Hypergeometric Euler numbers

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Abstract

For a nonnegative integer N, define hypergeometric Euler numbers $E_{N,n}$ by

$$\frac{1}{{}_1F_2(1;N+1,(2N+1)/2;t^2/4)}=\sum_{n=0}^\infty E_{N,n}\frac{t^n}{n!}\,,$$

where ${}_1F_2(a;b,c;z)$ is the hypergeometric function defined by

$${}_1F_2(a;b,c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}}\frac{z^n}{n!}\,.$$

Here, $(x)^{(n)}$ is the rising factorial, defined by $(x)^{(n)}=x(x+1)\cdots(x+n-1)$ $(n\ge 1)$ with $(x)^{(0)}=1$. When $N=0$, then $E_n=E_{0,n}$ are classical Euler numbers. Hypergeometric Euler numbers $E_{N,n}$ are analogues of hypergeometric Bernoulli numbers $B_{N,n}$ and hypergeometric Cauchy numbers $c_{N,n}$, defined by

$$\frac{1}{{}_1F_1(1;N+1;t)}=\sum_{n=0}^\infty B_{N,n}\frac{t^n}{n!}$$

and

$$\frac{1}{{}_2F_1(1,N;N+1;-t)}=\sum_{n=0}^\infty c_{N,n}\frac{t^n}{n!}\,,$$

respectively.

In this paper, we shall consider several expressions and sums of products of hypergeometric Euler numbers. We also introduce complementary hypergeometric Euler numbers and give some characteristic properties. There are strong reasons why these hypergeometric numbers are important. The hypergeometric numbers have one of the advantages that yield the natural extensions of determinant expressions of the numbers, though many kinds of generalizations of the Euler numbers have been considered by many authors.
2010 *Mathematics Subject Classification*: Primary 11B68; Secondary 11B37, 11R58, 33C20.
Key words and phrases: Hypergeometric Euler numbers, Euler numbers, Bernoulli numbers, Hasse-Teichüller derivative, sums of products, determinats.

1 Introduction

Euler numbers E_n are defined by the generating function

$$\frac{1}{\cosh t}=\sum_{n=0}^\infty E_n\frac{t^n}{n!}\,. \tag{1}$$

One of the different definitions is

$$\frac{2}{e^t+1}=\sum_{n=0}^\infty E_n\frac{t^n}{n!}$$

(see e.g. [3]). Generalizations of one or other of these definitions have previously been studied. For example, one kind of poly-Euler numbers is a typical generalization, in the aspect of L-functions ([18, 19, 20]). Other generalizations can be found in [4, 15] and the reference therein.

A different type of generalization is based upon hypergeometric functions. For $N\ge 1$, define hypergeometric Bernoulli numbers $B_{N,n}$ (see [8, 9, 10]) by

$$\frac{1}{{}_1F_1(1;N+1;t)}=\frac{t^N/N!}{e^t-\sum_{n=0}^{N-1}t^n/n!}=\sum_{n=0}^\infty B_{N,n}\frac{t^n}{n!}\,,$$

where

$$ {}_1F_1(a;b;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!} $$

is the confluent hypergeometric function with $(x)^{(n)}=x(x+1)\cdots(x+n-1)$ $(n\ge 1)$ and $(x)^{(0)}=1$. When $N=1$, $B_n=B_{1,n}$ are classical Bernoulli numbers defined by

$$ \frac{t}{e^t-1}=\sum_{n=0}^\infty B_n\frac{t^n}{n!}\,. $$

In addition, define hypergeometric Cauchy numbers $c_{N,n}$ (see [11]) by

$$ \frac{1}{{}_2F_1(1,N;N+1;-t)}=\frac{(-1)^{N-1}t^N/N}{\log(1+t)-\sum_{n=1}^{N-1}(-1)^{n-1}t^n/n}=\sum_{n=0}^\infty c_{N,n}\frac{t^n}{n!}\,, $$

where

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!} $$

is the Gauss hypergeometric function. When $N=1$, $c_n=c_{1,n}$ are classical Cauchy numbers defined by

$$ \frac{t}{\log(1+t)}=\sum_{n=0}^\infty c_n\frac{t^n}{n!}\,. $$

Our generalization is different from the generalizations in [13] and the references therein.

Now, for $N\ge 0$ define *hypergeometric Euler numbers* $E_{N,n}$ $(n=0,1,2,\dots)$ by

$$ \frac{1}{{}_1F_2(1;N+1,(2N+1)/2;t^2/4)}=\sum_{n=0}^\infty E_{N,n}\frac{t^n}{n!}\,, \tag{2} $$

where ${}_1F_2(a;b,c;z)$ is the hypergeometric function defined by

$$ {}_1F_2(a;b,c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}}\frac{z^n}{n!}\,. $$

It is seen that

$$\begin{aligned}\cosh t-\sum_{n=0}^{N-1}\frac{t^{2n}}{(2n)!}&=\frac{t^{2N}}{(2N)!}\sum_{n=0}^{\infty}\frac{(2N)!n!}{(2n+2N)!}\frac{(t^2)^n}{n!}\\&=\frac{t^{2N}}{(2N)!}{}_1F_2\bigl(1;N+1,\frac{2N+1}{2};\frac{t^2}{4}\bigr)\,.\end{aligned}\tag{3}$$

When $N=0$, then $E_n=E_{0,n}$ are classical Euler numbers defined in (1). In [16], the truncated Euler polynomial $E_{m,n}(x)$ is introcuded as a generalization of the classical Euler polynomial $E_n(x)$. The concept is similar but without hypergeometric functions.

We list the numbers $E_{N,n}$ for $0\le N\le 6$ and $0\le n\le 12$ in Table 1. From (3) we see that $E_{N,n}=0$ if n is odd. Similarly to poly-Euler numbers ([18, 19, 20]), hypergeometric Euler numbers are rational numbers, though the classical Euler numbers are integers.

Table 1: The numbers $E_{N,n}$ for $0 \leq N \leq 6$ and $0 \leq n \leq 14$

n	0	2	4	6	8	10	12	14
$E_{0,n}$	1	−1	5	−61	1385	−50521	2702765	−199360981
$E_{1,n}$	1	−1/6	1/10	−5/42	7/30	−15/22	7601/2730	−91/6
$E_{2,n}$	1	−1/15	13/1050	−1/350	−31/173250	1343/750750	−6137/2388750	3499/6693750
$E_{3,n}$	1	−1/28	17/5880	−29/362208	−863/6420960	6499/131843712	6997213/156894017280	−68936107/917226562560
$E_{4,n}$	1	−1/45	7/7425	53/2027025	−443/22052250	−10157/4873547250	558599021/126395447928750	39045649/62503243481250
$E_{5,n}$	1	−1/66	25/66066	47/2906904	−16945/5300012718	−475767/492312292472	71844089/268802511689712	1162911301/4483980359834976
$E_{6,n}$	1	−1/91	29/165620	1205/153728484	−2279/4467168888	−6430761/25339270989032	−17675104079/4917799642149532320	837165624457/2458899821074766160

From (2) and (3), we have

$$\begin{aligned}
\frac{t^{2N}}{(2N)!} &= \left(\sum_{n=N}^{\infty}\frac{t^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty}E_{N,n}\frac{t^n}{n!}\right)\\
&= t^{2N}\left(\sum_{n=0}^{\infty}\frac{\frac{1+(-1)^n}{2}t^n}{(n+2N)!}\right)\left(\sum_{n=0}^{\infty}E_{N,n}\frac{t^n}{n!}\right)\\
&= t^{2N}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{\frac{1+(-1)^{n-i}}{2}}{(2N+n-i)!}\frac{E_{N,i}}{i!}\right)t^n\,.
\end{aligned}$$

Hence, for $n\ge 1$, we have

$$\sum_{i=0}^{n}\frac{1+(-1)^{n-i}}{(2N+n-i)!i!}E_{N,i}=0\,.$$

Thus, we have the following proposition. Note that $E_{N,n}=0$ when n is odd.

Proposition 1.

$$\sum_{i=0}^{n/2}\frac{1}{(2N+n-2i)!(2i)!}E_{N,2i}=0\quad (n\ge 2 \textit{ is even})$$

and $E_{N,0}=1$.

By using the identity in Proposition 1 or the identity

$$E_{N,n}=-n!(2N)!\sum_{i=0}^{n/2-1}\frac{E_{N,2i}}{(2N+n-2i)!(2i)!}\,, \tag{4}$$

we can obtain the values of $E_{N,n}$ $(n=0,2,4,\dots)$. We record the first few values of $E_{N,n}$:

$$\begin{aligned}
E_{N,2} &= -\frac{2}{(2N+1)(2N+2)}\,,\\
E_{N,4} &= \frac{2\cdot 4!(4N+5)}{(2N+1)^2(2N+2)^2(2N+3)(2N+4)}\,,\\
E_{N,6} &= \frac{4\cdot 6!(8N^3-2N^2-65N-61)}{(2N+1)^3(2N+2)^3(2N+3)(2N+4)(2N+5)(2N+6)}\,,\\
E_{N,8} &= \frac{16\cdot 8!}{(2N+1)^4(2N+2)^4(2N+3)^2(2N+4)^2(2N+6)(2N+7)(2N+8)}\\
&\quad\times(16N^6-44N^5-516N^4-667N^3+1283N^2+3126N+1662)\,.
\end{aligned}$$

We have an explicit expression of $E_{N,n}$ for each even n:

Theorem 1. *For $N \geq 0$ and $n \geq 1$ we have*

$$E_{N,2n} = (2n)! \sum_{r=1}^{n} (-1)^r \sum_{\substack{i_1+\cdots+i_r=n \\ i_1,\dots,i_r \geq 1}} \frac{\bigl((2N)!\bigr)^r}{(2N+2i_1)! \cdots (2N+2i_r)!} \,.$$

Proof. The proof is done by induction for n. If $n=1$, then

$$E_{N,2} = 2!(-1)\frac{(2N)!}{(2N+2)!} = -\frac{2}{(2N+1)(2N+2)} \,.$$

Assume that the result is valid up to $n-1$. Then by Proposition 1

$$\begin{aligned}
E_{N,2n} &= -(2n)!(2N)!\sum_{i=0}^{n-1}\frac{E_{N,2i}}{(2N+2n-2i)!(2i)!}\\
&= -(2n)!(2N)!\sum_{i=1}^{n-1}\frac{1}{(2N+2n-2i)!}\sum_{r=1}^{i}(-1)^r\\
&\qquad\times\sum_{\substack{i_1+\cdots+i_r=i\\ i_1,\dots,i_r\ge 1}}\frac{\bigl((2N)!\bigr)^r}{(2N+2i_1)!\cdots(2N+2i_r)!}\\
&\quad-(2n)!(2N)!\frac{1}{(2N+2n)!}\\
&= -(2n)!(2N)!\sum_{r=1}^{n-1}(-1)^r\bigl((2N)!\bigr)^r\sum_{i=r}^{n-1}\frac{1}{(2N+2n-2i)!}\\
&\qquad\times\sum_{\substack{i_1+\cdots+i_r=i\\ i_1,\dots,i_r\ge 1}}\frac{1}{(2N+2i_1)!\cdots(2N+2i_r)!}\\
&\quad-\frac{(2n)!(2N)!}{(2N+2n)!}\\
&= -(2n)!(2N)!\sum_{r=2}^{n}(-1)^{r-1}\bigl((2N)!\bigr)^{r-1}\sum_{i=r-1}^{n-1}\frac{1}{(2N+2n-2i)!}\\
&\qquad\times\sum_{\substack{i_1+\cdots+i_{r-1}=i\\ i_1,\dots,i_{r-1}\ge 1}}\frac{1}{(2N+2i_1)!\cdots(2N+2i_{r-1})!}\\
&\quad-\frac{(2n)!(2N)!}{(2N+2n)!}\\
&= (2n)!\sum_{r=2}^{n}(-1)^r\bigl((2N)!\bigr)^r\sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r\ge 1}}\frac{1}{(2N+2i_1)!\cdots(2N+2i_r)!}\\
&\quad-\frac{(2n)!(2N)!}{(2N+2n)!}\quad(n-i=i_r)\\
&= (2n)!\sum_{r=1}^{n}(-1)^r\sum_{\substack{i_1+\cdots+i_r=n\\ i_1,\dots,i_r\ge 1}}\frac{\bigl((2N)!\bigr)^r}{(2N+2i_1)!\cdots(2N+2i_r)!}\,.
\end{aligned}$$

□

2 Determinant expressions of hypergeometric numbers

These hypergeometric numbers have one of the advantages that yield the natural extensions of determinant expressions of the numbers, though many kinds of generalizations of the Euler numbers have been considered by many authors.

By using Proposition 1 or the relation (4), we have a determinant expression of hypergeometric Euler numbers ([12]).

Proposition 2. *The hypergeometric Euler numbers $E_{N,2n}$ ($N \geq 0$, $n \geq 1$) can be expressed as*

$$E_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N)!}{(2N+2)!} & 1 & & \\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} \end{vmatrix} .$$

In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy and Euler numbers. When $N = 0$, the determinant in Proposition (2) is reduced to a famous determinant expression of Euler numbers (*cf.* [5, p.52]):

$$E_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} .$$

In [1], the hypergeometric Bernoulli numbers $B_{N,n}$ ($N \geq 1$, $n \geq 1$) can be expressed as

$$B_{N,n} = (-1)^n n! \begin{vmatrix} \frac{N!}{(N+1)!} & 1 & & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix} .$$

When $N = 1$, we have a determinant expression of Bernoulli numbers ([5, p.53]):

$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} . \tag{5}$$

In [2], the hypergeometric Cauchy numbers $c_{N,n}$ ($N \ge 1$, $n \ge 1$) can be expressed as

$$c_{N,n} = n! \begin{vmatrix} \frac{N}{N+1} & 1 & & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix} .$$

When $N = 1$, we have a determinant expression of Cauchy numbers ([5, p.50]):

$$c_n = n! \begin{vmatrix} \frac{1}{2} & 1 & & & \\ \frac{1}{3} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix} . \tag{6}$$

In [12], the complementary Euler numbers $\widehat{E}_n$ and their hypergeometric generalizations (defined below) have also determinant expressions.

3 Hasse-Teichmüller derivative

We define the Hasse-Teichmüller derivative $H^{(n)}$ of order n by

$$H^{(n)} \left(\sum_{m=R}^{\infty} c_m z^m \right) = \sum_{m=R}^{\infty} c_m \binom{m}{n} z^{m-n}$$

for $\sum_{m=R}^{\infty} c_m z^m \in \mathbb{F}((z))$, where R is an integer and $c_m \in \mathbb{F}$ for any $m \ge R$.

The Hasse-Teichmüller derivatives satisfy the product rule [21], the quotient rule [6] and the chain rule [7]. One of the product rules can be described as follows.

Lemma 1. *For $f_i \in \mathbb{F}[[z]]$ $(i=1,\dots,k)$ with $k \ge 2$ and for $n \ge 1$, we have*

$$H^{(n)}(f_1\cdots f_k) = \sum_{\substack{i_1,\dots,i_k\ge 0\\ i_1+\cdots+i_k=n}} H^{(i_1)}(f_1)\cdots H^{(i_k)}(f_k)\,.$$

The quotient rules can be described as follows.

Lemma 2. *For $f \in \mathbb{F}[[z]]\backslash\{0\}$ and $n \ge 1$, we have*

$$H^{(n)}\left(\frac{1}{f}\right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\dots,i_k\ge 1\\ i_1+\cdots+i_k=n}} H^{(i_1)}(f)\cdots H^{(i_k)}(f) \tag{7}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\dots,i_k\ge 0\\ i_1+\cdots+i_k=n}} H^{(i_1)}(f)\cdots H^{(i_k)}(f)\,. \tag{8}$$

By using the Hasse-Teichmüller derivative of order n, we shall obtain some explicit expressions of the hypergeometric Euler numbers.

Another proof of Theorem 1. Put

$$\begin{aligned} F :&= {}_1F_2\big(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}\big)\\ &= \sum_{n=0}^{\infty} \frac{(2N)!}{(2N+2n)!} t^{2n} \end{aligned}$$

for simplicity. Note that

$$H^{(i)}(F)\big|_{t=0} = \sum_{j=0}^{\infty} \frac{(2N)!}{(2N+2j)!}\binom{2j}{i} t^{2j-i}\Bigg|_{t=0} = \begin{cases} (2N)!/(2N+i)! & \text{if } i \text{ is even;}\\ 0 & \text{if } i \text{ is odd.}\end{cases}$$

Hence, by using Lemma 2 (7), we have

$$\begin{aligned} \frac{E_{N,n}}{n!} &= H^{(n)}\left(\frac{1}{F}\right)\Bigg|_{t=0}\\ &= \sum_{k=1}^{n} \frac{(-1)^k}{F^{k+1}}\Bigg|_{t=0} \sum_{\substack{i_1,\dots,i_k\ge 1\\ i_1+\cdots+i_k=n}} H^{(i_1)}(F)\big|_{t=0}\cdots H^{(i_k)}(F)\big|_{t=0}\\ &= \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\dots,i_k\ge 1\\ 2(i_1+\cdots+i_k)=n}} \frac{\big((2N)!\big)^k}{(2N+2i_1)!\cdots(2N+2i_k)!}\,. \end{aligned}$$

□

We can express the hypergeometric Euler numbers also in terms of the binomial coefficients. In fact, by using Lemma 2 (8) instead of Lemma 2 (7) in the above proof, we obtain a little different expression from one in Theorem 1.

Proposition 3. *For $N \ge 0$ and even $n \ge 2$,*

$$E_{N,n} = n! \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{\substack{i_1,\dots,i_k \ge 0 \\ i_1+\cdots+i_k = n/2}} \frac{\big((2N)!\big)^k}{(2N+2i_1)! \cdots (2N+2i_k)!}\,.$$

For example, when $n=4$, we get

$$\begin{aligned}
E_4 &= 4!\left(-\binom{5}{2}\frac{1}{4!}+\binom{5}{3}\left(\frac{2}{4!}+\frac{1}{2!2!}\right)\right.\\
&\qquad\left.-\binom{5}{4}\left(\frac{3}{4!}+\frac{3}{2!2!}\right)+\binom{5}{5}\left(\frac{4}{4!}+\frac{6}{2!2!}\right)\right)\\
&= 5\,,\\
E_{1,4} &= 4!\left(-\binom{5}{2}2\frac{1}{6!}+\binom{5}{3}2^2\left(\frac{2}{6!2!}+\frac{1}{4!4!}\right)\right.\\
&\qquad\left.-\binom{5}{4}2^3\left(\frac{3}{6!2!2!}+\frac{3}{4!4!2!}\right)+\binom{5}{5}2^4\left(\frac{4}{6!2!2!2!}+\frac{6}{4!4!2!2!}\right)\right)\\
&= \frac{1}{10}\,,\\
E_{2,4} &= 4!\left(-\binom{5}{2}4!\frac{1}{8!}+\binom{5}{3}(4!)^2\left(\frac{2}{8!4!}+\frac{1}{6!6!}\right)\right.\\
&\qquad\left.-\binom{5}{4}(4!)^3\left(\frac{3}{8!4!4!}+\frac{3}{6!6!4!}\right)+\binom{5}{5}(4!)^4\left(\frac{4}{8!4!4!4!}+\frac{6}{6!6!4!4!}\right)\right)\\
&= \frac{13}{1050}\,,\\
E_{3,4} &= 4!\left(-\binom{5}{2}6!\frac{1}{10!}+\binom{5}{3}(6!)^2\left(\frac{2}{10!6!}+\frac{1}{8!8!}\right)\right.\\
&\qquad\left.-\binom{5}{4}(6!)^3\left(\frac{3}{10!6!6!}+\frac{3}{8!8!6!}\right)+\binom{5}{5}(6!)^4\left(\frac{4}{10!6!6!6!}+\frac{6}{8!8!6!6!}\right)\right)\\
&= \frac{17}{5880}\,.
\end{aligned}$$

4 Some hypergeometric Euler numbers

If $N=1$, we have the following relation between hypergeometric Euler numbers and Bernoulli numbers.

Theorem 2. *For $n\ge 1$ we have*

$$E_{1,n}=-(n-1)B_n\,.$$

Proof. The result is clear for $n = 0, 1$ and odd numbers n. By using the following Lemma 3 and Proposition 1, we get the result. ☐

Lemma 3. *For $n \geq 1$ we have*

$$\sum_{i=0}^{n} \frac{(i-1)B_i}{(n-i+2)!i!} = \begin{cases} 0 & \text{if } n \text{ is even;} \\ -B_{n+1}/n! & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Firstly,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(i-1)B_i}{(n-i+2)!i!} x^n \\
&= \left(\sum_{k=0}^{\infty} \frac{x^k}{(k+2)!} \right) \left(\sum_{i=0}^{\infty} (i-1) B_i \frac{x^i}{i!} \right) \\
&= \left(\frac{1}{x^2} \sum_{k=0}^{\infty} \frac{x^{k+2}}{(k+2)!} \right) \left(-2 \sum_{i=0}^{\infty} B_i \frac{x^i}{i!} + \frac{d}{dx} \sum_{i=0}^{\infty} B_i \frac{x^{i+1}}{i!} \right) \\
&= \frac{e^x - 1 - x}{x^2} \left(-\frac{2x}{e^x - 1} + \frac{2x(e^x - 1) - x^2 e^x}{(e^x - 1)^2} \right) \\
&= \frac{e^x (x + 1 - e^x)}{(e^x - 1)^2} .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&-\frac{1}{2} - \sum_{n=0}^{\infty} B_{2n+2} \frac{x^{2n+1}}{(2n+1)!} \\
&= -\frac{1}{2} - \frac{d}{dx} \left(\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} - B_0 - B_1 x \right) \\
&= -\frac{1}{2} - \frac{d}{dx} \left(\frac{x}{e^x - 1} - 1 + \frac{x}{2} \right) \\
&= \frac{e^x (x + 1 - e^x)}{(e^x - 1)^2} .
\end{aligned}$$

Comparing the coefficients of x^n, we get the result. ☐

5 Sums of products of hypergeometric Euler numbers

It is known that

$$\sum_{i=0}^{n}\binom{2n}{2i}E_{2i}=0$$

with $E_0=1$, and $E_{2i-1}=0$ $(i\ge 1)$.

First, let us consider the sums of products of hypergeometric Euler numbers:

$$Y_{N,2}(n)=\sum_{i=0}^{n}\binom{2n}{2i}E_{N,2i}E_{N,2n-2i}\,.$$

It is clear that

$$\sum_{i=0}^{n}\binom{n}{i}E_{N,i}E_{N,n-i}=0$$

if n is odd.

If $N=0$, then

$$Y_{0,2}(n)=\frac{2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{2n+2}\quad (n\ge 0)\,.$$

Indeed,

$$\{Y_{0,2}(n)\}_{n\ge 0}=1,-2,16,-272,7936,-353792,22368256,-1903757312,\dots.$$

The numbers taking their absolute value are called the *tangent numbers* or the *zag numbers* ([17, A000182]). Thus, we also have

$$Y_{0,2}(n)=\sum_{k=1}^{2n+2}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{j+1}(k-2j)^{2n+2}}{2^k\sqrt{-1}^k k}\,.$$

In other words, they appear as numerators in the Maclaurin series of $\tan x$:

$$\tan x=\sum_{n=0}^{\infty}(-1)^n Y_{0,2}(n)\frac{x^{2n+1}}{(2n+1)!}\,.$$

Put

$$F := {}_1F_2\big(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}\big)$$
$$= \sum_{n=0}^{\infty} \frac{(2N)!}{(2N+2n)!} t^{2n}$$

for simplicity again. Then by

$$\frac{d}{dt}F = \sum_{n=0}^{\infty} \frac{(2n)(2N)!}{(2N+2n)!} t^{2n-1}\,,$$

we have

$$2NF + t\frac{t}{dt}F = 2N \cdot {}_1F_2\big(1; N, \frac{2N+1}{2}; \frac{t^2}{4}\big)\,. \tag{9}$$

For further simplicity, we put for $k = 1, 2, \ldots, 2N$

$$F_{(2N-k)} = {}_1F_2\left(1; \left\lfloor \frac{k+2}{2} \right\rfloor, \left\lfloor \frac{k+1}{2} \right\rfloor + \frac{1}{2}; \frac{t^2}{4}\right)$$

with $F_{(0)} = F$. Then, in general, we obtain for $k = 1, 2, \ldots, 2N$

$$kF_{(2N-k)} + t\frac{d}{dt}F_{(2N-k)} = kF_{(2N-k+1)}\,. \tag{10}$$

Proposition 4. *For $k = 0, 1, \ldots, 2N$ we have*

$$\cosh t = \sum_{i=0}^{k} \frac{t^i}{i!}\binom{k}{i}\frac{d^i}{dt^i}F_{(2N-k)}\,.$$

Proof. For $k = 0$, we get

$$F_{(2N)} = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \cosh t\,.$$

Assume that the result holds for some $k \ge 0$. Then by (10)

$$
\begin{aligned}
&\sum_{i=0}^k \frac{t^i}{i!}\binom{k}{i}\frac{d^i}{dt^i}F_{(2N-k)}\\
&=\sum_{i=0}^k \frac{t^i}{i!}\binom{k}{i}\frac{d^i}{dt^i}\left(F_{(2N-k-1)}+\frac{t}{k+1}\frac{d}{dt}F_{(2N-k-1)}\right)\\
&=\sum_{i=0}^k \frac{t^i}{i!}\binom{k}{i}\\
&\qquad\times\left(\frac{d^i}{dt^i}F_{(2N-k-1)}+\frac{i}{k+1}\frac{d^i}{dt^i}F_{(2N-k-1)}+\frac{t}{k+1}\frac{d^{i+1}}{dt^{i+1}}F_{(2N-k-1)}\right)\\
&=\sum_{i=0}^k \frac{t^i}{i!}\binom{k}{i}\frac{k+i+1}{k+1}\frac{d^i}{dt^i}F_{(2N-k-1)}\\
&\qquad+\sum_{i=1}^{k+1}\frac{t^{i-1}}{(i-1)!}\binom{k}{i-1}\frac{t}{k+1}\frac{d^i}{dt^i}F_{(2N-k-1)}\\
&=\sum_{i=0}^{k+1}\frac{t^i}{i!}\binom{k+1}{i}\frac{d^i}{dt^i}F_{(2N-k-1)}\,.
\end{aligned}
$$

□

We introduce the *complementary hypergeometric Euler numbers* $\widehat{E}_{N,n}$ by

$$
\frac{t^{2N+1}/(2N+1)!}{\sinh t-\sum_{n=0}^{N-1}t^{2n+1}/(2n+1)!}=\sum_{n=0}^\infty \widehat{E}_{N,n}\frac{t^n}{n!}
$$

as an analogue of (2). When $n=0$, $\widehat{E}_n=\widehat{E}_{0,n}$ are the *complementary Euler numbers* defined by

$$
\frac{t}{\sinh t}=\sum_{n=0}^\infty \widehat{E}_n\frac{t^n}{n!}
$$

as an analogue of (1). In [16], they are called *weighted Bernoulli numbers*, but this naming means different in other literatures. Since

$$
\begin{aligned}
F^* &:= {}_1F_2\big(1;N,\frac{2N+1}{2};\frac{t^2}{4}\big)\\
&=\sum_{n=0}^\infty \frac{(2N-1)!}{(2N+2n-1)!}t^{2n}
\end{aligned}
$$

and

$$\frac{d}{dt}F = -F^2 \frac{d}{dt}\frac{1}{F}\,, \tag{11}$$

by (9) we have

$$\frac{1}{F^2} = \frac{1}{F^*}\left(\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F}\right)\,. \tag{12}$$

Since

$$\begin{aligned}\frac{1}{F^*} &= \frac{t^{2N-1}}{(2N-1)!\sum_{n=0}^\infty t^{2N+2n-1}/(2N+2n-1)!}\\ &= \sum_{n=0}^\infty \widehat{E}_{N-1,n}\frac{t^n}{n!}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F} &= \sum_{n=0}^\infty E_{N,n}\frac{t^n}{n!} - \frac{t}{2N}\sum_{n=1}^\infty E_{N,n}\frac{t^{n-1}}{(n-1)!}\\ &= \sum_{n=0}^\infty \frac{2N-n}{2N}E_{N,n}\frac{t^n}{n!}\,,\end{aligned}$$

we have

$$\begin{aligned}\frac{1}{F^*}\left(\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F}\right) &= \left(\sum_{m=0}^\infty \widehat{E}_{N-1,m}\frac{t^m}{m!}\right)\left(\sum_{k=0}^\infty \frac{2N-k}{2N}E_{N,k}\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^\infty\sum_{k=0}^n \binom{n}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,n-k}\frac{t^n}{n!}\,.\end{aligned}$$

Comparing the coefficients, we obtain a result about the sums of products.

Theorem 3. *For $N \ge 1$ and $n \ge 0$,*

$$\sum_{i=0}^n \binom{n}{i}E_{N,i}E_{N,n-i} = \sum_{k=0}^n \binom{n}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,n-k}\,.$$

Using (11) and (12) again, we have

$$\begin{aligned}\frac{1}{F^3} &= \frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{2N}\frac{1}{F}\frac{d}{dt}\frac{1}{F}\right)\\ &= \frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2}\right).\end{aligned}$$

Since

$$\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2} = \sum_{n=0}^{\infty}\frac{4N-n}{4N}\sum_{k=0}^{n}\binom{n}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,n-k}\frac{t^n}{n!},$$

we have

$$\begin{aligned}&\frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2}\right)\\ &= \left(\sum_{i=0}^{\infty}\widehat{E}_{N-1,i}\frac{t^i}{i!}\right)\left(\sum_{m=0}^{\infty}\frac{4N-m}{4N}\sum_{k=0}^{m}\binom{m}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,m-k}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}\frac{(4N-m)(2N-k)}{8N^2}E_{N,k}\widehat{E}_{N-1,n-m}\widehat{E}_{N-1,m-k}\frac{t^n}{n!},\end{aligned}$$

Comparing the coefficients, we get a result about the sums of products for trinomial coefficients.

Theorem 4. *For $N \geq 1$ and $n \geq 0$,*

$$\begin{aligned}&\sum_{\substack{i_1+i_2+i_3=n\\ i_1,i_2,i_3\geq 0}}\binom{n}{i_1,i_2,i_3}E_{N,i_1}E_{N,i_2}E_{N,i_3}\\ &\qquad = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}\frac{(4N-m)(2N-k)}{8N^2}E_{N,k}\widehat{E}_{N-1,n-m}\widehat{E}_{N-1,m-k}.\end{aligned}$$

5.1 Complementary hypergeometric Euler numbers

By using the similar methods in previous sections, the complementary hypergeometric Euler numbers satisfy the recurrence relation for even n

$$\sum_{i=0}^{n/2}\frac{\widehat{E}_{N,2i}}{(2N+n-2i+1)!(2i)!} = 0$$

or

$$\widehat{E}_{N,n}=-n!(2N+1)!\sum_{i=0}^{n/2-1}\frac{\widehat{E}_{N,2i}}{(2N+n-2i+1)!(2i)!}\,.$$

By using the Hasse-Teichmüller derivative or by proving by induction, we have

Theorem 5. *For $N\ge 0$ and $n\ge 1$ we have*

$$\begin{aligned}\widehat{E}_{N,n}&=n!\sum_{k=1}^n(-1)^k\sum_{\substack{i_1,\dots,i_k\ge 1\\ i_1+\cdots+i_k=n/2}}\frac{\bigl((2N+1)!\bigr)^k}{(2N+2i_1+1)!\cdots(2N+2i_k+1)!}\\&=n!\sum_{k=1}^n(-1)^k\binom{n+1}{k+1}\sum_{\substack{i_1,\dots,i_k\ge 0\\ i_1+\cdots+i_k=n/2}}\frac{\bigl((2N+1)!\bigr)^k}{(2N+2i_1+1)!\cdots(2N+2i_k+1)!}\,.\end{aligned}$$

Some initial values of $\widehat{E}_{N,n}$ ($n=0,2,4,\dots$), we have

$$\begin{aligned}\widehat{E}_{N,2}&=-\frac{2}{(2N+2)(2N+3)}\,,\\ \widehat{E}_{N,4}&=\frac{2\cdot 4!(4N+7)}{(2N+2)^2(2N+3)^2(2N+4)(2N+5)}\,,\\ \widehat{E}_{N,6}&=\frac{4\cdot 6!(8N^3+10N^2-61N-93))}{(2N+2)^3(2N+3)^3(2N+4)(2N+5)(2N+6)(2N+7)}\,,\\ \widehat{E}_{N,8}&=\frac{8\cdot 8!}{(2N+2)^4(2N+3)^4(2N+4)^2(2N+5)^2(2N+7)(2N+8)(2N+9)}\\&\quad\times(32N^6+8N^5-1132N^4-3538N^3-1063N^2+7280N+6858)\,.\end{aligned}$$

Put

$$\widehat{F}=\sum_{n=0}^\infty\frac{(2N+1)!}{(2N+2n+1)!}t^{2n}$$

so that

$$\frac{1}{\widehat{F}}=\sum_{n=0}^\infty\widehat{E}_{N,n}\frac{t^n}{n!}\,.$$

Since

$$(2N+1)\widehat{F}+t\frac{d}{dt}\widehat{F}=(2N+1)F\,,$$

we have

$$\begin{aligned}\frac{1}{\widehat{F}^2} &= \frac{1}{F}\left(\frac{1}{\widehat{F}} - \frac{t}{2N+1}\frac{d}{dt}\frac{1}{\widehat{F}}\right)\\ &= \left(\sum_{m=0}^\infty E_{N,m}\frac{t^m}{m!}\right)\left(\sum_{k=0}^\infty \frac{2N-k+1}{2N+1}\widehat{E}_{N,k}\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\frac{t^n}{n!}\,.\end{aligned}$$

Hence, as an analogue of Theorem 3, we have the following.

Theorem 6. *For $N \ge 1$ and $n \ge 0$,*

$$\sum_{i=0}^n \binom{n}{i}\widehat{E}_{N,i}\widehat{E}_{N,n-i} = \sum_{k=0}^n \binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\,.$$

We then have

$$\frac{1}{\widehat{F}^3} = \frac{1}{F}\left(\frac{1}{\widehat{F}^2} - \frac{t}{2(2N+1)}\frac{d}{dt}\frac{1}{\widehat{F}^2}\right)\,.$$

Since

$$\frac{1}{\widehat{F}^2} - \frac{t}{2(2N+1)}\frac{d}{dt}\frac{1}{\widehat{F}^2} = \sum_{n=0}^\infty \frac{4N-n+2}{2(2N+1)}\sum_{k=0}^n \binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\frac{t^n}{n!}\,,$$

we have the following result as an analogue of Theorem 4.

Theorem 7. *For $N \ge 1$ and $n \ge 0$,*

$$\begin{aligned}&\sum_{\substack{i_1+i_2+i_3=n\\ i_1,i_2,i_3\ge 0}} \binom{n}{i_1,i_2,i_3}\widehat{E}_{N,i_1}\widehat{E}_{N,i_2}\widehat{E}_{N,i_3}\\ &\quad = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m}\binom{m}{k}\frac{(4N-m+2)(2N-k+1)}{2(2N+1)^2}\widehat{E}_{N,k}E_{N,n-m}E_{N,m-k}\,.\end{aligned}$$

One can continue to obtain the sum of four or more products, though the results seem to become more complicated.

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