Patterns in treeshelves

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Abstract

We study the distribution and the popularity of left children on sets of treeshelves avoiding a pattern of size three. (Treeshelves are ordered binary increasing trees where every child is connected to its parent by a left or a right link.) The considered patterns are sub-treeshelves, and for each such a pattern we provide exponential generating function for the corresponding distribution and popularity. Finally, we present constructive bijections between treeshelves avoiding a pattern of size three and some classes of simpler combinatorial objects.

1   Introduction and notation

The study of patterns in permutations was first introduced by Knuth [13], and continues to be an active area of research today. Recently, patterns have been studied in contexts other than permutations, see for instance [5, 16] where the combinatorial class under consideration are inversion sequences, which can be seen as an alternative representation for permutations. The present paper deals with treeshelves (formally defined below) which are still another class in bijection with permutations, and patterns are sub-treeshelves contained or avoided in a similar way as consecutive patterns do in permutations or in inversion sequences. More precisely, we consider the class of unrestricted treeshelves and of those avoiding a pattern of size 3 (treeshelves avoiding a pattern of size 2 collapse trivially to a singleton set). We not only enumerate these classes for any avoider of size 3, but also give bivariate generating functions with respect to the size and to the number of occurrences of a second pattern of size 2. As a byproduct we obtain the popularity among these classes of the pattern of size 2, obtaining counting sequences which are not yet recorded in Sloane's Encyclopedia of Integer Sequences [20].

Treeshelves are particular classes of binary increasing trees, considered for example in Françon's work [10] in the context of data structures for binary search methods. An *increasing tree* of size n, is a rooted tree with n nodes labeled by distinct integers in $\{1, 2, \ldots, n\}$, so that the sequences of labels are increasing along all branches starting at the root (and thus, the root is labeled by 1). A *binary increasing tree* (sometimes called 0-1-2 increasing tree) is an increasing tree where every node has at most two children. Many studies (*e.g.*, [1, 2, 3, 6, 15]) investigate binary increasing trees, but very few deal with such trees endowed with the additional property that every child (including those with no siblings) is connected to its parent by either a left or a right link. We call such a binary increasing tree *treeshelf* (or *t-shelf* for short), and its size is the number of its nodes, see Figure 1 for a size 7 t-shelf. We denote by $\mathcal{B}_n$ the set of size n t-shelves, and $\mathcal{B}_1$ consists of the single one-node t-shelf. Often it is more convenient to represent graphically t-shelves by trees where the integers labeling the nodes are proportional with the lengths of the branches. For example, the size 3 t-shelf

$$\overset{1}{\underset{2\ \ 3}{\wedge}} \text{ is represented by } \diagdown, \text{ and } \overset{1}{\underset{3\ \ 2}{\wedge}} \text{ is represented by } \diagup,$$

see also Figure 1. In this representation, $\mathcal{B}_2 = \{\nearrow, \searrow\}$, and $\mathcal{B}_3 = \{\nearrow, \wedge, \prec, \wedge, \succ, \searrow, \}$.

We denote $\cup_{n\geq 0}\mathcal{B}_n$ by $\mathcal{B}$, and $\cup_{n\geq 1}\mathcal{B}_n$ by $\mathcal{B}^\bullet$. The labeled tree rooted at the left child of the root of a t-shelf T becomes a t-shelf after appropriately relabeling its nodes, and in the following we refer to it as the *left t-shelf of T*, and similarly for the *right t-shelf of T*.

There is a bijection between $\mathcal{B}_n$ and the set of permutations of size n, and so the cardinality of $\mathcal{B}_n$ is $n!$. Indeed, to any t-shelf T in $\mathcal{B}_n$ we can uniquely associate the length n permutation $\pi = \alpha(n - r(T) + 1)\beta$, where $r(T)$ is the label of the root of T, and α (resp. β) is recursively defined from the left (resp. right) t-shelf of T (see again Figure 1). As mentioned by Bergeron, Flajolet, and Salvy [1], this construction appears in [10] and thereafter recalled in Stanley's book [21]. Additional information (including historical notes) about binary and other families of increasing trees can be found for example in [1, 4, 11].

Figure 1: The t-shelf corresponding to the permutation 5 3 7 4 6 2 1; dashed/dotted lines correspond to different patterns of size three.

In this paper we are interested in the sets of t-shelves avoiding a pattern $P \in \mathcal{B}_3$, *i.e.*, the sets of those that do not contain any occurrence of P. The containment/avoidance of a pattern in a t-shelf can most easily be explained with examples. The avoidance of

$\diagdown$ in a t-shelf T means that T does not contain any node where the label of its left child is less than that of its right child. The t-shelf in Figure 1 contains only one pattern $\diagup\diagdown$ (illustrated by dashed lines), one pattern $\diagdown$ (dotted) and avoids the pattern $\bullet\!\!-\!\!\bullet$.

Since the number of $\diagdown$ patterns in a t-shelf is equal to the size of the t-shelf minus the number of $\diagup$ patterns, minus one, in the following we will consider only $\diagup$ patterns. Moreover, an occurrence of the $\diagup$ pattern is equivalent to that of a left child in the underlying tree of the t-shelf, we will refer to this pattern as a left child (similarly the pattern $\diagdown$ corresponds to a right child). Also, since the patterns $\diagdown$ and $\diagup$ are equivalent by symmetry, and so are the patterns $\diagup$ and $\diagdown$, and the patterns $\diagup\diagdown$ and $\diagup\diagdown$, we will consider only avoiders P in $\{\diagdown, \diagup, \diagup\diagdown\}$.

T-shelves are labeled combinatorial objects, and so it is appropriate to use exponential generating functions (e.g.f.) for the enumerative analysis of them. In Section 2, for each of the avoiders P above mentioned, we consider the set $\mathcal{B}(P)$ of t-shelves avoiding P, or $\mathcal{B}^{\bullet}(P)$ when we restrict to non-empty t-shelves. We provide a bivariate exponential generating function for each $\mathcal{B}(P)$ with respect to the size and the number of left children, that is, function where the coefficient of $\frac{z^n y^k}{n!}$ in its series expansion is the number of t-shelves of size n having exactly k left children, and deduce the e.g.f. for $\mathcal{B}(P)$ with respect to the size. We also give the e.g.f. for the popularity of the left children among $\mathcal{B}(P)$, function where the coefficient of $\frac{z^n}{n!}$ in its series expansion is the total number of left children appearing in all size n t-shelves in $\mathcal{B}(P)$. These results are summarized in Tables 1 and 2.

Our method consists in constructing recursively the combinatorial class in question from two smaller classes, $\mathcal{A}_1$ and $\mathcal{A}_2$, using the usual labeled product $\mathcal{A}_1 \star \mathcal{A}_2$ and the boxed product $\mathcal{A}_1^{\square} \star \mathcal{A}_2$. The boxed product $\mathcal{A}_1^{\square} \star \mathcal{A}_2$ is a subset of $\mathcal{A}_1 \star \mathcal{A}_2$ where the smallest label appears in the $\mathcal{A}_1$ component. See [8] for more information about the boxed product and its application on labeled combinatorial structures.

Theorems 4-6 in Section 3 give constructive proofs of some results in Section 2, namely constructive bijections between: (i) t-shelves avoiding $\diagdown$ and set partitions, (ii) (unordered) binary increasing trees where every node of degree one has either a left or a right child and t-shelves avoiding the pattern $\diagup\diagdown$, and (iii) unordered binary increasing trees and t-shelves avoiding the pattern $\bullet\!\!-\!\!\bullet$.

2   T-shelves avoiding a size 3 pattern

We begin this section by considering unrestricted t-shelves, then we extend our approach to those avoiding a pattern in $\{\diagdown, \diagup, \diagup\diagdown\} \subset \mathcal{B}_3$.

A t-shelf is either empty or consists of a root with two (possibly empty) children. Thus, the set $\mathcal{B}$ of unrestricted t-shelves can be expressed as

$$\mathcal{B} = \epsilon + \mathcal{Z}^{\square} \star \mathcal{B}^2,$$

where $\mathcal{Z}$ corresponds to the atom, *i.e.*, the singleton formed by the unique object of size one.

As the boxed product $\mathcal{A}_1^{\square} \star \mathcal{A}_2$ has its exponential generating function given by $\int_0^z \partial_t A_1(t) \cdot A_2(t)\, \mathrm{d}t$, where $A_1(t)$ and $A_2(t)$ are the exponential generating functions of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively (see [8, Theorem II.5]), we obtain the differential equation

$$B(z) = 1 + \int_0^z B^2(t)\, \mathrm{d}t,$$

which, with the initial condition $B(0) = 1$, gives as expected $B(z) = \frac{1}{1-z}$, the e.g.f. for the sequence $n!$.

If we are interested in the bivariate exponential generating function $B(z, y)$ where the coefficient of $\frac{z^n y^k}{n!}$ is the number of t-shelves of size n having exactly k left children (or, equivalently by symmetry, k right children), then it is more convenient to consider the set $\mathcal{B}^{\bullet}$ of non-empty t-shelves. A t-shelf $T \in \mathcal{B}^{\bullet}$ can be in one of the following cases: the root of T either

- has no children (T is reduced to one root node), in this case the set of such T is $\mathcal{Z}$; or
- has only a left or only a right child, in both cases the set of such T is $\mathcal{Z}^{\square} \star \mathcal{B}^{\bullet}$; or
- has both left and right children, the set of such T is $\mathcal{Z}^{\square} \star \mathcal{B}^{\bullet} \star \mathcal{B}^{\bullet}$.

Thus, $\mathcal{B}^{\bullet}$ can be expressed as

$$\mathcal{B}^{\bullet} = \mathcal{Z} + \mathcal{Z}^{\square} \star \mathcal{B}^{\bullet} + \mathcal{Z}^{\square} \star \mathcal{B}^{\bullet} + \mathcal{Z}^{\square} \star (\mathcal{B}^{\bullet})^2,$$

and after multiplying by y whenever a new left child is created, we obtain the differential equation

$$B^{\bullet}(z, y) = z + \int_0^z B^{\bullet}(t, y)\, \mathrm{d}t + y \int_0^z B^{\bullet}(t, y)\, \mathrm{d}t + y \int_0^z (B^{\bullet}(t, y))^2\, \mathrm{d}t,$$

where $B(0, y)^{\bullet} = 0$, and its solution is $B^{\bullet}(z, y) = \frac{1 - \mathrm{e}^{z(y-1)}}{\mathrm{e}^{z(y-1)} - y}$. Finally,

$$B(z, y) = 1 + B^{\bullet}(z, y) = \frac{1 - y}{\mathrm{e}^{z(y-1)} - y},$$

and we retrieve two well known results, see [18, Exercise 1.9]:

- the distribution of the left children on the set $\mathcal{B}$ has the exponential generating function $B(z, y)$, and it is given by a shift of the Eulerian numbers (sequence A008292 in OEIS [20]); and
- the popularity of the left (or right) children among $\mathcal{B}$, which is the coefficient of $\frac{z^n}{n!}$ in $\partial_y B(z, y)|_{y=1} = \frac{z^2}{2z^2 - 4z + 2}$, is given by the Lah numbers (sequence A001286 in OEIS [20]).

In the following, for each t-shelf $P \in \{\textrm{⟨},\textrm{╱},\textrm{⋀}\}$ we will count the class $\mathcal{B}(P)$ (or $\mathcal{B}^{\bullet}(P)$) of t-shelves avoiding P, and explore the distribution and the popularity of left children (*i.e*, of the pattern ╱) among each class.

2.1 Pattern ⌄

Here we consider $\mathcal{B}(P)$ with $P = $ ⌄, that is, t-shelves having all they left children with no right child, we refer to Figure 2 for an illustration of the shape of such a t-shelf.

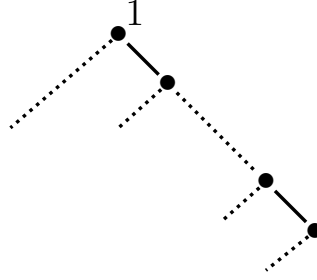Figure 2: The shape of a t-shelf avoiding pattern ⌄.

Theorem 1. *Let P be the pattern* ⌄. *The bivariate e.g.f. for $\mathcal{B}(P)$ with respect to the size of t-shelves and the number of left children is given by*

$$C(z, y) = e^{\frac{e^{zy} - 1}{y}}.$$

Proof. Let $\mathcal{C} = \mathcal{B}(P)$ and $T \in \mathcal{C}$. According to the shape of the t-shelves in $\mathcal{C}$ (see Figure 2), if T is non-empty, then it is obtained by a pair of t-shelves, namely a non-empty t-shelf with no right children containing the smallest label of T, and a second unrestricted t-shelf in $\mathcal{C}$. The set of such non-empty t-shelves is $\mathcal{D}^{\square} \star \mathcal{C}$, where $\mathcal{D}$ is the set of non-empty t-shelves with no right children. Thus we have

$$\mathcal{C} = \epsilon + \mathcal{D}^{\square} \star \mathcal{C}.$$

Since the bivariate exponential generating function for $\mathcal{D}$ is $D(z, y) = \frac{e^{zy} - 1}{y}$, we obtain the differential equation

$$C(z, y) = 1 + \int_0^z e^{ty} \cdot C(t, y) \cdot \mathrm{d}t$$

where $C(0, y) = 1$, with the solution $C(z, y) = e^{\frac{e^{zy} - 1}{y}}$. $\square$

By calculating $C(z, 1)$ we have the following corollary.

Corollary 1. *The exponential generating function for the set $\mathcal{B}(P)$ with respect to the size of t-shelves is $Bell(z) = e^{e^z - 1}$, which generates the Bell numbers (sequence A000110 in OEIS [20]).*

Corollary 2. *The popularity of the left children among the set $\mathcal{B}(P)$ is given by the exponential generating function*

$$PC(z) = (ze^z - e^z + 1) e^{e^z - 1}.$$

Moreover, the coefficient pc_n of $\frac{z^n}{n!}$ in $PC(z)$ satisfies $pc_n = (n+1)b_n - b_{n+1}$ where b_n is the nth Bell number. The asymptotic of pc_n is given by

$$\sqrt{n}\left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} e^{\frac{n}{W(n)}-n-1},$$

where W is the Lambert function [7, 19], that is, $W(n)$ is the unique solution of $W(n) \cdot e^{W(n)} = n$.

(The first terms of pc_n, $n \geq 2$, are $1, 5, 23, 109, 544, 2876, 16113, 95495$.)

Proof. The popularity is given by $\partial_y C(z,y)|_{y=1} = (ze^z - e^z + 1)e^{e^z-1}$. The recurrence for pc_n is directly obtain from the relation $(z-1)\partial_z \text{Bell}(z) + \text{Bell}(z) = PC(z)$.
Finally, the asymptotic follows from the asymptotic formula due to M. Klazar [12, Proposition 2.6] and D.E. Knuth [14, eq. (30), p. 69]:

$$\frac{b_{n+1}}{b_n} \sim \frac{n}{\ln(n)},$$

and from the well known asymptotic for the Bell numbers (see A.M. Odlyzko [17]):

$$\frac{1}{\sqrt{n}}\left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} e^{\frac{n}{W(n)}-n-1},$$

where $W(n)$ is the unique solution of $W(n) \cdot e^{W(n)} = n$. $\qquad\square$

2.2 Pattern ✏

Here we consider the set $\mathcal{B}(P)$ of t-shelves avoiding the pattern $P = $ ✏.

Theorem 2. *Let P be the pattern* ✏. *Then the bivariate e.g.f. for $\mathcal{B}(P)$ with respect to the size of t-shelves and the number of left children is given by*

$$E(z,y) = \frac{2y-1}{y\cosh\left(z\sqrt{-2y+1}+\ln\left(\frac{1}{y}\left(y+\sqrt{-2y+1}-1\right)\right)\right)+y}.$$

Proof. Let $\mathcal{E} = \mathcal{B}(P)$ and $T \in \mathcal{E}$. One of the following cases can occur.

- T is empty.
- T is not empty, and its root does not have a left child. In this case, the right t-shelf of T belongs to $\mathcal{E}$ and the set of such t-shelves T is $\mathcal{Z}^{\square} \star \mathcal{E}$.
- The root of T has a left child. In this case T is obtained from a pair of t-shelves satisfying the second point above, namely one formed by the root of T together with its right t-shelf, and the other one being the left t-shelf of T. See Figure 3 for an illustration of this case. So, T is the product of two t-shelves satisfying the second point above and, with the smallest label belonging to the first t-shelf. Thus, the set of such t-shelves T is $\mathcal{F}^{\square} \star \mathcal{F}$ where $\mathcal{F} = \mathcal{Z}^{\square} \star \mathcal{E}$.

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Combining these cases we have

$$\mathcal{E} = \epsilon + \mathcal{Z}^{\square} \star \mathcal{E} + \left(\mathcal{Z}^{\square} \star \mathcal{E}\right)^{\square} \star \left(\mathcal{Z}^{\square} \star \mathcal{E}\right),$$

which yields the differential equation

$$E(z, y) = 1 + \int_0^z E(t, y)\mathrm{d}t + y \cdot \int_0^z \left(E(u, y) \cdot \int_0^u E(t, y)\,\mathrm{d}t \right) \mathrm{d}u,$$

with the initial conditions $E(0, y) = 1$ and $\partial_z E(z, y)|_{z=0} = 1$. A simple calculation (using Maple for instance) gives the desired result. □
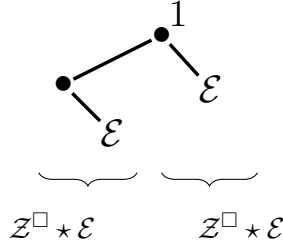
Figure 3: Illustration of a t-shelf satisfying the third case in the proof of Theorem 2.

The next corollary is obtained by calculating $E(z, 1)$.

Corollary 3. *The exponential generating function for the set $\mathcal{B}(P)$ with respect to the size of trees is given by*

$$Eul(z) = \frac{1}{1 - \sin z},$$

which yields a shift of the Euler numbers (sequence [A000111](https://oeis.org/A000111) in OEIS [20]– not to be confused with Eulerian numbers).

Corollary 4. *The popularity pe_n of the left children among the set $\mathcal{B}(P)$ is given by the exponential generating function*

$$PE(z) = \frac{-\sin z + 1 + (z - 1)\cos z}{(1 - \sin z)^2}.$$

Moreover, the coefficient pe_n of $\frac{z^n}{n!}$ satisfies $pe_n = (n+1)e_n - e_{n+1}$ where e_n is the shifted Euler number defined by the e.g.f. $Eul(z) = \frac{1}{1-\sin(z)}$. The asymptotic of pe_n is given by

$$\frac{8(\pi - 2)}{\pi^3}n^2 \left(\frac{2}{\pi}\right)^n.$$

(The first terms of pe_n, $n \geq 2$, are $1, 4, 19, 94, 519, 3144, 20903, 151418$).

Proof. Using Theorem 2, $PE(z)$ is obtained by calculating $\partial_y E(z, y)|_{y=1}$. The recurrence relation is directly obtained with the relation $PE(z) = (z - 1)\partial_z \mathrm{Eul}(z) + \mathrm{Eul}(z)$, and the asymptotic follows from the classical singularity analysis (see for instance [8]). □

2.3   Pattern $\bigwedge$

We conclude this section by considering the pattern $P = \bigwedge$ and the set $\mathcal{B}^\bullet(P)$ of non-empty t-shelves avoiding P.

Theorem 3. *Let P be the pattern $\bigwedge$. Then the bivariate e.g.f. for $\mathcal{B}^\bullet(P)$ with respect to the size of t-shelves and the number of left children is given by*

$$G(z,y) = \frac{-2}{1 + y - \sqrt{y^2+1}\coth\left(\frac{z\sqrt{y^2+1}}{2}\right)}.$$

Proof. For $P = \bigwedge$, a non-empty t-shelf T in $\mathcal{G} = \mathcal{B}^\bullet(P)$ is in one of the following cases:
  - T is reduced to one (root) node.
  - T has at least two nodes and its root does not have a left child. In this case the set of such t-shelves T is $\mathcal{Z}^\square \star \mathcal{G}$.
  - T has at least two nodes and its root does not have a right child. As above, the set of such t-shelves T is $\mathcal{Z}^\square \star \mathcal{G}$.
  - The root of T has both left and right children (see Figure 4). In this case T is obtained from a pair of t-shelves in $\mathcal{G}$ connected by a common root, with the smallest label of T in its right t-shelf. The set of such T is $\mathcal{Z}^\square \star (\mathcal{G}^\square \star \mathcal{G})$.

Combining these four cases we obtain

$$\mathcal{G} = \mathcal{Z} + \mathcal{Z}^\square \star \mathcal{G} + \mathcal{Z}^\square \star \mathcal{G} + \mathcal{Z}^\square \star (\mathcal{G}^\square \star \mathcal{G})$$

which induces the differential equation

$$G(z,y) = z + \int_0^z G(t,y)\,\mathrm{d}t + y \cdot \int_0^z G(t,y)\,\mathrm{d}t + y \cdot \int_0^z \int_0^u \partial_t G(t,y) \cdot G(t,y)\,\mathrm{d}t\,\mathrm{d}u,$$

with the initial conditions $G(0,y) = 0$ and $\partial_z G(z,y)|_{z=0} = 1$. A simple calculation (using Maple for instance) gives the desired result. $\square$
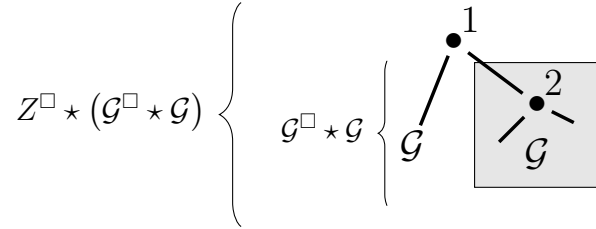

Figure 4: Illustration of a t-shelf satisfying fourth case in the proof of Theorem 3.

Corollary 5. *The exponential generating function for the set $\mathcal{B}(P)$ with respect to the size of t-shelves is given by*

$$1 + \frac{-2}{-\sqrt{2}\coth\left(\frac{z}{\sqrt{2}}\right) + 2}$$

which generates the sequence [A131178](#) *in OEIS [20].*

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Corollary 6. *The popularity pg_n of the left children among the set $\mathcal{B}(P)$ is given by the exponential generating function*

$$PG(z) = \frac{e^{\sqrt{2}z}\left(4\,z - 4\right) - \left(\sqrt{2} - 2\right)e^{2\sqrt{2}z} + \sqrt{2} + 2}{\left(\left(\sqrt{2} - 2\right)e^{\sqrt{2}z} + 2 + \sqrt{2}\right)^2}.$$

Moreover, the asymptotic of the coefficient pg_n of $\frac{z^n}{n!}$ is given by

$$n\left(\frac{\sqrt{2}}{\ln\left(2\sqrt{2} + 3\right)}\right)^{n+1}.$$

(The first terms of pg_n, $n \geq 2$, are $1, 5, 24, 128, 770, 5190, 38864, 320704$.)

Proof. Using Theorem 3, $PG(z)$ is obtained by calculating $\partial_y G(z,y)|_{y=1}$, and the asymptotic follows from the classical singularity analysis. $\qquad\qquad\square$

Pattern P	Sequence counting $\mathcal{B}(P)$	OEIS [20]	Comments
	$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, ...$	A000110 (Bell)	Cor. 1 and Th. 4
	$1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, ...$	A000111 (Euler)	Cor. 3 and Th. 6
	$1, 1, 2, 5, 16, 64, 308, 1730, 11104, 80176, ...$	A131178	Cor. 5 and Th. 5

Table 1: Number of t-shelves avoiding the pattern P.

Pattern P	Popularity of left children in $\mathcal{B}(P)$	Comments
	$1, 5, 23, 109, 544, 2876, 16113, ...$	Corollary 2
	$1, 4, 19, 94, 519, 3144, 20903, 151418, ...$	Corollary 4
	$1, 5, 24, 128, 770, 5190, 38864, 320704, ...$	Corollary 6

Table 2: Popularity of left children among t-shelves avoiding the pattern P. None of these sequences is yet recorded in OEIS [20].

3 Constructive bijections

The counting sequences for t-shelves avoiding a pattern of length 3 given in Corollaries 1, 3 and 5 are known (see Table 1), and these results deserve bijective proofs. Here, for each pattern $P \in \{$$,$$,$$\}$, we give an explicit bijection between $\mathcal{B}(P)$ and a simpler combinatorial class. These results are stated in the next three theorems, the first two of them are straightforward.

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Theorem 4. *There is a bijection between the set of partitions of $\{1, 2, \ldots, n\}$ and the set $\mathcal{B}_n(P)$ of t-shelves of size n avoiding the pattern $P = $* ⌄ *.*

Proof. For a partition $S_1, S_2, \ldots, S_k$ of a set $S \subseteq \{1, 2, \ldots, n\}$ with $\min S_1 < \min S_2 < \ldots < \min S_k$ we define a t-shelf T with nodes labeled by integers from S. If $k = 1$, then T is simply the t-shelf with no right children (and with labels in $S = S_1$). Elsewhere, T is defined recursively as:

- the root of T is labeled by $\min S_1$,
- the left t-shelf of T has size equal to card $S_1 - 1$ and does not have a right children; its nodes are labeled by integers in $S_1 \setminus \{\min S_1\}$,
- the right t-shelf of T is obtained recursively from the partition $S_2, \ldots, S_k$ of $S \setminus S_1$.

Clearly, the t-shelf T corresponding to a set partition of $\{1, 2, \ldots, n\}$ is a size n t-shelf avoiding P. See the recursive definition of $\mathcal{B}(P)$ in the proof of Theorem 1 and the shape of T given in Figure 2. This construction is reversible, and the statement holds □

Unordered binary increasing trees are the non-ordered counterpart of t-shelves: in an unordered binary increasing tree the sibling nodes are not longer ordered among themselves, and nodes with no sibling are not 'labeled' left/right. Thus each unordered binary increasing tree T can be expanded into 2^k different t-shelves of same size, where k is the number of nodes of T having at least one child. Theorem 6 below establishes a bijection between size $n + 1$ unordered binary increasing trees and size n t-shelves avoiding ⟋. An interesting intermediate ordered/unordered combinatorial class is that of binary increasing trees where, as above, the sibling nodes are not ordered, but nodes with no sibling are still 'labeled' left/right. We denote by $\mathcal{J}$ the set of these trees.

We define a transformation ϕ acting on unordered binary increasing trees and on trees in $\mathcal{J}$ by ordering the nodes having a sibling: if a node of a tree has two children, then we consider the child with the smaller label as the right one (and thus, that with the larger label as the left one). This configuration is depicted below.

$$\underset{y}{\overset{u}{\diagup}} \diagdown z, \text{ when } z < y.$$

Clearly, applying ϕ on a tree in $\mathcal{J}$ a t-shelf avoiding ⋀ is obtained. Moreover, this transformation is reversible, and since $\mathcal{J}$ is counted by the sequence [A131178](#) in [20] (see the references therein), the next theorem gives a constructive proof for the counting sequence of t-shelves avoiding ⋀.

Theorem 5. *There is a bijection between the set $\mathcal{J}$ and the set $\mathcal{B}(P)$ of t-shelves avoiding the pattern $P = $* ⋀ *.*

In order to obtain a bijection between unordered binary increasing trees and t-shelves that avoid ⟋ (next theorem), we apply the shift (defined below) on unordered binary increasing trees in standard representation. The *standard representation* of such a tree is the t-shelf obtained after ordering sibling nodes, which is obtained by performing

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the above transformation ϕ, together with considering as right child each node with no sibling, as depicted below.

$$u \searrow z.$$

The *shift* of a node y of a t-shelf has effect only if the following conditions are satisfied: (i) y is a left child and it has a right sibling, say z; and (ii) z in turn does not have a left child and its label is smaller than that of y. Otherwise the shift has no effect. With this notation, the shift of a node y satisfying the two conditions above consists of pruning y from its parent and grafting it as the left child of z, see Figure 5.
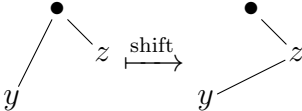
Figure 5: The shift operation. The label of z is smaller than that of y.

Finally, the shift of a t-shelf T is defined recursively by shifting, in order, the right t-shelf of T, the root of T, and then the left t-shelf of T. See the first part of Figure 6 for an illustration. Obviously, the shift of a t-shelf T is still a t-shelf, and if T is the standard representation of some unordered binary increasing tree, then the shift of T avoids $\diagup\!\!\!\bullet$, and its root does not have a left child.
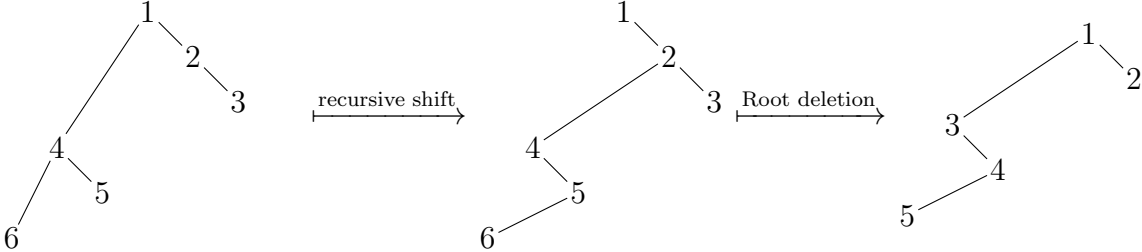
Figure 6: A unordered binary increasing tree in standard representation and its image after the recursive shift process, and after the deletion of the root.

Theorem 6. *There is a bijection between unordered binary increasing trees with $n+1$ nodes and the set $\mathcal{B}_n(P)$ of t-shelves of size n avoiding the pattern $P = \diagup\!\!\!\bullet$.*

Proof. Let S be an unordered binary increasing tree with $n+1$ nodes and T be the shift of its standard representation. As mentioned above, T is a t-shelf and its root does not have a left child. We define the mapping $S \mapsto U$, where U is the t-shelf obtained after deleting the root of T and decreasing by one each label of the obtained t-shelf, see Figure 6. This mapping is injective, and any t-shelf with n nodes that avoids $\diagup\!\!\!\bullet$ can be obtained by this mapping from an unordered binary increasing tree. $\square$

Let us remark that unordered binary increasing trees are equinumerous with alternating permutations that starts by a descent, as proved by Foata and Schützenberger [9].

The corresponding bijection is given by Donaghey [6]. Using Donaghey's bijection together with the bijection in Theorem 6, we obtain a one-to-one correspondence between alternating permutations starting with a descent and t-shelves avoiding ✦.

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