Modular forms, Schwarzian conditions, and symmetries of differential equations in physics

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Abstract.

We give examples of infinite order rational transformations that leave linear differential equations covariant. These examples are non-trivial yet simple enough illustrations of exact representations of the renormalization group. We first illustrate covariance properties on order-two linear differential operators associated with identities relating the same $_2F_1$ hypergeometric function with different rational pullbacks. These rational transformations are solutions of a differentially algebraic equation that already emerged in a paper by Casale on the Galoisian envelopes. We provide two new and more general results of the previous covariance by rational functions: a new Heun function example and a higher genus $_2F_1$ hypergeometric function example. We then focus on identities relating the same $_2F_1$ hypergeometric function with two different algebraic pullback transformations: such remarkable identities correspond to modular forms, the algebraic transformations being solution of another differentially algebraic Schwarzian equation that also emerged in Casale's paper. Further, we show that the first differentially algebraic equation can be seen as a subcase of the last Schwarzian differential condition, the restriction corresponding to a factorization condition of some associated order-two linear differential operator. Finally, we also explore generalizations of these results, for instance, to $_3F_2$, hypergeometric functions, and show that one just reduces to the previous $_2F_1$ cases through a Clausen identity. The question of the reduction of these Schwarzian conditions to modular correspondences remains an open question. In a $_2F_1$ hypergeometric framework the Schwarzian condition encapsulates all the modular forms and modular equations of the theory of elliptic curves, but these two conditions are actually richer than elliptic curves or $_2F_1$ hypergeometric functions, as can be seen on the Heun and higher genus example. This work is a strong incentive to develop more differentially algebraic symmetry analysis in physics.

PACS: 05.50.+q, 05.90.+m, 05.10.-a, 02.30.Hq, 02.30.lk,02.30.Gp, 02.40.Xx
AMS Classification scheme numbers: 34M55, 47Exx, 32Hxx, 32Nxx, 34Lxx, 34Mxx, 14Kxx, 14H52

Key-words: Square Ising model, Schwarzian derivative, infinite order rational symmetries of ODEs, Fuchsian linear differential equations, Gauss and generalized hypergeometric functions, Heun function, globally nilpotent linear differential operators, isogenies of elliptic curves, Hauptmoduls, elliptic functions, modular forms, modular equations, modular correspondences, mirror maps, renormalization group, Malgrange pseudo-group, Galoisian envelope, Lattès transformations.

1. Introduction: infinite order symmetries.

In its simplest form, the concept of symmetries in physics corresponds to a (univariate) transformation $x \rightarrow R(x)$ preserving some structures. Whether these structures are linear differential equations, or more complicated mathematical objects (systems of differential equations, functional equations, etc ...), they must be *invariant* or *covariant* under the previous transformations $x \rightarrow R(x)$. Of course, these transformation symmetries can be studied, per se, in a discrete dynamical perspective†. Along this iteration line, or more generally, *commuting transformations* line, there is no need to underline the success of the renormalization group revisited by Wilson [5, 6] seen as a fundamental symmetry in lattice statistical mechanics or field theory.

The renormalization of the one-dimensional Ising model without a magnetic field (even if it can also be performed with a magnetic field [7]), which corresponds to the simple (commuting) transformations $x \rightarrow x^n$ (where $x = \tanh(K)$), is usually seen as the heuristic "student" example of *exact* renormalization in physics, but it is trivial being one-dimensional. For less academical models one could think that no exact‡ closed form representation of the renormalization group exists, but can one hope to find anything better ? For Yang-Baxter integrable models [8, 9] with a canonical genus-one parametrization [10, 11, 12] (elliptic functions of modulus k) *exact* representations of the generators of the renormalization group happen to exist. Such exact symmetry transformations must have $k = 0$ and $k = 1$ as a fixed point, be compatible with the Kramers-Wannier duality $k \leftrightarrow 1/k$, and, most importantly, be compatible with the *lattice of periods* of the elliptic functions parametrizing the model. Thus, these exact generators must be the *isogenies* [13, 14] of the elliptic functions (of modulus k). The simplest example of a transformation carrying these properties is the *Landen transformation* [9, 13]

$$k \quad \longrightarrow \quad k_L = \frac{2\sqrt{k}}{1+k}, \qquad (1)$$

with the *critical point* of the square Ising model (resp. Baxter model) given by the fixed point of the transformation: $k = 1$.

This algebraic transformation corresponds to multiplying (*or dividing* because of the modular group symmetry $\tau \leftrightarrow 1/\tau$) the ratio τ of the two periods of the elliptic curves $\tau \longleftrightarrow 2\tau$. The other (isogeny) transformations†† correspond to $\tau \leftrightarrow N \cdot \tau$, for various integers N.

Setting out to find the precise covariance of some of the physical quantities related to the 2-D Ising model, like the partition function per site, the correlation functions, the n-fold correlations $\chi^{(n)}$ associated with the full susceptibility [16, 17, 18, 19], with respect to transformations of the Landen type (1), is a difficult task. An easier goal would be to find a covariance, not on the selected¶ linear differential operators that

† In their pioneering work Julia, Fatou and Ritt the theory of iteration of rational functions was seen as a method for investigating functional equations [1, 2, 3]. More generally, one can try to find all pairs of *commuting rational functions*, see [4].

‡ For instance, a Migdal-Kadanoff decimation can introduce, in a finite-dimensional parameter space of the model, rational transformations that can be seen as efficient approximations of the generators of the renormalization group, hoping that the basin of attraction of the fixed points of the transformation is "large enough".

†† See for instance (2.18) in [15].

¶ They are not only Fuchsian, the corresponding linear differential operators are globally nilpotent or G-operators [20, 21, 22].

these quantities satisfy, but on the different *factors* of these operators. Luckily the factors of the operators associated with these physical quantities are linear differential operators whose solutions can be expressed in terms of *elliptic functions, modular forms* [20] (and beyond ${}_4F_3$ hypergeometric functions associated with *Calabi-Yau ODEs* [23, 24], etc ...).

Let us give an illustration of the precise action of non-trivial symmetries like (1) on some elliptic functions that actually occur in the 2-D Ising model [23, 24, 25]: weight-one *modular forms*.

Let us introduce the j-invariant of the elliptic curve and its transform by the Landen transformation

$$j(k) \; = \; 256 \cdot \frac{(1-k^2+k^4)^3}{k^4 \cdot (1-k^2)^2}, \qquad j(k_L) \; = \; 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}. \tag{2}$$

and let us also introduce the two corresponding *Hauptmoduls* [13]

$$x \; = \; \frac{1728}{j(k)}, \qquad y \; = \; \frac{1728}{j(k_L)}, \tag{3}$$

with the two Hauptmoduls being related by the *modular equation* [26, 27, 28, 29, 30, 31]:

$$1953125\, x^3 y^3 \; - 187500\, x^2 y^2 \cdot \, (x+y) \; + 375\, xy \cdot \, (16\, x^2 - 4027\, xy + 16\, y^2)$$
$$- \, 64 \, (x+y) \cdot \, (x^2 + 1487\, xy + y^2) \; + 110592\, xy \; = \; 0. \tag{4}$$

The transformation $x \, \rightarrow \, y(x) \, = \, y$, where y is given by the modular equation (4), is an *algebraic* transformation *which corresponds to the Landen transformation* (as well as the inverse Landen transformation: it is *reversible* because of the $x \, \leftrightarrow \, y$ symmetry of (4)). The emergence of a *modular form* [23, 24, 25] corresponds to the remarkable identity on the *same* hypergeometric function but where the pullback x is changed $x \, \rightarrow \, y(x) \, = \, y$ according to the modular equation (4) corresponding to the Landen transformation, or inverse Landen transformation

$${}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, y\Big) \; = \; \mathcal{A}(x) \cdot \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, x\Big), \tag{5}$$

where $\mathcal{A}(x)$ is an algebraic function given by:

$$1024\, \mathcal{A}(x)^{12} \; - 1152\, \mathcal{A}(x)^8 \; + 132\, \mathcal{A}(x)^4 \; + 125\, x \; - 4 \; = \; 0. \tag{6}$$

The emergence of a modular form is thus associated with a selected hypergeometric function having an *exact covariance property* [32, 33] *with respect to an infinite order algebraic transformation*, corresponding here to the Landen transformation, which is precisely what we expect for an exact representation of the renormalization group of the square Ising model [7, 13].

With the example of the Ising model one sees that the exact representation of the renormalization group immediately requires considering the isogenies of *elliptic curves* [13], and thus transformations, corresponding to the *modular equations*, $x \rightarrow y(x)$ which are (multivalued) *algebraic* functions.

In a previous paper [7], we studied simpler examples of identities on ${}_2F_1$ hypergeometric functions where the transformations $x \; \rightarrow \; y(x)$ were *rational functions*. In that paper we found that the rational functions $y(x)$ are highly selected. They are *differentially algebraic* [34, 35]. They verify a (non-linear) differential equation

$$\Big(\frac{dy(x)}{dx}\Big)^2 \cdot A(y(x)) \; = \; \frac{dy(x)}{dx} \cdot A(x) \; + \frac{d^2y(x)}{dx^2}. \tag{7}$$

where $A(x)$ is a rational function (which is in fact a log-derivative [7]). This non-trivial condition coincides exactly with one of the conditions G. Casale obtained [36, 37, 38, 39, 40, 41, 42] in a classification of Malgrange's $\mathcal{D}$-envelope and $\mathcal{D}$-groupoïds on $\mathbb{P}_1$. Denoting $y'(x)$, $y''(x)$ and $y'''(x)$ the first, second and third derivative of $y(x)$ with respect to x, these conditions read respectively¶

$$\mu(y) \cdot y'(x) \; - \mu(x) \; + \; \frac{y''(x)}{y'(x)} \; = \; 0, \qquad (8)$$

$$\nu(y) \cdot y''(x)^2 \; - \nu(x) \; + \; \frac{y'''(x)}{y'(x)} \; - \frac{3}{2} \cdot \Big(\frac{y''(x)}{y'(x)}\Big)^2 \; = \; 0, \qquad (9)$$

together with $\gamma(y) \cdot y'(x)^n - \gamma(x) \; = \; 0$ and $h(y) \; = \; h(x)$, corresponding respectively to rank two, rank three, together with rank one and rank nul groupoïds, where $\nu(x)$, $\mu(x)$, $\gamma(x)$ are *meromorphic* functions ($h(x)$ is holomorph). Clearly Casale's condition (8) is *exactly the same condition as* the one we already found in [7], and this is not a coincidence ! In this paper we will refer to Casale's first condition (8) as the "rank-two condition", and to the Casale's second condition (9) as the "rank-three condition", or the "Schwarzian condition". When our paper [7] was published we had no example corresponding to a *Schwarzian condition* like (9).

Without going into the details of Malgrange's pseudo-groups [37, 42], Galoisian envelopes, $\mathcal{D}$-envelopes of a germ of foliation [41], and $\mathcal{D}$-groupoïds, let us just say that these concepts are built in order to generalize the idea of differential Galois groups to *non-linear* [43] ODEs‡ or *non-linear* functional equations† (see [44]). In an experimental mathematics pedagogical approach, we will provide more examples of *rational* transformations verifying rank-two condition (8), and new pedagogical examples of *algebraic* transformations verifying *Schwarzian conditions* like in (9). We hope that these (slightly obfuscated for physicists) Galoisian envelope conditions will become clearer in a framework of *identities on hypergeometric functions*. In a *modular form* perspective, we will show that the infinite number of algebraic transformations corresponding to the *infinite number of the modular equations*, are solutions of a *unique* Schwarzian condition (9) with $\nu(x)$ a *rational function*.

The paper is organized as follows. We first recall the ${}_2F_1$ results in [7] which correspond to rational transformations and rank-two condition (8) on these rational transformations. We then display a set of new results also corresponding to rational transformations with condition (8). Then focusing on a modular form hypergeometric identity, we show that it actually provides a first heuristic example of a Schwarzian condition (9) where $\nu(x)$ is a rational function and analyze them in detail. We then show that the rank-two condition (8) is a subcase of the rank-three Schwarzian condition (9), the restriction corresponding to a *factorization condition* of some associated order-two linear differential operator. We then explore generalizations of the hypergeometric identity to ${}_3F_2$, ${}_2F_2$ and ${}_4F_3$ hypergeometric functions, and show that the ${}_3F_2$ attempt, in fact, just reduces to the previous ${}_2F_1$ cases through a Clausen identity.

¶ More generally see the concept of differential algebraic invariant of isogenies in [14].
‡ In the case of linear ODEs the $\mathcal{D}$-envelope gives back the differential Galois group of the linear ODEs.
† The typical example is the (non-linear) functional equation $f(x+1) \; = \; y(f(x))$, which is such that its Malgrange pseudo-group (generalization of the Galois group) will be "small enough" if and only if, there exists a rational function $\nu(x)$, such that the Schwarzian condition (9) is satisfied.

2. Recalls: rational transformation and ${}_2F_1$ hypergeometric functions

We recall a few examples and results from [7] on the hypergeometric examples displayed in [45]. The hypergeometric function

$$Y(x) \;=\; x^{1/4} \cdot {}_2F_1\Bigl([\frac{1}{2}, \frac{1}{4}], [\frac{5}{4}]; \, x\Bigr) \;=\; \frac{1}{4} \cdot \int_0^x t^{-3/4} \cdot (1-t)^{-1/2} \cdot dt$$
$$= \; x^{1/4} \cdot (1-x)^{-1/2} \cdot \; {}_2F_1\Bigl([\frac{1}{2}, \frac{1}{4}], [\frac{5}{4}]; \, \frac{-4\,x}{(1-x)^2}\Bigr), \qquad (10)$$

is the integral of an algebraic function. It has a simple covariance property with respect to the *infinite order rational* transformation $x \, \rightarrow \, -4\,x/(1-x)^2$:

$$Y\Bigl(\frac{-4\,x}{(1-x)^2}\Bigr) \;=\; (-4)^{1/4} \cdot Y(x). \qquad (11)$$

This hypergeometric function can be seen as an 'ideal' example of physical functions, covariant by an exact (rational) transformation. Three other hypergeometric functions with similar covariant properties were analyzed in [7]:

$$Y(x) \;=\; x^{1/3} \cdot {}_2F_1\Bigl([\frac{1}{3}, \, \frac{2}{3}], \, [\frac{4}{3}], \, x\Bigr) \;=\; \frac{1}{3} \cdot \int_0^x t^{-2/3} \cdot (1-t)^{-2/3} \cdot dt \qquad (12)$$
$$= \; (-8)^{-1/3} \cdot R(x)^{1/3} \cdot {}_2F_1\Bigl([\frac{1}{3}, \, \frac{2}{3}], \, [\frac{4}{3}], \, R(x)\Bigr), \quad \hbox{with:} \quad R(x) \;=\; \frac{x \cdot (x\,-2)^3}{(1\,-2\,x)^2},$$

as well as

$$Y(x) \;=\; x^{1/6} \cdot {}_2F_1\Bigl([\frac{1}{2}, \, \frac{1}{6}], \, [\frac{7}{6}], \, x\Bigr) \;=\; \frac{1}{6} \cdot \int_0^x t^{-5/6} \cdot (1-t)^{-1/2} \cdot dt \qquad (13)$$
$$= \; (-27)^{-1/6} \cdot R(x)^{1/6} \cdot {}_2F_1\Bigl([\frac{1}{2}, \, \frac{1}{6}], \, [\frac{7}{6}], \, R(x)\Bigr), \quad \hbox{with:} \quad R(x) \;=\; \frac{-27\,x}{(1\,-4\,x)^3},$$

which can be seen as a particular subcase ($\alpha \, = \, 1/2$) of the identity on hypergeometric functions:

$${}_2F_1\Bigl([\alpha, \, \frac{1-\alpha}{3}], \, [\frac{4\,\alpha\,+5}{6}], \, x\Bigr)$$
$$= \; (1\,-4\,x)^{-\alpha} \cdot {}_2F_1\Bigl([\frac{\alpha}{3}, \, \frac{\alpha+1}{3}], \, [\frac{4\,\alpha\,+5}{6}], \, \frac{-\,27\,x}{(1\,-4\,x)^3}\Bigr), \qquad (14)$$

and, finally, the simple function $Y(x) \; = \; tanh^{-1}(x^{1/2})$ that one represents as a hypergeometric function:

$$Y(x) \;=\; x^{1/2} \cdot {}_2F_1\Bigl([1, \, \frac{1}{2}], [\frac{3}{2}], \, x\Bigr) \;=\; \frac{1}{2} \cdot \int_0^x t^{-1/2} \cdot (1-t)^{-1} \cdot dt \qquad (15)$$
$$= \; (4)^{-1/2} \cdot R(x)^{1/6} \cdot {}_2F_1\Bigl([1, \, \frac{1}{2}], [\frac{3}{2}], \; R(x)\Bigr), \quad \hbox{with:} \quad R(x) \;=\; \frac{4\,x}{(1\,+z)^2}.$$

Though not mentioned in [7], two other hypergeometric functions, also covariant under a rational transformation, could have been deduced from the previous hypergeometric examples using Goursat and Darboux identities (see Appendix B):

$$Y(x) \;=\; x^{1/4} \cdot (1\,-x)^{1/4} \cdot {}_2F_1\Bigl([\frac{1}{2}, \, 1], \, [\frac{5}{4}], \, x\Bigr) \;=\; \frac{1}{4} \cdot \int_0^x t^{-3/4} \cdot (1-t)^{-3/4} \cdot dt$$
$$= \; (-4)^{-1/4} \cdot R(x)^{1/4} \cdot (1\,-R(x))^{1/4} \cdot {}_2F_1\Bigl([\frac{1}{4}, \, \frac{1}{2}], \, [\frac{5}{4}], \, 4\,x\,(1\,-x)\Bigr) \qquad (16)$$
$$\hbox{where:} \qquad R(x) \;=\; \frac{-4 \cdot x \cdot (1\,-x)}{(1\,-2\,x)^2}, \qquad (17)$$

and

$$Y(x) \; = \; x^{1/6} \cdot {}_2F_1\Big([\frac{1}{6}, \, \frac{2}{3}], \, [\frac{7}{6}], \, x\Big) \; = \; \frac{1}{6} \cdot \int_0^x t^{-5/6} \cdot (1-t)^{-2/3} \cdot dt \qquad (18)$$

$$= \; (64)^{-1/6} \cdot R(x)^{1/6} \cdot {}_2F_1\Big([\frac{1}{6}, \, \frac{2}{3}], \, [\frac{7}{6}], \, R(x)\Big) \quad \hbox{with:} \quad R(x) \; = \; \frac{64 \, x}{(1 \, + 18 \, x \, - 27 \, x^2)^2}.$$

These six hypergeometric functions are incomplete integrals that are canonically associated with an algebraic curve $u^N \, - \, P(t) \; = \; 0$ of *genus one* for (10), (12), (13), (16) (18), and genus zero for (15)

$$Y(x) \; = \; \frac{1}{N} \cdot \int_0^x \frac{dt}{u(t)} \; = \; \frac{1}{N} \cdot \int_0^x \frac{dt}{\sqrt[N]{P(t)}}, \quad \hbox{or:} \quad N \cdot Y'(x) \; = \; \frac{1}{u(x)}, \qquad (19)$$

and are solutions of a second order linear differential operator:

$$\Omega \; = \; \omega_1 \cdot D_x, \qquad \hbox{with:} \qquad \omega_1 \; = \; D_x \, + A_R(x), \qquad (20)$$

where a rational function $A_R(x)$ is† the logarithmic derivative of a simple algebraic‡ function $u(x) \; = \; \sqrt[N]{P(x)}$. The expressions of the rational functions $A_R(x)$ read respectively for the four hypergeometric examples (10), (12), (13) and (15)

$$\frac{1}{4} \frac{3 \, - 5 \, x}{x \cdot (1 \, - x)}, \qquad \frac{2}{3} \cdot \frac{1 - 2 \, x}{x \cdot (1 \, - x)}, \qquad \frac{1}{6} \cdot \frac{5 - 8 \, x}{x \cdot (1 \, - x)}, \qquad \frac{1}{2} \cdot \frac{1 \, - 3 \, x}{x \cdot (1 \, - x)}. \qquad (21)$$

and for the two new examples (16) and (18):

$$\frac{3}{4} \cdot \frac{1 \, - 2 \, x}{x \cdot (1 \, - x)}, \qquad \frac{1}{6} \cdot \frac{5 \, - 9 \, x}{x \cdot (1 \, - x)}. \qquad (22)$$

In the interesting cases emerging in physics [9, 23, 24, 25], the operator Ω happens to be globally nilpotent [20], in which case $A_R(x)$ is the log-derivative of the N-th root of a rational function. At first we do not require Ω to be globally nilpotent††, then we will see what this assumption entails.

A rational transformation $x \; \rightarrow \; R(x)$, its corresponding cofactor $\gamma(x) \; = \; 1/R'(x)$, and the order-one operator $\omega_1 \; = \; D_x \, + \, A(x)$ give the identity:

$$\gamma(x) \cdot D_x \cdot \Big(\frac{1}{\gamma(x)}\Big) \; = \; D_x \; - \; \frac{d\ln(\gamma(x))}{dx}. \qquad (23)$$

The change of variable $x \, \rightarrow \, R(x)$ on ω_1 reads:

$$D_x \, + \, A_R(x) \quad \longrightarrow \quad \gamma(x) \cdot D_x \, + \, A_R(R(x)) \; = \; \gamma(x) \cdot \Big(D_x \, + \, B(x)\Big).$$

Now we want to impose this RHS expression to be written as

$$\gamma(x)^2 \cdot \Big(D_x \, + \, A_R(x)\Big) \cdot \frac{1}{\gamma(x)},$$

which, because of (23), occurs if:

$$B(x) \; = \; A_R(x) \; - \; \frac{d\ln(\gamma(x))}{dx},$$

† The fact that $A_R(x)$ is the log-derivative of the N-th root of a rational function, here a polynomial, is a consequence of the fact that Ω is a globally nilpotent linear differential operator [20].
‡ Note that $u(x)$ being an algebraic function, these examples being such that $N \cdot Y'(x) \, = \, 1/u(x)$, $Y'(x)$ is holonomic but also its reciprocal $1/Y'(x)$.
†† Imposing the global nilpotence generates additional relations (see section (2.2) below.

yielding a rank-two functional equation [7] on $A_R(x)$ and $R(x)$:

$$\Big(\frac{dR(x)}{dx}\Big)^2 \cdot A_R(R(x)) \,\, = \,\, \frac{dR(x)}{dx} \cdot A_R(x) \,\, + \frac{d^2R(x)}{dx^2}. \qquad (24)$$

This condition is *exactly* the first rank-two Casale's condition (8). Using the chain rule formula of derivatives for the composition of functions, one can show, for a given rational function $A_R(x)$, that the composition $R_1(R_2(x))$ *verifies condition* (24) if two rational functions $R_1(x)$ and $R_2(x)$ verify condition (24). In particular if $R(x)$ verifies condition (24), all the iterates of $R(x)$ also verify that condition¶: $R(x) \longrightarrow R(R(x)),\, R(R(R(x))),\, \cdots$

Keeping in mind the well-known example of the parametrization of the standard map $x \,\rightarrow\, 4\,x \cdot (1-x)$ with $x \,=\, \sin^2(\theta)$, yielding $\theta \,\rightarrow\, 2\,\theta$, let us seek a (*transcendental*) parametrization $x \,=\, P(u)$ such that‡

$$R_{a_1}\Big(P(u)\Big) \,\, = \,\, P(a_1\, u) \qquad \text{or:} \qquad R_{a_1} \,\, = \,\, P \cdot H_{a_1} \cdot Q, \qquad (25)$$

where H_{a_1} denotes the scaling transformation $x \,\rightarrow\, a_1 \cdot x$ and $Q \,=\, P^{-1}$ denotes the composition inverse of P. One can also verify an essential property that we expect to be true for a representation of the renormalization group, namely that two $R_{a_1}(x)$ for different values of a_1 commute, the result corresponding to the product of these two a_1:

$$R_{a_1}\Big(R_{b_1}(x)\Big) \,\, = \,\, R_{b_1}\Big(R_{a_1}(x)\Big) \,\, = \,\, R_{a_1\cdot b_1}(x). \qquad (26)$$

The neutral element of this abelian group corresponds to $a_1 \,=\, 1$, giving the identity transformation $R_1(x) \,=\, x$. Performing the composition inverse of $R_{a_1}(x)$ amounts to changing a_1 into its inverse $1/a_1$. The structure of the (one-parameter) group and the extension of the composition of n times a rational function $R(x)$ (namely $R(R(\cdots R(x)\cdots)))$ to *n any complex number*, is a straight consequence of this relation. For example, in the case of the ${}_2F_1$ hypergeometric function (12), the one-parameter series expansion of $R_{a_1}(x)$ reads:

$$R(a,\, x) \,\, = \,\, a \cdot x \,\, + a \cdot (a-1) \cdot S_a(x) \qquad \text{where:} \qquad (27)$$

$$S_a(x) \,\, = \,\, -\frac{1}{2} \cdot x^2 \,\, + \frac{1}{28} \cdot (5\,a-9) \cdot x^3 \,\, - \frac{(3\,a^2-12\,a+13)}{56} \cdot x^4 \,\, + \,\, \cdots$$

This one-parameter series (27) is a family of commuting one-parameter series solution of the rank-two condition (24), and these solution series have *movable singularities* (more details in Appendix A).

Along this line one can define some "infinitesimal composition" ($\epsilon \,\simeq\, 0$):

$$R_{1\,+\,\epsilon}(x) \,\, = \,\, P \cdot H_{1\,+\,\epsilon} \cdot P^{-1}(x) \,\, = \,\, x \,\, + \epsilon \cdot F(x) \,\, + \,\, \cdots \qquad (28)$$

From (26), we see that the function† $F(x)$ satisfies the following functional equations involving a rational function $R(x)$ (in the one-parameter family $R_{a_1}(x)$):

$$\frac{dR(x)}{dx} \cdot F(x) \,\, = \,\, F(R(x)), \qquad \frac{dR^{(n)}(x)}{dx} \cdot F(x) \,\, = \,\, F(R^{(n)}(x)),$$

$$\text{where:} \qquad R^{(n)}(x) \,\, = \,\, R(R(\cdots R(x))\cdots). \qquad (29)$$

¶ This is in agreement with the fact that (24) is the condition for $\Omega \,=\, (D_x \,+\, A(x)) \cdot D_x$ to be covariant by $x \,\rightarrow\, R(x)$: this condition is obviously preserved by the composition of $R(x)$'s (for $A(x)$ fixed).

‡ This is the idea of Siegel's linearization [46, 47, 48] (or Koenig's linearization theorem see [49]).

† Generically, $F(x)$ is a transcendental function, not a rational nor an algebraic function.

From (28) one deduces that

$$Q(x) \cdot P'(Q(x)) \, = \, F(x) \qquad \text{and thus:} \qquad x \cdot \frac{dP(x)}{dx} \, = \, F(P(x)), \qquad (30)$$

which is equivalent to:

$$\frac{d\,Q(x)}{dx} \cdot F(x) \, = \, Q(x). \qquad (31)$$

Inserting (28) in the rank-two condition (24) one immediately finds (at the first order in ϵ) that $F(x)$ is a *holonomic function*, solution of a second order linear differential operator Ω^* which can be seen to be the *adjoint* of the second order operator Ω defined by (20):

$$\Omega^* \, = \, D_x^2 \, - A(x) \cdot D_x \, - \frac{dA(x)}{dx} \, = \, D_x \cdot \Big(D_x \, - A(x)\Big). \qquad (32)$$

2.1. New results: $Q(x)$ and $P(x)$ as differentially algebraic functions

The two functions $Q(x)$ and its composition inverse $P(x) = Q^{-1}(x)$ are *differentially algebraic functions* [34, 35] as can be seen in [7]. The function $Q(x)$ is solution of the differentially algebraic equation:

$$A_R(x) \cdot G(x) \cdot \frac{d\,G(x)}{dx} - \frac{d\,A_R(x)}{dx} \cdot G(x)^2 \, + 2 \Big(\frac{d\,G(x)}{dx}\Big)^2 - G(x) \cdot \frac{d^2\,G(x)}{dx^2} \, = \, 0, \qquad (33)$$

where $G(x)$ is the log-derivative‡ of $Q(x)$, i.e. $G(x) \, = \, Q'(x)/Q(x)$. While equation (31) means that $F(x) \, = \, 1/G(x)$, equation (33) is immediately obtained by imposing $F(x) \, = \, 1/G(x)$ to be a solution of Ω^*.

One remarks that this non-linear differential equation corresponds to a *homogeneous* quadratic equation in $G(x)$ and its derivatives. In terms of $Q(x)$ this equation corresponds to a homogeneous cubic equation in $Q(x)$ and its derivatives:

$$A_R(x) \cdot \Big(\Big(\frac{d\,Q(x)}{dx}\Big)^2 - Q(x) \cdot \frac{d^2\,Q(x)}{dx^2}\Big) \cdot \frac{d\,Q(x)}{dx} \, + \frac{d\,A_R(x)}{dx} \cdot Q(x) \cdot \Big(\frac{d\,Q(x)}{dx}\Big)^2 \qquad (34)$$

$$+ \frac{d^2\,Q(x)}{dx^2} \cdot \Big(\frac{d\,Q(x)}{dx}\Big)^2 \, + Q(x) \cdot \frac{d^3\,Q(x)}{dx^3} \cdot \frac{d\,Q(x)}{dx} \, - 2\,Q(x) \cdot \Big(\frac{d^2 Q(x)}{dx^2}\Big)^2 \, = \, 0.$$

The function $P(x)$, being the composition inverse of a differentially algebraic function, is solution of the *differentially algebraic* [34, 35] equation¶:

$$A_R(P(x)) \cdot \Big(\frac{dP(x)}{dx}\Big)^2 \cdot \Big(x \cdot \frac{d^2 P(x)}{dx^2} + \frac{dP(x)}{dx}\Big) \, + \, x \cdot A_R'(P(x)) \cdot \Big(\frac{dP(x)}{dx}\Big)^4$$

$$+ x \cdot \Big(\frac{d^2 P(x)}{dx^2}\Big)^2 \, - x \cdot \frac{dP(x)}{dx} \cdot \frac{d^3 P(x)}{dx^3} \, - \frac{dP(x)}{dx} \cdot \frac{d^2 P(x)}{dx^2} \, = \, 0. \qquad (35)$$

For instance, for the hypergeometric function (10), one verifies straightforwardly that $P(x) \, = \, sn^4(x, (-1)^{1/2})$, given in [7], verifies (35) with $A_R(x)$ given by the first rational function in (21).

‡ With an extra log-derivative step equation (33) can be written in an even simpler form. Introducing $H(x) \, = \, G'(x)/G(x)$ (33) becomes $A_R(x)' \, - A_R(x) \cdot H(x) \, + H'(x) \, - \, H(x)^2 \, = \, 0$.

¶ Equation (35) can be obtained using the Faà di Bruno formulas for the higher derivatives of inverse functions.

2.2. Assuming that Ω is globally nilpotent

The rank-two condition (24) turns out to identify exactly with the first Casale condition (8), the only difference being that $A_R(x)$ is not meromorphic as in Casale's condition (8), but a *rational function*: in lattice statistical mechanics and enumerative combinatorics, the differential operators are linear differential operators with polynomial coefficients. In fact, the operators emerging in lattice models are not only Fuchsian, but *globally nilpotent* operators [20], or *G*-operators [21], thus their wronskians are the N-th root of a rational function [20]. This naturally leads us to examine the case where Ω is taken to be globally nilpotent. Given Ω globally nilpotent, there exists an algebraic function $u(x)$ (N-th root of a rational function) such that $A_R(x)$ is the log-derivative of $u(x)$. Consequently Ω and Ω^*, which read respectively $\Omega \, = \, u(x)^{-1} \cdot D_x \cdot u(x) \cdot D_x$ and $\Omega^* \, = \, D_x \cdot u(x) \cdot D_x \cdot u(x)^{-1}$, are related by the simple conjugation:

$$\Omega^* \cdot u(x) \quad = \quad u(x) \cdot \Omega. \qquad (36)$$

Thus, $F(x)$ and $Y(x)$ are related through the simple equation:

$$u(x) \cdot Y(x) \quad = \quad F(x). \qquad (37)$$

The fact that the holonomic function $Y(x)$ is solution of Ω, amounts to writing that the log-derivative of $Y'(x)$ is equal to $-A_R(x)$. If Ω is globally nilpotent then $-A_R(x)$ is the log-derivative of the reciprocal $1/u(x)$, and the logarithm of $Y'(x)$ is equal to the logarithm of $1/u(x)$, up to a constant of integration $\ln(\alpha)$, and thus:

$$\alpha \cdot {{dY(x)} \over {dx}} \quad = \quad {{1} \over {u(x)}} \qquad \hbox{or:} \qquad \alpha \cdot {{Y'(x)} \over {Y(x)}} \quad = \quad {{1} \over {u(x) \cdot Y(x)}}. \qquad (38)$$

Recalling the fact that the rank-two condition (24) gives (31), namely that the log-derivative of $Q(x)$ is equal to $1/F(x)$, one deduces by combining (37) with (38):

$${{Q'(x)} \over {Q(x)}} \quad = \quad {{1} \over {F(x)}} \quad = \quad \alpha \cdot {{Y'(x)} \over {Y(x)}} \qquad \hbox{i.e.} \qquad Q(x) \quad = \quad \lambda \cdot Y(x)^{\alpha}. \qquad (39)$$

Note that, without any loss of generality, one can restrict λ to $\lambda \, = \, 1$.

$F(x)$ is solution of Ω^* as a consequence of the rank-two condition (24). This second order linear differential equation can be integrated into $F'(x) - A_R(x) \cdot F(x) \, = \, u(x) \cdot Y'(x)$, and taking into account (38) this gives:

$${{dF(x)} \over {dx}} \, - A_R(x) \cdot F(x) \quad = \quad {{1} \over {\alpha}}. \qquad (40)$$

For the new results (see sections (3) and (4) below), corresponding to a rank-two condition (24) like the hypergeometric examples seen in the beginning of this section, the holonomic function $Y(x)$ is of the form (19). Thus the constant α is actually equal to a *positive integer* N (see the case where $N \, = \, 3$ in Appendix A for a worked example). Further one deduces from (39) that $Q(x)$ is *always a holonomic function*: $Q(x) \, = \, \lambda \cdot Y(x)^N$, for instance, for the hypergeometric functions (10), (12), (13), (15), (16) and (18), we have $Q(x) \, = \, Y(x)^N$ with $N = \, 4, \, 3, \, 6, \, 2, \, 4, \, 6$ respectively.

Without assuming (19), the constant α is not necessarily a positive integer, thus $Q(x)$ has no reason to be holonomic: it is just *differentially algebraic* (see (34)). The log-derivatives of $Q(x)$ and $Y(x)$ being equal up to a multiplicative factor α (see (39)), one deduces from the fact that (33) is a *homogeneous* (quadratic) condition in

$G(x)$ and its derivatives, that $Q(x)$ and $Y(x)$ verify necessarily the *same* differentially algebraic condition (34).

With this global nilpotence assumption, the differentially algebraic function $P(x)$ is, in fact, solution of much simpler non-linear ODEs. From $u(x) \cdot Y(x) \,=\, F(x)$ one gets using (30):

$$u(P(x))^\alpha \cdot Y(P(x))^\alpha \;\;=\;\; F(P(x))^\alpha \;\;=\;\; \Big(x \cdot \frac{dP(x)}{dx}\Big)^\alpha. \qquad (41)$$

Using $Q(x) = \lambda \cdot Y(x)^\alpha$, and $Q(P(x)) \,=\, x$, one deduces:

$$x \cdot u(P(x))^\alpha \;\;=\;\; \lambda \cdot \Big(x \cdot \frac{dP(x)}{dx}\Big)^\alpha. \qquad (42)$$

3. More rational transformations: an identity on a Heun function

In this section we write an identity similar to the ${}_2F_1$ hypergeometric identities (10), (12), (13), but, this time, on a *Heun function*, that is a holonomic function with *four* singularities instead of the well-known three singularities $0,\, 1,\, \infty$ of the hypergeometric functions.

Let us consider the rational transformation¶

$$x \quad \longrightarrow \qquad 4 \cdot \frac{x \cdot (1-x) \cdot (1\,-k^2\,x)}{(1\,-k^2\,x^2)^2}, \qquad (43)$$

where one recognizes the transformation‡ $\theta \,\rightarrow\, 2\,\theta$ on the square of the elliptic sine $x \,=\, sn(\theta,\, k)^2$:

$$sn(\theta,\, k)^2 \quad \longrightarrow \qquad sn(2\,\theta,\, k)^2 \;= $$
$$= \;\; 4 \cdot \frac{sn(\theta,\, k)^2 \cdot (1\,-sn(\theta,\, k)^2) \cdot (1\,-\,k^2 \cdot sn(\theta,\, k)^2)}{(1\,-k^2 \cdot sn(\theta,\, k)^4)^2}. \qquad (44)$$

Denoting $M \,=\, 1/k^2$, the transformation (43) yields:

$$R(x) \;\;=\;\; 4 \cdot \frac{x \cdot (1-x) \cdot (1\,-\,x/M)}{(1\,-\,x^2/M)^2}. \qquad (45)$$

For a given M, the transformations $\theta \,\rightarrow\, p\,\theta$ give rational transformations $x \,\rightarrow\, R_p(x)$ on the square of the elliptic sine, $x \,=\, sn(\theta,\, k)^2$, which are sketched for the first primes p in Appendix C. The series expansions of these rational transformations read $R_p(x) \,=\, p^2 \cdot x + \cdots$ With these rational functions $R_p(x)$ we have the following identity on a Heun function:

$$R_p(x) \cdot Heun\Big(M,\, \frac{M\,+1}{4},\, \frac{1}{2},\, 1,\, \frac{3}{2},\, \frac{1}{2},\, R_p(x)\Big)^2$$
$$=\;\; p^2 \cdot x \cdot Heun\Big(M,\, \frac{M\,+1}{4},\, \frac{1}{2},\, 1,\, \frac{3}{2},\, \frac{1}{2},\, x\Big)^2. \qquad (46)$$

Using the formalism introduced in section (2), we write

$$A_R(x) \;=\; \frac{du(x)}{dx} \;\;=\;\; -\frac{1}{2\,(M\,-x)} \,+\, \frac{2\,x\,-1}{2\,x\,(x-1)} \;\;=\;\; \frac{1}{2} \cdot \frac{3\,x^2\,-2\,(M+1)\,x\,+M}{x \cdot (1-x) \cdot (M\,-x)},$$
$$\text{where:} \qquad u(z) \;\;=\;\; (x \cdot (1-x) \cdot (1\,-x/M))^{1/2}. \qquad (47)$$

¶ Emerging as a symmetry of the complete elliptic integrals of the third kind in the anisotropic Ising model (see [50]).
‡ The general case $\theta \,\rightarrow\, p\,\theta$ is laid out in Appendix C.

The Liouvillian solution of the operator $\Omega \, = \, (D_x + A_R(x)) \cdot D_x$ corresponds to the *incomplete elliptic integral of the first kind* (introducing $u \, = \, sin^2(t)$):

$$F(x, \, m) \, = \, \int_0^x {{dt} \over {(1 \, - \, m \cdot \, sin^2(t))^{1/2}}} \, = \, {{1} \over {2}} \cdot \int_0^x {{du} \over {u^{1/2} \cdot \, (1 \, - u)^{1/2} \cdot \, (1 \, - \, m \cdot \, u)^{1/2}}}.$$

This corresponds to a Heun function, or equivalently to the *inverse Jacobi sine*:

$$Heun\Bigl(M, \, {{M \, +1} \over {4}}, \, {{1} \over {2}}, \, 1, \, {{3} \over {2}}, \, {{1} \over {2}}, \, x\Bigr) \, = \, InverseJacobiSN(x, \, M^{-1/2}). \qquad (48)$$

The Heun solution of Ω reads:

$$Y(x) \, = \, x^{1/2} \cdot \, Heun\Bigl(M, \, {{M \, +1} \over {4}}, \, {{1} \over {2}}, \, 1, \, {{3} \over {2}}, \, {{1} \over {2}}, \, x\Bigr) \, = \, {{2} \over {M^{1/2}}} \cdot \, F\Bigl(x, \, {{1} \over {M}}\Bigr)$$
$$= \, {{1} \over {2}} \cdot \int {{dx} \over {x^{1/2} \cdot \, (1 \, -x)^{1/2} \cdot \, (M \, -x)^{1/2}}}. \qquad (49)$$

The Heun identity (46) amounts to writing a covariance on this Heun function given by:

$$Y\Bigl(R_p(x)\Bigr) \, = \, p \, \cdot \, Y(x). \qquad (50)$$

The adjoint operator $\Omega^* \, = \, D_x \cdot (D_x \, - A(x))$ has the following Heun function solution:

$$F(x) \, = \, x \cdot \, (1 \, -x)^{1/2} \cdot \, \Bigl(1 \, - {{x} \over {M}}\Bigr)^{1/2} \cdot \, Heun\Bigl(M, \, {{M \, +1} \over {4}}, \, {{1} \over {2}}, \, 1, \, {{3} \over {2}}, \, {{1} \over {2}}, \, x\Bigr). \qquad (51)$$

All the rational transformations $R_p(x)$ verify a rank-two condition (24) with $A_R(x)$ given by (47). More generally, the one-parameter series solution of the rank-two condition (24) are, again, *commuting* series:

$$R(a, \, x) \, = \, a \cdot \, x \, \, + a \cdot (a-1) \cdot \, S_a(x) \qquad \hbox{where:} \qquad (52)$$
$$S_a(x) \, = \, - {{(M \, +1)} \over {3 \, M}} \cdot \, x^2 \, + \, {{(2 \, \cdot \, (M^2+1) \cdot \, (a \, -4) \, + \, (13 \, a \, -7) \cdot \, M)} \over {45 \cdot \, M^2}} \cdot \, x^3$$
$$- \, {{(M \, +1)} \over {315 \cdot \, M^3}} \cdot \Bigl((M^2+1) \cdot \, (a \, -4) \cdot \, (a \, -9) \, + \, (29 \, a^2 \, -62 \, a \, -6) \cdot \, M\Bigr) \cdot \, x^4 \, + \, \cdots$$

with $R(a_1, \, R(a_2, \, x)) \, = \, R(a_2, \, R(a_1, \, x)) \, = \, R(a_1 \, a_2, \, x)$. The one-parameter series (52) reduces to the series expansion of the rational functions $R_p(x)$ for $a \, = \, p^2$ *for every integer* p. One thus sees that the rank-two condition (24) with $A_R(x)$ given by (47), *encapsulates an infinite number of commuting rational transformations* $R_p(x)$.

Finally, as far as the Köenig-Siegel linearization [46, 47, 48, 49] of the one-parameter series is concerned, one has $Q(x) \, = \, Y(x)^2$ and:

$$P(x) \, = \, sn\Bigl(x^{1/2}, \, {{1} \over {M^{1/2}}}\Bigr)^2. \qquad (53)$$

One easily verifies that this exact expression (53) in terms of the elliptic sine is solution of the differentially algebraic equation (35) with $A_R(x)$ given by (47).

One can verify (though it is not totally straightforward) that the rational function (45), and more generally the $R_p(x)$, have the decomposition

$$4 \cdot \, {{x \cdot \, (1 \, -x) \cdot \, (1 \, - \, x/M)} \over {(1 \, - \, x^2/M)^2}} \, = \, P(4 \cdot Q(x)), \qquad R_p(x) \, = \, P(p^2 \cdot Q(x)), \qquad (54)$$

with $P(x)$ and $Q(x)$ given respectively by (53) and $Q(x) \, = \, Y(x)^2$.

3.1. $_2F_1$ hypergeometric functions deduced from the Heun example

We know from [51, 52] for example, that selected Heun functions can reduce to pullbacked $_2F_1$ hypergeometric functions. This is also the case for the Heun function (51) in section (3) for selected values of M. For $M = 2$ we have:

$$Heun\Big(M, \frac{M+1}{4}, \frac{1}{2}, 1, \frac{3}{2}, \frac{1}{2}, x\Big) = (1-x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [\frac{5}{4}], \frac{-x^2}{4 \cdot (1-x)}\Big), \quad (55)$$

for $M = -1$:

$$Heun\Big(M, \frac{M+1}{4}, \frac{1}{2}, 1, \frac{3}{2}, \frac{1}{2}, x\Big) = (1-x^2)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [\frac{5}{4}], \frac{-x^2}{1-x^2}\Big), \quad (56)$$

and for $M = 1/2$:

$$Heun\Big(M, \frac{M+1}{4}, \frac{1}{2}, 1, \frac{3}{2}, \frac{1}{2}, x\Big) = (1-2x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [\frac{5}{4}], \frac{-x^2}{1-2x}\Big). \quad (57)$$

Besides, the three previous values of $M = 1/k^2$ such that the Heun function (or the inverse Jacobi sine, InverseJacobiSN in Maple) reduces to pullbacked hypergeometric functions, correspond to a *complex multiplication value* of the j-function [53], namely [13] $j = (12)^3 = 1728$:

$$j = 256 \cdot \frac{(M^2 - M + 1)^3}{M^2 \cdot (M-1)^2}, \qquad j = 1728 \quad \longleftrightarrow \quad M = 2, \frac{1}{2}, -1. \quad (58)$$

The other complex multiplication values (Heegner numbers see [13]) do not seem to correspond to a reduction of the Heun function to pullbacked hypergeometric functions.

Recalling (55), (56), (57), and specifying the Heun identity (46), or (50), for $M = 2$, $M = -1$, and $M = 1/2$ respectively, one gets three identities on the hypergeometric function $_2F_1([1/4, 3/4], [5/4], x)$. These three identities are in fact consequences of the simple identity:

$$Y(x) = x^{1/4} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [\frac{5}{4}], x\Big) = \frac{1}{2} \cdot \mathcal{P}(x)^{1/4} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [\frac{5}{4}], \mathcal{P}(x)\Big), \quad (59)$$

where

$$\mathcal{P}(x) = 16 \cdot \frac{x \cdot (1-x)}{(1+4x-4x^2)^2}, \quad (60)$$

together with the "transmutation" relations

$$\mathcal{P}(p_k(x)) = p_k(\mathcal{P}_k(x)), \qquad k = 1, 2, 3, \quad (61)$$

where the pullbacks $\mathcal{P}(p_k(z))$ are transformation (45) for respectively $M = 2$, $M = -1$, and $M = 1/2$

$$\mathcal{P}_1(x) = 8 \cdot \frac{x \cdot (1-x) \cdot (2-x)}{(x^2-2)^2}, \qquad \mathcal{P}_2(x) = 4 \cdot \frac{x \cdot (1-x^2)}{(1+x^2)^2},$$

$$\mathcal{P}_3(x) = 4 \cdot \frac{x \cdot (1-x) \cdot (1-2x)}{(1-2x^2)^2}, \quad (62)$$

and where $p_k(x)$ are the pullbacks emerging in the ${}_2F_1$ representations (55), (56), (57) of the Heun function:

$$p_1(x) \,\, = \,\, -\frac{1}{4} \cdot \frac{x^2}{1-x}, \qquad p_2(x) \,\, = \,\, -\frac{x^2}{1-x^2}, \qquad p_3(x) \,\, = \,\, -\frac{x^2}{1-2\,x}. \qquad (63)$$

The hypergeometric function $Y(x)$ given by (59), is solution of the order-two linear differential operator $\Omega \, = \, (D_x \, + \, A_R(x)) \cdot D_x$ where

$$A_R(x) \,\, = \,\, \frac{3}{4} \cdot \frac{1\, - 2\,x}{x \cdot (1\, - x)}, \qquad (64)$$

verifies the rank-two condition (24):

$$A_R(\mathcal{P}(x)) \cdot \Big(\frac{d\mathcal{P}(x)}{dx}\Big)^2 \,\, = \,\, A_R(x) \cdot \frac{d\mathcal{P}(x)}{dx} \, + \frac{d^2\mathcal{P}(x)}{dx^2}. \qquad (65)$$

One notes that the hypergeometric functions (16) and (59) are associated with the same $A_R(x)$ given by (64): one can easily show that these two hypergeometric functions are equal. Therefore (59) shares the same rank-two condition (24) with (64), a condition that is also verified for the rational transformation (17) together with the pullback (60), with (17) and (60) *commuting*. The hypergeometric function (59) verifies an identity with the pullback (17), namely $Q(R(x)) \, = \, -4 \cdot Q(x)$ with $R(x)$ given by (17), where $Q(x) \, = \, Y^4(x)$.

Remark: For these selected values of M, one could be surprised that the function $Q(x)$ in the case of the Heun function is such that $Q(x) \, = \, Y(x)^2$, when the $Q(x)$ for the hypergeometric function (59) closely related to this Heun function (see identities (55), (56), (57)) is such that $Q(x) \, = \, Y(x)^4$. This difference comes from the pullbacks (63): the pullbacked hypergeometric functions (55), (56), (57) also correspond to $Q(x) \, = \, Y(x)^2$.

3.2. A comment on the non globally bounded character of the Heun function

Heun functions with generic parameters are generally not reducible to ${}_2F_1$ hypergeometric functions with one or several pullbacks¶.

Unlike ${}_2F_1$ functions the corresponding linear differential Heun operators are generally not globally nilpotent, and the series of Heun functions are not globally bounded. While, for Heun functions, the reductibility to pullbacked ${}_2F_1$ hypergeometric functions, the global nilpotence, and the global boundedness implicate each other in general, this is not true when the corresponding linear differential operator *factors*. Note that the series (51) as well as the series $Q(x) \, = \, Y(x)^N$ for the various hypergeometric functions ((10), (12), (13), ... with $N \, = \, 4, \, 3, \, 6, \, ...$)) are *not globally bounded*†.

In this light, the fact that the series (51) as well as the series $Q(x) \, = \, Y(x)^N$ are not globally bounded, does not seem to be in agreement with the previous modular form emergence and the previous remarkable identities (50), or $Q(R(x)) \, = \, 4 \cdot Q(x)$. The series

$$G(x) \,\, = \,\, Heun\Big(M, \, \frac{M\,+1}{4}, \, \frac{1}{2}, \, 1, \, \frac{3}{2}, \, \frac{1}{2}, \, 4\,M\,x\Big), \qquad (66)$$

¶ This corresponds to the emergence of a modular form represented as a ${}_2F_1$ hypergeometric functions with two possible pullbacks [25]: the series expansion can be recast into a series with *integer coefficients* [54].

† A globally bounded series is a series that can be recast into a series with *integer coefficients* [25].

might not be globally bounded, yet it is "almost globally bounded": the denominator of the coefficients of x^n are of the form $2\,n\,+1$. Therefore one finds that the closely related series

$$\tilde{G}(x) \;=\; 2\,x \cdot \frac{dG(x)}{dx} \;+\; G(x)$$
$$= \; 1 \,+2\,(M+1)\cdot\, x \,+2\,(3\,M^2+2\,M+3)\cdot\, x^2 \,+4\,(M+1)\,(5\,M^2-2\,M+5)\cdot\, x^3 \,+\, 2\,(35\,M^4+20\,M^3+18\,M^2+20\,M+35)\cdot\, x^4 \;+\; \cdots \qquad (67)$$

is actually globally bounded for any rational number value of M: the coefficient of x^n is a polynomial in M *with integer coefficients* of degree n. $\tilde{G}(x)$ is solution of an order-one linear differential operator and is an algebraic function: $\tilde{G}(x) \; = \; (1\,-\,4\,x)^{-1/2}\,\cdot\,(1\,-\,4M\,x)^{-1/2}$. Thus the series (67) is globally bounded for any rational number M.

Remark: To be globally bounded is a property that is preserved by operator homomorphisms: the transformation by a linear differential operator of a globally bounded series is also globally bounded, however, *it is not preserved by integration.*

4. $_2F_1$ hypergeometric function: a higher genus case

The $_2F_1$ hypergeometric examples (10), (12), (13), and (15) are associated with *elliptic or rational* (see (15)) *curves.* It is tempting to imagine the rank-two conditions (24) to be *only* associated with hypergeometric functions connected to *elliptic curves*, and with pullbacks given by *rational functions*†. This is not the case though, as we shall see in the next *genus-two* hypergeometric example with *algebraic* function pullbacks.

Let us consider the hypergeometric function

$$Y(x) \;=\; x^{1/6}\cdot{}_2F_1\Big([\frac{1}{6},\,\frac{1}{3}],\,[\frac{7}{6}],\,x\Big) \;=\; \frac{1}{6}\cdot\int^x (1\,-t)^{-1/3}\cdot t^{-5/6}\cdot dt, \qquad (68)$$

solution of the (factorized) order-two operator $\Omega \,=\, (D_x \,+\, A_R(x))\cdot D_x$ where:

$$A_R(x) \;=\; \frac{1}{6}\,\frac{7\,x\,-5}{x\cdot\,(x\,-1)} \;=\; \frac{u'(x)}{u(x)} \qquad \text{where:} \qquad u(x) \;=\; (1\,-x)^{1/3}\cdot\,x^{5/6}, \qquad (69)$$

and one gets $6\cdot\,Y'(x) \;=\; 1/u(x)$. Introducing $u \;=\; 6\cdot\,Y'(x)$, one can canonically associate to (69) the algebraic curve

$$u^6 \;-\,(1\,-x)^2\cdot\,x^5 \;=\; 0, \qquad (70)$$

which is a *genus-two algebraic curve.* We are seeking an identity on this hypergeometric function (68) of the form:

$$\mathcal{A}(x)\cdot{}_2F_1\Big([\frac{1}{6},\,\frac{1}{3}],\,[\frac{7}{6}],\,x\Big) \;=\; {}_2F_1\Big([\frac{1}{6},\,\frac{1}{3}],\,[\frac{7}{6}],\,y(x)\Big). \qquad (71)$$

Introducing the order-two linear differential operators annihilating respectively the LHS and RHS of (71), the identification of the wronskians of these two operators gives the algebraic function $\mathcal{A}(x)$ in terms of the pullback $y(x)$:

$$\mathcal{A}(x) \;=\; \Big(\frac{-27\cdot\,x}{y(x)}\Big)^{1/6}. \qquad (72)$$

† Casale showed in [36] that the only *rational* functions from $\mathbb{P}_1$ to $\mathbb{P}_1$ with a non-trivial $\mathcal{D}$-envelope are Chebyshev polynomials and *Lattès transformations*. Lattès transformations are rational transformations associated with elliptic curves (see for instance [55]).

The pullback $y(x)$ must be some symmetry (isogeny) of the genus-two algebraic curve (70). At first sight, this *seems to exclude rational function* pullbacks similar to the ones previously introduced. In fact, remarkably, there exists a simple identity on this (higher genus) hypergeometric function:

$$ {}_2F_1\Bigl([\frac{1}{6}, \frac{1}{3}], [\frac{7}{6}], \ -27 \cdot \frac{v \cdot (1-v) \cdot (1+v)^4}{(1+3v) \cdot (1-3v)^4}\Bigr)^6 $$
$$ = \frac{(1+3v)^2 \cdot (1-3v)^4}{(1-v)^2 \cdot (1+v)^4} \cdot {}_2F_1\Bigl([\frac{1}{6}, \frac{1}{3}], [\frac{7}{6}], \ \frac{v \cdot (1+3v)}{1-v}\Bigr)^6. \qquad (73) $$

The two pullbacks in this remarkable identity (73) yield the simple rational parametrization

$$ x \; = \; \frac{v \cdot (1+3v)}{1-v}, \qquad\qquad y \; = \; -27 \frac{v \cdot (1-v) \cdot (1+v)^4}{(1+3v) \cdot (1-3v)^4}, \qquad (74) $$

which parametrizes the following *genus-zero (i.e. rational) curve*††:

$$ -27 \cdot x \cdot (x-1)^4 \cdot (y^2+1) $$
$$ -\, (x^6 - 12x^5 + 807x^4 + 2504x^3 + 807x^2 - 12x + 1) \cdot y \; = \; 0. \qquad (75) $$

The algebraic function $y = y(x)$, defined by the genus-zero curve (75), is an example of a pullback¶ $y(x)$ occurring in the ${}_2F_1$ hypergeometric identity (71). This (multivalued) algebraic function $y = y(x)$ has the following series expansions:

$$ y_1 \; = \; -27x - 216x^2 - 648x^3 - 1944x^4 - 648x^5 - 27864x^6 + 203256x^7 $$
$$ -\, 2123928x^8 + 21844728x^9 - 233611992x^{10} \; + \; \cdots \qquad (76) $$
$$ y_2 \; = \; -\frac{1}{27x} + \frac{8}{27} - \frac{40x}{27} + \frac{200x^2}{27} - \frac{1192x^3}{27} + \frac{8456x^4}{27} - \frac{68264x^5}{27} $$
$$ + \frac{604360x^6}{27} - \frac{5722664x^7}{27} + \frac{6332872x^8}{3} \; + \; \cdots \qquad (77) $$

Note that the rational curve (75) has the obvious symmetry $y \leftrightarrow 1/y$ (as well as the $x \leftrightarrow 1/x$ symmetry, consequence of the palindromic form of (75)), therefore the series (77) is the reciprocal of (76): $y_2 = 1/y_1$. Clearly x and y are not on the same footing. The composition inverse of the previous series gives the series

$$ \frac{-y}{27} - \frac{8y^2}{729} - \frac{104y^3}{19683} - \frac{1672y^4}{531441} - \frac{30248y^5}{14348907} - \frac{196568y^6}{129140163} \; + \; \cdots \qquad (78) $$
$$ \frac{-27}{y} + 8 + \frac{40y}{27} + \frac{520y^2}{729} + \frac{8552y^3}{19683} + \frac{158344y^4}{531441} + \frac{3151144y^5}{14348907} \; + \; \cdots, \qquad (79) $$

the second series being the reciprocal of the first one‡.

Furthermore the two series (76) and (77) verify† the rank-two condition (24) with $A_R(x)$ given by (69):

$$ A_R(y(x)) \cdot \Bigl(\frac{dy(x)}{dx}\Bigr)^2 \; = \; A_R(x) \cdot \frac{dy(x)}{dx} + \frac{d^2y(x)}{dx^2}. \qquad (80) $$

†† One should not confuse these two algebraic curves: the genus-two curve (70) is associated with integrant of the hypergeometric integral (68), when the rational curve (75) is associated with the pullback in the hypergeometric identity (71).

¶ This is a consequence of identity (73).

‡ Note that the rational curve (75) provides additional Puiseux series.

† These two series are related by $y \leftrightarrow 1/y$. Note that $y \leftrightarrow 1/y$ is not a symmetry of (80) in general.

Do note that the series, corresponding to the composition inverse of these two series (76) and (77) (namely (78) and (79) where one changes y into x), also verify the rank-two condition (24) with $A_R(x)$ given by (69). For example, introducing $Q(x) \, = \, Y(x)^6$, one finds that $Q(y(x)) \, = \, -27 \cdot Q(x)$ for $y(x)$ the algebraic function corresponding to series (76). The composition inverse of series‡ (76) gives the (reversed) result: $Q(x) \, = \, -27 \cdot Q(y(x))$.

Remark: The rank-two condition (80) with $A_R(x)$ given by (69) has a one-parameter family of *commuting* solution series:

$$R(a, x) \, = \, a \cdot x \, + a \cdot (a-1) \cdot S_a(x) \qquad \text{where:} \tag{81}$$

$$S_a(x) \, = \, -\frac{2}{7} \cdot x^2 \, + \frac{17\,a\, -87}{637} \cdot x^3 \, + \frac{2 \cdot (113\,a^2 \, -856\,a \, + 3438)}{84721} \cdot x^4$$
$$- \frac{3674\,a^3 + 121194\,a^2 - 552261\,a + 2095059}{38548055} \cdot x^5 \, + \, \cdots \tag{82}$$

with $R(a_1, \, R(a_2, \, x)) \, = \, R(a_2, \, R(a_1, \, x)) \, = \, R(a_1\, a_2, \, x)$, where (81) reduces to the algebraic series (76) and (78) for $a \, = \, -27$ and $a \, = \, -1/27$ respectively. Consequently the occurrence of a *higher genus* curve like (70) is not an obstruction to the existence of a family of one-parameter *abelian* series.

5. Schwarzian condition on an algebraic transformation: ${}_2F_1$ representation of a modular form

The typical situation emerging in physics with *modular forms* [23, 24] is that some "selected" hypergeometric function ${}_2F_1([\alpha, \, \beta], \, [\gamma], \, x)$ verifies an identity with *two different pullbacks*¶ related by an *algebraic* curve, the modular equation curve $M(p_1(x), \, p_2(x)) \, = \, 0$:

$${}_2F_1\Big([\alpha, \, \beta], \, [\gamma], \, p_1(x)\Big) \, = \, \tilde{A}(x) \cdot {}_2F_1\Big([\alpha, \, \beta], \, [\gamma], \, p_2(x)\Big), \tag{83}$$

where $\tilde{A}(x)$ is an algebraic function.

This representation of *modular forms* in terms of hypergeometric functions with *many pullbacks*, is well described in Maier's papers [56, 57]. It is different from the "mainstream" mathematical definition of modular forms as (complex) analytic functions on the upper half-plane satisfying functional equations with respect to the group action of the modular group. *However, this hypergeometric representation is the one we do need in physics* [23, 24]. The reason why this hypergeometric function representation of modular forms exists is a consequence of a not very well-known equality between the Eisenstein [58] series E_4 (of weight four under the modular group), and a hypergeometric function of the (weight zero) j-invariant:

$$E_4 \, = \, 1 \, + \, 240 \sum_{n=0}^{\infty} n^3 \cdot \frac{q^n}{1-q^n} \, = \, {}_2F_1\Big([\frac{1}{12}, \, \frac{5}{12}], \, [1], \, \frac{1728}{j}\Big)^4. \tag{84}$$

One can rewrite a remarkable hypergeometric identity like (83) in the form

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\alpha, \, \beta], \, [\gamma], \, x\Big) \, = \, {}_2F_1\Big([\alpha, \, \beta], \, [\gamma], \, y(x)\Big), \tag{85}$$

‡ Namely series (78) where one changes y into x.
¶ The modular forms occuring in phyics often correspond to cases where the two different pulbacks $p_1(x)$ and $p_2(x)$ are rational functions, but they can also be algebraic functions [25, 30].

where $\mathcal{A}(x)$ is an algebraic function and where $y(x)$ is an algebraic function corresponding to the previous modular curve $M(x, y(x)) = 0$.

The Gauss hypergeometric function ${}_2F_1([\alpha, \beta], [\gamma], x)$ is solution of the second order linear differential operator†:

$$\Omega \, = \, D_x^2 \, + \, A(x) \cdot D_x \, + \, B(x), \qquad \hbox{where:}$$
$$A(x) \, = \, \frac{(\alpha+\beta+1) \cdot x \, - \gamma}{x \cdot (x \, - 1)} \, = \, \frac{u'(x)}{u(x)}, \qquad B(x) \, = \, \frac{\alpha \, \beta}{x \cdot (x \, - 1)}. \qquad (86)$$

We would like now, to identify the two order-two linear differential operators of the LHS and RHS of identity (83). A straightforward calculation enables us to find the algebraic function $\mathcal{A}(x)$ in terms of the algebraic function pullback $y(x)$ in (85):

$$\mathcal{A}(x) \, = \, \Bigl(\frac{u(x)}{u(y(x))} \cdot \frac{dy(x)}{dx} \Bigr)^{1/2}. \qquad (87)$$

Expression (87) for $\mathcal{A}(x)$ is such that the two order-two linear differential operators (of a similar form as (86)) have the same D_x coefficient. The identification of these two operators thus corresponds (beyond (87)) to just one condition that can be rewritten (after some algebra ...) in the following Schwarzian form:

$$W(x) \, - W(y(x)) \cdot y'(x)^2 \, + \{y(x), \, x\} \, = \, 0, \qquad (88)$$

or:

$$\frac{W(x)}{y'(x)} \, - W(y(x)) \cdot y'(x) \, + \frac{\{y(x), \, x\}}{y'(x)} \, = \, 0, \qquad (89)$$

where

$$W(x) \, = \, \frac{dA(x)}{dx} \, + \, \frac{A(x)^2}{2} \, - 2 \cdot B(x), \qquad (90)$$

and where $\{y(x), x\}$ denotes the *Schwarzian derivative* [59]:

$$\{y(x), \, x\} \, = \, \frac{y'''(x)}{y'(x)} \, - \, \frac{3}{2} \cdot \Bigl(\frac{y''(x)}{y'(x)}\Bigr)^2 \, = \, \frac{d}{dx}\Bigl(\frac{y''(x)}{y'(x)}\Bigr) \, - \frac{1}{2} \cdot \Bigl(\frac{y''(x)}{y'(x)}\Bigr)^2,$$
$$\hbox{where:} \qquad y'''(x) \, = \, \frac{d^3y(x)}{dx^3}, \quad y''(x) \, = \, \frac{d^2y(x)}{dx^2}, \quad y'(x) \, = \, \frac{dy(x)}{dx}.$$

In the identity (83), characteristic of modular forms, the two pullbacks $p_1(x)$ and $p_2(x)$ are clearly on the *same footing*, while identity (85) breaks this fundamental symmetry, seeing y as a function of x. We can perform the same calculations seeing the variable x as a function of y in (85). Despite the simplicity of condition (89) it is not clear whether x and y are on the same footing in condition (89). This is actually the case, since if one considers x as a function of y, we have the well-known classical result that the Schwarzian derivative of x with respect to y is simply related to (92), the Schwarzian derivative of y with respect to x:

$$\{y(x), \, x\} \, = \, -y'(x)^2 \cdot \{z(y), \, y\}. \qquad (91)$$

In other words, if one introduces the following Schwarzian bracket

$$[y, \, x] \, = \, \frac{\{y(x), \, x\}}{y'(x)} \, = \, \frac{y'''(x)}{y'(x)^2} \, - \, \frac{3}{2} \cdot \frac{y''(x)^2}{y'(x)^3}, \qquad (92)$$

† Note that $A(x)$ is the log-derivative of $u(x) \, = \, x^c \cdot (1 \, - x)^{\alpha+\beta+1-c}$.

it is antisymmetric: $[y, x] = -[x, y]$. With this appropriate notation, x and y *can be seen on the same footing*. With this in mind we can now rewrite condition (89) in a balanced way:

$$2 \cdot W(x) \cdot \frac{dx}{dy} \; + [y, x] \; = \; 2 \cdot W(y) \cdot \frac{dy}{dx} \; + [x, y]. \tag{93}$$

If one denotes by $L(x, y)$ the LHS of (93)

$$L(x, y) \; = \; \frac{2 \cdot W(x) + \{y, x\}}{y'}, \tag{94}$$

the Schwarzian condition (89), or (93), reads $L(x, y) = L(y, x)$.

Being the result of the covariance (85), a Schwarzian identity like (93) *has to be compatible* with the composition of functions. For instance, from (85) one immediately deduces

$$\begin{aligned} {}_2F_1\Big([\alpha, \beta], [\gamma], y(y(x))\Big) \; &= \; \mathcal{A}(y(x)) \cdot {}_2F_1\Big([\alpha, \beta], [\gamma], y(x)\Big) \\ &= \; \mathcal{A}(y(x)) \cdot \mathcal{A}(x) \cdot {}_2F_1\Big([\alpha, \beta], [\gamma], x\Big). \end{aligned} \tag{95}$$

One thus expects condition (89) to be compatible with the composition of function (similarly to the previous compatibility of the rank-two condition (24) with the iteration of $x \rightarrow R(x)$): this is actually the case. Recalling the (well-known) chain rule for the Schwarzian derivative of the composition of functions

$$\{z(y(x)), x\} \; = \; \{z(y), y\} \cdot y'(x)^2 \; + \; \{y(x), x\}, \tag{96}$$

it is straightforward to show directly (without referring to the covariance (85)) that condition (89) is actually compatible with the composition of functions (see Appendix D for a demonstration).

The Schwarzian derivative is the perfect tool [59] to describe the *composition of functions* and the *reversibility* of an iteration (the previously mentioned $x \longleftrightarrow y$ symmetry): it is not a surprise to see the emergence of a Schwarzian derivative in the description of the modular forms [60, 61, 62] corresponding to identities like (85). We are going to see, for a given (selected ...) hypergeometric function ${}_2F_1([\alpha, \beta], [\gamma], x)$, that the condition (89) "encapsulates" all the isogenies corresponding to all the modular equations associated to transformations on the ratio of periods $\tau \rightarrow N \cdot \tau$ (resp. $\tau \rightarrow \tau/N$), for various values of the integer N corresponding to the different modular equations.

5.1. Schwarzian condition and the simplest example of modular forms: a series viewpoint

Let us focus on an example of a modular form that emerged many times in the analysis of n-fold integrals of the square Ising model [8, 9, 13, 25]. Let us recall the simplest example of a modular form and of a modular equation curve

$$\mathcal{A}(x) \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], x\Big) \; = \; {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], y\Big), \tag{97}$$

where $\mathcal{A}(x)$ is an algebraic function and where $y = y(x)$ is an algebraic function corresponding to the modular equation (4). The algebraic function $y = y(x)$ is a multivalued function, but we can single out the series expansion†:

$$y \;=\; {{1} \over {1728}} \cdot x^2 \;+{{31} \over {62208}} \cdot x^3 \;+{{1337} \over {3359232}} \cdot x^4 \;+{{349115} \over {1088391168}} \cdot x^5 \;+{{20662501} \over {78364164096}} \cdot x^6 \;+{{1870139801} \over {8463329722368}} \cdot x^7 \;+\;\cdots \qquad (98)$$

One verifies easily that the Schwarzian condition (89) is verified with:

$$W(x) \;=\; -{{32\,x^2-41\,x+36} \over {72 \cdot\, x^2 \cdot\, (x\,-1)^2}},\; A(x) \;=\; {{3 \cdot\, x\; -2} \over {2\,x \cdot\, (x\,-1)}},\; B(x) \;=\; {{5} \over {144\,x \cdot\, (x\,-1)}}. \qquad (99)$$

5.1.1. *Other algebraic transformations: other modular equations*

The *modular equations* $\mathcal{M}_N(x,\,y) \;=\; 0$, corresponding to the transformation $\tau \;\rightarrow\; N \cdot\, \tau$, or $\tau \;\rightarrow\; \tau/N$, define algebraic transformations (isogenies) $x \;\rightarrow\; y$ for the identity (97) with $\mathcal{A}(x)$ given by an algebraic function.

Let us consider another important modular equation. The modular equation of order three corresponding to $\tau \rightarrow 3 \cdot\, \tau$, or $\tau \rightarrow \tau/3$, reads¶:

$$k^4 \;+12\,k^3\lambda \;+6\,k^2\lambda^2 \;+12\,k\lambda^3 \;+\lambda^4 \;\;-16\,k^3\lambda^3\,-16\,k\lambda \;\;=\;\; 0. \qquad (100)$$

Recalling that

$$x \;=\; {{27\,k^4 \cdot\, (1\,-k^2)^2} \over {4\,(k^4-k^2+1)^3}} \;=\; {{1728} \over {j(k)}}, \qquad y \;=\; {{27\,\lambda^4\,\left(1-\lambda^2\right)^2} \over {4\,(\lambda^4-\lambda^2+1)^3}} \;=\; {{1728} \over {j(\lambda)}}, \qquad (101)$$

the modular equation (100) becomes the modular curve:

$$262144000000000 \cdot\, x^3y^3 \cdot\, (x+y) \;+4096000000 \cdot\, x^2y^2 \cdot\, (27\,x^2-45946\,xy+27\,y^2) \;+15552000 \cdot\, xy \cdot\, (x+y) \cdot\, (x^2+241433\,xy+y^2) \;+729\,x^4 \;-779997924\,x^3y \;+1886592284694\,x^2y^2 \;-779997924\,xy^3 \;+729\,y^4 \;+2811677184 \cdot\, xy \cdot\, (x+y) \;-2176782336 \cdot\, x\,y \;\;=\;\; 0. \qquad (102)$$

which gives the series expansion:

$$y \;=\; {{x^3} \over {2985984}} \;+{{31\,x^4} \over {71663616}} \;+{{36221\,x^5} \over {82556485632}} \;+{{29537101\,x^6} \over {71328803586048}} \;+\;\cdots \qquad (103)$$

If one denotes by $\mathcal{M}_2(x,\,y)$ the LHS of the modular equation (102), the polynomial with integer coefficients $\mathcal{M}_4(x,\,y)$, corresponding to the transformation $\tau \rightarrow 4 \cdot\, \tau$, or $\tau \rightarrow \tau/4$, is straightforwardly obtained by calculating the resultant of $\mathcal{M}_2(x,\,z)$ and $\mathcal{M}_2(z,\,y)$ in z: this resultant factorizes in the form♯ $(x-y)^2 \cdot\, \mathcal{M}_4(x,\,y)$. The modular equation $\mathcal{M}_4(x,\,y) \;=\; 0$ defines several algebraic series corresponding to the different branches† of the (multivalued) algebraic function transformation $x \;\rightarrow\; y$. We find Puiseux series and two analytic series at $x \,=\, 0$ given by

$$y \;=\; {{x^4} \over {5159780352}} \;+{{31\,x^5} \over {92876046336}} \;+{{43909\,x^6} \over {106993205379072}} \;+\;\cdots \qquad (104)$$

† This series (98) has a radius of convergence 1, even if the discriminant of the modular equation (4) which vanishes at $x \,=\, 1$, vanishes for values inside the unit radius of convergence, for instance at $x \,=\, -64/125$.
¶ Legendre already knew (1824) this order three modular equation in the form $(k\lambda)^{1/2}+(k'\lambda')^{1/2} = 1$, where k and k', and λ, λ' are pairs of complementary moduli $k^2+k'^2 = 1$, $\lambda^2+\lambda'^2 = 1$, and Jacobi derived that modular equation [63, 64].
♯ The exact expression of $\mathcal{M}_4(x,\,y)$ is a bit too large to be given here.
† These series can be obtained using the command "algeqtoseries" in the "gfun" package of Maple.

which is clearly similar to the previous series (98) and (100), but also an *involutive*‡ series of radius of convergence 1, of the (quite unexpected) simple form $-x + \cdots$ namely:

$$y \,=\, -x \,-\, \frac{31\, x^2}{36} \,-\, \frac{961}{1296} \cdot x^3 \,-\, \frac{203713}{314928} \cdot x^4 \,-\, \frac{4318517}{7558272} \cdot x^5 \,-\, \frac{832777775}{1632586752} \cdot x^6$$
$$-\, \frac{729205556393}{1586874322944} \cdot x^7 - \frac{2978790628903}{7140934453248} \cdot x^8 - \frac{43549893886943}{114254951251968} \cdot x^9 \,+ \ldots \quad (105)$$

One easily verifies that all these series (98), (103), (104), (105) (as well as the other Puiseux series) are solutions of the Schwarzian condition (89). The series (98), (103), (104) also *commute* while (104) and (105) *do not* ! This is a consequence of the fact that they correspond to the various commuting isogenies $\tau \rightarrow N \cdot \tau$ (resp. $\tau \rightarrow \tau/N$).

5.1.2. A one-parameter solution series of the Schwarzian condition

Let us first seek solution-series of the Schwarzian condition (89) of the form $e \cdot x + \cdots$ with $W(x)$ given by (99). One finds that the Schwarzian condition (89) has a *one-parameter family* of solution-series as well of the form $e \cdot x + \cdots$ namely¶:

$$y(e, x) \,=\, e \cdot x \,+ e \cdot (e-1) \cdot S_e(x), \qquad \text{where:} \qquad (106)$$
$$S_e(x) \,=\, -\frac{31}{72} \cdot x^2 \,+ \frac{(9907\, e - 20845)}{82944} \cdot x^3$$
$$-\, \frac{(4386286\, e^2 - 20490191\, e + 27274051)}{161243136} \cdot x^4 \,+\, \cdots \qquad (107)$$

The series (106) is a one-parameter family of *commuting series*:

$$y(e, y(\tilde{e}, x)) \,=\, y(\tilde{e}, y(e, x)) \,=\, y(e\, \tilde{e}, x), \qquad (108)$$

and in the $e \rightarrow 1$ limit of the one-parameter family (106), one has:

$$y(e, x) \,=\, x \,+\, \epsilon \cdot F(x) \,+\, \epsilon^2 \cdot G(x) \,+\, \cdots \qquad \text{where:} \qquad (109)$$
$$F(x) \,=\, x \cdot (1-x)^{1/2} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], x\Big)^2, \quad G(x) \,=\, \frac{1}{2} \cdot F(x) \cdot \Big(\frac{dF(x)}{dx} - 1\Big).$$

5.1.3. Other one-parameter solution series of the Schwarzian condition

Clearly the analytic series (98), (103), (104) corresponding to the various isogenies $\tau \rightarrow N \cdot \tau$, are not series of the form $e \cdot x + \cdots$, instead they are solution-series of the Schwarzian condition (89) of the form $a \cdot x^N + \cdots$

In order to generalize the solution-series (98), we will first seek solution-series of the Schwarzian condition (89) of the form $a \cdot x^2 + \cdots$ A straightforward calculation gives a one-parameter family of solution-series of (89) of the form $a \cdot x^2 + \cdots$:

$$y_2 \,=\, a \cdot x^2 + \frac{31 \cdot a x^3}{36} - \frac{a \cdot (5952\, a - 9511)}{13824} \cdot x^4 - \frac{a \cdot (14945472\, a - 11180329)}{20155392} \cdot x^5$$
$$+ \frac{a \cdot \big(88746430464\, a^2 - 677409785856\, a + 338926406215\big)}{743008370688} \cdot x^6 \,+\, \cdots \qquad (110)$$

‡ The series (105) is the *only involutive series* of the form $-x + \cdots$ *which verifies the Schwarzian condition* (89).

¶ The one-parameter series (106) is *completely defined* by the fact that it is a series of the form $e \cdot x + \cdots$ commuting with the algebraic series (105) and the hypergeometric series (109), *without referring to the Schwarzian condition* (89).

which actually reduces to (98) for $a \, = \, 1/1728$. Similarly, one also finds a one-parameter family of solution-series of (89) of the form $b \cdot x^3 \, + \, \cdots$:

$$y_3 \, = \, b \cdot x^3 \, + \frac{31\,b}{24} \cdot x^4 \, + \frac{36221\,b}{27648} \cdot x^5 \, - \frac{b \cdot (23141376\,b - 66458485)}{53747712} \cdot x^6 \, - \frac{b \cdot (183649959936\,b - 187769367601)}{165112971264} \cdot x^7 \, + \, \cdots \tag{111}$$

which reduces to (100) for $b \, = \, 1/2985984 \, = \, 1/1728^2$, and another one-parameter family of solution-series of (89) of the form $c \cdot x^4 \, + \, \cdots$:

$$y_4 \, = \, c \cdot x^4 \, + \frac{31\,c}{18} \cdot x^5 \, + \frac{43909\,c}{20736} \cdot x^6 \, + \frac{46242779\,c}{20155392} \cdot x^7 \, + \frac{c \cdot (869687301215 - 159953190912\,c)}{371504185344} \cdot x^8 \, + \, \cdots \tag{112}$$

which reduces to (104) for $c \, = \, 1/5159780352 \, = \, 1/1728^3$. The series (110), (111), (112), *do not commute*. The composition of the one-parameter series (111) with the one-parameter series (110) gives the series‡:

$$y_2(y_3(x)) \, = \, d \cdot x^6 \, + \frac{31\,d \cdot x^7}{12} \, + \frac{59285\,d}{13824} \cdot x^8 \, + \frac{19676177\,d}{3359232} \cdot x^9 \, + \frac{197722802303\,d}{27518828544} \cdot x^{10} \, + \frac{8173747929317\,d}{990677827584} \cdot x^{11} \, + \, \cdots \tag{113}$$

where $d \, = \, a \cdot b^2$. The composition of the one-parameter series (110) with the one-parameter series (111) gives a similar result where, now, $d \, = \, b \cdot a^3$. These two one-parameter series commute when $a \cdot b^2 \, = \, b \cdot a^3$, i.e. $b \, = \, a^2$, and the modular equation series corresponds to $b \, = \, a^2$ with $a \, = \, 1/1728$.

The composition of the one-parameter series (110) with the one-parameter series (106) gives the series (110) for $a\,e$ and $a\,e^2$ respectively:

$$y_1(e, \, y_2(a, \, x)) \, = \, y_2(a\,e, \, x), \qquad y_2(a, \, y_1(e, \, x)) \, = \, y_2(a\,e^2, \, x). \tag{114}$$

In other words if one introduces the modular equation series $Y_2(x)$ given by (98), corresponding to $y_2(a, \, x)$ for $a \, = \, 1/1728$, the one-parameter series $y_2(a, \, x)$ given by (110), can be obtained as $y_1(1728\,a, \, Y_2(x))$ or as $Y_2(y_1((1728\,a)^{1/2}, \, x))$.

Therefore, all the one-parameter families (110), (111), (112), are nothing but the *isogeny-series* (98), (100), (104) *transformed by the one-parameter series* (106).

5.2. The equivalent of $P(z)$ and $Q(z)$ for the Schwarzian condition: the mirror maps

Let us recall the concept of *mirror map* [23, 24, 65, 66, 67, 68] relating the reciprocal of the j-function and the nome, with the well-known series with integer coefficients:

$$\tilde{X}(q) \, = \, q \, - 744\,q^2 \, + 356652\,q^3 \, - 140361152\,q^4 \, + 49336682190\,q^5 \, - 16114625669088\,q^6 \, + 4999042477430456\,q^7 \, - 1492669384085015040\,q^8 \, + 432762759484818142437\,q^9 \, + \, \cdots \tag{115}$$

‡ If one seeks for the solution series of the Schwarzian condition (89) of the form $d \cdot x^6 \, + \, \cdots$ one recovers the one-parameter family (113).

and† its composition inverse:

$$\tilde{Q}(x) \;=\; x \;+744\,x^2 \;+750420\,x^3 \;+872769632\,x^4 \;+1102652742882\,x^5 \\ +1470561136292880\,x^6 +2037518752496883080\,x^7 +2904264865530359889600\,x^8 \\ +4231393254051181981976079\,x^9 \;+\; \cdots \qquad (116)$$

These series correspond to x being the reciprocal of the j-function: $1/j$. In this paper, as a consequence of the (modular form) hypergeometric identities (97) (see (3), (4) and also (84)), we need x to be identified with the *Hauptmodul* $1728/j$. Consequently we introduce $X(q) \,=\, 1728 \cdot \tilde{X}(q)$ and $Q(x) \,=\, \tilde{Q}(x/1728)$. With these appropriate changes of variables one finds that the series (106) is nothing but $X(e \cdot Q(x))$.

Thus an interpretation of the one-parameter series (106) through the prism of the mirror map, is that the one-parameter series amounts to the multiplication of the nome of elliptic functions [13] by an arbitrary complex number e: $q \longrightarrow e \cdot q$. The isogenies correspond to $q \longrightarrow q^N$ (resp. $q \longrightarrow q^{1/N}$) for an integer N and the one parameter families we have encountered (namely (110), (111)) correspond to the composition of $q \longrightarrow e \cdot q$ and $q \longrightarrow q^N$ (resp. $q \longrightarrow q^{1/N}$), namely $q \longrightarrow e \cdot q^N$ (resp. $q \longrightarrow e \cdot q^{1/N}$).

The series $X(q) \,=\, 1728 \cdot \tilde{X}(q)$ (with $\tilde{X}(q)$ given by (115)) is solution of the Schwarzian equation

$$\{X(q),\, q\} \;-\frac{1}{2\,q^2} \;+\; \frac{1}{72} \cdot \frac{32\,X(q)^2 \,-41\,X(q)\,+36}{X(q)^2 \cdot\, (1\,-X(q))^2} \cdot \Bigl(\frac{dX(q)}{dq}\Bigr)^2 \;=\; 0. \qquad (117)$$

which is nothing but:

$$\{X(q),\, q\} \;-\frac{1}{2\,q^2} \;-\, W(X(q)) \cdot \Bigl(\frac{dX(q)}{dq}\Bigr)^2 \;=\; 0. \qquad (118)$$

The series $Q(x) \,=\, \tilde{Q}(x/1728)$ (with $\tilde{Q}(x)$ given by (116)) is solution of the Schwarzian equation

$$-\,\{Q(x),\, x\} \;-\frac{1}{2 \cdot Q(x)^2} \cdot \Bigl(\frac{dQ(x)}{dx}\Bigr)^2 \,+\, \frac{1}{72} \cdot \Bigl(\frac{32\,x^2 \,-41\,x+36}{x^2 \cdot (1-\,x)^2}\Bigr) \;=\; 0, \qquad (119)$$

equivalently written as:

$$\{Q(x),\, x\} \;+\frac{1}{2 \cdot Q(x)^2} \cdot \Bigl(\frac{dQ(x)}{dx}\Bigr)^2 \,+\, W(x) \;=\; 0. \qquad (120)$$

The two mirror map series (115), (116) thus correspond to differentially algebraic [34, 35] functions, and are solutions of simple Schwarzian equations like in (88).

These differentially algebraic mirror maps transformations $Q(x)$ and $X(q)$ are the well-suited changes of variables such that the transformation $x \longrightarrow y(x)$ verifying the Schwarzian equation (88) become simple transformations, "simple" meaning transformations like $q \longrightarrow S(q) \,=\, e \cdot q^N$ (or $S(q) \,=\, e \cdot q^{1/N}$) in the nome q of elliptic functions [13]. Generalizing the Köenig-Siegel linearization [46, 47, 48, 49], we thus decompose $y(x)$ as $y(x) \,=\, X(S(Q(x)))$.

The Schwarzian conditions (118), (120) are essentially the well-known Schwarzian equation discovered by Jacobi [63, 64] on the j-function (see for instance equation

† In Maple the series (115) can be obtained substituing $L \,=\, EllipticModulus(q^{1/2})^2$, in $1/j \,=\, L^2 \cdot (L-1)^2/(L^2-L+1)^3/256$. See https://oeis.org/A066395 for the series (115) and https://oeis.org/A091406 for the series (116).

(1.26) in [69]). The compatibility of the Schwarzian equations (118), (120) on the mirror maps with the Schwarzian condition (88) on $y(x)$ emerging from a more general Malgrange's pseudo-group perspective [36, 37, 38, 39], is shown in Appendix E. The fact that the *same function* $W(x)$ occurs in the Schwarzian conditions (118), (120) on the mirror maps, and on the Schwarzian condition (88), is crucial for this demonstration and is not a mere coincidence.

5.3. The general case: ${}_2F_1([\alpha,\, \beta],\, [\gamma], x)$ *hypergeometric function.*

5.3.1. The ${}_2F_1([1/6, 1/3], [1], x)$ *hypergeometric function.*

We have analyzed in some detail in section (5.1) the modular form example (97). For other values of the $[[\alpha,\, \beta],\, [\gamma]]$ parameters of the ${}_2F_1$ (see (85), (86)) one can easily find series expansions of the solution $y(x)$ of the Schwarzian condition. A set of values like $[[1/2,\, 1/2],\, [1]]$, $[[1/4,\, 1/4],\, [1]]$, $[[1/3,\, 1/3],\, [1]]$, $[[1/3,\, 2/3],\, [1]]$ or $[[1/3,\, 1/6],\, [1]]$ (see for instance [25, 56] and Ramanujan's cubic theory of alternative bases [70]) which are known to yield modular form hypergeometric identities like (97) with algebraic pullbacks $y(x)$ associated with modular equations. For these values of the $[[\alpha,\, \beta],\, [\gamma]]$ parameters one finds a set of one-parameter series totally similar to what is described in section (5.1). The example of the ${}_2F_1([1/6, 1/3], [1], x)$ hypergeometric function is sketched in Appendix F.

5.3.2. The general case: ${}_2F_1([\alpha,\, \beta],\, [\gamma], x)$ *hypergeometric function.*

Let us now consider arbitrary parameters of the Gauss hypergeometric function $[[\alpha,\, \beta],\, [\gamma]]$ that are not in the previous selected set, and are different from the cases given in sections (2), (3.1), and (4) corresponding to the rank-two condition.

A simple calculation shows that one always finds a series of the form $e \cdot x \, + \, \cdots$ (like (106) or (F.1)), solution of the Schwarzian condition, *but it is only for* $\gamma \, = \, 1$ that series of the form $a \cdot x^2 \, + \, \cdots$, $b \cdot x^3 \, + \, \cdots$, etc ... (like (110) or (111)) can be solutions of the Schwarzian condition.

When $\gamma \, = \, 1$ one gets the following series of the form $a \cdot x^2 \, + \, \cdots$ solution of the Schwarzian condition

$$y_2(u,\, x) \, = \, a \cdot x^2 \, - 2\,a \cdot (2\,\alpha\beta \, - \alpha - \beta) \cdot x^3 \, + \, \frac{a}{2} \cdot C_4 \cdot x^4 \, + \, \cdots \qquad \text{with:}$$

$$C_4 \, = \, 2\,(2\,\alpha\beta \, - \alpha - \beta) \cdot a \, + (\alpha\beta \, - 1)(\alpha\beta \, - \alpha - \beta) \, + \, 5\,(2\,\alpha\beta \, - \alpha - \beta)^2, \quad (121)$$

and one also gets the following series of the form $b \cdot x^3 + \cdots$ solution of the Schwarzian condition

$$y_3(v,\, x) \, = \, b \cdot x^3 \, - 3\,b \cdot (2\,\alpha\beta \, - \alpha - \beta) \cdot x^4 \qquad (122)$$
$$+ \, \frac{3\,b}{4} \cdot \Big((\alpha\beta \, - 1) \cdot (\alpha\beta \, - \alpha - \beta) \, + 7\,(2\,\alpha\beta \, - \alpha - \beta)^2 \Big) \cdot x^5 \, + \, \cdots$$

together with the one-parameter family of commuting series of the form $e \cdot x \, + \cdots$

$$y_1(e,\, x) \, = \, e \cdot x \, + e \cdot \, (e \, - 1) \cdot (2\,\alpha\beta \, - \alpha - \beta) \cdot x^2 \, + \, \frac{e \cdot (e-1)}{4} \cdot C_3 \cdot x^3 \, + \, \cdots$$

$$\text{with:} \qquad C_3 \, = \, (\alpha\beta - 1)(\alpha\beta - \alpha - \beta) \cdot (e \, + 1) \, + (2\,\alpha\beta - \alpha - \beta)^2 \cdot (5\,e - 3). \quad (123)$$

Again one has the equalities

$$y_1(e,\, y_2(a,\, x)) \, = \, y_2(a\,e,\, x), \qquad y_2(a,\, y_1(e,\, x)) \, = \, y_2(a\,e^2,\, x),$$
$$y_1(e,\, y_3(b,\, x)) \, = \, y_3(b\,e,\, x), \qquad y_3(b,\, y_1(e,\, x)) \, = \, y_3(b\,e^3,\, x), \qquad (124)$$

and, again, the two series $y_2(a, x)$ and $y_3(b, x)$ commute for $b = a^2$. As far as series analysis is concerned we have *exactly the same structure* (124) as the one previously described (see (5.1) and (5.3.1)) where *modular correspondences* [71] take place. However, it is not clear if such one-parameter series can reduce to algebraic functions for some selected values of the parameter $a, b, \cdots$ In other words, are these series modular correspondences, or are they just "similar" to modular correspondences ? *The question of the reduction of these Schwarzian conditions to modular correspondences remains an open question.*

When $\gamma \neq 1$ the situation is drastically different‡: one does not have solution of the Schwarzian equation of the form $a \cdot x^2 + \cdots$ or $b \cdot x^3 + \cdots$ etc ... One only has a one-parameter family of commuting series:

$$y(e, x) \, = \, e \cdot x \, - e \cdot (e-1) \cdot {{ \gamma^2 \, - (\alpha+\beta+1) \cdot \gamma \, +2 \, \alpha \, \beta } \over { \gamma \cdot (\gamma-2)}} \cdot x^2 \, + \, \cdots \qquad (125)$$

Again, it is not clear to see if such a one-parameter series can reduce to algebraic functions for some selected values of the parameter e.

6. Rank-two condition on the rational transformations as a subcase of the Schwarzian condition

6.1. Preliminary result: factorization of the order-two linear differential operator

When

$$B(x) \, = \, {{C(x)} \over {4}} \cdot (2 \, A(x) \, - C(x)) \, + \, {{1} \over {2}} \cdot {{dC(x)} \over {dx}}, \qquad (126)$$

the second order linear differential operator

$$\Omega \, = \, D_x^2 \, + \, A(x) \cdot D_x \, + \, B(x), \qquad (127)$$

factorizes as follows:

$$\Omega \, = \, \Bigl(D_x \, + \, A(x) \, - {{C(x)} \over {2}}\Bigr) \cdot \Bigl(D_x \, + \, {{C(x)} \over {2}}\Bigr). \qquad (128)$$

Let us assume that $C(x)$ is a log-derivative:

$$C(x) \, = \, 2 \cdot {{d \ln(\rho(x))} \over {dx}}, \qquad (129)$$

one immediately finds that a conjugation of (128) factors as follows:

$$\rho(x) \cdot \Omega \cdot {{1} \over {\rho(x)}} \, = \, \Bigl(D_x \, + \, A(x) \, - C(x)\Bigr) \cdot D_x. \qquad (130)$$

Therefore the $A_R(x)$ in the rank-two condition (24) is not the $A(x)$ in (127) but $A_R(x) \, = \, A(x) \, - C(x)$ where $B(x)$ is of the form (126).

The rank-two condition reads:

$${{d^2 y(x)} \over {dx^2}} \, = \, (A(y(x)) \, - \, C(y(x))) \cdot \Bigl({{dy(x)} \over {dx}}\Bigr)^2 \, - \, (A(x) \, - \, C(x)) \cdot {{dy(x)} \over {dx}}, \qquad (131)$$

to be compared with the Schwarzian condition

$$W(x) \, - W(y(x)) \cdot y'(x)^2 \, + \, \{y(x), \, x\} \, = \, 0, \qquad (132)$$

‡ Recall that *globally bounded* ${}_nF_{n-1}$ series of "weight zero" [80] (no "down" parameter is equal to 1 or to an integrer, i.e. in the case of globally bounded ${}_2F_1$ series, γ is different from an integer), are *algebraic functions*.

where:

$$W(x) \;=\; \frac{dA(x)}{dx} \;+\; \frac{A(x)^2}{2} \;\; - 2 \cdot B(x). \tag{133}$$

Remark: For a general Gauss hypergeometric function ${}_2F_1([\alpha, \beta], [\gamma], x)$, $A(x)$ and $B(x)$ are given by (86). The factorization condition (126) can be satisfied only for selected values of the $[[a, b], [c]]$ parameters†: $\gamma = \alpha + 1$, $\gamma = \beta + 1$, $\gamma = 1$, $\beta = 1$, $\gamma = \beta$, $\gamma = \alpha$, $\alpha = 0$ and $\beta = 0$.

6.2. Condition on the rational transformation as a subcase of the Schwarzian condition

Let us assume that the rank-two condition (131) is satisfied, then we can use it to express the second derivative $y''(x)$ in terms of $y(x)$ and the first derivative the $y'(x)$. One finds that the Schwarzian condition (132) is automatically verified provided $A(x)$, $B(x)$, $C(x)$ are related though the condition (126) which amounts to a factorization condition for the second order linear differential operator (127). The $A_R(x)$ in the rank-two condition:

$$\frac{d^2y(x)}{dx^2} \;=\; A_R(y(x)) \cdot \Bigl(\frac{dy(x)}{dx}\Bigr)^2 \;-\; A_R(x) \cdot \frac{dy(x)}{dx}, \tag{134}$$

is nothing but $A_R(x) \,=\, A(x) \,-\, C(x)$, or conversely $A(x) \,=\, A_R(x) + C(x)$. If one eliminates $A(x)$ and $B(x)$ in (133) from $A(x) \,=\, A_R(x) + C(x)$ and (126), one finds the simple expression for $W(x)$:

$$W(x) \;=\; \frac{dA_R(x)}{dx} \;+\; \frac{A_R(x)^2}{2}. \tag{135}$$

With this expression (135) of $W(x)$ the Schwarzian condition reads:

$$W(x) \;\; - W(y(x)) \cdot y'(x)^2 \;\; + \{y(x),\, x\} \;\;=\;\; 0, \tag{136}$$

One can rewrite the rank-two condition as

$$\frac{y''(x)}{y'(x)} \;=\; A_R(y(x)) \cdot y'(x) \;\; - A_R(x), \tag{137}$$

and, using (136), rewrite the Schwarzian derivative as

$$\begin{aligned}
\{y(x),\, x\} \;&=\; \frac{d}{dx}\Bigl(\frac{y''(x)}{y'(x)}\Bigr) \;-\frac{1}{2} \cdot \Bigl(\frac{y''(x)}{y'(x)}\Bigr)^2 \\
&=\; W(y(x)) \cdot y'(x)^2 \;\; - W(x) \;\; + \;\Delta, \qquad\qquad \text{with:} \\
\Delta \;=\; & \;\; A_R(y(x)) \cdot y''(x) \\
& - A_R(y(x))^2 \cdot y'(x)^2 \; + A_R(x) \cdot A_R(y(x)) \cdot y'(x),
\end{aligned} \tag{138}$$

where Δ is clearly zero if the rank-two condition is fullfilled.

Remark: The Heun function case of section (3) was a case where the rank-two condition was verified with $A_R(x)$ given by (47). One also verifies that the rational transformation (45), and more generally the rational transformations $R_p(x)$ (pullbacks

† For these conditions on the parameters the function $C(x)$ read respectively $C(x) \,=\, 2\,\alpha/x$, $C(x) = 2\,\beta/x$, $C(x) = 2\,(\beta\, x - \gamma + 1)/x/(x-1)$, $C(x) = 2\,(\alpha\, x - \gamma + 1)/x/(x-1)$, $C(x) = 2\,\alpha/(x-1)$, $C(x) = 2\,\gamma\,\beta/x/(x-1)$, $C(x) = 0$, $C(x) = 0$.

on the Heun function, see (46)), are solutions of a Schwarzian equation (136) with $W(x)$ deduced from (135) with $A_R(x)$ given by (47), namely:

$$W(x) \; = \; -\frac{3}{8 \cdot (x-M)^2} \; - \frac{1}{4} \cdot \frac{2\,x\,-1}{(M\,-x) \cdot x \cdot (x-1)} \; - \frac{1}{8} \cdot \frac{4\,x^2 - 4\,x + 3}{x^2 \cdot (x-1)^2}. \qquad (139)$$

In the previous case where the rank-two condition can be seen as a subcase of the Schwarzian condition (136) on $y(x)$, it is tempting to imagine, in a Köenig-Siegel linearization perspective, that the differentially algebraic function $Q(x)$ (see (34)) also verifies a Schwarzian condition similar to the Schwarzian condition (120) on $Q(x)$ now seen as a mirror map and we show in Appendix G that this is actually the case.

7. Schwarzian condition for generalized hypergeometric functions

7.1. Schwarzian condition and $_3F_2$ hypergeometric identities

Generalizing the modular form identity considered in section (1), let us seek a $_3F_2$ hypergeometric identity of the form

$$\mathcal{A}(x) \cdot {}_3F_2\Big([a,\, b,\, c],\, [d,\, e],\, x\Big) \; = \; {}_3F_2\Big([a,\, b,\, c],\, [d,\, e],\, y(x)\Big), \qquad (140)$$

where $\mathcal{A}(x)$ is an algebraic function. Similarly to what has been performed in section (1), we consider the two order-three linear differential operators associated respectively to the LHS and RHS of (140).

A straightforward calculation enables us to find (from the equality of the wronskians of these two operators) the algebraic function $\mathcal{A}(x)$ in terms of the algebraic function pullback $y(x)$ in (140):

$$\mathcal{A}(x) \; = \; \Big(\frac{p(x)^\eta \cdot (1\,-p(x))^\nu}{x^\eta \cdot (1\,-x)^\nu}\Big) \cdot \Big(\frac{dy(x)}{dx}\Big)^{-1},$$

$$\eta \; = \; \frac{d+e+1}{3}, \qquad \qquad \nu \; = \; \frac{a+b+c+2-d-e}{3}, \qquad (141)$$

The identification of the D_x coefficients of these two linear differential operators, gives (beyond (141)) a first condition that can be rewritten in the following Schwarzian form:

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x),\, x\} \quad = \quad 0, \qquad (142)$$

where $W(x)$ reads:

$$W(x) \; = \; \frac{1}{6} \cdot \frac{P_W(x)}{x^2 \cdot (1\,-x)^2}, \qquad \qquad \text{where:} \qquad (143)$$

$$P_W(x) \; = \; (a^2\, + b^2\, + c^2\, - ab\, - \, ac\, - bc\, - 3) \cdot x^2$$
$$+ \, (3\,(ab\, + \, ac\, + \, bc\, + \, de\, + 1) - 2\,(ad\, + \, ae\, + \, bd\, + \, be\, + \, cd\, + \, ce)\, + a + b + c) \cdot x$$
$$+ \, d^2\, + e^2\, - de\, - d\, - e\, - 2. \qquad (144)$$

The identification of the coefficients with no D_x of these two linear differential operators gives a second condition where the fourth derivative of $y(x)$ takes place. The analysis of this set of conditions corresponds to tedious but straightforward differential algebra calculations which are performed in Appendix H.

One finds that all the conditions on the parameters a, b, c, d, e of the $_3F_2$ hypergeometric function associated with $Q(x) \; = \; 0$, correspond to cases where the order-three operator is the symmetric square of a second order operator having $_2F_1$

solutions. In other words this situation correspond to the *Clausen identity*, the ${}_3F_2$ hypergeometric function reducing to the square of a ${}_2F_1$ hypergeometric function:

$$
\begin{aligned}
{}_3F_2\Bigl([2\,a,\, a\,+b,\, 2\,b],\, [a\,+b\,+\frac{1}{2},\, 2\,a\,+2\,b],\, x\Bigr) \\
=\ {}_2F_1\Bigl([a,\, b],\, [a\,+b\,+\frac{1}{2}],\, y(x)\Bigr)^2. \qquad (145)
\end{aligned}
$$

In that Clausen identity case, the Schwarzian condition (142) we found for the ${}_3F_2$ is nothing but the Schwarzian condition on the underlying ${}_2F_1$.

7.1.1. The intriguing ${}_3F_2([1/9, 4/9, 5/9], [1/3, 1], x)$ case

Beyond the trivial transformation $y(x) \,=\, x$ one hopes to find a condition (140) where the pullback $y \,=\, y(x)$ is an algebraic function.

For the intriguing hypergeometric function ${}_3F_2([1/9, 4/9, 5/9], [1/3, 1], x)$, known to be a globally bounded¶ series [25], one does not know if it is the *diagonal of a rational function*, or not. It is natural to apply the previous conditions to see if we could have an identity like (140) generalizing the identities one gets for modular forms. The occurrence of a series with *integer coefficients* is a strong argument for a "modular form interpretation" of this intriguing ${}_3F_2$ hypergeometric function. Therefore, it is tempting to imagine that a remarkable identity like (140) exists for this ${}_3F_2$ hypergeometric function.

The corresponding order-three operator has a differental Galois group that is an extension† of $SL(3,\, \mathbb{C})$. Therefore, this operator cannot be homomorphic to the symmetric square of an order-two operator: *an identity of the Clausen type is thus excluded for this ${}_3F_2$ hypergeometric function*. The previous calculations showing that an identity like (140) exists only when the ${}_3F_2$ hypergeometric function reduces to square of ${}_2F_1$ hypergeometric functions discards an identity like (140) for ${}_3F_2([1/9, 4/9, 5/9], [1/3, 1], x)$. This is easily seen: for this hypergeometric function the "invariant" $\mathcal{I}(x) \,=\, \mathcal{I}y(x)$ (see Appendix H), and the rational function $W(x)$ in the Schwarzian condition read respectively

$$
\mathcal{I}(x) \,=\, \frac{p_8^3}{(140\,x^3 + 81\,x^2 + 2403\,x - 864)^8}, \quad W(x) \,=\, -\frac{230\,x^2 - 261\,x + 207}{486\,x^2\,(x-1)^2}, \qquad (146)
$$

where:

$$
\begin{aligned}
p_8 \,=\, & 254800\,x^8 \,+ 7247520\,x^7 \,+ 223006266\,x^6 \,- 533339127\,x^5 \,- 62800191\,x^4 \\
& + 1082145339\,x^3 \,- 244855791\,x^2 \,- 290993472\,x \,+ 26873856. \qquad (147)
\end{aligned}
$$

Reinjecting the invariance condition $\mathcal{I}(x) \,=\, \mathcal{I}y(x)$ with (146) in the Schwarzian condition (142), one finds that there is no (algebraic) solution $y(x)$ except the trivial solution $y(x) \,=\, x$.

7.2. Schwarzian condition and other generalized hypergeometric functions

In Appendix I we seek an identity of the form (140) but where the ${}_3F_2$ hypergeometric function is replaced by a ${}_4F_3$ hypergeometric function known to correspond to a

¶ The series ${}_3F_2([1/9, 4/9, 5/9], [1/3, 1],\, 3^5\,x)$ is a series with *integer coefficients* [25].
† See the Boucher-Weil criterion [72]. The symmetric square and exterior square of a normalized order-three operator has no rational solutions. One sees also clearly that this order-three operator is not homomorphic to its adjoint.

Calabi-Yau ODE [23, 24], or a hypergeometric function with *irregular* singularities namely a simple $_2F_2$ hypergeometric function. One finds, unfortunately, that the only solution, for these two examples sketched respectively in Appendix I.1 and Appendix I.2, is the trivial solution $y(x) = x$. Keeping in mind the non trivial results previously obtained on a Heun function, or on a $_2F_1$ hypergeometric function associated with a higher genus curve, these two negative results should rather be seen as an incentive to find more non trivial examples of these extremely rich and deep Schwarzian equations.

8. Conclusion

In this paper we focus essentially on identities relating the same hypergeometric function with two different algebraic pullback transformations related by modular equations. This corresponds to the modular forms that emerged so many times in physics [23, 24, 25]: these algebraic transformations can be seen as simple illustrations of exact representations of the renormalization group [7]. Malgrange's pseudo-group approach aims at generalizing differential Galois theory to non-linear differential equations. In his analysis of Malgrange's pseudo-group Casale found two non-linear differential equations (8) and (9) yet these two conditions were presented separately with no explicit link. In a previous paper [7], where we gave simple examples of exact representations of the renormalization group, associated with selected linear differential operators covariant by rational pullbacks, we found simple exact examples of Casale's condition (8). Building on this work we revisited these previous examples and provided non-trivial new examples associated with a Heun function and a $_2F_1$ hypergeometric function associated with higher genus curves. Then we instantiated, for the first time, Casale's second condition (9) with the examples given in section (5). Furthermore we found that Casale's condition (8) can be seen as a subcase of the Schwarzian condition (9), corresponding to a factorization of a linear differential operator Ω. Seemingly, this Schwarzian condition (9) is seen to "encapsulate" in one differentially algebraic (Schwarzian) equation, all the *modular forms* and *modular equations* of the theory of elliptic curves. The Schwarzian condition (9) can thus be seen as some quite fascinating "pandora box", which encapsulates an infinite number of highly remarkable modular equations, and a whole "universe" of *Belyi-maps*‡. Furthermore we found, only when $\gamma = 1$, that one-parameter series starting with quadratic, cubic, or higher order terms satisfy the rank-three condition. The question of a modular correspondence interpretation of these series is an open question.

Recalling the two previous higher-genus and Heun examples, it is important to underline that these conditions (8) and (9) are actually richer than just elliptic curves, and go beyond "simple" restriction to $_2F_1$ hypergeometric functions.

This paper provides a simple and pedagogical illustration of such exact non-linear symmmetries in physics (exact representations of the renormalization group transformations like the Landen transformation for the square Ising model, ...) and is a strong incentive to discover more differentially algebraic equations involving fundamental symmetries, developping more differentially algebraic analysis in physics [34, 35], beyond obvious candidates like the full susceptibility of the square-lattice Ising model [35, 77].

‡ Belyi-maps [52, 73, 74, 75, 76] are central to Grothendieck's program of "dessins d'enfants".

Acknowledgments: We would like to thank S. Boukraa, G. Casale, S. Hassani, E. Paul, C. Penson and J-A. Weil for very fruitful discussions. This work has been performed without any ERC, ANR, PES or MAE financial support.

Appendix A. $_2F_1$ hypergeometric example: $N \, = \, 3$

Recalling Vidunas paper [45] one introduces the following hypergeometric function:

$$Y(x) \; = \; x^{1/3} \cdot {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [\frac{4}{3}], \, x\Big), \tag{A.1}$$

for which one has the following exact expressions for $A_R(x)$, $u(x)$ and $R(x)$:

$$A_R(x) \; = \; \frac{2}{3} \cdot \frac{2\,x\,-1}{x \cdot (x\,-1)} \; = \; \frac{u'(x)}{u(x)}, \qquad \text{where:} \qquad u(x) \; = \; x^{2/3} \cdot (1-x)^{2/3},$$

$$R(x) \; = \; \frac{x \cdot (x\,-2)^3}{(1\,-2\,x)^3}. \tag{A.2}$$

One verifies that $Q(x) \, = \, Y(x)^3$:

$$\frac{dQ(x)}{dx}/Q(x) \; = \; 3 \cdot \frac{dY(x)}{dx}/Y(x) \; = \; \frac{1}{F(x)}, \quad \text{where:} \quad F(x) \; = \; u(x) \cdot Y(x). \tag{A.3}$$

One has the identity:

$$Q(R(x)) \; = \; -8 \cdot Q(x) \; = \; -8 \cdot x \, \cdot {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [\frac{4}{3}], \, x\Big)^3$$

$$= \; \frac{x \cdot (x\,-2)^3}{(1\,-2\,z)^3} \cdot {}_2F_1\Big([\frac{1}{3}, \, \frac{2}{3}], \, [\frac{4}{3}], \, \frac{x \cdot (x\,-2)^3}{(1\,-2\,x)^3}\Big)^3.$$

The rational function†:

$$\tilde{R}(x) \; = \; \frac{27\,x \cdot (1\,-x)\,(1\,-x\,+x^2)^3}{(1\,+3\,x-6\,x^2\,+\,x^3)^3}, \tag{A.4}$$

commutes with $R(x)$ given by (A.2). Also note that $R(x)$ given by (A.2) commutes with the two known symmetries of the hypergeometric function, namely $R(x) \, = \, 1\,-\,x$ and $R(x) \, = \, 1/x$. These last two transformations yield the involution $R(x) \, = \, -x/(1\,-x)$ which commutes with the two previous rational transformations (A.2), (A.4), and corresponds to $Q(-x/(1\,-x)) \, = \, Q(x)$.

The composition of $R(x) \, = \, -x/(1\,-x)$ with (A.2) and (A.4) gives respectively:

$$\frac{x \cdot (2\,-x)}{(1\,-x) \cdot (1\,+x)^3}, \qquad \frac{-27\,x \cdot (1\,-x)\,(1\,-x\,+x^2)^3}{(1\,-6\,x+3\,x^2\,+\,x^3)^3}. \tag{A.5}$$

Note that $R(x) \, = \, 1/x$ and $R(x) \, = \, 1-x$ also verify the rank-two condition.

As we can see, the one-parameter familly of solution of

$$\Big(\frac{dR(a,\,x)}{dx}\Big)^2 \cdot A(R(a,\,x)) \; = \; \frac{dR(a,\,x)}{dx} \cdot A(x) \; + \frac{d^2R(a,\,x)}{dx^2}, \tag{A.6}$$

namely the differentially algebraic series

$$R(a,\,x) \; = \; a \cdot x \; - \frac{1}{2}\,a \cdot (a-1) \cdot x^2 \; + \frac{1}{28}\,a \cdot (a-1) \cdot (5\,a-9) \cdot x^3 \tag{A.7}$$

$$- \frac{a \cdot (a-1)\,(3\,a^2-12\,a+13)}{56} \cdot x^4 \; + \; \cdots \; + a \cdot (a-1) \cdot \frac{P_{18}(a)}{D_{20}} \cdot x^{20} \; + \; \cdots$$

† Note a typo in [45]: the $R(x)$ in equation (64) of [45] is $-R(x)$.

corresponds to *movable singularities*. For (an infinite number of) selected values of the parameter a, this series becomes a rational function, for instance (A.2) for $a = -8$, (A.4) for $a = 27$, (A.5) for $a = 8$ and $a = -27$. For a generic parameter a the series is much more complex, it is not globally bounded. For instance, $P_{18}(a)$ in (A.9) is a polynomial with integer coefficients of degree 18 in a, and the denominator $D_{20} = 1277610230161807653119590400$ is an integer that factors in many primes: $D_{20} = 2^{17} \cdot 5^2 \cdot 7^9 \cdot 13^4 \cdot 19^3 \cdot 31 \cdot 37 \cdot 43$.

One verifies easily on this series that the two differentially algebraic series $R(a, x)$ and $R(b, x)$ commute and that

$$R(a,\, R(b,\, x)) \; = \; R(b,\, R(a,\, x)) \; = \; R(a\, b,\, x). \qquad \text{(A.8)}$$

Note that the $a \rightarrow 1$ limit of the one-parameter series (A.9) gives as expected

$$R(1 \, + \, \epsilon \cdot x) \; = \; x \; + \; \epsilon \cdot F(x) \; + \; \cdots \qquad \text{(A.9)}$$

where:

$$F(x) \; = \; x \cdot (1 \, - x)^{2/3} \cdot {}_2F_1\Big([\frac{1}{3},\, \frac{2}{3}],\, [\frac{4}{3}],\, x\Big) \; = \; x \; - \frac{x^2}{2} \; - \frac{x^3}{7} \; + \; \cdots \qquad \text{(A.10)}$$

Appendix B. ${}_2F_1$ hypergeometric functions deduced from Goursat and Darboux identity

Appendix B.1. ${}_2F_1$ hypergeometric functions deduced from the quadratic identity

Using the quadratic identity

$${}_2F_1\Big([\alpha,\, \beta],\, [\frac{\alpha+\beta+1}{2}],\, x\Big) \; = \; {}_2F_1\Big([\frac{\alpha}{2},\, \frac{\beta}{2}],\, [\frac{\alpha+\beta+1}{2}],\, 4\, z\, (1 \, - x)\Big), \qquad \text{(B.1)}$$

one deduces:

$${}_2F_1\Big([\frac{1}{2},\, 1],\, [\frac{5}{4}],\, x\Big) \; = \; (1 \, - \, x/2)^{-\alpha} \cdot {}_2F_1\Big([\frac{1}{4},\, \frac{1}{2}],\, [\frac{5}{4}],\, 4\, z\, (1 \, - x)\Big). \qquad \text{(B.2)}$$

The previously described relations on ${}_2F_1([1/4,\, 1/2], [5/4], x)$, together with the rational function $R(x) = -4\, x/(1 - x)^2$, yields the new identity

$$(1 \, - 2\, x) \cdot {}_2F_1\Big([\frac{1}{2},\, 1],\, [\frac{5}{4}],\, x\Big) \; = \; {}_2F_1\Big([\frac{1}{2},\, 1],\, [\frac{5}{4}],\, -4\, \frac{x \cdot (1 \, - x)}{(1 \, - 2\, x)^2}\Big), \qquad \text{(B.3)}$$

where we have used the relation $R_3(R_1(x)) = R_2(R_3(x))$ with:

$$R_1(x) \; = \; -4\, \frac{x \cdot (1 \, - x)}{(1 \, - 2\, x)^2}, \quad R_2(x) \; = \; \frac{-4\, x}{(1 \, - \, x)^2}, \quad R_3(x) \; = \; 4\, x \cdot (1 \, - x). \qquad \text{(B.4)}$$

Introducing

$$Y(x) \; = \; x^{1/4} \cdot (1 \, - x)^{1/4} \cdot {}_2F_1\Big([\frac{1}{2},\, 1],\, [\frac{5}{4}],\, x\Big), \qquad \text{(B.5)}$$

one sees that it is solution of $\Omega = (D_x + A_R(x)) \cdot D_x$ with:

$$A_R(x) \; = \; \frac{3}{4} \cdot \frac{2\, x - 1}{x\, (x-1)} \; = \; \frac{u'(x)}{u(x)}, \qquad u(x) \; = \; x^{3/4} \cdot (1 \, - x)^{3/4}. \qquad \text{(B.6)}$$

The rank-two condition is verified with $A_R(x)$ given by (B.6) and $R(x)$ given by $R_1(x)$ in (B.11).

Appendix B.2. ${}_2F_1$ hypergeometric functions deduced from the Goursat identity

Using the Goursat identity

$$ {}_2F_1\Bigl([\alpha,\,\beta],\,[2\,\beta],\,x\Bigr) \;=\; (1\,-\,x/2)^{-a}\cdot\,{}_2F_1\Bigl([\frac{\alpha}{2},\,\frac{\alpha+1}{2}],\,[\beta\,+\,\frac{1}{2}],\,\frac{x^2}{(2\,-\,x)^2}\Bigr). \quad \text{(B.7)}$$

for $\alpha \,=\, 1/3,\ \beta \,=\, 2/3$, one gets:

$$ {}_2F_1\Bigl([\frac{1}{3},\,\frac{2}{3}],\,[\frac{4}{3}],\,x\Bigr) \;=\; (1\,-\,x/2)^{-1/3}\cdot\,{}_2F_1\Bigl([\frac{1}{6},\,\frac{2}{3}],\,[\frac{7}{6}],\,\frac{x^2}{(2\,-\,x)^2}\Bigr). \quad \text{(B.8)}$$

Combining this last identity with (A.4) one gets:

$$\begin{aligned} &\frac{x\cdot\,(x\,-2)^3}{(1\,-2\,x)^3}\cdot\,{}_2F_1\Bigl([\frac{1}{3},\,\frac{2}{3}],\,[\frac{4}{3}],\,\frac{x\cdot\,(x\,-2)^3}{(1\,-2\,x)^3}\Bigr)^3 \;=\; -8\cdot\,z\,\cdot\,{}_2F_1\Bigl([\frac{1}{3},\,\frac{2}{3}],\,[\frac{4}{3}],\,x\Bigr)^3 \\ &\quad=\; -8\cdot\,x\,\cdot\,{}_2F_1\Bigl([\frac{1}{3},\,\frac{2}{3}],\,[\frac{4}{3}],\,x\Bigr)^3 \;=\; \frac{16\,x}{x\,-2}\cdot\,{}_2F_1\Bigl([\frac{1}{6},\,\frac{2}{3}],\,[\frac{7}{6}],\,\frac{x^2}{(2\,-\,x)^2}\Bigr)^3 \qquad \text{(B.9)} \\ &\quad=\; \frac{-2\,x\,(x-2)^3}{x^4+10\,x^3-12\,x^2+4\,x-2}\cdot\,{}_2F_1\Bigl([\frac{1}{6},\,\frac{2}{3}],\,[\frac{7}{6}],\,\frac{x^2\,(x-2)^6}{(x^4+10\,x^3-12\,x^2+4\,x-2)^2}\Bigr)^3. \end{aligned}$$

It yields the identity on this new hypergeometric function:

$$\begin{aligned} &{}_2F_1\Bigl([\frac{1}{6},\,\frac{2}{3}],\,[\frac{7}{6}],\,\frac{64\,x}{(1\,+18\,x\,-27\,x^2)^2}\Bigr) \\ &\qquad=\; (1+18\,x-27\,x^2)^{1/3}\cdot\,{}_2F_1\Bigl([\frac{1}{6},\,\frac{2}{3}],\,[\frac{7}{6}],\,x\Bigr). \qquad \text{(B.10)} \end{aligned}$$

We have used the relation $R_3(R_1(x)) \,=\, R_2(R_3(x))$ with:

$$ R_1(x) \;=\; \frac{x\cdot\,(x\,-2)^3}{(1\,-2\,x)^3},\quad R_2(x) \;=\; \frac{64\,x}{(1\,+18\,x\,-27\,x^2)^2},\quad R_3(z) \;=\; \frac{x^2}{(2\,-x)^2}. \quad \text{(B.11)}$$

Appendix C. Miscellaneous rational functions for the covariance of a Heun function

Let us consider the well-known formula for the addition on elliptic sine:

$$ sn(u\,+\,v) \;=\; \frac{sn(u)\,cn(v)\,dn(v)\;+\;sn(v)\,cn(u)\,dn(u)}{1\,-k^2\,sn(u)^2\,sn(v)^2}. \quad \text{(C.1)}$$

Introducing the variables $x \,=\, sn(u)^2,\ y \,=\, sn(v)^2$ and $x \,=\, sn(u\,+\,v)^2$, and $M \,=\, 1/k^2$, the previous addition formula (C.1) for the elliptic sine reads:

$$\begin{aligned} &(M-xy)^2\cdot\,x^2\;\;+2\,M\cdot\,\Bigl(2\,xy\cdot\,(M\,+1)\,-\,(x+y)\cdot\,(xy\,+M)\Bigr)\cdot\,x \\ &\qquad +\,(x-y)^2\cdot\,M^2 \;=\; 0. \qquad \text{(C.2)} \end{aligned}$$

The rational transformation (45) corresponding to $\theta \,\rightarrow\, 2\,\theta$ is obtained by imposing $y \,=\, x$ in (C.2). Imposing in (C.2) y to be equal to (45) one deduces the rational transformation corresponding to $\theta \,\rightarrow\, 3\,\theta$, and one can deduce from the "master" equation (C.2) all the rational transformations corresponding to $\theta \,\rightarrow\, p\,\theta$. When p is a prime number different from $p \,=\, 2$, the corresponding rational transformations have a simple form.

Introducing the square of the elliptic sine $x = sn(\theta, k)^2$, the rational transformations corresponding to $\theta \rightarrow p\,\theta$ give for a given M:

$$R_p(x, M) \,=\, x \cdot \Bigl({{P_p(x, M)} \over {Q_p(x, M)}}\Bigr)^2, \qquad \text{where:} \tag{C.3}$$

$$Q_p(x, M) \,=\, x^{(p^2-1)/2} \cdot M^{(p^2-1)/4} \cdot P_p\Bigl({{1} \over {x}}, {{1} \over {M}}\Bigr),$$

where $P_p(x, M)$ are polynomials in x and M of degree $(p^2-1)/2$ in x and of degree $(p^2-1)/4$ in M. For instance, $P_3(x, M)$ reads:

$$P_3(x, M) \,=\, x^4 \,-6\,M \cdot x^2 \,+4 \cdot M \cdot (M+1) \cdot x \,-3\,M^2. \tag{C.4}$$

The polynomial $P_p(x, M)$ reads for $p = 5$:

$$\begin{aligned} P_5(z, M) \,=\, & x^{12} - 50\,M\,x^{10} + 140\,M\,(M+1) \cdot x^9 - 5\,M\,(32\,M^2 + 89\,M + 32) \cdot x^8 \\ & + 16\,M\,(M+1)\,(4\,M^2 + 31\,M + 4) \cdot x^7 \,- 60\,M^2\,(4\,M^2 + 13\,M + 4) \cdot x^6 \\ & + 360\,M^3\,(M+1) \cdot x^5 \,- 105\,M^4 \cdot x^4 \,- 80\,M^4\,(M+1) \cdot x^3 \\ & + 2\,M^4\,(8\,M^2 + 47\,M + 8) \cdot x^2 \,- 20\,M^5\,(M+1) \cdot x \,+ 5\,M^6, \end{aligned} \tag{C.5}$$

It is straightforward to calculate the next $P_p(z, M)$ for $p = 7, 11, 13, \cdots$, but the expressions become quickly too large to be given here.

As expected, the two rational functions (C.3) *commute* for different primes p. The series expansion of these rational transformations read:

$$R_p(x) \,=\, p^2 \cdot x \,-\, {{p^2 \cdot (p^2-1)} \over {3}} \cdot {{M+1} \over {M}} \cdot x^2 \,+\, \cdots \tag{C.6}$$

When p is not a prime the rational functions $R_p(x)$ corresponding to $\theta \rightarrow p\,\theta$, are no longer of the form (C.3) but they still have the series expansion (C.6).

We have the following identity on a Heun function where $R_p(x)$ are the previous rational functions (C.3):

$$\begin{aligned} & R_p(x) \cdot Heun\Bigl(M, {{M+1} \over {4}}, {{1} \over {2}}, 1, {{3} \over {2}}, {{1} \over {2}}, R(x)\Bigr)^2 \\ & \quad = \, p^2 \cdot x \cdot Heun\Bigl(M, {{M+1} \over {4}}, {{1} \over {2}}, 1, {{3} \over {2}}, {{1} \over {2}}, x\Bigr)^2. \end{aligned} \tag{C.7}$$

Note that the Heun identity (C.7) is valid even when the integer p is no longer a prime, $R_p(x)$ being a rational function representation of $\theta \rightarrow p \cdot \theta$, and that all these (commuting) rational transformations are solutions of the rank-two condition.

Appendix D. The Schwarzian conditions are compatible with the composition of functions

We want to have

$$W(x) \,-\, W(z(y(x))) \cdot \Bigl({{dz(y(x))} \over {dx}}\Bigr)^2 \,+\, \{z(y(x)), x\} \,=\, 0, \tag{D.1}$$

which reads using the derivative of composition of function and the previous chain rule (96):

$$\begin{aligned} & W(x) \,-\, W(z(y(x))) \cdot \Bigl({{dz(y)} \over {dy}}\Bigr)^2 \cdot y'(x)^2 \\ & \qquad + \{z(y), y\} \cdot y'(x)^2 \,+\, \{y(x), x\} \,=\, 0, \end{aligned} \tag{D.2}$$

from

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x),\, x\} \quad = \quad 0, \qquad \text{(D.3)}$$

and

$$W(y) \quad - W(z(y)) \cdot z'(y)^2 \quad + \{z(y),\, y\} \quad = \quad 0. \qquad \text{(D.4)}$$

Let us multiply the previous relation (D.4) by $y'(x)^2$ one gets:

$$W(y) \cdot y'(x)^2 \quad - W(z(y)) \cdot z'(y)^2 \cdot y'(x)^2 \quad + \{z(y),\, y\} \cdot y'(x)^2 \quad = \quad 0. \qquad \text{(D.5)}$$

Adding (D.3) to (D.6) one gets:

$$W(x) \quad + W(y) \cdot y'(x)^2 \quad - W(z(y)) \cdot z'(y)^2 \cdot y'(x)^2 \quad + \{z(y),\, y\} \cdot y'(x)^2 \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x),\, x\} \quad = \quad 0. \qquad \text{(D.6)}$$

which gives after simplification nothing but (D.2). Q. E. D.

Appendix E. Compatibility of the three Schwarzian conditions (118), (120) and (88)

The Schwarzian equation on the j-invariant are known to be invariant by the group of modular transformations (see for instance equation (1.26) in [69] or (1.13) in [78]). More remarkably (and less known) the Schwarzian equation (120) on the nome† is *invariant* under the transformations‡ $q \longrightarrow S(q) = e \cdot q^N$. Equation (120) is clearly invariant under the rescaling $Q(x) \to e \cdot Q(x)$, and one can verify easily, using the chain rule for the Schwarzian derivative of a composition, that the sum of the first two terms in the LHS of (120), namely $\{Q(x),\, x\} + Q'(x)^2/Q(x)^2/2$ is actually invariant by $Q(x) \to Q(x)^N$. Therefore we also have the equation:

$$\{S(Q(x)),\, x\} \quad + \frac{1}{2 \cdot S(Q(x))^2} \cdot \Big(\frac{dS(Q(x))}{dx}\Big)^2 \ + W(x) \quad = \quad 0. \qquad \text{(E.1)}$$

Equation (118) yields

$$\{X(S(Q(q))),\, S(Q(x))\} \ - \frac{1}{2\, S(Q(x))^2} \ - W(X(S(Q(q)))) \cdot \Big(\frac{dX(S(Q(x)))}{dS(Q(x))}\Big)^2 \ = \ 0,$$

and thus:

$$\{X(S(Q(x))),\, S(Q(x))\} \cdot \Big(\frac{dS(Q(x))}{dx}\Big)^2 \quad - \frac{1}{2\, S(Q(x))^2} \cdot \Big(\frac{dS(Q(x))}{dx}\Big)^2 \quad - W(X(S(Q(x)))) \cdot \Big(\frac{dX(S(Q(x)))}{dS(Q(x))}\Big)^2 \cdot \Big(\frac{dS(Q(x))}{dx}\Big)^2 \quad = \quad 0. \qquad \text{(E.2)}$$

Using the chain rule for Schwarzian derivative of the composition of functions

$$\{X(S(Q(x))),\, x\} \quad = \quad \{X(S(Q(x))),\, S(Q(x))\} \cdot \Big(\frac{dS(Q(x))}{dx}\Big)^2 \ + \quad \{S(Q(x)),\, x\},$$

we see immediately that the sum of (E.1) and (E.2) gives:

$$W(x) \quad - W(y(x)) \cdot y'(x)^2 \quad + \{y(x),\, x\} \quad = \quad 0, \qquad \text{(E.3)}$$

† In the Schwarzian equation (120) the nome is seen as a function of the Hauptmodul.
‡ See the concept of *replicable functions* [79].

Appendix F. The $_2F_1([1/6, 1/3], [1], x)$ hypergeometric function.

Let us consider the Schwarzian condition in the case of the $_2F_1([1/6, 1/3], [1], x)$ hypergeometric function.

The one-parameter family of commuting series solution of the Schwarzian condition reads:

$$y_1(e, x) \;=\; e \cdot x \;+ e \cdot (e-1) \cdot S_e(x), \qquad \text{where:}$$
$$S_e(x) \;=\; -\frac{7}{18} \cdot x^2 \;+ \frac{(109\, e - 283)}{1296} \cdot x^3 \;+\; \cdots \qquad (F.1)$$

The series of the form $a \cdot x^2 + \cdots$ reads

$$y_2(a, x) \;=\; a \cdot x^2 + \frac{7\, a}{9} \cdot x^3 - a \cdot \frac{84a - 127}{216} \cdot x^4 - a \cdot \frac{47628a - 36049}{78732} \cdot x^5 + \cdots \quad (F.2)$$

and is such that:

$$y_1(e, y_2(a, x)) \;=\; y_2(a\, e, x), \qquad y_2(a, y_1(e, x)) \;=\; y_2(a\, e^2, x). \qquad (F.3)$$

The series of the form $b \cdot x^3 + \cdots$ reads

$$y_3(b, x) \;=\; b \cdot x^3 \;+ \frac{7\, b}{6} \cdot x^4 \;+ \frac{479\, b}{432} \cdot x^5 \;+ b \cdot \frac{81648b - 210031}{209952} \cdot x^6 \;+\; \cdots \quad (F.4)$$

and is such that

$$y_1(e, y_3(b, x)) \;=\; y_3(b\, e, x), \qquad y_3(b, y_1(e, x)) \;=\; y_3(b\, e^3, x). \qquad (F.5)$$

The two series $y_2(a, x)$ and $y_3(b, x)$ commute for $b = a^2$. For $a = 1/108$ the series (F.4) becomes the series expansion

$$y \;=\; \frac{x^2}{108} + \frac{7\, x^3}{972} + \frac{71\, x^4}{13122} + \frac{4451\, x^5}{1062882} + \frac{63997\, x^6}{19131876} + \frac{1417505\, x^7}{516560652} + \cdots \quad (F.6)$$

which corresponds to the modular equation (A.3) in [24]:

$$4\, x^3\, y^3 \;- 12\, x^2\, y^2 \cdot (x+y) \;+ 3\, x\, y \cdot (4\, x^2 - 127\, x\, y + 4\, y^2)$$
$$- 4 \cdot (x+y) \cdot (x^2 + 83\, x\, y + y^2) \;+ 432\, x\, y \;=\; 0, \qquad (F.7)$$

This modular equation has a rational parametrization: it corresponds to the relation between two rational pullbacks in the hypergeometric identity (A.11) in [25]:

$$_2F_1\Bigl([\frac{1}{6}, \frac{1}{3}], [1], \; 108\, v^2 \cdot (1 + 4v)\Bigr)$$
$$=\; (1 \, - 12\, v)^{-1/2} \cdot \, _2F_1\Bigl([\frac{1}{6}, \frac{1}{3}], [1], \; -\frac{108\, v \cdot (1+4v)^2}{(1 \, - 12\, v)^3}\Bigr). \qquad (F.8)$$

Appendix G. The solutions $Q(x)$ of the non-linear conditions (34) seen as solutions of the Schwarzian conditions on the mirror maps

In all the cases recalled in section (2), the differentially algebraic function $Q(x)$ was of the form† $Y(x)^N$. From $Q(x) = Y(x)^N$ or even $Q(x) = \alpha \cdot Y(x)^N$, one can rewrite the Schwarzian derivative on $Q(x)$ with respect to x:

$$\{Q(x), \, x\} \;=\; \{Y(x)^N, \, x\} \;=\; -\frac{N^2 \, - 1}{2} \cdot \Bigl(\frac{Y'(x)}{Y(x)}\Bigr)^2 \;+\; \{Y(x), \, x\}. \qquad (G.1)$$

† The constant N being a positive integer $Q(x)$ was, in fact, holonomic.

Since $Y(x)$ is a solution of the operator Ω, the ratio $Z(x) = Y''(x)/Y'(x)$ (log-derivative of $Y'(x)$) is in fact a rational function, namely $-A_R(x)$. The Schwarzian derivative $\{Y(x),\, x\}$ can also be written as:

$$\{Y(x),\, x\} \,=\, \frac{dZ(x)}{dx} \,-\, \frac{Z(x)^2}{2} \,=\, -\frac{dA_R(x)}{dx} \,-\, \frac{A_R(x)^2}{2} \,=\, -W(x). \qquad (G.2)$$

From $Q(x) \,=\, \alpha \cdot Y(x)^N$ one deduces immediately the relation between the log-derivative of $Q(x)$ and $Y(x)$, namely $Q'(x)/Q(x) \,=\, N \cdot Y'(x)/Y(x)$. Equation (G.2) can be rewritten using $Q'(x)/Q(x) \,=\, N \cdot Y'(x)/Y(x)$ and (G.2), as‡:

$$\{Q(x),\, x\} \,+\, \frac{N^2\,-1}{2\,N^2} \cdot \Bigl(\frac{Q'(x)}{Q(x)}\Bigr)^2 \,+\, W(x) \,=\, 0. \qquad (G.3)$$

For instance, one verifies immediately that $Q(x)$ given by $Q(x) \,=\, Y(x)^N$ and $Y(x)$ given by (10), (12), (13), (15), (16) (which identifies with (59)) and (18) are actually solutions of the Schwarzian condition (G.3) for the corresponding $A_R(x)$ given in (21) for respectively $N = 4,\, 3,\, 6,\, 2,\, 4,\, 6$. Note that the higher-genus case hypergeometric function (68) is *also such that* $Q(x) \,=\, Y(x)^6$ *is solution of the Schwarzian condition* (G.3) with $N \,=\, 6$ and $W(x)$ deduced from $A_R(x)$ given by (69).

One gets immediately the Schwarzian condition for the composition inverse $P(x) \,=\, Q^{-1}(x)$, namely:

$$\{X(q),\, q\} \,-\, \frac{N^2\,-1}{2\,N^2} \cdot \frac{1}{q^2} \,-\, W(X(q)) \cdot \Bigl(\frac{dX(q)}{dq}\Bigr)^2 \,=\, 0. \qquad (G.4)$$

Remark: One verifies straightforwardly for the Heun function example of section (3) that

$$Q(x) \,=\, Y(x)^2 \,=\, x \cdot Heun\Bigl(M,\, \frac{M\,+1}{4},\, \frac{1}{2},\, 1,\, \frac{3}{2},\, \frac{1}{2},\, x\Bigr)^2, \qquad (G.5)$$

is *actually solution of the Schwarzian condition* (G.3) with $N \,=\, 2$, with $W(x)$ given by (139). The composition inverse of the holonomic function $Q(x)$ given by (G.5) is

$$P(x) \,=\, sn\Bigl(x^{1/2},\, \frac{1}{M^{1/2}}\Bigr)^2 \qquad (G.6)$$
$$= \, x \,-\frac{1}{3}\,\frac{M\,+1}{M} \cdot x^2 \,+\, \frac{1}{45}\,\frac{2\,M^2+13\,M+2}{M^2} \cdot x^3 \,+\, \cdots$$

It is solution of (G.4) with $N \,=\, 2$:

$$\{P(x),\, x\} \,-\frac{3}{8} \cdot \frac{1}{x^2} \,-\, W(P(x)) \cdot \Bigl(\frac{dP(x)}{dx}\Bigr)^2 \,=\, 0. \qquad (G.7)$$

Appendix G.1. From the Schwarzian condition (G.3) back to the differential algebraic condition (35) on $Q(x)$

If one compares the Schwarzian condition (G.3) with the differentially algebraic condition (35) on $Q(x)$, one finds that they both have a third derivative $Q'''(x)$ but one condition depends on a constant N, while the other one is "universal": let us try to understand the compatibility between these two conditions. If one eliminates the

‡ One recovers the Schwarzian condition (120) in the $N \,\rightarrow\, \infty$ limit.

third derivative $Q'''(x)$ between these two equations one finds a remarkably factorized condition $E_+ \cdot E_- = 0$ where:

$$E_\pm \, = \, \frac{d\mathcal{F}(x)}{dx} \, - \, A_R(x) \cdot \, \mathcal{F}(x) \, \pm \, \frac{1}{N} \qquad \hbox{with:} \qquad \mathcal{F}(x) \, = \, \frac{Q(x)}{Q'(x)}. \qquad \hbox{(G.8)}$$

Recalling (31) we see that $\mathcal{F}(x)$ is nothing but $F(x)$. The holonomic function $F(x)$ is known to be solution of Ω^*, which can be rewritten, after one integration step, as $F'(x) \, - \, A_R(x) \cdot F(x) \, = \, Cst$, which is actually (G.8). The compatibility of the Schwarzian condition (G.3) with the differentially algebraic condition (35) thus corresponds to $F(x)$ being annihilated by Ω^*.

Appendix H. Reduction of $_3F_2$ identities to $_2F_1$ Schwarzian conditions

Performing the derivative of the Schwarzian condition (142) one can eliminate this fourth derivative of $y(x)$, and then, in a second step, eliminate the third derivative of $y(x)$ between the previous result and the Schwarzian condition (142), and so on. One finally gets the following relation that can be seen as the compatibility condition between the two previous conditions:

$$x^3 \cdot \, (1 \, - x)^3 \cdot \, Q(y(x)) \cdot \Bigl(\frac{dy(x)}{dx} \Bigr)^3 \, \, = \, \, \, y(x)^3 \cdot \, (1 \, - y(x))^3 \cdot \, Q(x), \qquad \hbox{(H.1)}$$

where the polynomial $Q(x)$ reads:

$$Q(x) \, = \, \, -2 \, \cdot \, (b+c-2\,a)\,(a+c-2\,b)\,(a+b-2\,c) \cdot \, x^3 \\ + 3 \cdot \, q_2 \cdot \, x^2 \, \, + 3 \cdot \, q_1 \cdot \, x \, \, -2 \, \cdot \, (1+d-2\,e)\,(d+e-2)\,(2\,d-e-1), \qquad \hbox{(H.2)}$$

where

$$q_2 \, = \, \, 6\,a^2b \, + 6\,a^2c \, - 4\,a^2d \, \, - 4\,a^2e \, \, + 6\,ab^2 \, \, - 18\,abc \, \, - 2\,abd \, - 2\,abe \, \, + 6\,ac^2 \\ - 2\,acd \, \, - 2\,ace \, \, + 6\,ade \, \, + 6\,b^2c \, \, - 4\,b^2d \, \, - 4\,b^2e \, \, + 6\,bc^2 \, \, - 2\,bcd \, \, - 2\,bce \\ + 6\,bde \, \, - 4\,c^2d \, - 4\,c^2e \, \, + 6\,cde \, \, + 2\,a^2 \, \, + ab \, + \, ac \, + 2\,b^2 \, + bc \, + 2\,c^2 \\ - 9\,de \, \, - 3\,a \, - 3\,b \, - 3\,c \, + 6\,d \, + 6\,e \, \, - 3, \qquad \hbox{(H.3)}$$

$$q_1 \, = \, 18\,abc \, - 6\,abd \, - 6\,abe \, - 6\,acd \, - 6\,ace \, + 4\,ad^2 \, + 2\,ade \, + 4\,ae^2 \, - 6\,bcd \, - 6\,bce \\ + 4\,bd^2 \, + 2\,bde + 4\,be^2 + 4\,cd^2 + 2\,cde + 4\,ce^2 \, - 6\,d^2e \, - 6\,de^2 + 3\,ab + 3\,ac \, - 4\,ad \\ - 4\,ae \, + 3\,bc \, - 4\,bd \, - 4\,be - 4\,cd - 4\,ce + 21\,de \, + a + b + c \, - 6\,d - 6\,e + 3. \quad \hbox{(H.4)}$$

The condition (H.1) can be seen as a an equality on a one-form and the same one-form where x has been changed into:

$$Q(x)^{1/3} \cdot \, \frac{dx}{x \cdot \, (1 \, - x)} \, \, = \, \, \frac{dx}{u} \, \, = \, \, Q(y)^{1/3} \cdot \, \frac{dy}{y \cdot \, (1 \, - y)}. \qquad \hbox{(H.5)}$$

This one-form is clearly associated with the algebraic curve:

$$Q(x) \cdot u^3 \, \, = \, \, \, x \cdot \, (1 \, - x). \qquad \hbox{(H.6)}$$

One actually finds that this algebraic curve (H.6) is a *genus-one curve*.

One can go a step further by eliminating all the derivatives $y'(x), y''(x), y'''(x)$, from the confrontation of the Schwarzian condition (142) with the compatibility condition (H.1). One gets that way (after some calculation) a condition reading

$$\mathcal{I}(x) \, = \, \mathcal{I}(y(x)) \qquad \hbox{where:} \qquad \mathcal{I}(x) \, = \, \frac{Q(x)^8}{P_8(x)^3}, \qquad \hbox{(H.7)}$$

where $P_8(x)$ is a (quite large) polynomial of degree 8 in x, sum of 4724 terms.

We are seeking for non-trivial pullbacks $y(x)$ being different from the obvious solution $y(x) = x$. The interesting cases for physics are the one where $x \rightarrow y(x)$ is an infinite order transformation. In such cases one has

$$\mathcal{I}(x) \, = \, \mathcal{I}(y(x)) \, = \, \mathcal{I}(y(y(x))) \, = \, \mathcal{I}(y(y(y(x)))) \, = \, \cdots \qquad \text{(H.8)}$$

which amounts to saying that $\mathcal{I}(x)$ must be a constant. The cases where $Q(x)^8 = \lambda \cdot P_8(x)^3$ correspond to a set of extremely large conditions on the parameters a, b, c, d, e of the ${}_3F_2$ hypergeometric function, that is difficult to study because of the size of polynomial $P_8(x)$. However a simple case can fortunately be analyzed, namely $\mathcal{I}(x) = 0$, which corresponds to $Q(x) = 0$. In such a case the two conditions are compatible, and one just has one condition: the Schwarzian condition (142) with the additional condition being automatically verified (see (H.1)).

One finds that all the conditions on the parameters a, b, c, d, e of the ${}_3F_2$ hypergeometric function associated with $Q(x) = 0$ in fact correspond to cases where the order-three operator is exactly the symmetric power of a second order operator have ${}_2F_1$ solutions. In other words this situation corresponds to the *Clausen identity*, the ${}_3F_2$ hypergeometric function reducing to the square of a ${}_2F_1$ hypergeometric function:

$$ {}_3F_2\Bigl([2\,a,\, a\,+b,\, 2\,b],\, [a\,+b\,+\frac{1}{2},\, 2\,a\,+2\,b],\, x\Bigr) \\ = \, {}_2F_1\Bigl([a,\, b],\, [a\,+b\,+\frac{1}{2}],\, x\Bigr)^2. \qquad \text{(H.9)}$$

In this Clausen identity case, we found that the Schwarzian condition (142) for ${}_3F_2$ is nothing but the Schwarzian condition for the underlying ${}_2F_1$.

If one considers the other case $P_8(x) = 0$ the vanishing condition of the x^8 coefficient and the vanishing condition of the constant coefficient in x read respectively:

$$(a^2 - ab - ac + b^2 - bc + c^2) \cdot (a\,+c - 2\,b)^2 \cdot (a + b\,-2\,c)^2 \cdot (2\,a\,-c - b)^2 \, = \, 0,$$
$$(d^2 - de + e^2 - d - e + 1) \cdot (1 + d\,-2\,e)^2 \cdot \,(d + e\,-2)^2 \cdot (2\,d - e - 1)^2 \, = \, 0.$$

These two conditions are, respectively, very similar to the vanishing condition of the x^3 and constant coefficient of $Q(x)$, the other coefficients of $P_8(x)$ being more involved. The vanishing condition of all the x^n coefficients of $P_8(x)$ yields more relations on the a, b, c, d, d, e parameters. All these miscellaneous cases correspond to cases where the order-three linear differential operator reduces to the symmetric square of an order-two operator, and to the Clausen identities of the form (H.9). More simply on can verify that for parameters such that $Q(x) = 0$ (for which a Clausen reduction take place (H.9)) are also such that $P_8(x) = 0$ (the invariant $\mathcal{I}(x)$ in (H.7) is thus of the form $0/0$).

Appendix I. Schwarzian condition and other generalised hypergeometric functions

Appendix I.1. Schwarzian condition and ${}_4F_3$ hypergeometric functions

Let us consider a ${}_4F_3$ hypergeometric known to correspond to a Calabi-Yau ODE [23, 24] and seek an identity of the form:

$$\mathcal{A}(x) \cdot {}_4F_3\Bigl([\frac{1}{2},\, \frac{1}{2},\, \frac{1}{2},\, \frac{1}{2}],\, [1,\, 1,\, 1],\, x\Bigr)$$

$$= \quad {}_4F_3\Big([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}], [1, 1, 1], y(x)\Big) \qquad \text{(I.1)}$$

where $\mathcal{A}(x)$ is an algebraic function. Again we introduce the order-four linear differential operator annihilating the LHS and RHS of identity (I.5). The equality of the wronskians of these two linear differential operators enables us to get the expression of $\mathcal{A}(x)$ in terms of $y(x)$, namely:

$$\mathcal{A}(x) \;=\; \Big(\frac{(1-y(x))\cdot y(x)^3}{(1-x)\cdot x^3\cdot y'(x)^3}\Big)^{1/2}. \qquad \text{(I.2)}$$

After eliminating $\mathcal{A}(x)$ from (I.2), the identification of the D_x^2 coefficients for these two linear differential operators of order four gives the Schwarzian condition

$$W(x) \;\; - W(y(x))\cdot y'(x)^2 \;\; + \{y(x), x\} \;\; = \;\; 0, \qquad \text{(I.3)}$$

where $W(x)$ reads:

$$W(x) \;=\; -\frac{1}{10}\cdot\frac{5\,x^2\, - 7\,x\, + 5}{x^2\cdot(1-x)^2}. \qquad \text{(I.4)}$$

The condition corresponding to the identification of the D_x coefficient can be seen to be compatible with the previous Schwarzian condition: it can be seen to be a consequence of condition (I.3), corresponding to a combination of (I.3) with the derivative of (I.3). One finds, unfortunately, that the only solution is the trivial solution $y(x) \; = \; x$, the other solutions being spurious solutions $y\, + 5 \; = \; 0$, $5\,y\, + 1 \, = \, 0$, $y\, - 1 \, = \, 0$, etc ...

Remark: Similarly to was has been performed in section (Appendix I.1) one can imagine to seek for an identity (I.5) but, now, for the general ${}_4F_3$ hypergeometric function ${}_4F_3([a, b, c, d], [e, f, g], x)$. These calculations are really too large.

Appendix I.2. Schwarzian condition and hypergeometric functions with irregular singularities

The n-fold integrals emerging in lattice statistical mechanics or enumerative combinatorics are naturally diagonal of rational functions [25], the corresponding linear differential operators being globally nilpotent, and in particular Fuchsian. In such a lattice framework only ${}_nF_{n-1}$ hypergeometric functions [80] with *regular* singularities occur. Of course *irregular* singularities can also occur in physics [81, 82, 83], in particular in the *scaling limit* of lattice models [84, 85] (modified Bessel functions, etc ...).

Let us consider a hypergeometric function with an *irregular* singularity, namely a simple ${}_2F_2$ hypergeometric function solution of an order-three linear differential operator. We seek an identity of the form:

$$\mathcal{A}(x)\cdot {}_2F_2\Big([\frac{1}{2}, \frac{1}{2}], [1, 1], x\Big) \;\;=\;\; {}_2F_2\Big([\frac{1}{2}, \frac{1}{2}], [1, 1], y(x)\Big). \qquad \text{(I.5)}$$

The calculations are the same as in section (7.1). The identification of the wronskians of the two operators (the pullbacked order-three linear differential operator and the conjugated one) gives

$$\mathcal{A}(x) \;=\; \exp\Big(\frac{x - y(x)}{3}\Big)\cdot\frac{y(x)}{x\cdot y'(x)}, \qquad \text{(I.6)}$$

and the Schwarzian equation:

$$W(x) \; - W(y(x)) \cdot y'(x)^2 \; + \{y(x), x\} \; = \; 0, \quad \text{with:} \quad W(x) \; = \; \frac{1}{6} \cdot \frac{x^2 \; - 3}{x^2}. \qquad \text{(I.7)}$$

However, combining equation (I.7) with the last condition emerging from the identification of the terms with no D_x in the two operators, one finds that there is no pullback $y(x)$ for (I.5) except the trivial solution $y(x) \; = \; x$.

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