Network coding and spherical buildings

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ABSTRACT. We develop a network coding technique based on flags of subspaces and a corresponding network channel model. To define error correcting codes we introduce a new distance on the flag variety, the Grassmann distance on flags and compare it to the commonly used gallery distance for full flags.

Keywords: Network coding, spherical building, flag variety, error correcting codes, Grassmann distance on flags

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1   Introduction

To transmit information over network channels, the currently used method consists of routing, i.e. simply forwarding the packets through each node. Network coding assumes that the packets that are sent through the network are elements of a vector space and the nodes in the network forward linear combinations of the received packets with randomly selected coefficients. It is well known (see [2]) that network coding allows multicast capacity achievability, which is not possible with packet forwarding only. When errors exist, the design of error correction codes for network-coded communication substantially differs from bit-level coding design. This is not only because of the richer algebraic structure but also because networking protocols exist between the physical transmission channel and the packet-level communication. Such protocols motivate packet-level error models.

In the seminal work [5] a novel solution to the problem of error and erasure correction is tackled over a linear network-coded packet flow. The randomly selected coding coefficients are assumed unknown (i.e. incoherent transmission) and

a novel framework of subspace coding is proposed. If the packets are elements of the vector space V then the solution by Kötter and Kschischang stems from the observation that information that is preserved after being linearly transformed by the network is the subspace generated by the input vectors, which is an element of the Grassmannian space $\mathcal{G}(V)$, the set of all subspaces of V.

In this work, we extend the applicability of such framework under the assumption that in-network nodes can keep track of packet sequence numbering, as it is the case on the Internet. Under such assumption, we propose encoding information over flags, i.e. chains of subspaces of V, that are network coded by the in-network nodes with the stabilizers of the flags as they traverse the network [9]. The set of all flags in V forms a simplicial complex, known as the spherical building of the general linear group of V. We modify the well known geometry of this spherical building to develop a minimum distance decoding scheme in the new geometry. As the geometry of spherical buildings is governed by the symmetric group $\mathcal{S}_n$, Section 2 is mostly devoted to summarize the relevant facts about symmetric groups. In Section 3 we introduce the basics of the flag variety of V and the associated spherical building. The major goal of the next section is to define a new distance on flags, the Grassmann distance (see Definition 4.3), which is more appropriate to measure errors and erasures in the transmission of flags through the network. The Grassmann distance seems to be much easier to compute than the commonly used gallery distance. In the last section, Section 5, we present a model for network transmission including errors and erasures which allows for the derivation of conditions for code constructions based on the Grassmann distance on flags. To set up a benchmark for comparing new flag codes to the classical situation of subspace codes we transfer and generalize the rank metric codes from [6] to our situation. Other examples for flag codes of smaller minimum distance are given which allow for easy decoding.

2 Symmetric groups

The symmetric group $\mathcal{S}_n$ is the group of all bijective mappings from $\{1, \ldots, n\}$ to itself. As this group will govern the geometry of the flag variety we collect some relevant facts about symmetric groups in this section.

2.1 The length and the depth of a permutation

Definition 2.1. *(see for instance [3, Section 1.6])*
The **length** *of a permutation $\pi \in \mathcal{S}_n$ is*

$$\ell(\pi) := \sum_{i=1}^{n} |\{k \in \{1, \ldots, i\} \mid \pi(k) > \pi(i)\}|.$$

Then the identity is the unique element of $\mathcal{S}_n$ having length 0 and the elements $\pi \in \mathcal{S}_n$ of length 1 are exactly the transpositions $t_i = (i, i+1)(1 \le i \le n-1)$ interchanging i and $i+1$ and leaving the other points fixed. It is well known that any element of $\mathcal{S}_n$ is a product of these t_i. The length of $\pi \in \mathcal{S}_n$ is the number of factors in such a minimal expression of π as a product of the t_i ([3, Section 1.7]). As all the t_i have order 2, this shows that $\ell(\pi) = \ell(\pi^{-1})$.

Remark 2.2. *There is a unique* **longest element** $w_0 \in \mathcal{S}_n$ *with maximal* $\ell(w_0)$. *This element is* $w_0 = (1, n)(2, n-1)(3, n-2) \cdots$ *and has length*

$$\frac{n(n-1)}{2} = \binom{n}{2} = |\{(i,j) \in \{1, \ldots, n\}^2 \mid i < j\}|$$

The most commonly used distance on the flag variety is the gallery distance (see Definition 4.8). This distance is defined using the length of a permutation. For our purposes it seems to be more appropriate to work with the Grassmann distance on flags defined in Definition 4.3 below. Here we replace the length function by a slightly different function called the depth function (see [10, Theorem 1.1]).

Definition 2.3. *For* $\pi \in \mathcal{S}_n$ *we define*

$$\mathfrak{dp}(\pi) := \sum_{i=1}^{n-1} |\{k \in \{1, \ldots, i\} \mid \pi(k) > i\}|.$$

Then $\mathfrak{dp} : \mathcal{S}_n \to \mathbb{N}_0$ *is called the* **depth function**.

It is easy to see that $\mathfrak{dp}(\pi) = 0$ if and only if $\pi = \mathrm{id}$. Also $\mathfrak{dp}(\pi) = 1$ if and only if $\ell(\pi) = 1$ if and only if $\pi = t_i = (i, i+1)$ for some $1 \le i \le n-1$. More generally we get

Theorem 2.4. *(see Observation 2.2 in [10]) For any permutation* $\pi \in \mathcal{S}_n$ *we have*

$$\frac{\ell(\pi) + \ell_{tr}(\pi)}{2} \le \mathfrak{dp}(\pi) \le \ell(\pi)$$

where $\ell_{tr}(\pi)$ *is the smallest number of transpositions needed to write* π.

Remark 2.5.   *(a) It is easy to see that*

$$\mathfrak{dp}(\pi) = \sum_{k=1, k < \pi(k)}^{n} \pi(k) - k.$$

(b) For the longest element w_0 *from Remark 2.2 we compute*

$$\mathfrak{dp}(w_0) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}(n - 2k + 1) = \begin{cases} (n/2)^2 & n \text{ even} \\ (n-1)(n+1)/4 & n \text{ odd} \end{cases}$$

(see Sequence A002620 in [8])

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(c) *The function*

$$s(\pi) := \mathfrak{dp}(\pi) + \mathfrak{dp}(\pi^{-1}) = \sum_{k=1}^{n} |\pi(k) - k|$$

is known as the sum of distances function. As we will see in Corollary 4.7 $\mathfrak{dp}(\pi) = \mathfrak{dp}(\pi^{-1})$ *so* $s(\pi) = 2\,\mathfrak{dp}(\pi)$.

(d) *The number of permutations* $\pi \in \mathcal{S}_n$ *such that* $\mathfrak{dp}(\pi) = k$ *is denoted by* $T(n, k)$ *in the Sequence A062869 in [8]. Sequence A062870 in [8] gives*

$$T(n, k_0) = \begin{cases} (\frac{n}{2}!)^2 & n \ even \\ (\frac{n(n-1)}{2}!)^2 & n \ odd \end{cases}$$

where $k_0 = \mathfrak{dp}(w_0) = \max\{\mathfrak{dp}(\pi) \mid \pi \in \mathcal{S}_n\}$.

2.2   Young subgroups

Definition 2.6. *Let* $T' := (k_1, \ldots, k_{m+1})$ *be a sequence of* $m + 1$ *natural numbers* $k_i \geq 1$ *with* $\sum_{i=1}^{m+1} k_i = n$. *The* **Young subgroup** $\mathcal{Y}_{T'} \cong \mathcal{S}_{k_1} \times \mathcal{S}_{k_2} \times \ldots \times \mathcal{S}_{k_{m+1}}$ *is the stabilizer in* $\mathcal{S}_n$ *of the sequence*

$$(\{1, \ldots, k_1\}, \{k_1 + 1, \ldots, k_1 + k_2\}, \ldots, \{k_1 + \ldots + k_m + 1, \ldots, n\}).$$

Clearly $|\mathcal{Y}_{T'}| = \prod_{i=1}^{m+1} k_i!$. For $m = 0$ we get $\mathcal{Y}_{T'} = \mathcal{Y}_{(n)} = \mathcal{S}_n$. Also if $k_i = 1$ for all i then $\mathcal{Y}_{T'} = \mathcal{Y}_{(1,\ldots,1)} = \{\mathrm{id}\}$.

It is well known (see for instance [4]) that any double coset $\mathcal{Y}_{T'}\pi\mathcal{Y}_{T'}$ contains a unique element of minimal length. So these double cosets have canonical representatives which we collect in the set $\Sigma_{T'}$:

Definition 2.7. *Let* $\Sigma_{T'}$ *denote the set of representatives of minimal length such that*

$$\mathcal{S}_n = \dot{\cup}_{\pi \in \Sigma_{T'}} \mathcal{Y}_{T'}\pi\mathcal{Y}_{T'}.$$

3   Spherical buildings

This section provides a constructive approach to the relevant facts about the spherical building of the general linear group of a finite dimensional vector space. For most of the proofs and more details we refer to the textbooks [1], [3], and [7].

3.1   The flag variety

Let K be a field and V an n-dimensional vector space over K. The general linear group of V, $\mathrm{GL}(V)$, is the group of all linear automorphisms of V (invertible linear maps from V to itself).

A **flag** is a set of subspaces $\Lambda := \{W_i \mid 1 \leq i \leq m\}$ of V with

$$\{0\} < W_1 < \ldots < W_m < V.$$

The **type** of Λ is the set of dimensions

$$\text{type}(\Lambda) := \{\dim(W_i) \mid W_i \in \Lambda\} \subseteq \{1, \ldots, n-1\}.$$

Let

$$\mathcal{F}(V) := \{\Lambda \mid \Lambda \text{ is a flag in } V\}$$

denote the set of all flags in V and for $T \subset \{1, \ldots, n-1\}$ let

$$\mathcal{F}_T(V) := \{\Lambda \in \mathcal{F}(V) \mid \text{type}(\Lambda) = T\}$$

be the set of all flags in V of type T. Note that the intersection of two flags is again a flag. The empty set is the unique minimal flag, its type is $\emptyset$. The second minimal flags $\{W_1\}$ are the proper subspaces W_1 of V. So the Grassmannian of all k-dimensional subspaces

$$\mathcal{G}_k(V) = \{0 < W_1 < V \mid \dim(W_1) = k\}$$

is in bijection with the set of flags $\mathcal{F}_{\{k\}}(V)$ of type $\{k\}$. A flag is called **full**, if its type is $\{1, \ldots, n-1\}$. The set of full flags in V is denoted by $\mathcal{F}_f(V)$.

To construct a set of canonical representatives of the orbits of $\text{GL}(V)$ on $\mathcal{F}(V)$ we choose and fix once and for all a full flag

$$\Delta_0 = \{V_1, \ldots, V_{n-1}\} \in \mathcal{F}_f(V)$$

such that $\dim(V_i) = i$ and call

$$B := \{g \in \text{GL}(V) \mid V_i g = V_i \text{ for all } 1 \leq i \leq n-1\} = \text{Stab}_{\text{GL}(V)}(\Delta_0)$$

the standard **Borel subgroup**. For $T = \{d_1, \ldots, d_m\} \subseteq \{1, \ldots, n-1\}$ define

$$\Delta_T := \{V_i \mid i \in T\} \in \mathcal{F}_T(V) \text{ and } P_T := \text{Stab}_{\text{GL}(V)}(\Delta_T).$$

The groups P_T are called the standard **parabolic subgroups** of $\text{GL}(V)$.

Remark 3.1. *If $T_1 \subseteq T_2$, then $P_{T_2} \subseteq P_{T_1}$. We have $P_\emptyset = \text{GL}(V)$ and $P_{\{1,\ldots,n-1\}} = B$.*

We summarize the situation by listing some important points:

Fact 3.2. *(a) The group $\text{GL}(V)$ acts on the set $\mathcal{F}(V)$.*

(b) *The* $\mathrm{GL}(V)$-*orbits are separated by the type, so the partition*

$$\mathcal{F}(V) = \bigcup_{T \subseteq \{1,\ldots,n-1\}} \mathcal{F}_T(V)$$

is a partition of $\mathcal{F}(V)$ *into* $\mathrm{GL}(V)$-*orbits. In particular*

$$\mathcal{F}_T(V) = \{\Delta_T g \mid g \in \mathrm{GL}(V)\}.$$

(c) *For a given type* $T \subseteq \{1, \ldots, n-1\}$, *the map*

$$\mathcal{F}_T(V) \to P_T \backslash \mathrm{GL}(V), \Delta_T g \mapsto P_T g$$

is a bijection between the set of all flags of type T *and the set*

$$P_T \backslash \mathrm{GL}(V) = \{P_T g \mid g \in \mathrm{GL}(V)\}$$

of right cosets of P_T *in* $\mathrm{GL}(V)$.

To define a geometry on the flag variety we want to study $\mathrm{GL}(V)$-invariant distance functions on $\mathcal{F}_T(V)$.

Remark 3.3. *Let* T *be some type and* $P_T = \mathrm{Stab}_{\mathrm{GL}(V)}(\Delta_T)$ *the standard parabolic subgroup of* $\mathrm{GL}(V)$. *If* d *is some* $\mathrm{GL}(V)$-*invariant function on* $\mathcal{F}_T(V) \times \mathcal{F}_T(V)$, *(so* $d(\Lambda g, \Lambda' g) = d(\Lambda, \Lambda')$ *for all* $g \in \mathrm{GL}(V)$ $\Lambda, \Lambda' \in \mathcal{F}_T(V))$, *then* $d(\Delta_T g, \Delta_T h) = \overline{d}(hg^{-1})$ *where*

$$\overline{d}(g) = d(\Delta_T, \Delta_T g) \text{ for all } g \in \mathrm{GL}(V).$$

Moreover $\overline{d}$ *is constant on the double coset* $P_T g P_T$.

<u>Proof.</u> As d is $\mathrm{GL}(V)$-invariant we see that

$$d(\Delta_T g, \Delta_T h) = d(\Delta_T, \Delta_T h g^{-1}) = \overline{d}(hg^{-1}).$$

To see the second assertion let $b_1, b_2 \in P_T$, $g \in \mathrm{GL}(V)$. Then

$$\overline{d}(b_1 g b_2) = d(\Delta_T, \Delta_T(b_1 g b_2)) = d(\Delta_T b_2^{-1}, \Delta_T b_1 g) = d(\Delta_T, \Delta_T g) = \overline{d}(g).$$

$\square$

As different double cosets are disjoint, we obtain a partition

$$\mathrm{GL}(V) = \dot{\cup} \, P_T g P_T$$

of the group $\mathrm{GL}(V)$ into a disjoint union of double cosets. The number of these double cosets does not depend on the field K. For $T = \{1, \ldots, n-1\}$ this number is always $n!$ and there is a canonical bijection between these double cosets and the group $\mathcal{S}_n$ of permutations of $\{1, \ldots, n\}$ where $n = \dim(V)$. Here we embed $\mathcal{S}_n$ as the set of permutation matrices into $\mathrm{GL}(V)$. This is captured by the Gauß-Bruhat decomposition. For any type T, the P_T double cosets in $\mathrm{GL}(V)$ are in bijection with the double cosets of the Young subgroup $\mathcal{Y}_{T'}$ in the symmetric group $\mathcal{S}_n$.

Theorem 3.4. *(see [3], [4])*

$$\mathrm{GL}(V) = \dot{\cup}_{\pi \in \mathcal{S}_n} \, B\pi B.$$

More generally for a given type $T = \{d_1, \ldots, d_m\} \subseteq \{1, \ldots, n-1\}$ with $0 < d_1 < \ldots < d_m < n$ we define $T' := (k_1, \ldots, k_m, k_{m+1})$ with

$$k_1 := d_1, k_i = d_i - d_{i-1} \text{ for } 2 \le i \le m \text{ and } k_{m+1} := n - d_m.$$

Then

$$P_T = \dot{\cup}_{\pi \in \mathcal{Y}_{T'}} \, B\pi B$$

for the Young subgroup $\mathcal{Y}_{T'}$ and

$$\mathrm{GL}(V) = \dot{\cup}_{\pi \in \Sigma_{T'}} \, P_T \pi P_T$$

where $\Sigma_{T'}$ is defined in Definition 2.7

The Gauß-Bruhat decomposition has a very nice property, as described in [7, Theorem 5.10]: For each $\pi \in \mathcal{S}_n$ there is a subgroup $U_\pi \le B$ such that any element in $B\pi B$ has a unique expression as $b\pi u$ with $b \in B$ and $u \in U_\pi$. If K is a finite field, then the order of U_π is $|K|^{\ell(\pi)}$ where ℓ is the length function on $\mathcal{S}_n$ (see Definition 2.1).

3.2 Buildings and apartments

To get a more precise model of the geometry of all flags, the so called spherical building of the group $\mathrm{GL}(V)$, we fix a basis $E := \{e_1, \ldots, e_n\}$ of V and put

$$\Delta_0 = \{V_1, \ldots, V_{n-1}\} \text{ with } V_i = \langle e_1, e_2, \ldots, e_i \rangle.$$

We identify $\mathrm{GL}(V)$ with the group of invertible $n \times n$-matrices $\mathrm{GL}_n(K)$ using coordinate rows with respect to this basis. Then $B = \mathrm{Stab}_{\mathrm{GL}(V)}(\Delta_0)$ is identified with the group of all lower triangular matrices in $\mathrm{GL}_n(K)$ and the parabolic subgroup P_T with all lower block triangular matrices

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{m1} & \ldots & \ldots & A_{mm} & 0 \\ A_{m+1,1} & \ldots & \ldots & A_{m+1,m} & A_{m+1,m+1} \end{pmatrix}$$

where $A_{ij} \in K^{k_i \times k_j}$ and $A_{ii} \in \mathrm{GL}_{k_i}(K)$ for $1 \le j \le i \le m+1$ if $T' = (k_1, \ldots, k_{m+1})$. For a permutation $\pi \in \mathcal{S}_n$ we denote by

$$\Delta_\pi := \{\langle e_{\pi(1)} \rangle, \langle e_{\pi(1)}, e_{\pi(2)} \rangle, \ldots \langle e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(n-1)} \rangle\}$$

the full flag constructed by reordering the basis vectors in E according to π. For $\pi \in \mathcal{S}_n$ let $\tilde{\pi} \in \mathrm{GL}_n(K)$ denote the corresponding permutation matrix so that $\Delta_\pi = \Delta_0 \tilde{\pi}$. Then the set of all full flags that can be constructed by reordering the basis vectors in E is

$$\mathcal{A} := \{\Delta_\pi \mid \pi \in \mathcal{S}_n\} = \{\Delta_0 \tilde{\pi} \mid \pi \in \mathcal{S}_n\}.$$

Definition 3.5. *The set $\mathcal{A}$ is called the* **standard apartment**.

The standard apartment $\mathcal{A}$ has a very nice property that it contains a system of representatives of the B-orbits on $\mathcal{F}_f(V)$. This follows directly from the Gauß-Bruhat decomposition:

Corollary 3.6. *For all $\Delta \in \mathcal{F}_f(V)$ there is a unique $\pi(\Delta) =: \pi \in \mathcal{S}_n$ such that*

$$\Delta b = \Delta_\pi \in \mathcal{A}$$

for some $b \in B$.

The next lemma expresses the well known fact that any two flags have a compatible basis.

Lemma 3.7. *For any two $\Delta, \Delta' \in \mathcal{F}_f(V)$ there is some $g \in \mathrm{GL}(V)$ such that $\Delta g = \Delta_0$ and $\Delta' g = \Delta_\pi \in \mathcal{A}$ for some $\pi \in \mathcal{S}_n$, uniquely determined by Δ and Δ'. In particular any $\mathrm{GL}(V)$-invariant distance function d on $\mathcal{F}_f(V)$ satisfies $d(\Delta, \Delta') = d(\Delta_0, \Delta_\pi)$.*

<u>Proof.</u> As the action of $\mathrm{GL}(V)$ on $\mathcal{F}_f(V)$ is transitive, there is some $h \in \mathrm{GL}(V)$ such that $\Delta h = \Delta_0$. By Corollary 3.6 there is some $b \in B$ such that $(\Delta' h)b = \Delta_\pi$ (with $\pi = \pi(\Delta' h) \in \mathcal{S}_n$). Then $g := hb$ satisfies $\Delta g = \Delta_0 b = \Delta_0$ and $\Delta' g = \Delta_\pi$ as desired. $\qquad\square$

As any flag can be refined to be a full flag, Lemma 3.7 holds equally for non full flags.

Corollary 3.8. *For any two flags $\Lambda, \Lambda' \in \mathcal{F}(V)$ (not necessarily of the same type) there is some $g \in \mathrm{GL}(V)$ such that Λg and $\Lambda' g$ are contained in full flags lying in $\mathcal{A}$.*

4 Distance functions on spherical buildings

4.1 The $\mathcal{S}_n$-valued distance function

In this section we want to study $\mathrm{GL}(V)$-invariant distance functions on $\mathcal{F}_T(V)$. We have seen in Remark 3.3 that such functions are constant on the double cosets. In particular for the full flags $\mathcal{F}_f(V)$ they factor through the $\mathcal{S}_n$-valued distance function:

Definition 4.1. *The $\mathcal{S}_n$-valued distance function of $\mathcal{F}_f(V)$ is defined as*

$$d_W(\Delta_0 g, \Delta_0 h) = \pi \in \mathcal{S}_n \ \ if \ hg^{-1} \in B\pi B.$$

For two flags Δ_π and Δ_σ in the standard apartment $\mathcal{A}$ we have

$$d_W(\Delta_\pi, \Delta_\sigma) = \sigma \pi^{-1}.$$

Remark 4.2. *For $\pi \in \mathcal{S}_n$ the π-circle around Δ_0 is defined as*

$$C_\pi(\Delta_0) := \{\Delta \in \mathcal{F}(V) \mid d_W(\Delta_0, \Delta) = \pi\}.$$

Then

$$C_\pi(\Delta_0) = \{\Delta_0 \tilde{\pi} b \mid b \in B\} = \Delta_\pi B.$$

*The π-circle $C_\pi(\Delta_0)$ is in bijection with the subgroup U_π mentioned in the end of Section **??**, i.e. for each $\Delta \in C_\pi(\Delta_0)$ there is a unique $u \in U_\pi$ such that*

$$\Delta = \Delta_0 \pi u.$$

In particular if K is a finite field with q elements, then $C_\pi(\Delta_0)$ contains exactly $q^{\ell(\pi)}$ elements. For each $\pi \in \mathcal{S}_n$ the intersection of the π-circle around Δ_0 with the standard apartment $\mathcal{A}$ defined in Definition 3.5 is

$$C_\pi(\Delta_0) \cap \mathcal{A} = \{\Delta_\pi\}.$$

4.2   The Grassmann distance of flags

In this section we define a new distance on the set of all flags of a given type in V, which we call the **Grassmann distance of flags**, because it is a direct generalization of the Grassmann distance on $\mathcal{G}_k(V)$ the set of subspaces of V of dimension k.

Definition 4.3. *Let $\Lambda = \{W_1, \ldots, W_m\}$ and $\Lambda' = \{W_1', \ldots, W_m'\}$ be two flags in V of the same type $T = \{d_1, \ldots, d_m\}$ with $d_i = \dim(W_i) = \dim(W_i') \in \{1, \ldots, n-1\}$ for all i. Then the Grassmann distance is defined as*

$$\mathfrak{E}(\Lambda, \Lambda') := \sum_{i=1}^{m} (d_i - \dim(W_i \cap W_i')).$$

Theorem 4.4. *For any type T, the Grassmann distance $\mathfrak{E}$ is a $\mathrm{GL}(V)$-invariant distance function on the set $\mathcal{F}_T(V)$ of all flags of type T.*

<u>Proof.</u> Let $\Lambda = \{W_1, \ldots, W_m\}$, $\Lambda' = \{W_1', \ldots, W_m'\}$ and $\Lambda'' = \{W_1'', \ldots, W_m''\}$ be flags in V of the same type $\{d_1, \ldots, d_m\}$ with $d_i = \dim(W_i) = \dim(W_i') = \dim(W_i'') \in \{1, \ldots, n-1\}$ for all i.

We clearly have that $\mathfrak{E}(\Lambda, \Lambda') = 0$ if and only if $\Lambda = \Lambda'$. Also the symmetry $\mathfrak{E}(\Lambda, \Lambda') = \mathfrak{E}(\Lambda', \Lambda)$ is clear. The triangle inequality

$$\mathfrak{E}(\Lambda, \Lambda'') \leq \mathfrak{E}(\Lambda, \Lambda') + \mathfrak{E}(\Lambda', \Lambda'')$$

follows from the well known triangle inequality of the Grassmann distance on subspaces: For all $1 \leq i \leq m$ we have

$$d_i - \dim(W_i \cap W_i'') \leq (d_i - \dim(W_i \cap W_i')) + (d_i - \dim(W_i' \cap W_i''))$$

so this also holds for the sum. That the function $\mathfrak{E}$ is $GL(V)$-invariant follows directly from the definition. $\qquad\square$

For two subspaces W_i, W_i' of dimension i we have

$$\dim(W_i \cap W_i') + \dim(W_i + W_i') = \dim(W_i) + \dim(W_i') = 2i.$$

In particular $\dim(W_i \cap W_i') \geq 2i - n$ so we have

Remark 4.5. *(cf. Remark 2.5) For two full flags Δ, Δ' we have that*

$$\mathfrak{E}(\Delta, \Delta') \leq \begin{cases} (n/2)^2 & n \text{ even} \\ (n-1)(n+1)/4 & n \text{ odd} \end{cases}$$

The Gauß-Bruhat decomposition shows that every $GL(V)$-invariant distance on the set of all full flags in V factors through d_W. This also holds for the Grassmann distance $\mathfrak{E}$, where d_W is composed with the depth function $\mathfrak{dp}$ from Definition 2.3.

Corollary 4.6. *For any pair $\Delta, \Delta' \in \mathcal{F}_f(V)$ of full flags in V, we have*

$$\mathfrak{E}(\Delta, \Delta') = \mathfrak{dp}(d_W(\Delta, \Delta')).$$

Proof. By Lemma 3.7 it is enough to consider the standard apartment $\mathcal{A}$ and to show that for all $\pi \in \mathcal{S}_n$
$$\mathfrak{E}(\Delta_0, \Delta_\pi) = \mathfrak{dp}(\pi).$$
So let $V_i = \langle e_1, \ldots, e_i \rangle$ and $V_i' = \langle e_{\pi(1)}, \ldots, e_{\pi(i)} \rangle$ for all $1 \leq i \leq n-1$. Then

$$V_i \cap V_i' = \langle e_j \mid j \leq i \text{ and } \pi^{-1}(j) \leq i \rangle = \langle e_{\pi(k)} \mid \pi(k) \leq i \text{ and } k \leq i \rangle$$

in particular
$$i - \dim(V_i \cap V_i') = |\{k \in \{1, \ldots, i\} \mid \pi(k) > i\}|.$$

$\qquad\square$

Corollary 4.7. *As $d_W(\Delta', \Delta) = d_W(\Delta, \Delta')^{-1}$ and the function $\mathfrak{E}$ is symmetric we obtain that $\mathfrak{dp}(\pi) = \mathfrak{dp}(\pi^{-1})$ for all $\pi \in \mathcal{S}_n$.*

4.3 The gallery distance

In the theory of spherical buildings, the most commonly used distance function is the the gallery distance. This section compares the Grassmann distance to the gallery distance.

Definition 4.8. *Two full flags Δ and Δ' are said to have gallery distance 1*

$$d_G(\Delta, \Delta') = 1$$

if and only if their intersection $\Delta \cap \Delta$ has cardinality $n - 2$,

$$\Delta \cap \Delta' = \Delta \setminus \{W_k\} = \Delta' \setminus \{W_k'\}$$

*for some $W_k \in \Delta$, $W_k' \in \Delta'$. A **gallery** is a sequence $\mathcal{G} = (\Delta_1, \Delta_2, \ldots, \Delta_m)$ of full flags Δ_i such that $d(\Delta_i, \Delta_{i+1}) = 1$ for all $1 \leq i < m$. The **length** of the gallery $\mathcal{G}$ is $m - 1$. It is well known (and follows from elementary linear algebra) that any two flags Δ and Δ' can be joined by some gallery $\mathcal{G} = (\Delta, \Delta_1, \ldots, \Delta_{m-1}, \Delta')$. Then the **gallery distance** $d_G(\Delta, \Delta')$ is the minimal length m of such a gallery.*

Theorem 4.9. *([1, Section 4.8]) For all $\Delta \in \mathcal{F}(V)$ we have $d_G(\Delta_0, \Delta) = \ell(\pi(\Delta))$. In particular if $\Delta, \Delta' \in \mathcal{F}_f(V)$ then*

$$d_G(\Delta, \Delta') = \ell(d_W(\Delta, \Delta')).$$

From Theorem 2.4 we now immediately obtain the following Corollary.

Corollary 4.10. *If $\Delta \neq \Delta' \in \mathcal{F}_f(V)$ are full flags in V, then*

$$2\,\mathfrak{E}(\Delta, \Delta') > d_G(\Delta, \Delta') \geq \mathfrak{E}(\Delta, \Delta').$$

5 The channel model

Throughout this section we will work with a fixed type $T = \{d_1, d_2, \ldots, d_m\}$ with $0 < d_1 < \ldots < d_m < n$, and put $k_i := d_i - d_{i-1}$, $i = 2, \ldots, m$, $k_1 := d_1$.

We will model our network as a finite directed, acyclic multigraph with a single source and possibly multiple receivers. Every edge gets a capacity of 1, but we allow multiple edges between nodes to model different capacities. The source and the receivers agree on a set $\mathcal{C} \subset \mathcal{F}_T(V)$ of flags of type T, the error correcting code. Information is encoded as a flag $\Lambda \in \mathcal{C}$.

Assume now that the source has a flag $\Lambda = \{V_1, V_2, \ldots, V_m\} \in \mathcal{C}$ with $d_i = \dim(V_i)$. Fixing a basis of V and therewith identifying V with the space of rows, K^n, we may think of Λ as a sequence of row vectors $x_1, x_2, \ldots x_{d_m} \in K^n$ such that $x_1, x_2, \ldots, x_{d_i}$ form a basis of V_i. For $1 \leq j \leq m$ let $X_j \in K^{d_j \times n}$ be the matrix whose i-th row is x_i. Of course X_j is a submatrix of X_{j+1} and so all the information is contained in

the matrix X_m.

At every time step $1 \leq i \leq m$ and for every outgoing edge the source chooses random coefficients $y \in K^{1 \times d_i}$ and sends $y \cdot X_i \in V$ through that edge.

Furthermore every intermediate node forms a random linear combination of everything received up to this point for every edge.

So at time i the receiver receives many (say a_i) random linear combinations of the rows $x_1, \ldots, x_{d_i}$, i.e. $Z_i = Y_i \cdot X_i$ with $Y_i \in K^{a_i \times d_i}$. The receiver defines spaces

$$W_i := \langle \text{ rows of } Z_j \mid 1 \leq j \leq i \rangle.$$

Then by definition $W_i \leq W_{i+1}$ for all i. Put $\Gamma := (W_1, W_2, \ldots, W_m)$.

Remark 5.1. *If $Z_i = Y_i \cdot X_i$ and the rank of the matrix formed by the last k_i columns of Y_i equals k_i for all i, then $W_i = V_i$ for all i and $\Lambda = \{W_1, \ldots, W_m\}$. This is the case if there are no errors or erasures in the transmission. Note that a necessary condition is that each Y_i has at least k_i rows, so all the k_i need to be at most the capacity of the network.*

Note however that due to erasures or errors the receiver gets some matrix $\tilde{Z}_i = Y_i X_i + E_i$ where the rank of Y_i is smaller than d_i (due to erasures) and $E_i \neq 0$ (due to errors). We then might have that $W_i \neq V_i$, and Γ might not even be a flag of length m, but only a stuttering flag in the sense of the following definition.

Definition 5.2. *(a) A* stuttering flag *of length m is a sequence $\Gamma := (W_1, W_2, \ldots, W_m)$ of subspaces of V such that $W_1 \leq W_2 \leq \ldots \leq W_m \leq V$.*

(b) Let $\Lambda = \{V_1, \ldots, V_m\} \in \mathcal{C}$ be the sent flag and $\Gamma = (W_1, W_2, \ldots, W_m)$ the received stuttering flag. In analogy to [5, Definition 1] we define

$$\rho_i = \rho_i(\Lambda, \Gamma) := \dim(V_i) - \dim(W_i \cap V_i)$$

to be the number of erasures in step i and

$$f_i = f_i(\Lambda, \Gamma) := \dim(W_i) - \dim(W_i \cap V_i)$$

the number of errors in step i.

(c) The final error count between Λ and Γ is

$$E(\Lambda, \Gamma) := \sum_{i=1}^{m} \dim(V_i + W_i) - \dim(V_i \cap W_i).$$

Remark 5.3. *The final error count satisfies $E(\Lambda, \Gamma) = \sum_{i=1}^{m}(\rho_i(\Lambda, \Gamma) + f_i(\Lambda, \Gamma))$.*

<u>Proof.</u> This follows from the famous Grassmann identity:
$\dim(V_i + W_i) + \dim(V_i \cap W_i) = \dim(V_i) + \dim(W_i)$. $\qquad\square$

Note that if Λ and Γ are both flags of type T then $E(\Lambda, \Gamma) = 2\,\mathfrak{E}(\Lambda, \Gamma)$. The error count also originates from the Grassmannian distance on subspaces and is thus a metric satisfying the triangle inequality. Hence in analogy to [5, Theorem 2] we get the following corollary.

Corollary 5.4. *Let $\mathcal{C}$ be a set of flags of type T and*

$$d(\mathcal{C}) := \min\{\mathfrak{E}(\Lambda', \Lambda) \mid \Lambda' \neq \Lambda \in \mathcal{C}\}$$

be the minimum distance of $\mathcal{C}$. Using the code $\mathcal{C}$ for transmission through the network we can correct all errors as long as the error count satisfies $E(\Lambda, \Gamma) < d(\mathcal{C})$, meaning that in this case Λ is the unique element of $\mathcal{C}$ such that $E(\Lambda, \Gamma)$ is minimal.

<u>Proof.</u> Let $e := E(\Lambda, \Gamma)$. For another flag $\Lambda \neq \Delta \in \mathcal{C}$ set $f := E(\Delta, \Gamma)$. Then the triangle inequality gives us

$$E(\Lambda, \Delta) \leq E(\Lambda, \Gamma) + E(\Gamma, \Delta) = e + f.$$

On the other hand Λ and Δ are elements of $\mathcal{C}$ and thus we can use the observation from above to get

$$d(\mathcal{C}) \leq \mathfrak{E}(\Lambda, \Delta) = \frac{E(\Lambda, \Delta)}{2}.$$

Putting these together we get $2d(\mathcal{C}) \leq e + f$. But as we assumed that $e < d(\mathcal{C})$ we thus have $f > d(\mathcal{C}) > e$, hence Λ is indeed the unique element of $\mathcal{C}$ having minimal distance to Γ. $\qquad\square$

5.1   Error correcting codes

For good error correcting codes, as in the classical situation, $|\mathcal{C}|$ and $d(\mathcal{C})$ both should be large.

One idea is to construct $\mathcal{C}$ as an orbit $\Delta_T S$ of some subgroup $S \leq \mathrm{GL}(V)$ with $S \cap P_T = \{1\}$. Then, using Remark 3.3 we can compute the minimum distance on $\mathcal{C} = \Delta_T S$ as follows:

Remark 5.5. *For $g, h \in \mathrm{GL}(V)$ we have*

$$\mathfrak{E}(\Delta_T g, \Delta_T h) = \mathfrak{E}(\Delta_T(gh^{-1}), \Delta_T) =: \overline{\mathfrak{E}}_T(gh^{-1}).$$

In particular if $S \leq \mathrm{GL}(V)$ with $S \cap P_T = \{1\}$, then

$$d(\Delta_T S) = d_T(S) := \min\{\overline{\mathfrak{E}}_T(g) \mid 1 \neq g \in S\}.$$

As usually we abbreviate $\overline{\mathfrak{E}}_{\{1,\dots,n-1\}}$ by $\overline{\mathfrak{E}}_f$.

Lemma 5.6. *Let* $T = \{d_1, \ldots, d_m\}$, $k_i := d_i - d_{i-1}$ $(2 \le i \le m)$, $k_1 := 1$, $k_{m+1} := n - d_m$, *and*

$$
g = \begin{pmatrix}
I_{k_1} & A_{12} & \cdots & \cdots & A_{1m} \\
0 & I_{k_2} & A_{23} & \cdots & A_{2m} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & I_{k_m} & A_{mm} \\
0 & \cdots & \cdots & 0 & I_{k_{m+1}}
\end{pmatrix}
$$

where $A_{ij} \in K^{k_i \times k_j}$. *For* $i = 1, \ldots, m$ *put*

$$
g_i := \begin{pmatrix}
A_{1,i+1} & \cdots & A_{1m} \\
\vdots & \vdots & \vdots \\
A_{i,i+1} & \cdots & A_{im}
\end{pmatrix}
$$

the upper right $d_i \times (n - d_i)$ *submatrix of* g *and* $r_i = \mathrm{rk}(g_i)$. *Then* $\overline{\mathfrak{E}}_T(g) = \sum_{i=1}^{m} r_i$.

<u>Proof.</u> Let $\Delta_T = (V_1, V_2, \ldots, V_m)$ and $\Lambda = \Delta_T g = (W_1, W_2, \ldots, W_m)$ where $\dim(V_i) = \dim(W_i) = d_i$ for all i. We claim that $r_i = d_i - \dim(V_i \cap W_i)$, again for all i. The proof of the lemma follows directly from that claim by writing a sum on both sides. To prove the claim fix one i and consider the matrix

$$
M = \begin{pmatrix}
I_{k_1} & A_{12} & \cdots & \cdots & A_{1,i+1} & \cdots & A_{1m} \\
0 & I_{k_2} & A_{23} & \cdots & A_{2,i+1} & \cdots & A_{2m} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & I_{k_i} & A_{i,i+1} & \cdots & A_{im} \\
& & I_{d_i} & & & 0 &
\end{pmatrix}.
$$

Then the row space of M equals $V_i + W_i$ and thus the rank of M equals

$$
\mathrm{rk}(M) = \dim(V_i + W_i) = 2d_i - \dim(V_i \cap W_i).
$$

To compute the rank of M we use Gaußian elemination. As we have a big identity matrix on the bottom we can use that to reduce M to the matrix

$$
\begin{pmatrix}
0 & g_i \\
I_{d_i} & 0
\end{pmatrix}.
$$

Now we see $\mathrm{rk}(M) = r_i + d_i$ and this gives us

$$
r_i + d_i = \mathrm{rk}(M) = 2d_i - \dim(V_i \cap W_i) \quad \Rightarrow \quad r_i = d_i - \dim(V_i \cap W_i).
$$

$\square$

In fact we retrieve the idea of [6] here:

Example 5.7. *Let $T := \{k\}$, so $\mathcal{F}_T(V)$ correspond to the Grassmannian $\mathcal{G}_k(V)$. Choose some subspace $C \leq K^{k \times n-k}$ and put*

$$U_C := \{u(c) := \begin{pmatrix} I_k & c \\ 0 & I_{n-k} \end{pmatrix} \mid c \in C\}.$$

Then U_C is a subgroup of $\mathrm{GL}(V)$ isomorphic to the additive group of C. In particular for $c, c' \in C$ we compute $u(c)u(c')^{-1} = u(c - c')$. So Lemma 5.6 is a direct generalization of [6, Proposition 4] that the Grassmann distance on the subspaces with basis matrix $(I_k|c)$ is the rank metric on $K^{k \times n-k}$.

To compare such commonly used subspace codes with our new flag codes assume for convenience that $n = 4m$ and $C \leq K^{2m \times 2m}$ is an MRD code of dimension $2m$ with rank metric distance

$$d(C) := \min\{\mathrm{rk}(c) \mid 0 \neq c \in C\} = 2m$$

(see for instance [6, Section C]). Then

$$\dim(C) = 2m = \dim(U_C) \text{ and } d(C) = d_{\{2m\}}(U_C) = 2m.$$

Using flags of type $T = \{m, 2m, 3m\}$ we can improve on the dimension of the flag code (we get dimension $3m$) keeping the minimum distance to be $2m$:

Proposition 5.8. *Assume that $n = 4m$ and put $T := \{m, 2m, 3m\}$. Given two MRD codes C_m and C_{2m} with*

$$C_i \leq K^{i \times i}, d(C_i) = i, \dim(C_i) = i$$

we put

$$\mathcal{C}(C_m, C_{2m}) := \{\Delta_T u(x, y) \mid x \in C_m, y \in C_{2m}\} \subseteq \mathcal{F}_T(V)$$

where

$$u(x, y) := \begin{pmatrix} I_m & x & & \\ 0 & I_m & & y \\ & & I_m & x \\ & 0 & & 0 & I_m \end{pmatrix} \in \mathrm{GL}_{4m}(K)$$

Then $\dim(\mathcal{C}(C_m, C_{2m})) = 3m$ and $d(\mathcal{C}(C_m, C_{2m})) = 2m$.

For the proof we need the following elementary fact about multiplication of block triangular matrices.

Lemma 5.9. *Let $A \in \mathrm{GL}_m(K)$, $B, D \in K^{m \times m}$. Then*

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BA^{-1} \\ 0 & A^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} A & D \\ 0 & A \end{pmatrix}\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}^{-1} = \begin{pmatrix} I_m & (D - B)A^{-1} \\ 0 & I_m \end{pmatrix}.$$

<u>Proof.</u> (of Proposition 5.8) For $0 \neq x \in C_m$, the rank of $x \in K^{m \times m}$ is m and hence $\overline{\mathfrak{E}}_T(u(x,y)) \geq 2m$ for any $y \in K^{2m \times 2m}$ by Lemma 5.6. Now $u(x,y)^{-1} = u(-x, y'')$ for some $y'' \in K^{2m \times 2m}$, hence $u(x', y')u(x,y)^{-1} = u(x' - x, y''')$ for some y''' so by Remark 5.5.

$$\mathfrak{E}(\Delta_T u(x,y), \Delta_T u(x',y')) = \overline{\mathfrak{E}}_T(u(x'-x, y''')) \geq 2m \text{ if } x \neq x'.$$

Assume that $x = x'$ then by Lemma 5.9 $u(x,y')u(x,y)^{-1} = u(0, y'')$ with

$$y'' = (y' - y) \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

in particular the rank of y'' is the same as the one of $y - y'$. If $y \neq y'$ then this is a non zero element of C_{2m} so it has rank $2m$. Using Remark 5.5 we again find $\mathfrak{E}(\Delta_T u(x,y), \Delta_T u(x,y')) = 2m$. $\qquad\square$

5.2 Checkerboard codes

We now want to iterate the idea from Proposition 5.8. Assume that we have a sequence of MRD codes $C_i \leq K^{2^i \times 2^i}$ $(i = 0, \ldots, t)$ such that

$$\dim_K(C_i) = 2^i, \quad \text{rk}(c) = 2^i \text{ for all } 0 \neq c \in C_i.$$

Definition 5.10. *For $x_i \in C_i$ $(0 \leq i \leq t)$ we define*

$$u(x_0, x_1, \ldots, x_t) \in \text{GL}_{2^{t+1}}(K)$$

recursively as

$$u(x_0) = \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix}, \quad u(x_0, \ldots, x_t) = \begin{pmatrix} u(x_0, \ldots, x_{t-1}) & x_t \\ 0 & u(x_0, \ldots, x_{t-1}) \end{pmatrix}.$$

*Then the **checkerboard code** associated to the MRD codes C_i is*

$$\mathcal{C}(C_0, C_1, \ldots, C_t) = \{\Delta_0 u(x_0, x_1, \ldots, x_t) \mid x_i \in C_i \text{ for } 0 \leq i \leq t\} \subset \mathcal{F}_f(V).$$

Note that the dimension of $\mathcal{C}(C_0, C_1, \ldots, C_t)$ is $\sum_{i=0}^{t} 2^i = 2^{t+1} - 1$.

Proposition 5.11. *Let $\mathcal{C} := \mathcal{C}(C_0, C_1, \ldots, C_t)$.*
Then $\dim(\mathcal{C}) = 2^{t+1} - 1$ and $d(\mathcal{C}) = 2^t$.

<u>Proof.</u> We show by induction on t that $d(\mathcal{C}(C_0, C_1, \ldots, C_t)) \geq 2^t$. For $t = 0$ there is nothing to show. So let

$$g := \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}, h := \begin{pmatrix} A' & B' \\ 0 & A' \end{pmatrix} \in \text{GL}_{2^{t+1}}(K)$$

where $A, A' \in \{u(x_0, \ldots, x_{t-1}) \mid x_i \in C_i\}$, $B, B' \in C_t$. By Remark 5.5 we need to show that $\overline{\mathfrak{E}}_f(hg^{-1}) \geq 2^t$, if $g \neq h$. By Lemma 5.9

$$hg^{-1} = \begin{pmatrix} A'A^{-1} & B'' \\ 0 & A'A^{-1} \end{pmatrix}$$

with $B'' = (B' - B)A^{-1}$ if $A = A'$. If $A \neq A'$ then

$$\overline{\mathfrak{E}}_f(hg^{-1}) \geq 2\overline{\mathfrak{E}}_f(A'A^{-1}) \geq 2 \cdot 2^{t-1} = 2^t$$

by induction. If $A = A'$ then $B' \neq B \in C_t$ (because $g \neq h$) and hence $\mathrm{rk}(B' - B) = 2^t$ so $\mathrm{rk}(B'') = 2^t$ and $\overline{\mathfrak{E}}_f(hg^{-1}) \geq 2^t$ by Lemma 5.6. Note that $\overline{\mathfrak{E}}_f(u(x_0, 0, \ldots, 0)) = 2^t$ by Lemma 5.6 so $d(\mathcal{C}(C_0, C_1, \ldots, C_t)) \leq 2^t$ and hence we get the equality as claimed. $\qquad\square$

5.3 Derived subgroup codes

Take D to be the subgroup of all upper uni-triangular matrices in $\mathrm{GL}_n(K)$:

$$D = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid * \in K \right\} \leq \mathrm{GL}_n(K).$$

Then the derived subgroups of D are of the form

$$D^{(k)} = \{g \in D \mid g_{ij} = 0 \text{ for } 0 < j - i \leq k\},$$

thus having exactly k secondary diagonals filled with zeros.

Proposition 5.12. *The code* $\mathcal{C}(n, k) := \Delta_0 D^{(k)}$ *consists of fine flags and has parameters*

$$d(\mathcal{C}(n, k)) = k + 1, \ \dim(\mathcal{C}(n, k)) = \frac{(n - k)(n - k - 1)}{2}.$$

Proof. To compute the dimension of $\mathcal{C}(n, k)$ we just count the number of free parameters to be

$$\sum_{j=1}^{n-k-1} j = \frac{(n - k)(n - k - 1)}{2}.$$

For computing the minimal distance we use the fact that $D^{(k)}$ is a group. So it suffices to compute $\overline{\mathfrak{E}}_f(g)$ for $1 \neq g \in D^{(k)}$. Then g has at least one non-zero entry at a position (i, j) with $j \geq i + k + 1$. Using Lemma 5.6 this gives us $j - i \geq k + 1$ matrices with rank at least one, hence $\overline{\mathfrak{E}}_f(g) \geq k + 1$.
On the other hand taking a $g \in D^{(k)}$ that only has one non-zero entry at position $(i, i + k + 1)$ for some i yields a matrix with $\overline{\mathfrak{E}}_f(g) = k + 1$. $\qquad\square$

Remark 5.13. *The code $\mathcal{C}(n, k)$ allows for decoding erasures using only Gaußian elimination. If we receive a stuttering flag*

$$\Lambda = (U_1, \ldots, U_{n-1})$$

we can uniquely recompute the corresponding matrix $g \in D^{(k)}$ as long as the longest subchain $U_i, U_{i+1}, \ldots,$ of subspaces such that $\dim(U_{i+j}) < i + j$ has length at most k.

<u>Proof.</u> Let g_j be the submatrix of g consisting of the first $j \leq n$ rows. Then due to the zeros on the secondary diagonals the last $k + 1$ rows of g_j are not changed when computing the reduced row echelon form of g_j. If we receive a subspace U_j with $\dim(U_j) = j$ we can hence compute a reduced row echelon form of a generator matrix of U_j and by the uniqueness of that form we get the i-th row of g for all $j - k \leq i \leq j$. Thus we can recompute g as long as we have that at least every $k-$th space in Λ has the correct dimension. $\qquad\square$

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