Rises in forests of binary shrubs

Jeffrey Remmel
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
remmel@math.ucsd.edu

Sai-nan Zheng
School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, PR China
zhengsainandlut@hotmail.com

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Abstract

The study of patterns in permutations associated with forests of binary shrubs was initiated by D. Bevan et al.. In this paper, we study five different types of rise statistics that can be associated with such permutations and find the generating functions for the distribution of such rise statistics.

1 Introduction

In [1], Bevan, Levin, Nugent, Pantone, Pudwell, Riehl, and Tlachac introduced the study of patterns in forests of binary shrubs. A k-ary heap H is a k-ary tree labeled with $\{1, \ldots, n\}$ such that every child has a larger label than its parent. Given a k-ary heap H, we associate a permutation σ_H with H by recording the vertex labels as they are encountered in the breadth-first search of the tree. For example, in Figure 1, we picture a 3-ary heap H whose associated permutation is $\sigma_H = 1\ 6\ 2\ 3\ 7\ 10\ 8\ 9\ 5\ 4$.

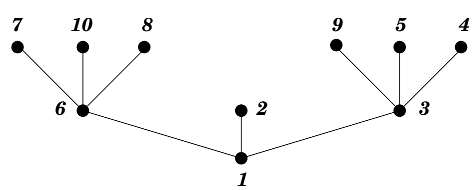

Figure 1: A 3-ary Heap.

A shrub is a heap whose leaves are all at most distance 1 from the root. A binary shrub is a heap whose underlying tree is a shrub with three vertices. A binary shrub forest is an ordered sequence of binary shrubs and we let $\mathcal{F}_n^2$ denote the set of all forests $F = (F_1, \ldots, F_n)$ of n binary shrubs whose set of labels is $\{1, \ldots, 3n\}$. For example, in Figure 2, we picture an element of $\mathcal{F}_5^2$. Given a forest $F = (F_1, \ldots, F_n) \in \mathcal{F}_n^2$, we let σ_F denote the permutation that results by concatenating the permutations $\sigma_{F_1} \ldots \sigma_{F_n}$. For example, the permutation σ_F for the $F \in \mathcal{F}_5^2$ pictured in Figure 2 is

$$\sigma_F = 5\ 12\ 9\ 6\ 13\ 15\ 1\ 4\ 10\ 7\ 11\ 8\ 2\ 14\ 3.$$

For any $n \geq 1$, we let $\mathcal{SF}_n^2$ denote the set of all σ_F such that $F \in \mathcal{F}_n^2$.

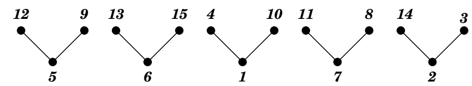

Figure 2: An element of $\mathcal{F}_n^2$.

The goal of this paper is to study generating functions for various types of rises in $\mathcal{SF}_n^2$. For example, given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ in the symmetric group S_n, we let

$$Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\} \text{ and } \mathrm{ris}(\sigma) = |Rise(\sigma)|.$$

Now suppose that we are given $F = (F_1, \ldots, F_n) \in \mathcal{F}_n^2$, then we let $\mathrm{ris}(F) = \mathrm{ris}(\sigma_F)$. However, given the structure of F, there are many other natural notions of rises in a forest of binary shrubs. That is, suppose that $\sigma_{F_i} = abc$ and $\sigma_{F_{i+1}} = def$ as pictured in Figure 3. Then we shall consider the following four types of rises.

1. $F_i <_T F_{i+1}$ if every element of $\{a, b, c\}$ is less than every element of $\{d, e, f\}$. We will refer to this type of rise as *total rise*.

2. $F_i <_B F_{i+1}$ if $a < d$. We will refer to this type of rise as *base rise*.

3. $F_i <_L F_{i+1}$ if $a < d$, $b < e$, and $c < e$. We will refer to this type of rise as *lexicographic rise*.

4. $F_i <_A F_{i+1}$ if $c < e$. We refer to this type of rise as an *adjacent rise* because when we look at the pictures of F_i and F_{i+1}, the rightmost element of F_i is less then the leftmost element of F_{i+1}.

Then we define

$$\begin{aligned}
RiseT(F) &= \{i : F_i <_T F_{i+1}\} \quad && \mathrm{risT}(F) = |RiseT(F)|, \\
RiseB(F) &= \{i : F_i <_B F_{i+1}\} \quad && \mathrm{risB}(F) = |RiseB(F)|, \\
RiseL(F) &= \{i : F_i <_L F_{i+1}\} \quad && \mathrm{risL}(F) = |RiseL(F)|, \text{ and} \\
RiseA(F) &= \{i : F_i <_A F_{i+1}\} \quad && \mathrm{risA}(F) = |RiseA(F)|.
\end{aligned}$$

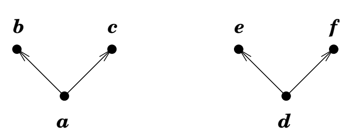

Figure 3: Two consecutive binary shrubs.

Our goal is to study the following generating functions.

$$
\begin{aligned}
\mathcal{R}(x,t) &= 1+\sum_{n\geq 1}\frac{t^{3n}}{(3n!)}\sum_{\sigma\in\mathcal{SF}_n^2} x^{\mathrm{ris}(\sigma)},\\
\mathcal{RT}(x,t) &= 1+\sum_{n\geq 1}\frac{t^{3n}}{(3n!)}\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risT}(F)},\\
\mathcal{RB}(x,t) &= 1+\sum_{n\geq 1}\frac{t^{3n}}{(3n!)}\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risB}(F)},\\
\mathcal{RA}(x,t) &= 1+\sum_{n\geq 1}\frac{t^{3n}}{(3n!)}\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risA}(F)}, \text{ and}\\
\mathcal{RL}(x,t) &= 1+\sum_{n\geq 1}\frac{t^{3n}}{(3n!)}\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risL}(F)}.
\end{aligned}
$$

For example, we shall prove that

$$
\mathcal{R}(x,t)=\frac{1-x}{1-x+\sum_{n\geq 1}\frac{(x(x-1)t^3)^n}{(3n)!}\prod_{k=1}^n(x+3k-2)}. \tag{1}
$$

For $Z\in\{T,A,B,L\}$, let

$$
\begin{aligned}
\mathcal{IZF}_n^2 &= \{(F_1,\ldots,F_n)\in\mathcal{F}_n^2 : F_1 <_Z F_2 <_Z \cdots <_Z F_n\},\\
\mathrm{IZF}_n^2 &= |\mathcal{IZF}_n^2|, \text{ and}\\
\mathcal{IZSF}_n^2 &= \{\sigma_F : F\in\mathcal{IZF}_n^2\}.
\end{aligned}
$$

Then for $Z\in\{T,A,B,L\}$, we shall show that

$$
\begin{aligned}
\mathcal{RZ}(x,t) &= 1+\sum_{n\geq 1}\frac{t^{3n}}{(3n)!}\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risZ}(F)}\\
&= \frac{1}{1-\sum_{n\geq 1}\frac{t^{3n}}{(3n)!}(x-1)^{n-1}\mathrm{IZF}_n^2}.
\end{aligned} \tag{2}
$$

Thus to find the generating functions $\mathcal{RT}(x,t)$, $\mathcal{RB}(x,t)$, $\mathcal{RL}(x,t)$, and $\mathcal{RA}(x,t)$, we need only compute ITF_n^2, IBF_n^2, ILF_n^2, and IAF_n^2. We shall show that

$$
\begin{aligned}
\mathrm{ITF}_n^2 &= 2^n,\\
\mathrm{IBF}_n^2 &= \frac{(3n)!}{3^n n!}, \text{ and}\\
\mathrm{ILF}_n^2 &= \frac{4^n(3n)!}{(n+1)!(2n+1)!}.
\end{aligned}
$$

Of these three formulas, the most interesting is the formula for ILF_n^2 which equals the number of paths of length n in the plane that start and end at the origin and which stay in the first quadrant that consists only of steps of the form $(1,1)$, $(0,-1)$ and $(-1,0)$. This number was first computed by Kreweras in [6]. We shall prove our formula by providing a bijection between $\mathcal{ILF}_n^2$ and the collection of such paths. We have not been able to find an explicit formula for IAF_n^2 but we shall show that we can develop a system of recursions that will allows us to compute IAF_n^2.

The main tool that we will use to compute these generating functions is the homomorphism method as described in [8]. The homomorphism method derives generating functions for various permutation statistics by applying a ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to simple symmetric function identities such as

$$H(t) = 1/E(-t) \tag{3}$$

where $H(t)$ and $E(t)$ are the generating functions for the homogeneous and elementary symmetric functions, respectively:

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1} \frac{1}{1-x_i t}, \quad E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1} 1 + x_i t. \tag{4}$$

The outline of the this paper is as follows. First in Section 2, we shall briefly review the background on symmetric functions that we need. In Section 3, we shall prove (1). In Section 4, we shall prove (2). In Section 5, we will compute ITF_n^2, IBF_n^2, ILF_n^2, and IAF_n^2 which combined with the results of Section 4 will allow us to compute the generating functions $\mathcal{RT}(x,t)$, $\mathcal{RB}(x,t)$, $\mathcal{RL}(x,t)$, and $\mathcal{RA}(x,t)$.

2 Symmetric functions

In this section, we give the necessary background on symmetric functions that will be used in our proofs.

A partition of n is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_s)$ such that $0 < \lambda_1 \leq \cdots \leq \lambda_s$ and $n = \lambda_1 + \cdots + \lambda_s$. We shall write $\lambda \vdash n$ to denote that λ is partition of n and we let $\ell(\lambda)$ denote the number of parts of λ. When a partition of n involves repeated parts, we shall often use exponents in the partition notation to indicate these repeated parts. For example, we will write $(1^2, 4^5)$ for the partition $(1, 1, 4, 4, 4, 4, 4)$.

Let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$. The n^{th} elementary symmetric function $e_n = e_n(x_1, x_2, \ldots)$ and n^{th} homogeneous symmetric function $h_n = h_n(x_1, x_2, \ldots)$ are defined by the generating functions given in (4). For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$. It is well known that $e_0, e_1, \ldots$ is an algebraically independent set of generators for Λ, and hence, a ring homomorphism θ on Λ can be defined by simply specifying $\theta(e_n)$ for all n.

If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, then a λ-brick tabloid of shape (n) is a filling of a rectangle consisting of n cells with bricks of sizes $\lambda_1, \ldots, \lambda_k$ in such a way that no two bricks overlap. For example, Figure 4 shows the six $(1^2, 2^2)$-brick tabloids of shape (6).

Let $\mathcal{B}_{\lambda,n}$ denote the set of λ-brick tabloids of shape (n) and let $B_{\lambda,n}$ be the number of λ-brick tabloids of shape (n). If $B \in \mathcal{B}_{\lambda,n}$, we will write $B = (b_1, \ldots, b_{\ell(\lambda)})$ if the lengths of the bricks in B, reading from left to right, are $b_1, \ldots, b_{\ell(\lambda)}$. For example, the brick

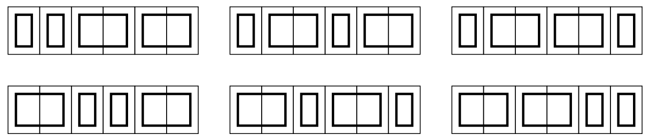

Figure 4: The six $(1^2, 2^2)$-brick tabloids of shape (6).

tabloid in the top right position in Figure 4 is denoted as $(1,2,2,1)$. Eğecioğlu and the second author [3] proved that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n}\, e_\lambda. \tag{5}$$

3 The generating function $\mathcal{R}(x,t)$.

It this section, we shall prove the following theorem.

Theorem 1.

$$\mathcal{R}(x,t) = 1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \sum_{\sigma \in \mathcal{SF}_n^2} x^{\mathrm{ris}(\sigma)} = \frac{1-x}{1 - x + \sum_{n\geq 1} \frac{(x(x-1)t^3)^n}{(3n)!} \prod_{k=1}^{n}(x+3k-2)}. \tag{6}$$

Proof. Let $\mathbb{Q}[x]$ denote the polynomial ring over the rational numbers $\mathbb{Q}$.

Let $\theta : \Lambda \to \mathbb{Q}[x]$ be the ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables determined by setting $\theta(e_0) = 1$, $\theta(e_{3n+1}) = \theta(e_{3n+2}) = 0$ for all $n \geq 0$, and

$$\theta(e_{3n}) = \frac{(-1)^{3n-1}}{(3n)!} x^n (x-1)^{n-1} \prod_{k=1}^{n} (x+3k-2)$$

for all $n \geq 1$. We claim that for $n \geq 0$, $\theta(h_{3n+1}) = \theta(h_{3n+2}) = 0$ and that for $n \geq 1$,

$$(3n)!\theta(h_{3n}) = \sum_{\sigma \in \mathcal{SF}_n^2} x^{\mathrm{ris}(\sigma)}. \tag{7}$$

First it is easy to see that our definitions ensure that $\theta(e_\lambda) = 0$ if λ has a part which is equivalent to either 1 or 2 mod 3. Since

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda, \tag{8}$$

it follows that $\theta(h_n) = 0$ if n is equivalent to 1 or 2 mod 3 since every partition of λ of n must contain a part which is equivalent to 1 or 2 mod 3. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, we let 3λ denote the partition $(3\lambda_1, \ldots, 3\lambda_k)$. It follows that in the expansion $\theta(h_{3n})$, we need only consider partitions λ of $3n$ of the form 3μ where μ is a partition of n. Thus

$$(3n)!\theta(h_{3n}) = (3n)!\sum_{\mu\vdash n}(-1)^{3n-\ell(\mu)}B_{3\mu,3n}\theta(e_{3\mu}) =$$

$$(3n)!\sum_{\mu\vdash n}(-1)^{3n-\ell(\mu)}\sum_{(3b_1,\ldots,3b_\mu)\in\mathcal{B}_{3\mu,3n}}\prod_{i=1}^{\ell(\mu)}\frac{(-1)^{3b_i-1}}{(3b_i)!}x^{b_i}(x-1)^{b_i-1}\prod_{k_i=1}^{b_i}(x+3k_i-2) =$$

$$\sum_{\mu\vdash n}\sum_{(3b_1,\ldots,3b_\mu)\in\mathcal{B}_{3\mu,3n}}\binom{3n}{3b_1,\ldots,3b_{\ell(\mu)}}\prod_{i=1}^{\ell(\mu)}x^{b_i}(x-1)^{b_i-1}\prod_{k_i=1}^{b_i}(x+3k_i-2). \tag{9}$$

Next our goal is to give a combinatorial interpretation to the right-hand side of (9). First we are interested in the set of permutations $\mathcal{ISF}_n^2$ which consists of all permutations $\sigma=\sigma_1\ldots\sigma_{3n}\in\mathcal{SF}_n^2$ such that $\sigma_{3i}<\sigma_{3i+1}$ for $i=1,\ldots,n-1$. One way to think of this set is that it is the set of permutations that arise from a forest $F=(F_1,\ldots,F_n)\in\mathcal{F}_n^2$ such that the label of the right-most element in F_i is less than the label of the root of F_{i+1}. For example, if $n=5$, then we are asking for labellings of the poset whose Hasse diagram is pictured at the top Figure 5. We want to find the set of all labellings of the nodes of this poset such that when there is an arrow from a node x to a node y, then the label of node x is less than label of node y. This is equivalent to finding the set of all linear extensions of the poset. We have given an example of such a labeling on the second line of Figure 5 and its corresponding permutation in $\mathcal{SF}_5^2$ in the third line of Figure 5. Given an element of $\sigma=\sigma_1\ldots\sigma_{3n}\in\mathcal{ISF}_n^2$, we let

$$\mathrm{ris}_{1,2}(\sigma)=|\{i:\sigma_i<\sigma_{i+1}\ \&\ i\equiv 1,2 \mod 3\}|.$$

That is, $\mathrm{ris}_{1,2}(\sigma)$ keep track of the number of rises between pairs of the form $\sigma_{3j+1}\sigma_{3j+2}$ and $\sigma_{3j+2}\sigma_{3j+3}$.

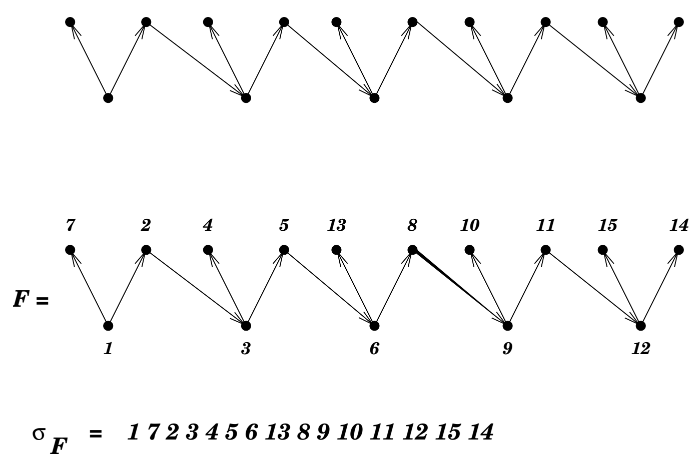

Figure 5: The poset for $\mathcal{ISF}_5^2$.

We claim that

$$x^n\prod_{k=1}^{n}(x+3k-2)=\sum_{\sigma\in\mathcal{ISF}_n^2}x^{\mathrm{ris}_{1,2}(\sigma)}.$$

This is easy to prove by induction. First, it easy to check that there are exactly two permutations in $\mathcal{ISF}^2_1$, namely, 123 and 132 so that $\sum_{\sigma\in\mathcal{ISF}^2_1} x^{\mathrm{ris}_{1,2}(\sigma)} = x(1+x)$ as claimed. Now suppose that our formula holds for $k<n$. Then consider Figure 6 where we have redrawn the poset so that the positions correspond to the elements in σ_F. It is easy to see that the label of the left-most element must be one since there is a directed path from that element to any other element in the poset. There must be a rise from σ_1 to σ_2 so we add a label x below that position. Next consider node which has label 2. If 2 is the label of the second element, then the label of the third element must be 3 since there is a directed path from that element to any of the other unlabeled elements in the poset at this point. In this case $2=\sigma_2<\sigma_3=3$ so we add a label x below that position. If the label of the second element is a where $a>2$, then the label of the third element must be 2 since there is a directed path from that element to any of the other unlabeled elements in the poset at this point. We have $3n-2$ ways to choose a. In this case the pair $\sigma_2\sigma_3$ is not a rise so that that we do not add a label x below that position. Thus our choices of labels for the binary shrub F_1 gives rise to a factor of $x(x+3n-2)$ in our sum. Note that once we have placed the labels on F_1, the remaining labels are completely free. Thus it follows that

$$\begin{aligned}\sum_{\sigma\in\mathcal{ISF}^2_n} x^{\mathrm{ris}_{1,2}(\sigma)} &= x(x+3n-2)\sum_{\sigma\in\mathcal{ISF}^2_{n-1}} x^{\mathrm{ris}_{1,2}(\sigma)}\\ &= x^n\prod_{k=1}^{n}(x+3k-2).\end{aligned}$$

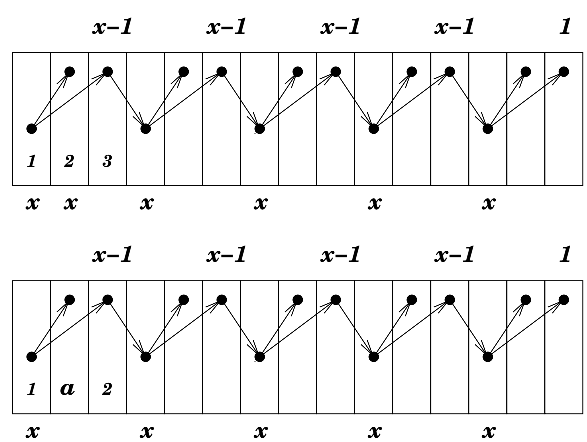

Figure 6: The recursive construction of elements of $\mathcal{ISF}^2_n$.

This given, we can interpret the extra factor of $(x-1)^{n-1}$ in $\theta(e_{3n})$ as adding a label $(x-1)$ on every third element except the last one. In Figure 6, we indicate this by putting such labels at the top of the diagram.

We are now in a position to give a combinatorial interpretation to the right-hand side of (9). That is, we first choose a brick tabloid $B=(3b_1,\ldots,3b_{\ell(\mu)})$ consisting of bricks whose size is a multiple of 3. Then we use the multinomial coefficient $\binom{3n}{3b_1,\ldots,3b_{\ell(\mu)}}$ to pick an ordered sequence of sets $S_1,\ldots,S_{\ell(\mu)}$ such that $|S_i|=3b_i$ and $S_1,\ldots,S_{\ell(\mu)}$ partition the elements $\{1,\ldots,3n\}$. For each brick $3b_i$, we interpret the factor $x^{b_i}\prod_{k=1}^{b_i}(x+3k-2)$ as all ways $\gamma_1^{(i)}\ldots\gamma_{3b_i}^{(i)}$ of arranging the elements of S_i in the cells of the brick $3b_i$ such that $\mathrm{red}(\gamma_1^{(i)}\ldots\gamma_{3b_i}^{(i)}) \in \mathcal{ISF}_{b_i}^2$ where we place a label x below the cell containing $\gamma_j^{(i)}$ if $j = 1,2 \mod 3$ and $\gamma_j^{(i)} < \gamma_{j+1}^{(i)}$. Finally, we can label the cells containing the elements $\gamma_3^{(i)},\ldots,\gamma_{3b_i-3}^{(i)}$ with either x or -1 and we label the last cell of a brick with 1. Let $\mathcal{O}_{3n}$ denote the set of all objects created in this way. Then $\mathcal{O}_{3n}$ consists of all triples (B,σ,L) such that $B=(3b_1,\ldots,3b_k)$ is a brick tabloid all of whose bricks have length a multiple of 3, σ is a permutation in S_{3n}, and L is labeling of the cells of B such that the following four conditions hold.

1. For each $i=1,\ldots,k$, the reduction of the sequence of elements obtained by reading the elements in the brick $3b_i$ from left to right is an element is in $\mathcal{ISF}_{b_i}^2$.
2. The cell containing a σ_i such that $i \equiv 1,2 \mod 3$ is labeled with an x if and only if $i \in Rise(\sigma)$.
3. The label of a cell at the end of any brick is 1.
4. The cells containing elements of the form σ_{3i} which are not at the end of brick are labeled with either -1 or x.

For each such $(B,\sigma,L) \in \mathcal{O}_{3n}$, we let the weight of (B,σ,L), $w(B,\sigma,L)$, be the product of all its x labels and we let the sign of (B,σ,L), $sgn(B,\sigma,L)$, be the product of all its -1 labels. For example, at the top of Figure 7, we picture an element $(B,\sigma,L) \in \mathcal{O}_{18}$ such that $w(B,\sigma,L)=x^{11}$ and $sgn(B,\sigma,L)=-1$. It follows that

$$(3n)!\theta(h_{3n}) = \sum_{(B,\sigma,L)\in\mathcal{O}_{3n}} sgn(B,\sigma,L)w(B,\sigma,L). \tag{10}$$

Next we will define a sign-reversing involution $J:\mathcal{O}_{3n}\to\mathcal{O}_{3n}$ which we will use to simplify the right-hand side of (10). Given a triple $(B,\sigma,L)\in\mathcal{O}_{3n}$, where $B=(3b_1,\ldots,3b_k)$ and $\sigma=\sigma_1\ldots\sigma_{3n}$, scan the cells from left to right looking for the first cell c such that either

Case 1. $c=3s$ for some $1\le s\le n-1$ and cell the label on cell c is -1 or

Case 2. c is that last cell of brick $3b_i$ for some $i<k$ and $\sigma_c<\sigma_{c+1}$.

In Case 1, suppose that c is in brick $3b_i$. Then $J(B,\sigma,L)$ is obtained from (B,σ,L) by splitting brick $3b_i$ into two bricks $3b_i^*$ and $3b_i^{**}$, where $3b_i^*$ contains the cells of $3b_i$ up to and including cell c and $3b_i^{**}$ contains the remaining cells of $3b_i$, and changing the label on cell c from -1 to 1. In Case 2, $J(B,\sigma,L)$ is obtained from (B,σ,L) by combining bricks $3b_i$ and $3b_{i+1}$ into a single brick $3b$ and changing the label on cell c from 1 to -1. If neither Case 1 or Case 2 applies, then we define $J(B,\sigma,L)=(B,\sigma,L)$.

For example, if (B,σ,L) is the element of $\mathcal{O}_{18}$ pictured at the top of Figure 7, then $B=(3b_1,3b_2,3b_3)$ where $b_1=2$, $b_2=1$ and $b_3=3$. Note that we cannot combine bricks $3b_1$ and $3b_2$ since $18=\sigma_6>\sigma_7=2$ and we cannot combine bricks $3b_2$ and $3b_3$ since $17=\sigma_9>\sigma_{10}=1$. Thus the first cell c where either Case 1 or Case 2 applies is cell

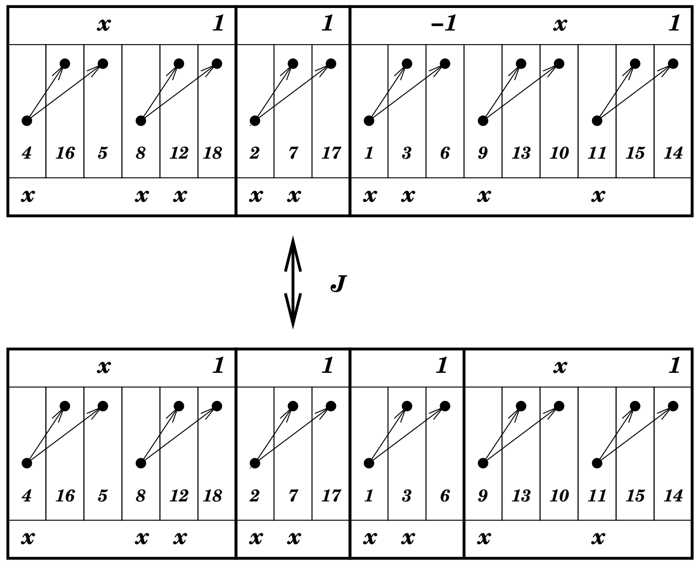

Figure 7: An example of the involution J

$c = 12$. Thus we are in Case 1 and $J(B,\sigma,L)$ is obtained from (B,σ,L) by splitting brick $3b_3$ into two bricks, the first one of size 3 and second one of size 6, and changing the label on cell 12 from -1 to 1. Thus $J(B,\sigma,L)$ is pictured at the bottom of Figure 7.

It is easy to see that J is an involution. That is, if we are in Case I using cell c to define $J(B,\sigma,L)$, then we will be in Case II using cell c when we apply J to $J(B,\sigma,L)$ so that $J(J(B,\sigma,L)) = (B,\sigma,L)$. Similarly, if we are in Case II using cell c to define $J(B,\sigma,L)$, then we will be in Case I using cell c when we apply J to $J(B,\sigma,L)$ so that $J(J(B,\sigma,L)) = (B,\sigma,L)$. Moreover it is easy to see that if $J(B,\sigma,L) \neq (B,\sigma,L)$, then

$$sgn(B,\sigma,L)w(B,\sigma,L) = -sgn(J(B,\sigma,L))w(J(B,\sigma,L)).$$

It follows that

$$\begin{aligned}(3n)!\theta(h_{3n}) &= \sum_{(B,\sigma,L)\in\mathcal{O}_{3n}} sgn(B,\sigma,L)w(B,\sigma,L) \\ &= \sum_{(B,\sigma,L)\in\mathcal{O}_{3n}, J(B,\sigma,L)=(B,\sigma,L)} sgn(B,\sigma,L)w(B,\sigma,L). \qquad (11)\end{aligned}$$

Thus we must examine the fixed points of J on $\mathcal{O}_{3n}$. It is easy to see that if (B,σ,L), where $B = (3b_1,\ldots,3b_k)$ and $\sigma = \sigma_1\ldots\sigma_{3n}$, is a fixed point of J, then there can be no cells labeled -1 and for $1 \leq i \leq k-1$, the element in the last cell of brick $3b_i$ must be greater than the element in the first cell of $3b_{i+1}$. It follows that if $c = 3i$ for some $1 \leq i \leq n-1$, then cell c is labeled with an x if and only if $\sigma_c < \sigma_{c+1}$. Thus for a fixed point (B,σ,L) of J, $wt(B,\sigma,L) = x^{\mathrm{ris}(\sigma)}$ and $sgn(B,\sigma,L) = 1$. Vice versa, given any $\sigma \in \mathcal{SF}_n^2$, we can create a fixed point of J, (B,σ,L) by having the bricks end at those cells $c = 3i$ such that $3i \notin Rise(\sigma)$ and labeling all the cells j such that $j \in Rise(\sigma)$ with an x. For example, if

$$\sigma = 4\ 16\ 5\ 8\ 12\ 18\ 2\ 7\ 17\ 1\ 3\ 6\ 9\ 13\ 10\ 11\ 15\ 14,$$

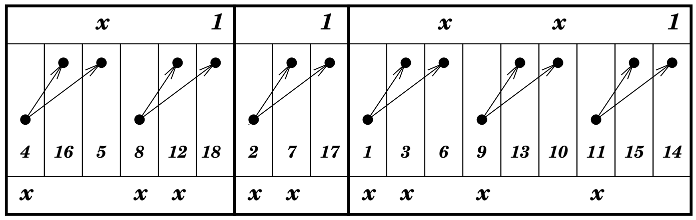

Figure 8: A fixed point of J.

then the fixed point corresponding to σ is pictured in Figure 8.

Hence, we have proved that

$$(3n)!\theta(h_{3n}) = \sum_{\sigma \in \mathcal{SF}_n^2} x^{\mathrm{ris}(\sigma)}$$

as desired.

It follows that

$$\begin{aligned}
\theta(H(t)) &= 1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \sum_{\sigma \in \mathcal{SF}_n^2} x^{\mathrm{ris}(\sigma)} \\
&= \frac{1}{\theta(E(-t))} = \frac{1}{1+\sum_{n\geq 1}(-t)^n \theta(e_n)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^{3n}\frac{(-1)^{3n-1}}{(3n)!}x^n(x-1)^{n-1}\prod_{k=1}^n(x+3k-2)} \\
&= \frac{1-x}{1-x+\sum_{n\geq 1}\frac{(x(x-1)t^3)^n}{(3n)!}\prod_{k=1}^n(x+3k-2)}.
\end{aligned}$$

□

We have used this generating function to compute the initial terms of the sequence $(\sum_{\sigma \in \mathcal{SF}_n^2} x^{\mathrm{ris}(\sigma)})_{n\geq 1}$.

$x(1+x)$

$x^2\left(16+39x+24x^2+x^3\right)$

$x^3\left(1036+4183x+5506x^2+2536x^3+178x^4+x^5\right)$

$x^4\left(174664+992094x+2054131x^2+1896937x^3+726622x^4+67768x^5+1383x^6+x^7\right)$

$x^5\left(60849880+446105914x+1272918569x^2+1800188609x^3+1307663949x^4+\right.$
$\left.442673265x^5+49244651x^6+1720211x^7+10951x^8+x^9\right)$

We note that if $\sigma = \sigma_1 \ldots \sigma_{3n} \in \mathcal{SF}_n^2$, then we are forced to have $\{3k+1 : k = 0,\ldots,n-1\} \subseteq Rise(\sigma)$ by our definition of the permutation associated with a forest of binary shrubs. It follows that

$$\begin{aligned}
\mathcal{R}(x, \frac{t}{x^{1/3}}) &= 1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \sum_{\sigma \in \mathcal{SF}_n^2} x^{\mathrm{ris}(\sigma)-n} \\
&= \frac{1-x}{1-x+\sum_{n\geq 1}\frac{((x-1)t^3)^n}{(3n)!}\prod_{k=1}^n(x+3k-2)}.
\end{aligned}$$

We can then set $x = 0$ in this expression to get the generating function of $\sigma \in \mathcal{SF}_n^2$ such that $\text{ris}(\sigma) = n$ which is the minimal number of rises that an element $F \in \mathcal{SF}_n^2$ can have. That is,

$$\begin{aligned}
1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} |\{\sigma \in \mathcal{SF}_n^2 : \text{ris}(\sigma) = n\}| &= \frac{1}{1 + \sum_{n\geq 1} \frac{(-t^3)^n}{(3n)!} \prod_{k=1}^n (3k-2)} \\
&= \frac{1}{1 + \sum_{n\geq 1} \frac{(-1)^n t^{3n}}{(3n)!} \prod_{k=1}^n (3k-2)}.
\end{aligned}$$

4 The generating functions $\mathcal{RZ}(x,t)$ for $Z \in \{T, B, L, A\}$

In this section, we shall give a general method for computing the generating functions $\mathcal{RT}(x,t)$, $\mathcal{RB}(x,t)$, $\mathcal{RA}(x,t)$, and $\mathcal{RTL}(x,t)$. For $Z \in \{T, A, B, L\}$, let

$$\begin{aligned}
\mathcal{IZF}_n^2 &= \{(F_1, \ldots, F_n) \in \mathcal{F}_n^2 : F_1 <_Z F_2 <_Z \cdots <_Z F_n\}, \\
\text{IZF}_n^2 &= |\mathcal{IZF}_n^2|, \text{ and} \\
\mathcal{IZSF}_n^2 &= \{\sigma_F : F \in \mathcal{IZF}_n^2\}.
\end{aligned}$$

Then we have the following theorem.

Theorem 2. *For $Z \in \{T, B, A, L\}$,*

$$\mathcal{RZ}(x,t) = 1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \sum_{F \in \mathcal{F}_n^2} x^{\text{risZ}(F)} = \frac{1}{1 - \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} (x-1)^{n-1} \text{IZF}_n^2}. \tag{12}$$

Proof. Let $Z \in \{T, B, A, L\}$ and let $\theta_Z : \Lambda \to \mathbb{Q}[x]$ be the ring homomorphism determined by setting $\theta_Z(e_0) = 1$, $\theta_Z(e_{3n+1}) = \theta_Z(e_{3n+2}) = 0$ for all $n \geq 0$, and

$$\theta_Z(e_{3n}) = \frac{(-1)^{3n-1}}{(3n)!} \text{IZF}_n^2 (x-1)^{n-1}$$

for all $n \geq 1$. We claim that for $n \geq 0$, $\theta_Z(h_{3n+1}) = \theta_Z(h_{3n+2}) = 0$ and that for $n \geq 1$,

$$(3n)! \theta_Z(h_{3n}) = \sum_{F \in \mathcal{F}_n^2} x^{\text{risZ}(F)}. \tag{13}$$

It is easy to see that our definitions ensure that $\theta_Z(e_\lambda) = 0$ if λ has a part which is equivalent to either 1 or 2 mod 3. Since

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda, \tag{14}$$

it follows that $\theta_Z(h_n) = 0$ if n is equivalent to 1 or 2 mod 3 since every partition of λ of n must contain a part which is equivalent to 1 or 2 mod 3. Moreover, it follows that in the expansion $\theta(h_{3n})$, we need only consider partitions λ of $3n$ of the form 3μ where μ is a partition of n. Thus

$$\begin{aligned}
(3n)!\theta_Z(h_{3n}) &= (3n)!\sum_{\mu\vdash n}(-1)^{3n-\ell(\mu)}B_{3\mu,3n}\theta_Z(e_{3\mu}) \\
&= (3n)!\sum_{\mu\vdash n}(-1)^{3n-\ell(\mu)}\sum_{(3b_1,\ldots,3b_\mu)\in\mathcal{B}_{3\mu,3n}}\prod_{i=1}^{\ell(\mu)}\frac{(-1)^{3b_i-1}}{(3b_i)!}\mathrm{IZF}^2_{b_i}(x-1)^{b_i-1} \\
&= \sum_{\mu\vdash n}\sum_{(3b_1,\ldots,3b_\mu)\in\mathcal{B}_{3\mu,3n}}\binom{3n}{3b_1,\ldots,3b_{\ell(\mu)}}\prod_{i=1}^{\ell(\mu)}\mathrm{IZF}^2_{b_i}(x-1)^{b_i-1}. \qquad (15)
\end{aligned}$$

Next our goal is to give a combinatorial interpretation to the right-hand side of (15). We first choose a brick tabloid $B=(3b_1,\ldots,3b_{\ell(\mu)})$ whose bricks have size a multiple of 3. Then we use the multinomial coefficient $\binom{3n}{3b_1,\ldots,3b_{\ell(\mu)}}$ to pick an ordered sequence of sets $S_1,\ldots,S_{\ell(\mu)}$ such that $|S_i|=3b_i$ and $S_1,\ldots,S_{\ell(\mu)}$ partition the elements $\{1,\ldots,3n\}$. For each brick $3b_i$, we interpret the factor IZF^2_n as all ways $\gamma_1^{(i)}\ldots\gamma_{3b_i}^{(i)}$ of arranging the elements of S_i in the cells of the brick $3b_i$ such that $\mathrm{red}(\gamma_1^{(i)}\ldots\gamma_{3b_i}^{(i)})\in\mathcal{IZSF}^2_{b_i}$. Finally, we can label the cells containing the elements $\gamma_3^{(i)},\ldots,\gamma_{3b_i-3}^{(i)}$ with either x or -1 and we label the last cell of a brick with 1. Let $\mathcal{OZ}_{3n}$ denote the set of all objects created in this way. Then $\mathcal{OZ}_{3n}$ consists of all triples (B,σ,L) such that $B=(3b_1,\ldots,3b_k)$ is a brick tabloid all of whose bricks have length a multiple of 3, $\sigma=\sigma_1\ldots\sigma_{3n}$ is a permutation in S_{3n}, and L is labeling of the cells of B such that the following three conditions hold.

1. For each $i=1,\ldots,k$, the reduction of the sequence of elements obtained by reading the elements in the brick $3b_i$ from left to right is an element is in $\mathcal{IZSF}^2_{b_i}$.
2. The label of a cell at the end of any brick is 1.
3. The cells containing elements of the form σ_{3i} which are not at the end of brick are labeled with either -1 or x.

For each such $(B,\sigma,L)\in\mathcal{OZ}_{3n}$, we let the weight of (B,σ,L), $w(B,\sigma,L)$, be the product of all its x labels and we let the sign of (B,σ,L), $sgn(B,\sigma,L)$, be the product of all its -1 labels. For example, suppose that $Z=B$. Then at the top of Figure 9, we picture an element $(B,\sigma,L)\in\mathcal{OB}_{18}$ such that $w(B,\sigma,L)=x^2$ and $sgn(B,\sigma,L)=-1$.

It follows that

$$(3n)!\theta_Z(h_{3n})=\sum_{(B,\sigma,L)\in\mathcal{OZ}_{3n}}sgn(B,\sigma,L)w(B,\sigma,L). \qquad (16)$$

Next we will define a sign-reversing involution $J_Z:\mathcal{OZ}_{3n}\to\mathcal{OZ}_{3n}$ which we will use to simplify the right-hand side of (16). Given a triple $(B,\sigma,L)\in\mathcal{OZ}_{3n}$, where $B=(3b_1,\ldots,3b_k)$ and $\sigma=\sigma_1\ldots\sigma_{3n}$, scan the cells from left to right looking for the first cell c such that either

Case 1. $c=3s$ for some $1\le s\le n-1$ and cell the label on cell c is -1 or

Case 2. c is that last cell of brick $3b_i$ for some $i < k$ and the binary shrub F corresponding to the cells $3b_i - 2, 3b_i - 1, 3b_i$ is $<_Z$ the binary shrub G corresponding to the cells $3b_i + 1, 3b_i + 2, 3b_i + 3$.

In Case 1, suppose that c is in brick $3b_i$. Then $J_Z(B, \sigma, L)$ is obtained from (B, σ, L) by splitting brick $3b_i$ into two bricks $3b_i^*$ and $3b_i^{**}$, where $3b_i^*$ contains the cells of $3b_i$ up to and including cell c and $3b_i^{**}$ contains the remaining cells of $3b_i$, and changing the label on cell c from -1 to 1. In Case 2, $J_Z(B, \sigma, L)$ is obtained from (B, σ, L) by combining bricks $3b_i$ and $3b_{i+1}$ into a single brick $3b$ and changing the label on cell c from 1 to -1. If neither Case 1 or Case 2 applies, then we define $J + Z(B, \sigma, L) = (B, \sigma, L)$.

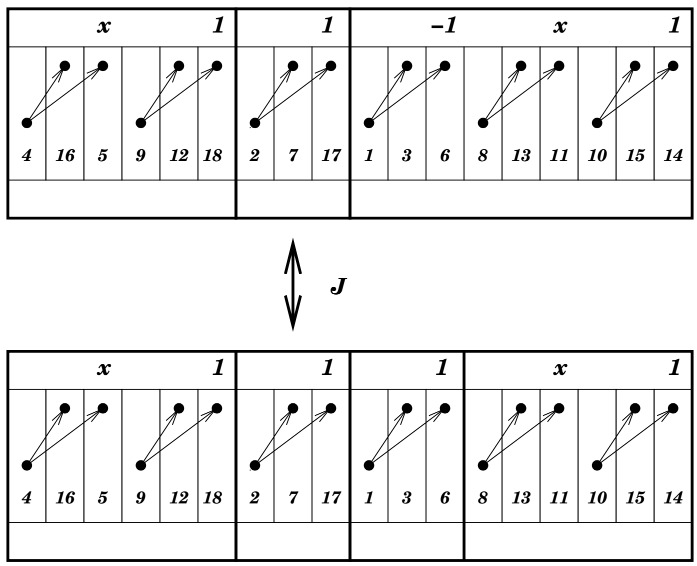

Figure 9: An example of the involution J_Z when $Z = B$.

For example, if (B, σ, L) is the element of $\mathcal{OB}_{18}$ pictured at the top of Figure 9, then $B = (3b_1, 3b_2, 3b_3)$ where $b_1 = 2$, $b_2 = 1$ and $b_3 = 3$. Note that we cannot combine bricks $3b_1$ and $3b_2$ since $9 = \sigma_4 > \sigma_7 = 2$ and we cannot combine bricks $3b_2$ and $3b_3$ since $2 = \sigma_7 > \sigma_{10} = 1$. Thus the first cell c where either Case 1 or Case 2 applies is cell $c = 12$. Thus we are in Case 1 and $J_B(B, \sigma, L)$ is obtained from (B, σ, L) by splitting brick $3b_3$ into two bricks, the first one of size 3 and second one of size 6, and changing the label on cell 12 from -1 to 1. Thus $J_B(B, \sigma, L)$ is pictured at the bottom of Figure 9.

We can use the same reasoning as in Theorem 1 to show that J_Z is an involution. Moreover it is easy to see that if $J_Z(B, \sigma, L) \neq (B, \sigma, L)$, then

$$sgn(B, \sigma, L)w(B, \sigma, L) = -sgn(J_Z(B, \sigma, L))w(J_Z(B, \sigma, L)).$$

It follows that

$$\begin{aligned}(3n)!\theta_Z(h_{3n}) &= \sum_{(B,\sigma,L)\in\mathcal{OZ}_{3n}} sgn(B, \sigma, L)w(B, \sigma, L) \\ &= \sum_{(B,\sigma,L)\in\mathcal{OZ}_{3n}, J_Z(B,\sigma,L)=(B,\sigma,L)} sgn(B, \sigma, L)w(B, \sigma, L). \qquad (17)\end{aligned}$$

Thus we must examine the fixed points of J_Z on $\mathcal{OZ}_{3n}$. It is easy to see that if (B,σ,L), where $B=(3b_1,\ldots,3b_k)$ and $\sigma=\sigma_1\ldots\sigma_{3n}$, is a fixed point of J_Z, then there can be no cells labeled -1 and for $1\leq i\leq k-1$, the binary shrub F determined by the last three cells of $3b_i$ is not $<_Z$ the binary shrub determined by the first three cells of $3b_{i+1}$. It follows that if $c=3i$ for some $1\leq i\leq n-1$, then cell c is labeled with an x if and only if the binary shrub F corresponding to the cells $3b_i-2, 3b_i-1, 3b_i$ is $<_Z$ the binary shrub G corresponding to the cells $3b_i+1, 3b_i+2, 3b_i+3$. Thus for a fixed point (B,σ,L) of J_Z, $wt(B,\sigma,L)=x^{\mathrm{risZ}(\sigma)}$ and $sgn(B,\sigma,L)=1$. Vice versa, given any σ_F where $F=(F_1,\ldots,F_n)\in\mathcal{F}_n^2$, we can create a fixed point of J_Z, (B,σ,L) by having the bricks end at those cell $c=3i$ such that $i\notin RiseZ(F)$ and labeling all the cells $3j$ such that $j\in RiseZ(F)$ with an x. For example, if $Z=B$ and

$$\sigma = 4\ 16\ 5\ 8\ 12\ 18\ 2\ 7\ 17\ 1\ 3\ 6\ 9\ 13\ 10\ 11\ 15\ 14,$$

then the fixed point corresponding to σ is pictured in Figure 10.

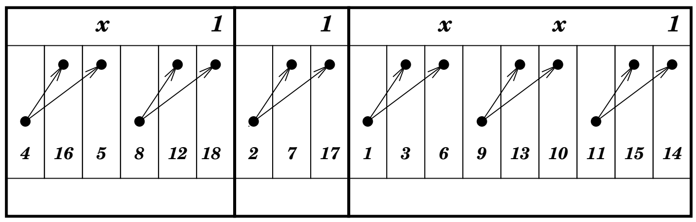

Figure 10: A fixed point of J_B.

Hence, we have proved that

$$(3n)!\theta_Z(h_{3n}) = \sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risZ}(F)}$$

as desired.

Hence for all $Z\in\{T,B,A,L\}$,

$$\begin{aligned}
\theta_Z(H(t)) &= 1+\sum_{n\geq 1}\frac{t^{3n}}{(3n)!}\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risZ}(F)}\\
&= \frac{1}{\theta_Z(E(-t))} = \frac{1}{1+\sum_{n\geq 1}(-t)^n\theta_Z(e_n)}\\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^{3n}\frac{(-1)^{3n-1}}{(3n)!}\mathrm{IZF}_n^2(x-1)^{n-1}}\\
&= \frac{1-x}{1-x+\sum_{n\geq 1}\frac{((x-1)t^3)^n}{(3n)!}\mathrm{IZF}_n^2}.
\end{aligned}$$

□

5 Computing IZF_n^2 for $Z\in\{T,B,L,A\}$

Based on our results from the last section, all we need to do is to compute the generating functions $\mathcal{RZ}(x,t)$ for $Z\in\{T,B,L,A\}$ is to compute IZF_n^2 for $Z\in\{T,B,L,A\}$.

5.1 ITF_n^2

It is easy to see that if $F = (F_1, \ldots, F_n)$ is such that $F_1 <_T F_2 <_T \cdots <_T F_n$, then the labels on F_i must be $3i-2$, $3i-1$ and $3i$ for $i = 1, \ldots, n$. We have exactly 2 ways to arrange these labels to make a binary shrub which are pictured in Figure 11. It follows that $\mathrm{ITF}_n^2 = 2^n$ for all $n \geq 1$. Thus by Theorem 2,

$$\begin{aligned}\mathcal{RT}(x,t) &= 1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risT}(F)} \\ &= \frac{1}{1 - \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} 2^n (x-1)^{n-1}} \\ &= \frac{1-x}{1 - x + \sum_{n\geq 1} \frac{(2(x-1)t^3)^n}{(3n)!}}.\end{aligned}$$

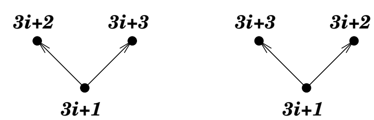

Figure 11: The two ways to label F_i for $F \in \mathcal{ITF}_n^2$.

We can use this generati ng function to compute the initial terms of the sequence $(\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risT}(F)})_{n\geq 1}$.

2

$76 + 4x$

$12104 + 1328x + 8x^2$

$5048368 + 843440x + 21776x^2 + 16x^3$

$4354721312 + 977383552x + 48921792x^2 + 349312x^3 + 32x^4$

$6736719017152 + 1898498010432x + 144468007808x^2 + 2715004544x^3 + 5592000x^4 + 64x^5$

5.2 IBF_n^2

One way to think of the set $\mathcal{IBF}_n^2$ is that it is the set of permutations that arise from a forest $F = (F_1, \ldots, F_n) \in \mathcal{F}_n^2$ such that the root elements are increasing from left to right. For example, if $n = 5$, then we are asking for labellings of the poset whose Hasse diagram is pictured at the top Figure 12 where, when there is an arrow from a node x to a node y, then the label of node x to be less than label of node y. We have given an example of such a labeling on the second line of Figure 12 and its corresponding permutation in $\mathcal{SF}_5^2$ in the third line of Figure 12. Thus we can think of $\mathcal{IBF}_n^2$ as the set of linear extensions of the poset whose Hasse diagram is of the form pictured in Figure 12.

We claim that

$$\mathrm{IBF}_n^2 = \prod_{k=1}^{n} 2\binom{3k-1}{2} = \frac{(3n)!}{3^n (n!)}.$$

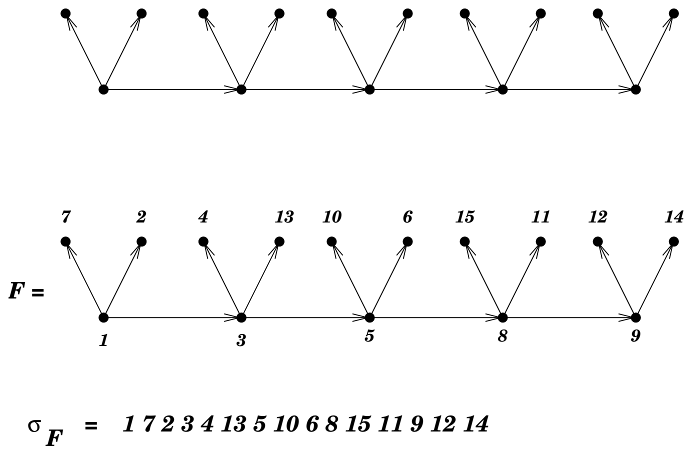

Figure 12: The poset for $\mathcal{IBF}_5^2$.

This is easy to prove by induction. First, it is easy to see from Figure 11 that

$$\mathrm{IBF}_1^2 = 2 = \frac{3!}{3}.$$

Thus the base case of our induction holds.

Now suppose that our formula holds for $k < n$. Let $F = (F_1, \ldots, F_n) \in \mathcal{IBF}_n^2$. Then consider Figure 12. It is easy to see that the label of the left-most root element must be 1 since there is a directed path from that element to any other element in the poset. Then we can choose the remaining two elements in F_1 in $\binom{3n-1}{2}$ ways and we have two ways to order the leaves of F_1. Thus we have $(3n-1)(3n-2)$ ways to pick F_1. Once we have picked the labels of F_1, the remaining labels for F are completely free. Thus if follows that

$$\begin{aligned}
\mathrm{IBF}_n^2 &= (3n-1)(3n-2)\mathrm{IBF}_{n-1}^2 \\
&= \prod_{k=1}^{n} (3k-1)(3k-2) = \frac{(3n)!}{3^n(n!)}.
\end{aligned}$$

Thus by Theorem 2,

$$\begin{aligned}
\mathcal{RB}(x,t) &= 1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \sum_{F \in \mathcal{F}_n^2} x^{\mathrm{risB}(F)} \\
&= \frac{1}{1 - \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \frac{(3n)!}{(3^n(n!))} (x-1)^{n-1}} \\
&= \frac{1-x}{1 - x + \sum_{n\geq 1} \frac{(\frac{1}{3}(x-1)t^3)^n}{n!}} \\
&= \frac{1-x}{-x + e^{\frac{1}{3}(x-1)t^3}}.
\end{aligned}$$

We can use this generating function to compute the initial terms of the sequence

$(\sum_{F\in\mathcal{F}^2_n} x^{\mathrm{risB}(F)})_{n\geq 1}$.

2
$40(1+x)$
$2240\left(1+4x+x^2\right)$
$246400\left(1+11x+11x^2+x^3\right)$
$44844800\left(1+26x+66x^2+26x^3+x^4\right)$
$12197785600\left(1+57x+302x^2+302x^3+57x^4+x^5\right)$

It follows from the generating function $RB(x,t)$ that

$$\sum_{F\in\mathcal{F}^2_n} x^{\mathrm{risB}(F)} = \frac{(3n)!}{3^n n!}\sum_{\sigma\in S_n} x^{\mathrm{ris}(\sigma)}. \tag{18}$$

In fact this is easy to see directly. Suppose that we are given a permutation $\tau=\tau_1\ldots\tau_n\in S_n$. Then we claim that there are $\frac{(3n)!}{3^n n!}$ ways to create an $F=F_1\ldots F_n\in\mathcal{F}^2_n$ such that if $\sigma_F=\sigma_1\ldots\sigma_{3n}$, then $\mathrm{red}(\sigma_1\sigma_4\ldots\sigma_{3n-2})=\tau$. That is, suppose that $\sigma_{t_k}=k$ for $k=1,\ldots,n$. We let 1 be the label of the root of F_{j_1} and then we have $(3n-1))(3n-2)$ ways to pick the right and left leaves of F_{j_1}. Once we have fixed F_{j_1}, we let c_2 be the smallest element c in $\{1,\ldots,3n\}$ such that c is not a label in F_{j_1}. We label the root of F_{j_2} with c_2 and then we have $(3n-4)(3n-5)$ ways to pick the right and left leaves of F_{j_2}. Once we have fixed F_{j_1} and F_{j_2}, we let c_3 be the smallest element c in $\{1,\ldots,3n\}$ such that c is not a label in F_{j_1} or F_{j_2}. We label the root of F_{j_3} with c_3 and then we have $(3n-7)(3n-8)$ ways to pick the right and left leaves of F_{j_3}. Continuing on in this way, we see that there are $\prod_{i=0}^{n-1}(3n-(3k+1))(3n-(3k+2)=\frac{(3n)!}{3^n n!}$ ways to create an $F=F_1\ldots F_n\in\mathcal{F}^2_n$ such that if $\sigma_F=\sigma_1\ldots\sigma_{3n}$, then $\mathrm{red}(\sigma_1\sigma_4\ldots\sigma_{3n-2})=\tau$. Observe that for any F created in this way, $\mathrm{risB}(F)=\mathrm{ris}(\tau)$. Thus (18) easily follows.

5.3 ILF^2_n

The set $\mathcal{ILF}^2_n$ is the set of forests $F=(F_1,\ldots,F_n)\in\mathcal{F}^2_n$ such that $F_1<_L F_2<_L<_L\cdots<_L F_n$. Such a forest can be considered to a be labeling of a poset $\mathbb{L}_{3n}$ of the type whose Hasse diagram is pictured in Figure 13. For example, at the bottom of Figure 13, we have redrawn the poset in a nicer form. Here when we draw an arrow from node x to a node y, then we want the label of node x to be less than label of node y in $\mathbb{L}_{3n}$. Thus the Hasse diagram of $\mathbb{L}_{3n}$ consists of 3 rows of n nodes such that there are arrows connecting the nodes in each row which go from left to right and, in each column, there are arrows going from the node in the middle row to the nodes at the top and bottom of that column. Let $\mathcal{L}_{3n}$ denote the set of all linear extensions of $\mathbb{L}_{3n}$, that is, the set of all labellings of $\mathbb{L}_{3n}$ with the numbers $\{1,\ldots,3n\}$ such that if there is an arrow from node x to y, then the label on node x is less than the label on node y. Thus $\mathrm{ILF}^2_n=|\mathcal{L}_{3n}|$.

We then have the following theorem.

Theorem 3. $\mathrm{ILF}^2_n=\frac{4^n(3n)!}{(n+1)!(2n+1)!}$.

Proof. G. Kreweras [6] proved that $\frac{4^n(3n)!}{(n+1)!(2n+1)!}$ is the number of paths $P = (p_1, \ldots, p_{3n})$ in the plane which start at (0,0) and end at (0,0), stay entirely in the first quadrant, and uses only northeast steps $(1,1)$, west steps $(-1,0)$, and south steps $(0,-1)$. See also [2] and [4]. The fact that the P starts and ends at $(0,0)$ means that P has n northeast steps, n west steps, and n south steps. For any $1 \leq i \leq 3n$, let $NE_i(P)$ equal the number of northeast steps in $(p_1, \ldots, p_i)$, $W_i(P)$ equal the number of west steps in $(p_1, \ldots, p_i)$, and $S_i(P)$ equal the number of south steps in $(p_1, \ldots, p_i)$. The fact that P stays in the first quadrant is equivalent to the conditions that $NE_i(P) \geq W_i(P)$ and $NE_i(P) \geq S_i(P)$ for $i = 1, \ldots, 3n$. Let $\mathcal{P}_{3n}$ denote the set of all such paths P of length $3n$.

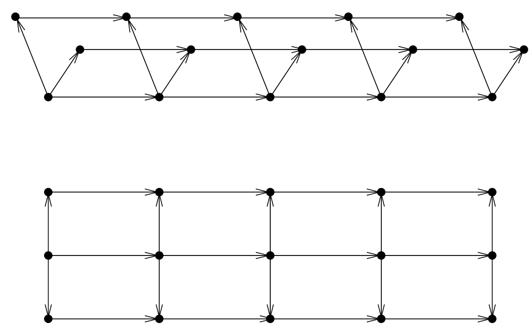

Figure 13: The poset for $\mathcal{IBF}_5^2$.

To prove our theorem, we shall define a bijection from $\Gamma : \mathcal{L}_{3n} \rightarrow \mathcal{P}_{3n}$. The map Γ is quite simple, given a labeling $L \in \mathcal{L}_{3n}$, we let $\Gamma(L) = (p_1, \ldots, p_{3n})$ be the path which starts at (0,0) and where p_i is a northeast step if the label i is in the middle row of L, p_i is west step if the label i is in the top row of L, and p_i is a south step if the label i is in the bottom row L. An example of this map is given in Figure 14 where is have put a label i on the i^{th}-step of the $\Gamma(L)$.

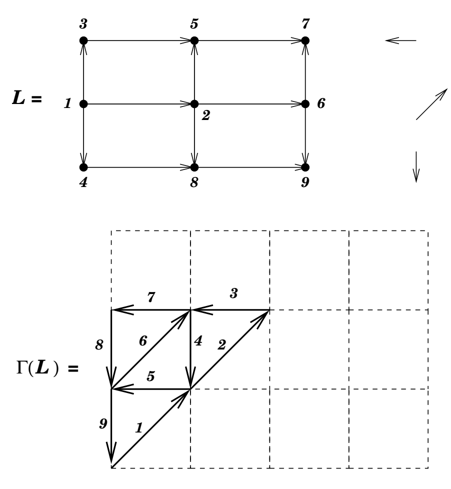

Figure 14: The bijection Γ.

First we must check that if $L \in \mathcal{L}_{3n}$, then $\Gamma(L) = (p_1, \ldots, p_{3n})$ is an element of $\mathcal{P}_{3n}$. It is easy to see that $\Gamma(L)$ starts and ends at $(0,0)$ since $\Gamma(L)$ has n northeast steps, n west steps, and n south steps. Let LT_i, LM_i, and LB_i denote the label in L of the i^{th} element of the top row, middle row, and bottom row, reading from left to right, respectively. Suppose for a contradiction that there is an t such that $i = W_t(P) > NE_t(P) = j$. This is impossible since this would imply that $LT_i \leq t$ and $LM_i > t$ which violates that fact that there is an arrow from the element in the middle row of the i^{th}-column to the element in the top row of i^{th}-column in $\mathbb{L}_{3n}$. Similarly, suppose that there is an t such that $i = S_t(P) > NE_t(P) = j$. This is impossible since this would imply that $LB_i \leq t$ and $LM_i > t$ which violates that fact that there is an arrow from the element in the middle row of the i^{th}-column to the element in the bottom row of i^{th}-column in $\mathbb{L}_{3n}$. Thus for all t, $NE_t(\Gamma(L)) \geq W_t(\Gamma(L))$ and $NE_t(\Gamma(L)) \geq S_t(\Gamma(L))$ which means that $\Gamma(L)$ stays in the first quadrant.

It is easy to see that Γ is one-to-one. That is, if L and L' are two different labellings in $\mathcal{L}_{3n}$, then let i be the least j such that j is not in the same position in the labellings L and L'. Then clearly, $\Gamma(L) \neq \Gamma(L')$ since the i^{th} step of $\Gamma(L)$ will not be the same as the i^{th} step of $\Gamma(L')$. To see that Γ maps onto $\mathcal{P}_{3n}$, suppose that we are given $P = (p_1, \ldots, p_{3n})$ in $\mathcal{P}_{3n}$. Let L be the labeling of $\mathbb{L}_{3n}$ which is increasing in the rows of $\mathbb{L}_{3n}$ such that i is label in the top row of $\mathbb{L}_{3n}$ if p_i is a west step, i is label in the middle row of $\mathbb{L}_{3n}$ if p_i is a northeast step, and i is label in the bottom row of $\mathbb{L}_{3n}$ if p_i is a south step. It is easy to see from our definitions that $\Gamma(L) = P$. Hence the only thing that we have to do is to check that $L \in \mathcal{L}_{3n}$. Since L is increasing in rows, we need only check that the for each column i, the label x of the element in the middle row of column i is less than the label y of the element in the top row of column i and and less than the label z of the element of the bottom row of column i. But this follows from the fact that P stays in the first quadrant. That is, if $y < x$, then in $(p_1, \ldots, p_y)$, we would have more west steps than northeast steps which would mean that the y^{th} step of P is not in the first quadrant. Similarly if $z < x$, then in $(p_1, \ldots, p_z)$, we would have more south steps than northeast steps which would mean that the z^{th} step of P is not in the first quadrant. Thus Γ is a bijection from $\mathcal{L}_{3n}$ onto $\mathcal{P}_{3n}$.

$\square$

Thus by Theorem 2,

$$
\begin{aligned}
\mathcal{RL}(x,t) &= 1 + \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risL}(F)} \\
&= \frac{1}{1 - \sum_{n\geq 1} \frac{t^{3n}}{(3n)!} \frac{4^n((3n)!)}{(n+1)!(2n+1)!}(x-1)^{n-1}} \\
&= \frac{1-x}{1 - x + \sum_{n\geq 1} \frac{(4(x-1)t^3)^n}{(n+1)!(2n+1)!}}.
\end{aligned}
$$

We can use this generating function to compute the initial terms of the sequence

$(\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risL}(F)})_{n\geq 1}$.

$$2$$
$$16(4+x)$$
$$192\left(43+26x+x^2\right)$$
$$2816\left(983+975x+141x^2+x^3\right)$$
$$46592\left(41141+57086x+16506x^2+766x^3+x^4\right)$$
$$835584\left(2848169+5084786x+2311247x^2+261973x^3+4324x^4+x^5\right)$$

5.4 IAF_n^2.

As with our other examples, we can think of IAF_n^2 as the number of linear extensions of a poset of the type whose Hasse diagram is pictured at the top of Figure 15. That is, the Hasse diagram of A_n consists of n binary shrubs where there is an arrow from the right most element of each shrub to the left-most element of the next shrub. In fact, we will want to consider three related posets, E_n, S_n, and B_n. E_n is the poset whose Hasse diagram starts with the Hasse diagram of A_n and adds one extra node which is connected to the Hasse diagram of A_n by an arrow that goes from the right-most node of the right-most binary shrub to the new node. S_n is the poset whose Hasse diagram starts with Hasse diagram of A_n and adds one extra node which is connected to the Hasse diagram of A_n by an arrow that goes from the new node to the left-most node of the left-most binary shrub. B_n is the poset whose Hasse diagram starts with the Hasse diagram of A_n and adds two extra nodes, one which is connected as in E_n and one which is connected as in S_n. Thus the Hasse diagram of E_n starts with Hasse diagram of A_n and adds an extra node at the end, the Hasse diagram of S_n starts with the Hasse diagram of A_n and adds a extra node at the start, and the Hasse diagram of B_n starts with the Hasse diagram of A_n and adds both an extra node at the end and a extra node at the start. For example, Figure 15 pictures A_5, E_5, S_5, and B_5.

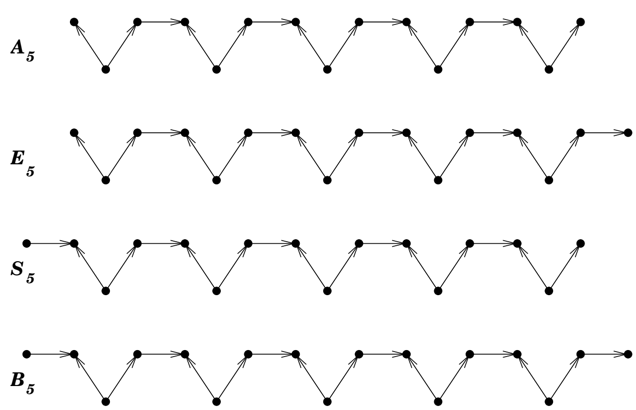

Figure 15: The posets A_5, E_5, S_5, and B_5.

For $Z \in \{A,E,S,B\}$, we let $\mathcal{LZ}_n$ denote the set of linear extensions of Z_n and $\mathrm{LZ}_n = |\mathcal{LZ}_n|$. We claim that we can develop simple recursions for LA_n, LE_n, LS_n, and

LB_n. First in Figures 16 and 17, we have listed all the elements of $\mathcal{LA}_1$, $\mathcal{LE}_1$, $\mathcal{LS}_1$, and $\mathcal{LB}_1$. Thus

$$\mathrm{LA}_1 = 2,\ \mathrm{LE}_1 = 3,\ \mathrm{LS}_1 = 5,\ \text{and } \mathrm{LB}_1 = 9.$$

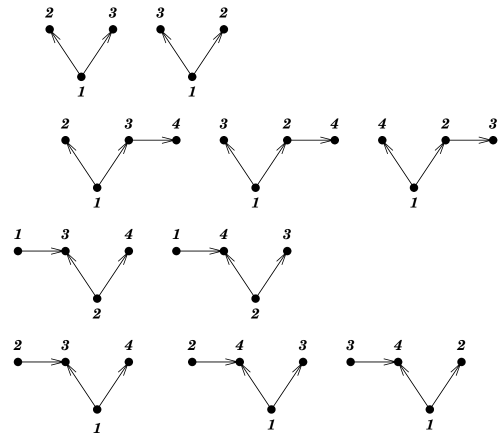

Figure 16: The elements of $\mathcal{LA}_1$, $\mathcal{LE}_1$, and $\mathcal{LS}_1$.

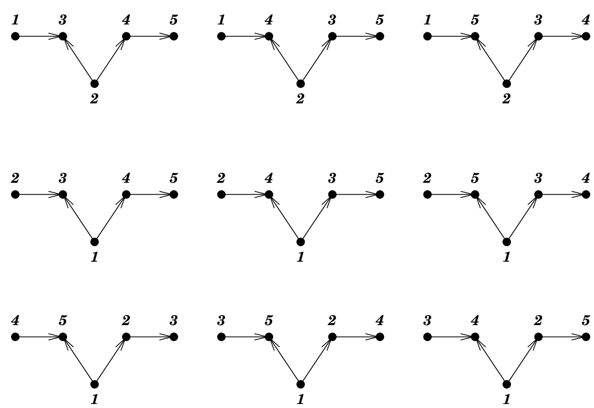

Figure 17: The elements of $\mathcal{LB}_1$.

We start with the recursion for LB_n. Now suppose that $n > 1$. Then consider where the label 1 can be in an element of $\mathcal{LB}_n$. There are four cases to consider. First, 1 could be the label of the left-most element in which case the remaining labels must correspond to a linear extension of E_n. Otherwise 1 is the label of the root of the k^{th} binary shrub for some $k = 1, \ldots, n$. If $1 < k < n$, then there is no relation that is forced between the labels to left of 1 which correspond to a linear extension of B_{k-1} and the labels to the right of 1 which correspond to a linear extension of B_{n-k}. In the special case where $k = 1$, the Hasse diagram of the poset to the left of the node labeled 1 is just a 2 element chain which we call B_0. Similarly, in special case where $k = n$, the Hasse diagram of the poset to the right of the node labeled 1 is just B_0. Clearly, $\mathrm{LB}_0 = 1$. These four cases are pictured in Figure 18. For each $k = 1, \ldots, n$, we have $\binom{3n+1}{3(k-1)+2}$ ways to choose the labels of the elements to the left of 1. It follows that

$$\mathrm{LB}_n = \mathrm{LE}_n + \sum_{k=1}^{n} \binom{3n+1}{3(k-1)+2} \mathrm{LB}_{k-1}\mathrm{LB}_{n-k}. \tag{19}$$

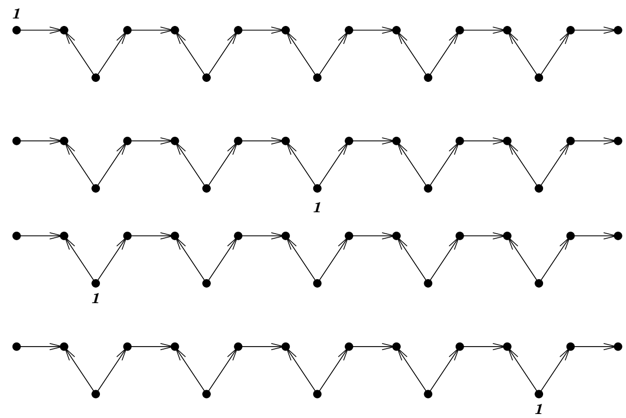

Figure 18: The recursion for LB_n.

Next consider the recursion for LS_n. Now suppose that $n > 1$. Then consider where the label 1 can be in an element of $\mathcal{LS}_n$. Again there are four cases to consider. First, 1 could be the label of the left-most element in which case the remaining labels must correspond to a linear extension of A_n. Otherwise 1 is the label of the root of the k^{th} binary shrub for some $k = 1, \ldots, n$. If $1 < k < n$, then there is no relation that is forced between the labels to left of 1 which correspond to a linear extension of B_{k-1} and the labels to the right of 1 which correspond to a linear extension of S_{n-k}. In the special case where $k = 1$, the Hasse diagram of the poset to the left of the node labeled 1 is just B_0. Similarly, in special case where $k = n$, the Hasse diagram of the poset to the right of the node labeled 1 is a one element poset which we call S_0. Clearly, $\mathrm{LS}_0 = 1$. These four cases are pictured in Figure 19. For each $k = 1, \ldots, n$, we have $\binom{3n}{3(k-1)+2}$ ways to choose the labels of the elements to the left of 1. It follows that got $n \geq 2$,

$$\mathrm{LS}_n = \mathrm{LA}_n + \sum_{k=1}^{n} \binom{3n}{3(k-1)+2} \mathrm{LB}_{k-1} \mathrm{LS}_{n-k}. \tag{20}$$

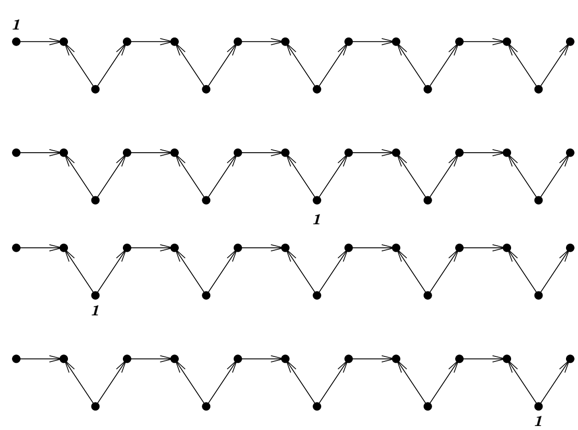

Figure 19: The recursion for LS_n.

Next consider the recursion for LE_n. Now suppose that $n > 1$. Then consider where the label 1 can be in an element of $\mathcal{LS}_n$. In this case, there are three cases to consider.

That is, 1 must be the label of the root of the k^{th} binary shrub for some $k = 1, \ldots, n$. If $1 < k < n$, then there is no relation that is forced between the labels to left of 1 which correspond to a linear extension of E_{k-1} and the labels to the right of 1 which correspond to a linear extension of B_{n-k}. In the special case where $k = 1$, the Hasse diagram of the poset to the left of the node labeled 1 is just a one element poset which we will also call E_0. Clearly, $\mathrm{LE}_0 = 1$. Similarly, in special case where $k = n$, the Hasse diagram of the poset to the right of the node labeled 1 is just B_0. These three cases are pictured in Figure 20. For each $k = 1, \ldots, n$, we have $\binom{3n}{3(k-1)+1}$ ways to choose the labels of the elements to the left of 1. It follows that

$$\mathrm{LE}_n = \sum_{k=1}^{n} \binom{3n}{3(k-1)+1} \mathrm{LE}_{k-1} \mathrm{LB}_{n-k}. \tag{21}$$

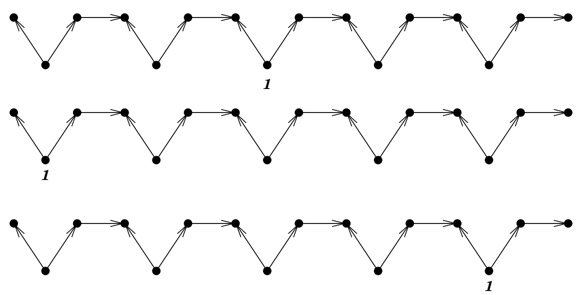

Figure 20: The recursion for LE_n.

Finally consider the recursion for LA_n. Now suppose that $n > 1$. Then consider where the label 1 can be in an element of $\mathcal{LS}_n$. In this case, there are three cases to consider. That is, 1 must be the label of the root of the k^{th} binary shrub for some $k = 1, \ldots, n$. If $1 < k < n$, then there is no relation that is forced between the labels to left of 1 which correspond to a linear extension of E_{k-1} and the labels to the right of 1 which correspond to a linear extension of S_{n-k}. In the special case where $k = 1$, the Hasse diagram of the poset to the left of the node labeled 1 is E_0. Similarly, in special case where $k = n$, the Hasse diagram of the poset to the right of the node labeled 1 is just S_0. These three cases are pictured in Figure 21. For each $k = 1, \ldots, n$, we have $\binom{3n1}{3(k-1)+1}$ ways to choose the labels of the elements to the left of 1. It follows that for $n \geq 2$,

$$\mathrm{LA}_n = \sum_{k=1}^{n} \binom{3n-1}{3(k-1)+1} \mathrm{LE}_{k-1} \mathrm{LS}_{n-k}. \tag{22}$$

One can check directly that (19), (21), (20), and (22) also hold for $n = 1$. One can then iterate these recursions to obtain the first few terms of the sequences $(\mathrm{LA}_n)_{n\geq 0}$, $(\mathrm{LB}_n)_{n\geq 0}$, $(\mathrm{LE}_n)_{n\geq 0}$, and $(\mathrm{LS}_n)_{n\geq 0}$. For example, the first few terms of $(\mathrm{LA}_n)_{n\geq 0}$ are

$$1, 2, 40, 3194, 666160, 287316122, 222237912664, 280180369563194,$$
$$537546603651987424, 1490424231594917313242, 5735930050702709579598280, \ldots$$

The first few terms of $(\mathrm{LB}_n)_{n\geq 0}$ are

$$1, 9, 477, 74601, 25740261, 16591655817, 17929265150637, 30098784753112329,$$
$$74180579084559895221, 256937013876000351610089, 1208025937371403268201735037, \ldots$$

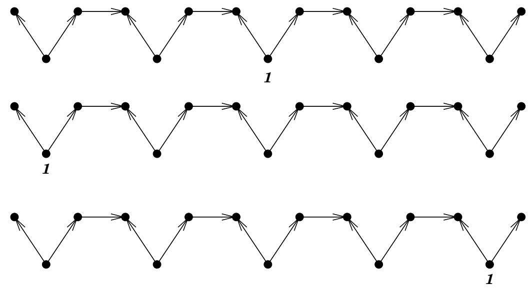

Figure 21: The recursion for LA_n.

The first few terms of $(\mathrm{LE}_n)_{n\geq 0}$ are

$$1, 3, 99, 11259, 3052323, 1620265923, 1488257158851, 2172534146099019,$$
$$4736552519729393091, 14708695606607601165843, 62671742039942099631403299, \ldots$$

The first few terms of $(\mathrm{LS}_n)_{n\geq 0}$ are

$$1, 5, 169, 19241, 5216485, 2769073949, 2543467934449, 3712914075133121,$$
$$8094884285992309261, 25137521105896509819605, 107107542395866078895709049 \ldots$$

None of these sequences appear in the OEIS [9].

One can also study the generating functions

$$\begin{aligned}
\mathcal{A}(t) &= 1 + \sum_{n\geq 1} \frac{\mathrm{LA}_n t^{3n}}{(3n)!},\\
\mathcal{E}(t) &= \sum_{n\geq 0} \frac{\mathrm{LE}_n t^{3n+1}}{(3n+1)!},\\
\mathcal{S}(t) &= \sum_{n\geq 0} \frac{\mathrm{LS}_n t^{3n+1}}{(3n+1)!}, \text{ and}\\
\mathcal{B}(t) &= \sum_{n\geq 0} \frac{\mathrm{LB}_n t^{3n+2}}{(3n+2)!}.
\end{aligned}$$

It is straightforward to show that the recursions (19), (21), (20), and (22) imply that the following differential equations hold:

$$\begin{aligned}
\mathcal{A}'(t) &= \mathcal{E}(t)\mathcal{S}(t),\\
\mathcal{E}'(t) &= 1 + \mathcal{E}(t)\mathcal{B}(t),\\
\mathcal{S}'(t) &= \mathcal{A}(t) + \mathcal{B}(t)\mathcal{S}(t), \text{ and}\\
\mathcal{B}'(t) &= t + \mathcal{E}(t) + (\mathcal{B}(t))^2.
\end{aligned}$$

Note that it follows from the last differential equation that

$$\mathcal{B}'(t) - t - (\mathcal{B}(t))^2 = \mathcal{E}(t),$$

which can be plugged into the second differential equation to show that

$$\mathcal{B}''(t) = 2 + 3\mathcal{B}'(t)\mathcal{B}(t) - t\mathcal{B}(t) - (\mathcal{B}(t))^3. \tag{23}$$

Thus in principle, we can obtain a recursion for the LB_n in terms of $\mathrm{LB}_0, \ldots, \mathrm{LB}_{n-1}$ which in turn can lead to more direct recursions for LE_n, LS_n, and LA_n. However, all such recursions are more complicated than the family of recursions described above.

One can use the initial terms of the sequence $(\mathrm{LA}_n)_{n\geq 0}$ to compute the initial terms of the sequence $(\sum_{F\in\mathcal{F}_n^2} x^{\mathrm{risA}(F)})_{n\geq 1}$.

$$2$$
$$40(1+x)$$
$$3194 + 7052x + 3194x^2$$
$$880\left(757 + 2603x + 2603x^2 + 757x^3\right)$$
$$2\left(143658061 + 671012156x + 1061347566x^2 + 671012156x^3 + 143658061x^4\right)$$
$$136\left(1634102299 + 9646627503x + 21007526198x^2 + 21007526198x^3 + 9646627503x^4 + 1634102299x^5\right)$$

6 Conclusions

In this paper, we computed the generating function of 5 different kinds of rises in forests of binary shrubs. Our work can be viewed as the first step in studying consecutive patterns in forests of binary shrubs. We will study such patterns in a subsequent paper.

In addition, we can also study the analogues of up-down permutation relative total rises, base rises, lexicographic rises, and adjacent rises. For example, we say that an $F = F_1 \ldots F_n \in \mathcal{F}_n^2$ is an up-down forest with respect to the $<_T$ is $RiseT(F)$ equals the set of odd numbers less than n. We also will study such analogues of up-down permutations in a subsequent paper.

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References

[1] D. Bevan, D. Levin, P. Nugent, J. Pantone, L. Pudwell, M. Riehl, and ML Talchac, Pattern avoidance in forests of binary shrubs, Discrete Mathematics and Theoretical Computer Science, **18:2** (2016), article no. 8.

[2] M. Bousquet-Mélou, Walks in the quarter plane: Kerweras' algebraic model, The Annals of Applied Probability, **15** no. 2 (2005) 1451-1491.

[3] O. Eğecioğlu and J. B. Remmel, Brick tabloids and the connection matrices between bases of symmetric functions, *Discrete Appl. Math.*, **34** (1991), no. 1-3, 107–120, Combinatorics and theoretical computer science (Washington, DC, 1989).

[4] I.M. Gessel, A probabilistic method for lattice path enumeration, Journal of Statistical Planning and Inference, **14** (1986), 49-58.

[5] S. Kiteav, *Patterns in permutations and words*, Springer-Verlag, 2011.

[6] G. Kreweras, Sur une class de problèmes li'es au triellis des partitions d'entries, Cahiers du B.U.R.O., **6** (1965), 5-105.

[7] D. Levin, L. Pudwell, M. Riehl, and A. Sandberg, pattern avoidance in k-ary heaps, Australasion Journal of Combinatorics, **64.1** (2016), 120-139.

[8] A. Mendes and J. Remmel, *Counting with Symmetric Functions*, Development in Mathematics Vol. 43, Springer, (2015).

[9] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at `http://www.research.att.com/~njas/sequences/`.

[10] R.P. Stanley, *Enumerative Combinatorics, vol. 2*, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, (1999).