Touchard's Drunkard

Nachum Dershowitz*
School of Computer Science
Tel Aviv University
Ramat Aviv, Israel
nachum@cs.tau.ac.il

December 14, 2016

> You're a baby and as stupid as a Frenchman. You persist in thinking that it's the same as it was at Touchard's, and that I'm as stupid as at Touchard's. . . . But I'm not so silly as I was at Touchard's. . . . I was drunk yesterday, but not from wine, but because I was excited.
>
> —Fyodor Dostoyevsky, *The Raw Youth* (1875)

Abstract

Based on Touchard's identity, a simple derivation is given for the enumeration of the N/S/E/W walks that remain on the north side of the origin.

1 Introduction: Drunken walks

An inebriated person in Nice (see Figure 1) takes a walk, each step in one of the four cardinal directions, north (N), south (S), east (E), and west (W). We are interested in those walks beginning at the center of the Promenade des Anglais (at the southern end of town[1]) and ending anywhere on the promenade—all the while remaining on land (in other words, not venturing south of the promenade). In how many possible ways can such walks meander?

*Corresponding author. This research benefitted from a fellowship at the Paris Institute for Advanced Studies (France), with the financial support of the French state, managed by the French National Research Agency's "Investissements d'avenir" program (ANR-11-LABX-0027-01 Labex RFIEA+).

[1]And site of recent carnage.

Figure 1: Center of Nice, France, with the promenade at its southern end.

Valid				Invalid			
N	N	S	S	N	S	S̲	N
N	S	N	S	S̲	N	N	S
N	S	E	E	S̲	N	E	E
N	S	E	W	S̲	N	E	W
N	S	W	E	S̲	N	W	E
N	S	W	W	S̲	N	W	W
N	E	S	E	S̲	E̲	N	E
N	E	S	W	S̲	E̲	N	W
N	W	S	E	S̲	W̲	N	E
N	W	S	W	S̲	W̲	N	W
N	E	E	S	S̲	E̲	E̲	N
N	E	W	S	S̲	E̲	W̲	N
N	W	E	S	S̲	W̲	E̲	N
N	W	W	S	S̲	W̲	W̲	N

Valid				Invalid			
				S̲	N	S̲	N
				S̲	S̲	N̲	N
E	N	S	E	E	S̲	N	E
E	N	S	W	E	S̲	N	W
W	N	S	E	W	S̲	N	E
W	N	S	W	W	S̲	N	W
E	N	E	S	E	S̲	E̲	N
E	N	W	S	E	S̲	W̲	N
W	N	E	S	W	S̲	E̲	N
W	N	W	S	W	S̲	W̲	N
E	E	N	S	E	E	S̲	N
E	W	N	S	E	W	S̲	N
W	E	N	S	W	E	S̲	N
W	W	N	S	W	W	S̲	N

Table 1: All Touchard walks of length 4 with equal quantities of N-steps and S-steps, 26 valid and 28 not, besides 16 valid E/W walks with no N/S-steps at all. The illegal steps in the Mediterranean are underlined. Walks with unequal numbers of N- and S-steps are always invalid.

Let

$$D_n = \begin{cases} \text{the number of } \textit{Touchard walks}\text{, consisting of a sequence of } n \\ \text{steps, each of which is one of N/S/E/W, such that at each} \\ \text{point along the way the number of N-steps that have been} \\ \text{taken is never less than the number of S-steps, and, further-} \\ \text{more, in the end they are equal (with no restrictions on the} \\ \text{distribution of E- or W-steps).} \end{cases}$$

These are walks that remain in the half-plane and return to the boundary. Table 1 lists valid and invalid walks of length $n = 4$. For an example of a longer walk, see Figure 2.

Theorem. *The number of Touchard walks with a total of n steps is*

$$D_n = C_{n+1},$$

where C_i is the ith Catalan number, $\frac{1}{i+1}\binom{2i}{i}$ (sequence A000108 in Sloane's *Encyclopedia of Integer Sequences* [22]).

There are, for instance, $C_5 = 42$ valid 4-step walks, listed in Table 1.

Guy [11] points out that this equality "is not well known! ... nor can we immediately see any correspondence between [Touchard] walks and any of the manifestations [of Catalan objects]." We aim to fill this lacuna.

For a history of Catalan enumerations, see [23, Appendix B].

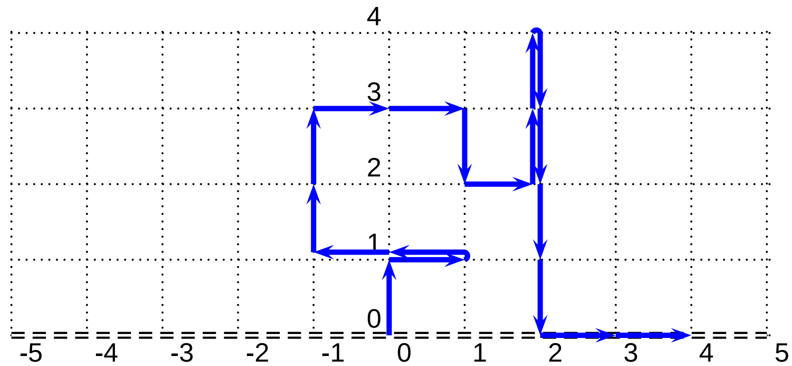

Figure 2: A valid walk, N E W W N N E E S E N N S S S S E E, consisting of 18 steps, 5 N, 5 S, 6 E, and 2 W.

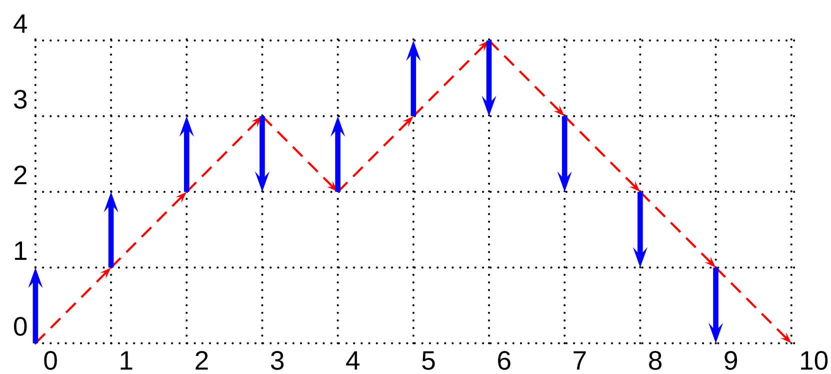

Figure 3: The ten N/S-steps of Figure 2, stretched out on a timeline: N N N S N N S S S S. Connecting the tails of the steps yields a Dyck path of NE/SE steps.

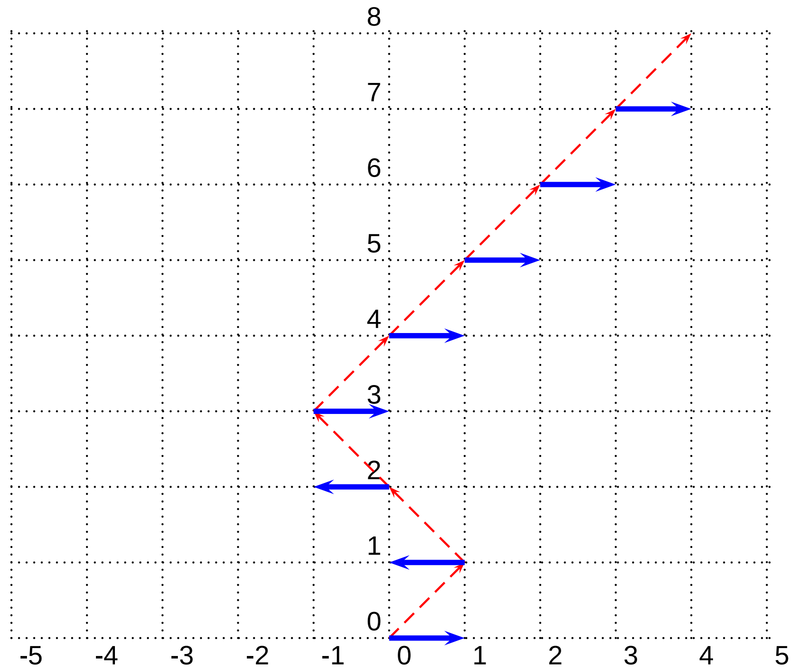

Figure 4: The eight E/W-steps of Figure 2: E W W E E E E E.

2 Enumeration: Touchard's identity

Touchard's [26] identity (see, for example, [14, p. 319]) states that

$$C_{n+1} = \sum_i C_i 2^{n-2i} \binom{n}{2i}.$$

For a nice proof of this, see [21].

Considering the above theorem and this identity, we can understand the drunken walks in Table 1, for $n = 4$, as comprising $\frac{1}{1}\binom{0}{0}2^4\binom{4}{0} = 16$ valid walks with no ($i = 0$) north-south steps, $\frac{1}{2}\binom{2}{1}2^2\binom{4}{2} = 24$ walks containing one ($i = 1$) north-step followed at some point by one south-step, and $\frac{1}{3}\binom{4}{2}2^0\binom{4}{4} = 2$ walks with two ($i = 2$) north-steps and two matching south-steps.

With Touchard's identity, the proof of the theorem is immediate:

1. Suppose there are i N-steps and i S-steps, for some i in the range $[0 \,..\, n/2]$, leaving $n - 2i$ steps of type E or W.

2. The factor C_i counts the patterns consisting of i N-steps and an equal number of S-steps, starting and ending on the promenade, and never venturing further south (see Figure 3). This is one of the many well-known instances of Catalan enumerations, and is a special case of the famous "ballot problem," stated and solved by Whitworth back in 1878 [27].

N E W W N N E E S E N N S S S S E E

⇔

N NN NS SN SN NN NN NS NS SS NS NN NN SS SS SS SS NS NS S

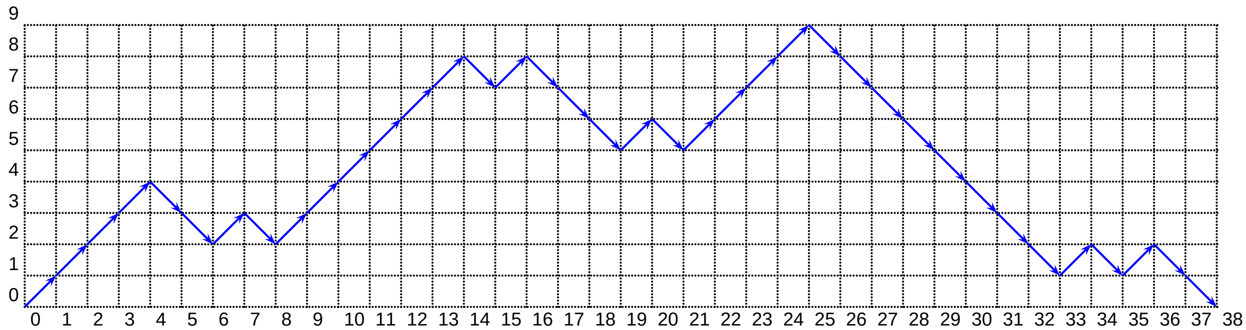

Figure 5: The Dyck path corresponding to the Touchard walk of Figure 2.

3. The factor 2^{n-2i} counts the patterns of the remaining unconstrained E/W-steps (see Figure 4).

4. The factor $\binom{n}{2i}$ is the number of ways of interspersing $2i$ N/S-steps among $n - 2i$ E/W-steps.

3 Bijection: Dyck paths

Walks whose steps are only north or south and stay on land (as in step 2 in the above proof) correspond to the well-known Dyck (monotonic lattice) paths [9, 14, pp. 151–153], which spread out the steps over a timeline that runs from west to east. Dyck paths are usually depicted as consisting of equal numbers of NE- and SE-steps, staying the whole time north of the origin. They are counted by the Catalan numbers. For an example, see the dashed line in Figure 3. Alternatively, such paths may be viewed as consisting of N- and E-steps, never going below the $y = x$ diagonal; see [16, pp.1–4].

Alternatively, Dyck paths may be viewed as consisting of N/S-steps meeting the requirements that (a) the number of south steps—throughout the walk—never exceeds the number of north ones and (b) that—at the end—they be equal. The enumeration D_n counts walks in any of the four directions (N/S/E/W) abiding by the identical constraints.

To relate the two kinds of walks, consider a Dyck path of length $2n + 2$, of which there are C_{n+1}. It must start with N and end with S. Forget those two steps. Then start from the beginning and replace as follows: NN $\mapsto$ N, SS $\mapsto$ S, NS $\mapsto$ E, SN $\mapsto$ W. The result is a Touchard walk of length n. The reverse direction of this bijection is straightforward. See Figure 5.

This construction is similar to one used in [6, 12]. Touchard walks are also easily seen to be in bijection with two-colored Motzkin paths [3] (the two colors being E and W). These in turn are in bijection with ballot sequences [25] or Dyck paths [9] in a manner similar to the above; see [23, item 40].

4 Extension: More dimensions

Walks can be entertained in more dimensions with varying degrees of restriction.

a. For each dimension with its two opposing directions (like N and S above) that must stay to one side of the origin and return to zero (the promenade in our example) at the end, there is a Catalan factor

$$C_i = \frac{1}{i+1}\binom{2i}{i},$$

accounting for $2i$ steps, i in each direction (A000108). This component represents ballot sequences or Dyck paths.

b. For each dimension that must return to zero at the end (but need not stay on one side of the origin), there is a central binomial factor

$$A_{2i} = \binom{2i}{i}$$

for i steps in each of the two directions, in any order (A000984). These are the linear "drunken walks" studied by Polyá [19], also called "grand-Dyck" paths [18].

c. For each dimension that must stay on one side of the origin (but need not return to zero at the end), there is a central binomial factor

$$A_j = \binom{j}{\lfloor j/2 \rfloor}$$

for a total of j steps (A001405). This component represents ballot sequences with an uneven number of votes for the two candidates or prefixes of Dyck paths.

d. For any remaining r unrestricted directions (like E and W above), there is an exponential factor

$$r^{n-m}$$

covering the $n-m$ steps that are not yet accounted for, where $m = 2i_1 + 2i_2 + \cdots + j_1 + \cdots$, the i_k for each case (*a*) or (*b*) and the j_k for cases (*c*). When all directions are accounted for by cases (*a*–*c*) and $r = 0$, we must have $m = n$ and this factor is 1.

- To fix which of the n steps belong to which category, there is a multinomial choice

$$\binom{n}{2i_1, \ldots, j_1, \ldots, n-m}.$$

The steps in dimensions adhering to cases (*a*) and (*b*) have an even number $2i_k$ of steps; cases (*c*) and (*d*) can have an odd number j_k of steps.

- All the factors are summed for all possible values of the indices:

$$\sum_{i_1,i_2,\ldots,j_1,j_2,\ldots} r^{n-m} C_{i_1} C_{i_2} \cdots A_{2i_\ell} \cdots A_{j_1} \cdots \binom{n}{2i_1, 2i_2, \ldots, j_1, \ldots, n-m},$$

  where $m = 2i_1 + 2i_2 + \cdots + j_1 + j_2 + \cdots$. If $r = 0$, however, the sum is

$$\sum_{m=n} C_{i_1} C_{i_2} \cdots A_{2i_\ell} \cdots A_{j_1} \cdots \binom{n}{2i_1, 2i_2, \ldots, j_1, \ldots}.$$

We can indicate the type of walk by a multiset of letters for the relevant cases. Each dimension contributes a letter a–e, where e is short for dd, meaning that there are no restrictions on steps in that dimension, whereas d means that the dimension is one-way only. Our drunkard's walk, then, is of type ae, being confined to the northern half of the plane but unrestricted longitudinally.

One-dimensional paths of types a, b, c, ee are classified in [2] as *excursions, bridges, meanders, and walks*, respectively, based on terminology of the theory of Brownian motion.

Whenever there are only dimensions of types a and b, the number of walks is 0 for an odd number n of steps.

Motzkin paths [1, pp. 300–301] are like Dyck paths but allow arbitrary horizontal E-steps in addition to NE and SE. They are equivalent to walks of type ad and are enumerated by the Motzkin numbers (A001006),

$$M_n = \sum_i C_i \binom{n}{2i}.$$

Were we to insist that our drunkard return to the origin at the end of an evening of wanderings, then those would be walks of type ab, which are counted by

$$\sum_i C_{\frac{n}{2}-i} \binom{n}{2i} \binom{2i}{i} = C_{\frac{n}{2}} \binom{n+1}{n/2}$$

for even n [24]. This is A000891$(n/2)$. When n is odd, there is—of course—no way home. (Nagy [17] finds related formulæ for the case when an N/S-walk crosses the abscissa an even number of times going south.)

The simplification of the above sum for walks of type ab, as well as the next three, may be seen as the result of a few applications of binomial cancellation (the "subset of subsets

	a	b	c	d	e
a	A005568*	A000891*	A001700	A001006	A000108
b		A002894*	A018224	A002426	A000984
c			A005566	A005773	A001700
d				A000079	A000244
e					A000302

Table 2: Two-dimensional walks. The types a–e are as explained in the text. Each square gives the enumeration of walks with one dimension according to the row and the other according to column. (*The three starred sequences enumerate walks of even length only, returning to the point of origin.)

equation") [13, eq. 1.2.6(20)] followed by Vandermonde's convolution [10, eq. 3.1]:

$$\begin{aligned}\sum_i \frac{1}{\frac{n}{2}-i+1}\binom{n-2i}{n/2-i}\binom{n}{2i}\binom{2i}{i} &= \sum_i \frac{1}{\frac{n}{2}-i+1}\binom{n-2i}{n/2-i}\binom{n}{i}\binom{n-i}{i} \\ &= \sum_i \frac{1}{\frac{n}{2}-i+1}\binom{n}{i}\binom{n-i}{n/2-i}\binom{n/2}{i} \\ &= \sum_i \frac{1}{\frac{n}{2}-i+1}\binom{n}{n/2}\binom{n/2}{i}\binom{n/2}{i} \\ &= \frac{1}{\frac{n}{2}+1}\binom{n}{n/2}\sum_i \binom{n/2}{i}\binom{n/2+1}{n/2-i} \\ &= \frac{1}{\frac{n}{2}+1}\binom{n}{n/2}\binom{n+1}{n/2}.\end{aligned}$$

Walks of type aa stay in one quadrant and return to the origin. They are counted by A005568 [11, §4],

$$\sum_i C_i C_{\frac{n}{2}-i}\binom{n}{2i} = C_{\frac{n}{2}} C_{\frac{n}{2}+1},$$

again for even n.

Walks of type ac stay in one quadrant but return to the abscissa (the promenade) and are counted by A001700,

$$\sum_i C_i A_{n-2i}\binom{n}{2i} = \binom{2n+1}{n}.$$

They are discussed at length in [11, §4].

Walks of type ce are just restricted to the half-plane. These are Guy's "Sandsteps" [11], introduced by Sands [20], and are also counted by A001700:

$$\sum_i C_i A_{n-2i} \binom{n}{2i} = \binom{2n+1}{n}.$$

These and the remaining two-dimensional cases are summarized in Table 2. Most of these were investigated in [8, 7, 12]. Their asymptotics were derived in [4]. More complicated walks involving diagonal steps have also been considered in the literature (e.g. [15, 5]).

5 Restriction: Three dimensions

Suppose that the swaggering pedestrian (or an intoxicated bird) can also move up (U) or down (D) at any point (and continue moving on those levels), never venturing underground. Suppose further that the path taken need only end up on the ground, not necessarily on the promenade. The number of n-step walks of this type (ace) is, by the general formula of the previous section,

$$\sum_{i,j} \frac{2^{n-2i-j}}{i+1} \binom{2i}{i} \binom{j}{\lfloor j/2 \rfloor} \binom{n}{2i, j, n-2i-j} = \sum_{i,j} \frac{2^{n-2i-j}}{i+1} \binom{n}{i, i, \lfloor j/2 \rfloor, \lceil j/2 \rceil, n-2i-j}.$$

Table 3 provides computed initial terms for all three-dimensional walks with steps in all six directions (N/S/E/W/U/D) and with some requirement or other to return towards the origin.

Acknowledgement

I thank Jeffrey Shallit and a referee for helpful suggestions.

References

[1] M. Aigner, *A Course in Enumeration*, Graduate Texts in Mathematics, Vol. 238, Springer, 2007.

[2] C. Banderier and P. Flajolet, Basic analytic combinatorics of directed lattice paths, *Theoret. Comput. Sci.* **281** (2002): 37–80.

[3] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani, A construction for enumerating k-coloured Motzkin paths, *Proc. 1st Annual Intl. Conf. on Computing and Combinatorics*, Springer, 1995, pp. 254–263.

[4] A. Bostan and M. Kauers, Automatic classification of restricted lattice walks, *Proc. 21st Intl. Conf. on Formal Power Series and Algebraic Combinatorics*, *Discrete Mathematics and Theoretical Computer Science*, **AK**, pp. 201–215, 2009, http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAK0117/2679.

Type	Space	Back	Sequence
aaa	octant	origin	1, 0, 3, 0, 24, 0, 285, 0, 4242, 0, 73206, 0, 1403028, 0, 29082339, ... (A064037*)
aab	quad.	origin	1, 0, 4, 0, 40, 0, 570, 0, 9898, 0, 195216, 0, 4209084, 0, 96941130, ...
aac	octant	axis	1, 1, 4, 9, 40, 120, 570, 1995, 9898, 38178, 195216, 805266, 4209084, ...
aad	octant	axis	1, 1, 3, 7, 23, 71, 251, 883, 3305, 12505, 48895, 193755, 783355, 3205931, ...
aae	quad.	axis	1, 2, 6, 20, 74, 292, 1214, 5252, 23468, 107672, 505048, 2413776, ... (A145867)
abb	half	origin	1, 0, 5, 0, 62, 0, 1065, 0, 21714, 0, 492366, 0, 12004740, 0, 308559537, ...
abc	quad.	axis	1, 1, 5, 12, 62, 200, 1065, 3990, 21714, 89082, 492366, 2147376, 12004740, ...
abd	quad.	axis	1, 1, 4, 10, 39, 131, 521, 1989, 8149, 33205, 139870, 592120, 2552155, ...
abe	half	axis	1, 2, 7, 26, 108, 472, 2159, 10194, 49396, 244328, 1229308, 6273896, ...
acc	octant	plane	1, 2, 7, 24, 98, 400, 1785, 7980, 37674, 178164, 874146, 4294752, 21667932, ...
acd	octant	plane	1, 2, 6, 19, 67, 246, 947, 3746, 15213, 62950, 264920, 1129965, ... (A145847)
ace	quad.	plane	1, 3, 11, 44, 188, 842, 3911, 18692, 91412, 455540, 2306028, 11829424, ...
add	octant	plane	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, ... (A000108)
ade	quad.	plane	1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369, 751236, 3328218, ... (A002212)
aee	half	plane	1, 4, 17, 76, 354, 1704, 8421, 42508, 218318, 1137400, 5996938, ... (A005572)
bbb	full	origin	1, 0, 6, 0, 90, 0, 1860, 0, 44730, 0, 1172556, 0, 32496156, ... (A002896*)
bbc	half	axis	1, 1, 6, 15, 90, 310, 1860, 7455, 44730, 195426, 1172556, ... (|A138547|)
bbd	half	axis	1, 1, 5, 13, 61, 221, 1001, 4145, 18733, 82381, 375745, 1703945, 7858225, ...
bbe	full	axis	1, 2, 8, 32, 148, 712, 3584, 18496, 97444, 521096, 2820448, ... (A202814)
bcc	quad.	plane	1, 2, 8, 30, 138, 620, 3060, 14910, 76650, 390852, 2063376, 10832052, ...
bcd	quad.	plane	1, 2, 7, 25, 101, 416, 1787, 7792, 34645, 155722, 707795, 3242515, ... (A150500)
bce	half	plane	1, 3, 12, 53, 252, 1252, 6416, 33609, 178996, 965660, 5263728, 28936404 ...
bdd	quad.	plane	1, 3, 11, 45, 195, 873, 3989, 18483, 86515, 408105, 1936881, ... (A000984)
bde	half	plane	1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, 705432, ... (A026375)
bee	full	plane	1, 4, 18, 88, 454, 2424, 13236, 73392, 411462, 2325976, ... (A081671)

Table 3: Three-dimensional walks, required to return to the origin (three dimensions of type a or b), axis of origin (two), or plane of origin (one). They may be constrained to a fraction of the space—octant (three of a, c, or d), quadrant (two), or half-space (one), or else allowed the full space (zero). The types are as explained in the text. (*The four starred sequences enumerate walks of even length only. In one case, *bbc*, the cited sequence, A138547, has alternating signs.)

[5] M. Bousquet-Mélou, Counting walks in the quarter plane, in B. Chauvin, P. Flajolet, D. Gardy, and A. Mokkadem, eds., *Mathematics and Computer Science II: Algorithms, Trees, Combinatorics and Probabilities*, Birkhäuser, 2002, pp. 49–67.

[6] W. Breckenridge, H. Gastineau-Hills, A. Nelson, P. Bos, G. Calvert, and K. Wehrhahn, Lattice paths and Catalan numbers, *Bull. Inst. Comb. and its App.* **1** (1991), pp. 41–55.

[7] E. Csáki, S. G. Mohanty, and S. Saran, On random walks in a plane, *Ars Combin.* **29** (1990), 309–318.

[8] D. W. DeTemple and J. M. Robertson, Equally likely fixed length paths in graphs, *Ars Combin.* 17 (1984), 243–254.

[9] E. Deutsch, Dyck path enumeration, *Discr. Math.* **204** (1999), 167–202.

[10] H. W. Gould, *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, rev. ed., self-published, Morgantown, WV, 1972.

[11] R. K. Guy, Catwalks, Sandsteps and Pascal pyramids, *J. Integer Seq.* **3** (2000), Article 00.1.6.

[12] R. K. Guy, C. Krattenthaler, and B. E. Sagan, Lattice paths, reflections, & dimension-changing bijections, *Ars Combin.* **34** (1992), 3–15.

[13] D. E. Knuth, *The Art of Computer Programming*, Vol. 1: *Fundamental Algorithms*, 3rd. ed., 1997.

[14] T. Koshy, *Catalan Numbers with Applications*, Oxford Univ. Press, 2009.

[15] G. Kreweras, Sur une classe de problèmes liés au treillis des partitions d'entiers, *Cahiers du B.U.R.O.* **6** (1965) 5–105.

[16] G. Mohanty, *Lattice Path Counting and Applications*, Academic Press, 1979.

[17] G. V. Nagy, A combinatorial proof of Shapiro's Catalan convolution, *Adv. in Appl. Math.* **49** (2012), 391–396.

[18] E. Pergola, Two bijections for the area of Dyck paths, *Discr. Math.* **241** (2001) 435–447.

[19] G. Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassennetz, *Mathematische Annalen* **84** (1921), 149–160. Also in *George Pólya: Collected Papers*, Vol. 4: Probability; Combinatorics; Teaching and Learning in Mathematics, G.-C. Rota, ed., MIT Press, 1984, 69–80, 609.

[20] B. Sands, Problem 1517*, *Crux Mathematicorum* **16** (1990), 44.

[21] L. W. Shapiro, A short proof of an identity of Touchard's concerning Catalan numbers, *J. Combin. Theory, Series A* **20** (1976), 375–376.

[22] N. J. Sloane, ed., The on-line encyclopedia of integer sequences, http://oeis.org, 1964–.

[23] R. P. Stanley, *Catalan Numbers*, Cambridge Univ. Press, 2015. Appendix B (History of Catalan numbers) by I. Pak.

[24] A. V. Sutherland, Comment to Seq. A000891 in [22] (2008).

[25] L. Takács, Ballot problems, *Z. Wahrsoheinliohkeitstheorie* **1**, 154–158 (1962).

[26] J. Touchard, Sur certaines équations fonctionnelles, in *Proc. Intl. Math. Congress* (Toronto, 1924), Vol. 1, pp. 465–472, 1928, http://www.mathunion.org/ICM/ICM1924.1/Main/icm1924.1.0465.0472.ocr.pdf.

[27] W. A. Whitworth, Arrangements of m things of one sort and n things of another sort, under certain conditions of priority, *Messenger of Math.* **8** (1878):105–114.

2010 *Mathematics Subject Classification*: Primary 05A15.
Keywords: path enumerations, drunken walks, random walks, Dyck paths, lattice paths, Touchard's identity, Catalan numbers, Motzkin numbers, central binomial factors.

(Concerned with sequences A000079, A000108, A000244, A000302, A000891, A000984, A001006, A001405, A001700, A002212, A002426, A002894, A002896, A005566, A005568, A005572, A005773, A018224, A026375, A064037, A081671, A138547, A145847, A145867, A150500, A202814.)
