A Generalization of the Eulerian Numbers

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Abstract

In the present paper we generalize the Eulerian numbers (also of the second and third orders). The generalization is connected with an autonomous first-order differential equation, solutions of which are used to obtain integral representations of some numbers, including the Bernoulli numbers.

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1 Introduction

While working on alternating sums of the form $\sum_{k=1}^{m}(-1)^k k^n$, and then more generally on $\sum_{k=1}^{m} j^k k^n$, in 1736 Euler [9, Ch. 7] discovered polynomials and numbers, which now are known as the Eulerian polynomials and the Eulerian numbers. The Eulerian numbers, denoted by $\left\langle {n \atop k} \right\rangle$, are defined by the recurrence formula

$$\left\langle {n+1 \atop k} \right\rangle = (k+1)\left\langle {n \atop k} \right\rangle + (n-k+1)\left\langle {n \atop k-1} \right\rangle, \tag{1}$$

with boundary conditions $\left\langle {0 \atop 0} \right\rangle = 1$, $\left\langle {n \atop k} \right\rangle = 0$ for $k < 0$ or $k \geq n$. Many authors use slightly different definition and different notation for Eulerian numbers. In their notation $A(n,k) = \left\langle {n \atop k-1} \right\rangle$. Some initial terms of the Eulerian numbers ID in OEIS (A008292) are shown in Table 1.

In the 1950s Riordan [19] found an interesting combinatorial property of the Eulerian numbers, which now is often used as their definition. It states that $\left\langle {n \atop k} \right\rangle$ is the number of all permutations of the set $(1, 2, \ldots, n)$, with exactly k ascents. The ascent appears in a given permutation if for two consecutive terms of the permutation the first term is smaller than the second one.

The second-order Eulerian numbers ID in (A008517), denoted by $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$, are defined by the following recurrence formula:

$$\left\langle\!\left\langle {n+1 \atop k} \right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle + (2n-k+1)\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle, \tag{2}$$

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Table 1: First few Eulerian numbers $\left\langle {n \atop k} \right\rangle$

$n\backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	4	1	0	0	0	0	0
4	1	11	11	1	0	0	0	0
5	1	26	66	26	1	0	0	0
6	1	57	302	302	57	1	0	0
7	1	120	1191	2416	1191	120	1	0

Table 2: First few second-order Eulerian numbers $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$

$n\backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	2	0	0	0	0	0	0
3	1	8	6	0	0	0	0	0
4	1	22	58	24	0	0	0	0
5	1	52	328	444	120	0	0	0
6	1	114	1452	4400	3708	720	0	0
7	1	240	5610	32120	58140	33984	5040	0

with the same boundary conditions as for the Eulerian numbers. Table 2 shows the first few of these numbers.

Eulerian numbers have been well described in books by Comtet [7], Graham et al. [12] and Riordan [19] or by Foata in his review article [11]. Some applications and generalizations of Eulerian numbers are given in many articles. For example, Carlitz [4, 5] introduced q-Eulerian numbers, Lehmer [16] defined a generalization based on the iteration of a differential operator, Butzer and Hauss [3] described Eulerian numbers with fractional order parameters. Barbero et al. [2] essentially solved the problem 6.94, stated in Graham et al. [12], about a solution of the general linear recurrence equation. Chung et al. [6] gave three proofs of an identity for Eulerian numbers, He [14] demonstrated their application to B-splines and Banaian [1] used them in the theory of juggling. The last application of the Eulerian numbers is especially interesting for the first author because his son Stanisław is able to juggle with seven balls.

In Sec. 2 we define a sequence $(G(n, k))$, which is a natural generalization of Eulerian numbers. In Sec. 3 we show that the generalized Eulerian numbers are associated with an autonomous differential equation. Then in Sec. 4 we use, on some examples, solutions of the equation to get integral representations for some numbers, including the Bernoulli numbers.

2 Definition and properties

Let us define a sequence $G(n,k)$ by the recurrence formula

$$G(n+1,k) = (nw_1 - n + k + 1)G(n,k) + (nw_2 - k + 1)G(n,k-1), \tag{3}$$

for integer numbers $n \geq 0, k$ and real parameters w_1, w_2, with boundary conditions $G(0,0) = 1$, $G(n,k) = 0$ for $k < 0$ or $k \geq n$.

For $w_1 = w_2 = 1$ formula (3) gives Eulerian numbers ID in OEIS (A008292), for $w_1 = 1, w_2 = 2$, the second-order Eulerian numbers ID in OEIS (A008517) and if $w_1 = 1, w_2 = 3$, the third-order Eulerian numbers ID in OEIS (A219512). Thus sequence (3) is a natural generalization of Eulerian numbers.

Lemma 1. *The sum of the nth row ($n = 1, 2, 3\ldots$) in the array of coefficients $G(n,k)$, for $k = 0,\ 1,\ 2, \ldots,\ n-1$ is a polynomial of $(w_1 + w_2)$ of order $n-1$, and the following formula holds:*

$$\sum_{k=0}^{n-1} G(n,k) = \prod_{m=0}^{n-1} (m(w_1 + w_2) - m + 1). \tag{4}$$

Proof. It is easy to check that adding by sides, for a given n, all $(n+1)$ identities (3) respectively, for $k = 0, 1, 2, \ldots, n$ we arrive at the following recurrence formula:

$$\sum_{k=0}^{n} G(n+1,k) = (n(w_1 + w_2) - n + 1) \sum_{k=0}^{n-1} G(n,k). \tag{5}$$

Since $G(1,0) = 1$ and $G(2,0) + G(2,1) = w_1 + w_2$ then, by induction, from identity (5) we get formula (4). $\square$

Lemma 1 is a particular case of a similar result, obtained by Neuwirth [18] for a more general sequence

$$G(n+1,k) = (\alpha n + \beta k + \lambda)G(n,k) + (\alpha' n + \beta' k + \lambda')G(n,k-1), \quad \beta + \beta' = 0.$$

Spivey [23] also gives some remarks connected with this result.

Let us denote by $P_{n-1}(x)$ the polynomial of a variable x of order $(n-1)$, which we get by substituting $x = w_1 + w_2$ on the right hand side of identity (4), i.e.,

$$P_{n-1}(x) = \prod_{m=0}^{n-1} (mx - m + 1). \tag{6}$$

Table 3: First few numbers of the sequence $(M(n,k))$

$n\backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	0	-1	2	0	0	0	0	0
4	0	2	-7	6	0	0	0	0
5	0	-6	29	-46	24	0	0	0
6	0	24	-146	329	-326	120	0	0
7	0	-120	874	-2521	3604	-2556	720	0

Expanding $P_{n-1}(x)$ for $n = 2, 3 \ldots$ into powers of x and denoting a coefficient of x^k by $M(n,k)$, we obtain

$$P_{n-1}(x) = M(n,1)x + M(n,2)x^2 + \cdots + M(n,n-1)x^{n-1}.$$

It is easy to see that the coefficients $(M(n,k))$, by the formula for generating polynomial (6), fulfill the following recurrence formula (with boundary conditions $M(1,0) = 1$, $M(n,0) = 0$ for $n \geq 2$, $M(n,k) = 0$ for $k \geq n$):

$$M(n+1,k) = (1-n)M(n,k) + nM(n,k-1).$$

Table 3 shows the first few elements of the sequence $(M(n,k))$. The sequence $(|M(n,k)|)$ appears in OEIS with ID (A059364), however without any information about a generating polynomial of the nth row of the array. The polynomial, in this case, has the following form:

$$|M(n,1)|x + |M(n,2)|x^2 + \cdots + |M(n,n-1)|x^{n-1} = \prod_{m=1}^{n-1}(mx + m - 1).$$

3 Main results

Let $u = u(z)$ be a holomorphic function defined in a domain $D \subset \mathbb{C}$ which fulfills the following autonomous first-order differential equation with constant coefficients:

$$u'(z) = r(u-a)^{w_1}(u-b)^{w_2}, \tag{7}$$

where r, a, b are real or complex numbers, $r \neq 0$, $a \neq b$ and w_1, w_2 are real numbers. We understand the powers $(u-a)^{w_1}$ and $(u-b)^{w_2}$ as $e^{w_1 \log(u-a)}$, $e^{w_2 \log(u-b)}$ for a branch of the logarithm function.

Theorem 1. *If a function $u(z)$ satisfies equation (7), then the nth derivative of $u(z)$ equals*

$$u^{(n)}(z) = r^n \sum_{k=0}^{n-1} G(n,k)(u-a)^{nw_1-n+k+1}(u-b)^{nw_2-k} \tag{8}$$

where $n = 2, 3, \ldots$.

Proof. We will proceed by induction. For $n = 1$ formula (8) becomes definition (7), therefore is true. Let us assume that for a positive integer n formula (8) holds. By the chain rule and recurrence formula (3) we get

$$\begin{aligned}
u^{(n+1)}(z) =& r^n \frac{d}{dz} \sum_{k=0}^{n-1} G(n,k)(u-a)^{nw_1-n+k+1}(u-b)^{nw_2-k} \\
=& r^n \sum_{k=0}^{n-1} G(n,k) \big[(nw_1 - n + k + 1)(u-a)^{nw_1-n+k}(u-b)^{nw_2-k} \\
& + (nw_2 - k)(u-a)^{nw_1-n+k+1}(u-b)^{nw_2-k-1}\big]\, r(u-a)^{w_1}(u-b)^{w_2} \\
=& r^{n+1} \big[G(n,0)(nw_1 - n + 1)(u-a)^{(n+1)w_1-n}(u-b)^{(n+1)w_2} \\
& + \sum_{k=1}^{n-1}((nw_1 - n + k + 1)G(n,k) + (nw_2 - k + 1)G(n,k-1))(u-a)^{(n+1)w_1-n+k} \\
& \times (u-b)^{(n+1)w_2-k} + G(n,n-1)(nw_2 - n + 1)(u-a)^{(n+1)w_1}(u-b)^{(n+1)w_2-n}\big] \\
=& r^{n+1} \sum_{k=0}^{n} G(n+1,k)(u-a)^{(n+1)w_1-(n+1)+k+1}(u-b)^{(n+1)w_2-k}
\end{aligned}$$

which ends the proof. ∎

Remark 1. Formula (8) may also be formulated in a real sense. Let us rewrite equation (7) as

$$v'(z) = s(v-a)^{w_1}(b-v)^{w_2}, \tag{9}$$

where s, a, b, w_1, w_2 are real numbers, $s \neq 0$ and $a < b$. We are looking for a real solution $v(z)$ of (9), which fulfills the condition $a \leq v(z) \leq b$, where $z \in D \subset \Re$ (D interval). Then if $v(z)$ be such a solution then its nth derivative is given by the following formula:

$$v^{(n)}(z) = s^n \sum_{k=0}^{n-1} (-1)^k G(n,k)(v-a)^{nw_1-n+k+1}(b-v)^{nw_2-k}. \tag{10}$$

The proof of (10) is analogous to that of Theorem 1.

Let us denote by $G_n(u)$ and $H_n(u)$ the sums on the right hand side of formulas (8) and (10) respectively, i.e.,

$$G_n(u) = \sum_{k=0}^{n-1} G(n,k)(u-a)^{nw_1-n+k+1}(u-b)^{nw_2-k}, \tag{11}$$

$$H_n(v) = \sum_{k=0}^{n-1} (-1)^k G(n,k)(v-a)^{nw_1-n+k+1}(b-v)^{nw_2-k}. \tag{12}$$

When $u(z)$ is a solution of (7), given in an explicit form, then we are able to compute the exponential generating function for the functions $(G_n(u))_{n\geq 1}$ in the following form:

$$g(u,w) = u + rG_1(u)w + r^2 G_2(u)\frac{w^2}{2!} + r^3 G_3(u)\frac{w^3}{3!} + \cdots .$$

In order to do it let us observe that $g(u(z), w)$ is the Taylor expansion of $u(z)$ at the point z. Therefore we obtain

$$\begin{aligned} g(u(z), w) =& u(z) + rG_1(u(z))w + r^2 G_2(u(z))\frac{w^2}{2!} + r^3 G_3(u(z))\frac{w^3}{3!} + \cdots \\ =& u(z) + u'(z)w + u''(z)\frac{w^2}{2!} + u'''(z)\frac{w^3}{3!} + \cdots = u(z+w). \end{aligned} \tag{13}$$

Assuming moreover that $u(z)$ is invertible on the set D we obtain from (13) the following formula for the generating function $g(u,w)$:

$$g(u,w) = u(z(u) + w). \tag{14}$$

Similarly we can express the exponential generating function for functions $(H_n(v))_{n\geq 1}$ in the form

$$h(v,w) = v + sH_1(v)w + s^2 H_2(v)\frac{w^2}{2!} + s^3 H_3(v)\frac{w^3}{3!} + \cdots ,$$

associated with a solution $v(z)$ of equation (9). With the same assumptions as before we have

$$h(v,w) = v(z(v) + w). \tag{15}$$

4 Examples

Example 1. Putting $w_1 = w_2 = 1$ into (3) we obtain recurrence (1) for theEulerian numbers. Equation (7) in this case is the Riccati differential equation with constant coefficients

$$u'(z) = r(u-a)(u-b), \tag{16}$$

and formula (8) takes the following form:

$$u^{(n)}(z) = r^n \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle (u-a)^{k+1}(u-b)^{n-k}, \tag{17}$$

where $n = 2, 3, \ldots$.

Formula (17) was discussed during the Conference ICNAAM 2006 in Greece and it appeared, with an inductive proof, in paper [20] (see also [21]). Independently Franssens [11] considered and proved formula (17), giving a proof based on generating functions. Hoffman [15] introduced so called derivative polynomials and stated similar problems for the first time.

In paper [22, Th. 2.2, p. 124] we have proved (taking in (17): $r = 1$, $a = 0$ and $b = 1$)

$$\int_0^1 \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle u^{k+1}(u-1)^{n-k} du = -B_{n+1}, \tag{18}$$

where B_n is the nth Bernoulli number. Bernoulli numbers are well described in books by Graham et al. [12] or Duren [8]. Let us mention that for such parameters one of the solutions of the Riccati equation (16) is $u(z) = 1/(1+e^z)$.

From equation (18) we can obtain at least two remarkable formulas. The first one we get by substituting in (18) $u = u(z) = 1/(1+e^z)$, which yields

$$\int_{-\infty}^{\infty} \left(\frac{1}{1+e^z}\right)^{(n)} \left(\frac{1}{1+e^z}\right)' dz = B_{n+1}. \tag{19}$$

Since

$$\left(\frac{1}{1+e^z}\right)' = \frac{-e^z}{(1+e^z)^2} = -\frac{1}{4\cosh^2(\frac{z}{2})},$$

then, assuming that $n+1$ is an even number, say $n+1 = 2m$, we substitute in (19) $x = z/2$ and integrate $m-1$ times by parts. In this way we arrive at the Grosset-Veselov formula

$$\frac{(-1)^{m-1}}{2^{2m+1}} \int_{-\infty}^{\infty} \left(\frac{d^{m-1}}{dx^{m-1}} \frac{1}{\cosh^2(x)}\right)^2 dx = B_{2m}.$$

Grosset and Veselov [13] obtained it while examining soliton solutions of the KdV equation.

The second formula we get, directly integrating the left hand side of equation (18). Since

$$\int_0^1 u^{k+1}(u-1)^{n-k} du = (-1)^{n-k} \frac{(k+1)!(n-k)!}{(n+2)!} = \frac{(-1)^{n-k}}{n+2} \frac{1}{\binom{n+1}{k+1}},$$

then we can rewrite (18) in the following form:

$$\sum_{k=0}^{n-1} (-1)^{n-k+1} \frac{\left\langle {n \atop k} \right\rangle}{\binom{n+1}{k+1}} = (n+2)B_{n+1}, \tag{20}$$

valid for $n = 1, 2, 3, \ldots$.

Example 2. Putting $w_1 = w_2 = \frac{1}{2}$ into (3) we obtain the following recurrence formula:

$$G(n+1,k) = \big(-\frac{n}{2} + k + 1\big)G(n,k) + \big(\frac{n}{2} - k + 1\big)G(n,k-1). \tag{21}$$

Table 4: First few numbers of the sequence $(G(n,k))$ for $w_1 = w_2 = \frac{1}{2}$

$n\backslash k$	0	1	2	3	4	5
1	1	0	0	0	0	0
2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
3	0	1	0	0	0	0
4	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
5	0	0	1	0	0	0
6	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
7	0	0	0	1	0	0
8	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0

From (21) it follows that if n is an odd positive integer then $G(n,k)$ differs from zero only for $k = \frac{n-1}{2}$ with $G(n, \frac{n-1}{2}) = 1$. When $n \geq 2$ is an even integer then only $G(n, \frac{n}{2}) = G(n, \frac{n}{2} - 1) = \frac{1}{2}$ are different from zero (see Table 4).

Let us consider a differential equation of type (9) associated with the sequence (21), say for $s = 1,\ a = -1,\ b = 1,\ w_1 = w_2 = \frac{1}{2}$

$$v'(z) = (v+1)^{\frac{1}{2}}(1-v)^{\frac{1}{2}} = \sqrt{1-v^2}. \tag{22}$$

One of the solutions of (22), for the initial condition $v(0) = 0$, is $v(z) = \sin z$ for $z \in [-\pi/2, \pi/2]$. Using formulas (12) and (10), or directly from derivatives of $v(z) = \sin z$, it follows that $H_1(v) = \sqrt{1-v^2}$, $H_2(v) = -v$, $H_3(v) = -\sqrt{1-v^2}$, $H_4(v) = v$ and $H_n(v) = H_{n-4}(v)$ for $n \geq 5$.

The generating function (15) is easy to find in the present case. Since

$$h(v(z), w) = v(z+w) = \sin(z+w) = \sin z \cos w + \cos z \sin w,$$

then

$$h(v,w) = u \cos w + \sqrt{1-v^2} \sin w.$$

Formulas corresponding to (18) and (19) are trivial in this case. Since $\int_{-1}^{1} \sqrt{1-v^2} dv = \pi/2$ we have

$$\int_{-1}^{1} h(v,w) dv = \frac{\pi}{2} \sin w = \frac{\pi}{2}\big(w - \frac{w^3}{3!} + \frac{w^5}{5!} - \frac{w^7}{7!} + \cdots\big),$$

and then

$$\int_{-1}^{1} H_n(v) dv = \begin{cases} 0, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}} \frac{\pi}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore the formula corresponding to (19) is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin z)^{(n)} (\sin z)' dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin z)^{(n)} \cos z dz = \begin{cases} 0, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}} \frac{\pi}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Table 5: First few numbers of the sequence $(G(n,k))$ for $w_1 = \frac{1}{2}, w_2 = 1$

$n\backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	$\frac{1}{2}$	1	0	0	0	0	0	0
3	0	2	1	0	0	0	0	0
4	0	1	$\frac{11}{2}$	1	0	0	0	0
5	0	0	$\frac{17}{2}$	13	1	0	0	0
6	0	0	$\frac{17}{4}$	45	$\frac{57}{2}$	1	0	0
7	0	0	0	62	192	60	1	0
8	0	0	0	31	536	726	$\frac{247}{2}$	1

Example 3. Let us put $w_1 = \frac{1}{2}, w_2 = 1$ into (3). We have

$$G(n+1,k) = \big(-\frac{n}{2} + k + 1\big)G(n,k) + (n-k+1)G(n,k-1). \tag{23}$$

Some initial terms of the sequence are shown in Table 5.

The numerical evidence shows that every odd row of the above sequence corresponds to a certain row of the sequence ID (A160468) in OEIS.

Let us consider also a differential equation of type (7) associated with sequence (23), say for parameters $r = 1,\ a = 0,\ b = 1,\ w_1 = \frac{1}{2},\ w_2 = 1$

$$u'(z) = \sqrt{u}(u-1). \tag{24}$$

One of the solutions of the equation (24) is

$$u(z) = \left(\frac{1-e^z}{1+e^z}\right)^2, \tag{25}$$

which we will consider only in the interval $(-\infty, 0]$, where $u(z)$ is a monotonic (decreasing) function. The inverse function of the function (25) is

$$z(u) = \log\frac{1-\sqrt{u}}{1+\sqrt{u}}, \tag{26}$$

defined in the interval $[0,1)$.

Now we will find the generating function (14). Since

$$g(u(z),w) = u(z+w) = \left(\frac{1-e^{z+w}}{1+e^{z+w}}\right)^2,$$

then using (26) we have

$$g(u,w)=\left(\frac{1-e^{w}\frac{1-\sqrt{u}}{1+\sqrt{u}}}{1+e^{w}\frac{1-\sqrt{u}}{1+\sqrt{u}}}\right)^2=\left(\frac{1+\sqrt{u}-e^{w}(1-\sqrt{u})}{1+\sqrt{u}+e^{w}(1-\sqrt{u})}\right)^2. \tag{27}$$

Let us denote by $f(w)$ the integral of the generating function (27)

$$\begin{aligned}f(w)=\int_0^1 g(u,w)du=\int_0^1\left(\frac{1+\sqrt{u}-e^{w}(1-\sqrt{u})}{1+\sqrt{u}+e^{w}(1-\sqrt{u})}\right)^2du\\ =\frac{16(e^{2w}+4e^{w}+1)e^{w}\log(\frac{2}{e^{w}+1})+(e^{w}-1)(e^{3w}+33e^{2w}+15e^{w}-1)}{(e^{w}-1)^4}.\end{aligned}$$

Thus, in this case, the formula corresponding to (18) is

$$\int_0^1 G_n(u)du=f_n,$$

and this corresponding to (19) is as follows:

$$\int_{-\infty}^0 (u(z))^{(n)}u'(z)dz=\int_{-\infty}^0\frac{d^n}{dz^n}\left(\frac{1-e^{z}}{1+e^{z}}\right)^2\cdot\frac{d}{dz}\left(\frac{1-e^{z}}{1+e^{z}}\right)^2dz=-f_n,$$

where f_n is the coefficient of $w^n/n!$ in the Taylor expansion of the function $f(w)$

$$f(w)=\frac{1}{2}-\frac{4}{15}w+\frac{4}{21}\frac{w^3}{3!}+\frac{1}{8}\frac{w^4}{4!}-\frac{4}{15}\frac{w^5}{5!}-\frac{1}{2}\frac{w^6}{6!}+\frac{20}{33}\frac{w^7}{7!}+\frac{21}{8}\frac{w^8}{8!}-\frac{2764}{1365}\frac{w^9}{9!}+O(w^{10}).$$

Example 4. By substituting in (3) $w_1=1,\ w_2=2$ we get the recurrence (2) for the second-order Eulerian numbers $G(n,k)=\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$.

The following differential equation:

$$u'(z)=u(u-1)^2 \tag{28}$$

associated with (2), unfortunately cannot be explicitly solved. Denoting by $u=L(z)$ the solution of (28) with the initial condition $L(0)=\frac{1}{2}$, we can only find its inverse function

$$z(u)=L^{-1}(u)=\log\frac{u}{1-u}+\frac{u}{1-u}-1,\qquad u\in(0,1).$$

The numerical evidence shows (as checked for $n\le 9$) that

$$\int_0^1 G_n(u)du=\int_0^1\sum_{k=0}^{n-1}\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle u^{k+1}(u-1)^{2n-k}du=\frac{B_{n+1}}{n+1}, \tag{29}$$

where B_n is the nth Bernoulli number.

If the hypothesis (29) were true, it would lead to the following formula:

$$\begin{aligned}
&\int_0^1 g(u,w)du = \int_0^1 L\big(\log\frac{u}{1-u} + \frac{u}{1-u} - 1 + w\big)du \\
&= \frac{1}{2} + \frac{B_2}{2}w + \frac{B_3}{3}\frac{w^2}{2!} + \frac{B_4}{4}\frac{w^3}{3!} + \cdots = \frac{1}{w}\big(-1 + w + 1 + B_1 w + B_2\frac{w^2}{2!} + B_3\frac{w^3}{3!} + \cdots\big) \\
&= \frac{1}{w}\big(-1 + w + \frac{w}{e^w - 1}\big) = -\frac{1}{w} + \frac{e^w}{e^w - 1},
\end{aligned}$$

where we have employed the generating function for Bernoulli numbers, often used as their definition

$$\frac{w}{e^w - 1} = 1 + B_1 w + B_2\frac{w^2}{2!} + B_3\frac{w^3}{3!} + \cdots .$$

Moreover, since

$$\int_0^1 u^{k+1}(u-1)^{2n-k}du = (-1)^k\frac{(k+1)!(2n-k)!}{(2n+2)!} = \frac{(-1)^k}{2(n+1)}\frac{1}{\binom{2n+1}{k+1}},$$

then from (29) we would also get

$$\sum_{k=0}^{n-1}(-1)^k\frac{\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle}{\binom{2n+1}{k+1}} = 2B_{n+1}.$$

The Reader may find an interesting, but unfinished, discussion on a similar topic on the web page MathOverflow [17].

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