13}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 21 & \frac{36}{5} & 1 & 0 & 0 & 0 & 0 & 0 \\ 107 & \frac{219}{5} & \frac{91}{9} & 1 & 0 & 0 & 0 & 0 \\ 561 & \frac{1272}{5} & \frac{226}{3} & \frac{222}{17} & 1 & 0 & 0 & 0 \\ \frac{8989}{3} & 1453 & \frac{4510}{9} & \frac{1970}{17} & \frac{529}{33} & 1 & 0 & 0 \\ 16213 & 8244 & 3155 & \frac{14886}{17} & \frac{1821}{11} & \frac{1236}{65} & 1 & 0 \\ \frac{265729}{3} & \frac{233303}{5} & \frac{57799}{3} & \frac{103299}{17} & \frac{46403}{33} & \frac{14581}{65} & \frac{2839}{129} & 1 \end{pmatrix}$$

Note that the first column contains the numbers $E_{n,0}(2)=\frac{M_{n+1}(2)}{3}.$ By [7] , Theorem 5.8, the Delannoy numbers $M_n(2)$ are multiples of $3,$ i.e. $E_{n-1,0}(2)\in\mathbb{N},$ if and only if the base 3 representation of n contains at least one $1.$ This is sequence OEIS [12], A081606, $\left(1,3,4,5,7,9,\cdots\right).$

3. Convolutions of Narayana polynomials.

Finally we want to derive some convolution formulae. By (1.41) we have

$$C(t,z)=\sum_{n\geq 0}C_n(t)z^n=\frac{1+z(t-1)-\sqrt{1-2z(t+1)+z^2(t-1)^2}}{2tz}$$

or equivalently

$$tzC(t,z)^2=C(t,z)-1-zC(t,z)+tzC(t,z). \tag{3.1}$$

We will show that

$$C(t,z)^m=\sum_{n\geq 0}c_n(m,t)z^n \tag{3.2}$$

with

$$c_n(m,t)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+m}{k+m}\frac{m}{n+m}t^k \tag{3.3}$$

and $c_0(m,t)=1.$

Note that $c_n(1,t)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+1}{k+1}\frac{1}{n+1}t^k=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n}{k}\frac{1}{k+1}t^k=C_n(t).$

It suffices to show that

$tzC(t,z)^m=C(t,z)^{m-1}\left(1+z(t-1)\right)-C(t,z)^{m-2}$

holds if we replace $C(t,z)^m$ by $\sum_{n\geq 0}c_n(m,t)z^n.$

The coefficient of z^{n+1} is

$tc_n\left(m,t\right)=c_{n+1}(m-1,t)+(t-1)c_n(m-1,t)-c_{n+1}(m-2,t).$

The coefficient of t^{k+1} is

$$\binom{n-1}{k}\binom{n+m}{k+m}\frac{m}{n+m}=\binom{n}{k+1}\binom{n+m}{k+m}\frac{m-1}{n+m}+\binom{n-1}{k}\binom{n+m-1}{k+m-1}\frac{m-1}{n+m-1}$$
$$-\binom{n-1}{k+1}\binom{n+m-1}{k+m}\frac{m-1}{n+m-1}-\binom{n}{k+1}\binom{n+m-1}{k+m-1}\frac{m-2}{n+m-1}$$

Dividing by $\binom{n-1}{k}\binom{n+m-1}{k+m-1}$ this gives

$$\frac{m}{k+m}=\frac{n}{k+1}\frac{m-1}{k+m}+\frac{m-1}{n+m-1}-\frac{n-k-1}{k+1}\frac{n-k}{k+m}\frac{m-1}{n+m-1}-\frac{n}{k+1}\frac{m-2}{n+m-1}$$

which is easily verified.

More generally we want to show that

$$\frac{\partial^m}{\partial t^m}\sum_{n\geq 0}\frac{D_{n+m,k}(t)}{(n+m)\cdots(n+1)}z^n=C(t,z)^m\sum_{n\geq 0}D_{n,k}(t)z^n. \tag{3.4}$$

The coefficient of z^n of the left-hand side is

$$v(n,m,k)=\sum_{j=0}^{n}\frac{\binom{n+m}{j}\binom{n+m}{j+k}\binom{j}{m}}{\binom{n+m}{m}}t^{j-m}$$

As above it suffices to verify that

$$tzC(t,z)^m \sum_{n\ge 0} D_{n,k}(t)z^n = C(t,z)^{m-1}\sum_{n\ge 0} D_{n,k}(t)z^n\left(1+z(t-1)\right) - C(t,z)^{m-2}\sum_{n\ge 0} D_{n,k}(t)z^n$$

or

$$tv(n,m,k) = v(n+1,m-1,k)+(t-1)v(n,m-1,k)-v(n+1,m-2,k).$$

This can easily be verified.

For $t=1$ formula (3.2) reduces to the well-known formula

$$C(1,z)^m = \left(\frac{1-\sqrt{1-4z}}{2z}\right)^m = \sum_{n\ge 0}\frac{m}{2n+m}\binom{2n+m}{n}z^n. \qquad (3.5)$$

A well-known convolution formula for the central binomial coefficients is

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n. \qquad (3.6)$$

A computational proof follows immediately by squaring the generating function (2.4).

For the $m-$fold convolution we get

$$u_m(n) = \sum_{i_1+\cdots+i_m=n}\binom{2i_1}{i_1}\binom{2i_2}{i_2}\cdots\binom{2i_m}{i_m} = 4^n\binom{\frac{m}{2}+n-1}{n} \qquad (3.7)$$

since

$$\left(\sum_{n\ge 0}\binom{2n}{n}x^n\right)^m = \left(1-4x\right)^{-\frac{m}{2}} = \sum_k \binom{-\frac{m}{2}}{k}(-4)^k x^k = \sum_k \binom{\frac{m}{2}+k-1}{k}4^k x^k.$$

A combinatorial proof has been given in [8].

I want now to compute the corresponding convolutions of the polynomials $M_n(t)$.

Their generating function is

$$\sum_{n\ge 0} M_n(t)x^n = \frac{1}{\sqrt{(1+(1-t)x)^2-4x}}. \qquad (3.8)$$

Let

$$\left(\frac{1}{\sqrt{(1+(1-t)x)^2-4x}}\right)^m = \sum_{n\ge 0} u_m(n,t)x^n. \qquad (3.9)$$

Then we get

Theorem 5

$$u_m(n,t)=\sum_{k\ge 0}\binom{n+m-1}{m-1}\binom{n}{k}\frac{\prod_{j=0}^{k-1}(2n+m-1-2j)}{\prod_{j=0}^{k-1}(2k+m-1-2j)}t^k. \tag{3.10}$$

To prove these identities by induction observe that

$$u_{m-2}(n,t)=u_m(n,t)-(1+t)u_m(n-1,t)+(1-t)^2u_m(n-2,t)$$

holds for all n.

The first 5 terms of $u_1(n,t),u_2(n,t),\cdots,u_5(n,t)$ are

1	1 + t	1 + 4 t + t^2	1 + 9 t + 9 t^2 + t^3	1 + 16 t + 36 t^2 + 16 t^3 + t^4
1	2 + 2 t	3 + 10 t + 3 t^2	4 + 28 t + 28 t^2 + 4 t^3	5 + 60 t + 126 t^2 + 60 t^3 + 5 t^4
1	3 + 3 t	6 + 18 t + 6 t^2	10 + 60 t + 60 t^2 + 10 t^3	15 + 150 t + 300 t^2 + 150 t^3 + 15 t^4
1	4 + 4 t	10 + 28 t + 10 t^2	20 + 108 t + 108 t^2 + 20 t^3	35 + 308 t + 594 t^2 + 308 t^3 + 35 t^4
1	5 + 5 t	15 + 40 t + 15 t^2	35 + 175 t + 175 t^2 + 35 t^3	70 + 560 t + 1050 t^2 + 560 t^3 + 70 t^4

All these polynomials are palindromic and gamma-nonnegative:

$$u_m(n,t)=\sum_{k=0}^{n}\binom{n+m-1}{m-1}\binom{2k}{k}\binom{n}{2k}\frac{(2k)!!}{\prod_{i=0}^{k-1}(m+2i+1)}t^k(1+t)^{n-2k}. \tag{3.11}$$

For the proof we make use of Gauss's theorem for hypergeometric polynomials

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix},1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.12}$$

By comparing coefficients of t^k in (3.10) and (3.11) it suffices to show that

$$\sum_{j=0}^{k}\frac{\binom{2j}{j}\binom{n}{2j}}{\binom{n}{k}}\frac{(2j)!!\binom{n-2j}{k-j}}{\prod_{i=0}^{j-1}(m+2i+1)}=\frac{\prod_{j=0}^{k-1}(2n+m-1-2j)}{\prod_{j=0}^{k-1}(2k+m-1-2j)}.$$

The left-hand side can we written as ${}_2F_1\left(\begin{matrix}-k,k-n\\ \frac{m+1}{2}\end{matrix},1\right)$ which by Gauss's Theorem equals

$$\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+1}{2}+n\right)}{\Gamma\left(\frac{m+1}{2}+k\right)\Gamma\left(\frac{m+1}{2}+n-k\right)}=\frac{\prod_{j=0}^{k-1}\left(2n+m-1-2j\right)}{\prod_{j=0}^{k-1}\left(2k+m-1-2j\right)}.$$

Let us finally consider two special cases in detail.

For $m=2$ we get

$$u_2(n,t)=\sum_{k=0}^{n}M_k(t)M_{n-k}(t)=\frac{1}{2}\sum_{k=0}^{n}\binom{2n+2}{2k+1}t^k=\sum_k\binom{n+1}{2k}t^k\sum_k\binom{n+1}{2k+1}t^k. \qquad (3.13)$$

For the generating function of $u_2(n,t^2)$ is

$$\sum_{n\geq 0}u_2(n,t^2)x^n=\frac{1}{(1+(1-t^2)x)^2-4x}=\frac{1}{4t}\left(\frac{(1+t)^2}{1-(1+t)^2x}-\frac{(1-t)^2}{1-(1-t)^2x}\right).$$

This implies

$$u_2\left(n,t^2\right)=\frac{(1+t)^{2n+2}-(1-t)^{2n+2}}{4t}=\frac{1}{2}\sum_{k=0}^{n}\binom{2n+2}{2k+1}t^{2k}.$$

The right-hand side follows from $(1+t)^{2n}-(1-t)^{2n}=\left((1+t)^n+(1-t)^n\right)\left((1+t)^n-(1-t)^n\right).$

For $m=3$ we get

$$u_3(n,t)=\sum_k\binom{n+2}{2}\binom{n}{k}\frac{\binom{n+1}{k}}{\binom{k+1}{1}}t^k=\binom{n+2}{2}\sum_k\binom{n}{k}\binom{n+1}{k}\frac{1}{k+1}t^k=\binom{n+2}{2}C_{n+1}(t).$$

It would be interesting to find combinatorial interpretations of these results.

References

[1] Paul Barry, Generalized Narayana polynomials, Riordan arrays, and lattice paths, J. Integer Sequences 15 (2012), Article 12.4.8

[2] Naiomi T. Cameron and Asamoah Nkwanta, *On some (pseudo) involutions in the Riordan group*, J. Integer Sequences 8 (2005), Article 05.3.7

[3] M. Vauchassade de Chaumont and Gérard Viennot, *Polynomes orthogonaux et problèmes d'énumeration en biologie moléculaire*, Sém. Lothar.Combin. B081(1984)

[4] Johann Cigler, *How to guess and prove explicit formulas for some Hankel determinants,* Preprint 2010, https://homepage.univie.ac.at/johann.cigler/preprints/simplehankel4.pdf

[5] Johann Cigler, *Some nice Hankel determinants,* arXiv:1109.1449

[6] Sylvie Corteel, Jang Soo Kim and Dennis Stanton, *Moments of orthogonal polynomials and combinatorics,* Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications 159, 2016.

[7] Emeric Deutsch and Bruce E. Sagan, *Congruences for Catalan and Motzkin numbers and related sequences,* arXiv:math/0407326

[8] Rui Duarte and Antonio Guedes de Oliveira, *New developments of an old identity*, arXiv:1203.5424

[9] Richard K. Guy, *Catwalks, Sandsteps and Pascal Pyramids*, J. Integer Sequences 3 (2000),

[10] Richard K. Guy, Christian Krattenthaler and Bruce E. Sagan, *Lattice paths, reflections, and dimension-changing bijections*, Ars combin. 34 (1992), 3-15

[11] Dongsu Kim and Jiang Zeng, *Combinatorics of generalized Tchebycheff polynomials,* European J. Combin. 24 (2003), no. 5, 499-509

[12] OEIS, http://oeis.org/

[13]Qiongqiong Pan and Jiang Zeng, *On total positivity of Catalan-Stieltjes matrices*, submitted to EJC

[14] T. Kyle Petersen, *Eulerian numbers*, Birkhäuser 2015

[15] Amitai Regev, Nathaniel Shar, and Doron Zeilberger, *A very short (bijective!) proof of Touchard's Catalan identity,* http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/touchard.pdf

[16] Rodica Simion and Dennis Stanton, *Octabasic Laguerre polynomials and permutation statistics,* J. Comput. Appl. Math. 68 (1996), no. 1-2, 297-329

[17]Gérard Viennot, Une Théorie combinatoire des polynômes orthogonaux généraux", UQAM 1983,

[18] Bao-Xuan Zhu, *Log-convexity and strong q-log-convexity for some triangular arrays,* Adv. Appl. Math. 50 (2013), 595-606