On the Number of Conjugate Classes of Derangements

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December 28, 2016

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Abstract

The number of conjugate classes of derangements of order n is the same as the number $h(n)$ of the restricted partitions with every portion greater than 1. It is also equal to the number of isotopy classes of $2 \times n$ Latin rectangles. In this paper, a recursion formula of $h(n)$ will be obtained, also will some elementary approximation formulae with high accuracy for $h(n)$ be presented. These estimation formulae can be used to obtain the approximate value of $h(n)$ by a pocket calculator without programming function.

Key Words: Enumeration, Conjugate class, derangements, Latin rectangles, Restricted Partition number, Estimation formula, Functional approximation, Accuracy

AMS2000 Subject Classification: 05A17, 11P81, 65D15

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1 Introduction

Below n is a positive integer.

A *permutation* of a sequence $[x_1, x_2, \cdots x_n]$ is the reordering of it. For example [2, 4, 2, 3] is a permutation of [2, 2, 3, 4], so is [4, 2, 3, 2]. Let S_n be the symmetry group of the set $\mathrm{X} = \{1,\ 2,\ \cdots,\ n\}$, i.e., the set (together with the operation of combination) of the bijections from X to itself. An element σ in the symmetry group S_n is also called a *permutation* (of order n). For any $\sigma \in \mathrm{S}_n$, if $\sigma(i) = a_i$, $(i = 1,\ 2,\ \cdots,\ n;$ $\{a_1,\ a_2,\ \cdots,\ a_n\} = \mathrm{X}$), then σ is usually denoted by $\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$. When $x_1, x_2, \cdots x_n$ are distinct pairwise, the two definitions are equivalent in essence. [1] If $\sigma \in \mathrm{S}_n$, $\sigma(i) \neq i$ $(\forall i \in \mathrm{X})$, σ will be called a *derangement* of order n. When σ transforms no element to itself, the sequence $[\sigma(1), \sigma(2), \cdots, \sigma(n)]$ will also be called a *derangement*. The number of derangements of order n is denoted by D_n (or $!n$ in some literatures). It is mentioned in nearly every combinatorics textbook that,

$$D_n = (n-1)\left(D_{n-1} + D_{n-2}\right) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor, \quad n \geqslant 1.$$

Here $\lfloor x \rfloor$ is the floor function, it stands for the maximum integer that will not exceed the real x.

For $x, y \in \mathrm{S}_n$, if $\exists z \in \mathrm{S}_n$, s.t. $x = zyz^{-1}$, then x and y will be called *conjugate,* and y is called the *conjugation* of x. Of course the conjugacy relation is an equivalence

[1] When $x_i = i$, $(i = 1,\ 2,\ \cdots,\ n)$, $\sigma(x_i) = a_i$, then $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$ can be written by the sequence $[a_1, a_2, \cdots, a_n]$ for short without difficulties.

relation. So the set of derangements of order n can be divided into some conjugate classes. This paper is mainly concerned on the number of conjugate classes of derangements of order n. The main method is the same as described in reference [34].

A matrix of size $k \times n$ $(1 \leqslant k \leqslant n-1)$ with every row being a reordering of a fixed set of n elements and every column being a part of a reordering of the same set of n elements, is called a *Latin rectangle*. Usually, the set of the n elements is assumed to be $\{1, 2, 3, \cdots, n\}$. (in some literatures, the members in a Latin rectangle is assumed in the set $\{0, 1, 2, \cdots, n-1\}$.) A Latin rectangle will be called *reduced* when the first row is in increasing order and the first column is 1, 2, 3, $\cdots$, k. A Latin rectangle will be called *normalized* (*normalised*) if the first row is the sequence $[1, 2, 3, \cdots, n]$ and the first column is in increasing order. But in some references, such as [37] or [17], a Latin rectangle is called normalised if it is reduced. In references by an excellent expert on Latin squares, Douglas S. Stone, such as [49], [50], [52], a normalized Latin rectangle matches only the condition that the first row is in natural order. [2] Here the conception "*normalized*" is defined a little differently from some other references.

A normalized $2 \times n$ Latin rectangle can be considered as a derangement. An isotopy class of $2 \times n$ Latin rectangles will correspond to a unique conjugate class of derangements. So it is naturally to find out that the number of isotopy classes of $2 \times n$ Latin rectangles is the same as the number of conjugate classes of derangements of order n.

All the members in a conjugate class share the same cycle structure. A cycle structure of a derangement can be considered as an integer solution of the equation

$$s_1 + s_2 + \cdots + s_q = n, \qquad (2 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_q). \tag{1}$$

For a fixed q, designate the number of integer solutions of the equation (1) as $H_q(n)$, where q is less than $\left\lfloor \dfrac{n}{2} \right\rfloor + 1$ (otherwise $H_q(n)$ is defined by 0), and denote $h(n)$ the number of all the integer solutions of Equation (1) for all the possible q, i.e.,

$$h(n) = \sum_{q=1}^{\lfloor \frac{n}{2} \rfloor} H_q(n).$$

So the number of conjugate classes of derangements of order n is $h(n)$. Since $h(n)$ is the number of a type of restricted partitions, it is tightly connected with the partition number.

Following the notation of [36], denote by $P_q(n)$ the number of integer solutions of equation

$$s_1 + s_2 + \cdots + s_q = n, \qquad (1 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_q) \tag{2}$$

[2] Besides the 3 papers on the number of Latin rectangles by Douglas, more information on the number of Latin rectangles can be found in [51] and [53].

for a fixed q, where $1 \leqslant q \leqslant n$, and by $p(n)$ the number of all the (unrestricted) partitions of n. It is clear that [3]

$$p(n) = \sum_{q=1}^{n} P_q(n). \tag{3}$$

There is a brief introduction of the important results on the partition number (or partition function) $p(n)$ and $P_q(n)$ in reference [36], such as the recursion formula of $p(n)$ and $P_q(n)$. More information about the partition number $p(n)$ may be found in reference [55]. There is a list of some important papers and book chapters on the partition number in [46] (including the “LINKS” and “REFERENCES ”) and [3].

There are also a lot of literatures on the number of some types of restricted partitions of n (such as [41], [29], [30], [31], [18], [38], [32], [8], [33], [10], [1], [35], [44], [22], [23], [27], [15], [5]) or on the congruence properties of (restricted) partition function (such as [54], [20], [16], [13], [7], [12], [7], [11], [25], [26], [6], [2], [28]).

In [45], we can find many cases of Restricted Partitions (some of them are introduced in [9], [43] or [42]). One class are concerned on the restriction of the sizes of portions, such as portions restricted to Fibonacci numbers, powers (of 2 or 3), unit, primes, non-primes, composites or non-composites; another class are related to the restriction of the number of portions, such as the cases that the number of parts will not exceed k; the third class are about the restrictions for both, for example, the cases that the number of parts is restricted while the parts restricted to powers or primes.

But the author has not found too much information on the number $h(n)$, especially on the approximate calculation, although we can find a lot of information on other restricted partition numbers.

2 Some Formulae for $h(n)$

In this section, a recursion formula will be obtained by the method mentioned in reference [36] (page 53~55). [4]

It is mentioned in [36] (page 52) that in 1942 Auluck gave a estimation of $P_q(n)$ by $P_q(n) \approx \frac{1}{q!}\begin{pmatrix} n-1 \\ q-1 \end{pmatrix}$ when n is large enough. By the same method shown in reference [36] (page 53, 57), we can obtain the generation function of $h(n)$:

$$G(x) = \sum_{n=0}^{\infty} h(n)x^n = \frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^4}\cdots\frac{1}{1-x^i}\cdots\cdots = \prod_{i=2}^{\infty}\left(1-x^i\right)^{-1}, \tag{4}$$

[3] In a lot of articles, $p(n,q)$ is used in stead of $P_q(n)$, but in some other literatures, $p(n,q)$ stands for some other number.

[4] This section was first written in 2012, contained in the Ph. D. thesis of the author.

and a formula

$$h(n)=\frac{1}{2\pi i}\oint_C \frac{G(x)}{x^{n+1}}\mathrm{d}x, \tag{5}$$

where $h(0)=1$, $h(1)=0$, and C is a contour around the original point. The original integral formula in [36] (page 57) for $p(n)$ is

$$p(n)=\frac{1}{2\pi i}\oint_C \frac{F(x)}{x^{n+1}}\mathrm{d}x, \tag{6}$$

where $F(x)=\prod\limits_{i=1}^{\infty}(1-x^i)^{-1}$ is the generation function of $p(n)$, i.e., $F(x)=\sum\limits_{n=0}^{\infty}p(n)x^n$.

It is difficult to get a simple formula to count the solutions of Equation (1) in general. But for a fixed integer q, the number $H_q(n)$ of solutions is 0 (when $q>\left\lfloor\frac{n}{2}\right\rfloor$) or

$$\sum_{s_1=2}^{\lfloor\frac{n}{q}\rfloor}\sum_{s_2=s_1}^{\lfloor\frac{n-s_1}{q-1}\rfloor}\cdots\sum_{s_{q-1}=s_{q-2}}^{\left\lfloor\frac{n-s_1-s_2\cdots-s_{q-2}}{2}\right\rfloor}1$$
$$=\sum_{s_1=2}^{\lfloor\frac{n}{q}\rfloor}\sum_{s_2=s_1}^{\lfloor\frac{n-s_1}{q-1}\rfloor}\cdots\sum_{s_{q-2}=s_{q-3}}^{\left\lfloor\frac{n-s_1-s_2\cdots-s_{q-3}}{3}\right\rfloor}\left(\frac{n-s_1-s_2\cdots-s_{q-2}}{2}-s_{q-2}+1\right)$$
$$=P_q(n-q)\ (\text{when } q\leqslant\left\lfloor\frac{n}{2}\right\rfloor).$$

Here $H_q(n)=P_q(n-q)$ (when $q\leqslant\left\lfloor\frac{n}{2}\right\rfloor$) holds because

$$s_1+s_2+\cdots+s_q=n \quad (2\leqslant s_1\leqslant s_2\leqslant\cdots\leqslant s_q)$$
$$\Longleftrightarrow (s_1-1)+(s_2-1)+\cdots+(s_q-1)=n-q \quad (2\leqslant s_1\leqslant s_2\leqslant\cdots\leqslant s_q)$$
$$\Longleftrightarrow t_1+t_2+\cdots+t_q=n-q \quad (1\leqslant t_1\leqslant t_2\leqslant\cdots\leqslant t_q,\ \text{where } t_i=s_i-1,\ i=1,2,\cdots,q),$$

hence, for a fixed q, there is a 1-1 correspondence between the solutions of Equation (1) for (n,q) and the solutions of Equation (2) for $(n,n-q)$. So

$$h(n)=\sum_{q=1}^{\lfloor\frac{n}{2}\rfloor}H_q(n)=\sum_{q=1}^{\lfloor\frac{n}{2}\rfloor}P_q(n-q). \tag{7}$$

And there is a recursion for $P_q(n)$ in reference [36] (page 51)

$$P_q(n)=\sum_{j=1}^{t}P_j(n-q), \tag{8}$$

where $t=\min\{q,\ n-q\}$, so there is no difficulty to obtain the values of $P_q(n)$ and $h(n)$ when n is small.

For the value of $p(n)$ there is a recursion,

$$\begin{aligned} p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots + \\ (-1)^{k-1} p\left(n - \frac{3k^2 \pm k}{2}\right) + \cdots\cdots \\ = \sum_{k=1}^{k_1} (-1)^{k-1} p\left(n - \frac{3k^2 + k}{2}\right) + \sum_{k=1}^{k_2} (-1)^{k-1} p\left(n - \frac{3k^2 - k}{2}\right), \end{aligned} \tag{9}$$

where

$$k_1 = \left\lfloor \frac{\sqrt{24n+1} - 1}{6} \right\rfloor, \; k_2 = \left\lfloor \frac{\sqrt{24n+1} + 1}{6} \right\rfloor, \tag{10}$$

and assume that $p(0) = 1$. (Refer [36], page 55)

We can obtain the same recursion for $h(n)$,

$$\begin{aligned} h(n) = h(n-1) + h(n-2) - h(n-5) - h(n-7) + \cdots + \\ (-1)^{k-1} h\left(n - \frac{3k^2 \pm k}{2}\right) + \cdots\cdots \\ = \sum_{k=1}^{k_1} (-1)^{k-1} h\left(n - \frac{3k^2 + k}{2}\right) + \sum_{k=1}^{k_2} (-1)^{k-1} h\left(n - \frac{3k^2 - k}{2}\right), \end{aligned} \tag{11}$$

where k_1 and k_2 are determined by Equation (10) and assume that $h(0) = 1$.

The proof of Equation (11) is easy to understand.

By Equation (4), we have

$$\left(\sum_{n=0}^{\infty} h(n) x^n\right) \left(\prod_{i=2}^{\infty} (1 - x^i)\right) = 1. \tag{12}$$

Since $F(x) = \sum\limits_{n=0}^{\infty} p(n) x^n = \prod\limits_{i=1}^{\infty} (1 - x^i)^{-1}$, where $p(0) = 1$. So $\left(\sum\limits_{n=0}^{\infty} p(n) x^n\right) \left(\prod\limits_{i=1}^{\infty} (1 - x^i)\right) = 1$, or

$$\left(\sum_{n=0}^{\infty} p(n) x^n\right) (1 - x) \left(\prod_{i=2}^{\infty} (1 - x^i)\right) = 1. \tag{13}$$

Compare Equation (12) and Equation (13), we have

$$\sum_{n=0}^{\infty} h(n) x^n = \left(\sum_{n=0}^{\infty} p(n) x^n\right) (1 - x) = \sum_{n=0}^{\infty} \big(p(n) - p(n-1)\big) x^n,$$

assume that $h(k) = p(k) = 0$ when $k < 0$. Hence, [5]

$$h(n) = p(n) - p(n-1), \quad (n = 0, 1, 2, \cdots). \tag{14}$$

By Equation (9), we have

$$\begin{aligned} p(n-1) &= p(n-2) + p(n-3) - p(n-6) - p(n-8) + \cdots + \\ &\quad (-1)^{k-1} p\left(n - 1 - \frac{3k^2 \pm k}{2}\right) + \cdots\cdots \\ &= \sum_{k=1}^{k_1} (-1)^{k-1} p\left(n - 1 - \frac{3k^2 + k}{2}\right) + \\ &\quad \sum_{k=1}^{k_2} (-1)^{k-1} p\left(n - 1 - \frac{3k^2 - k}{2}\right), \end{aligned} \tag{15}$$

where k_1 and k_2 are described in Equation (10).

By assumption, $p\left(n - 1 - \frac{3k^2 - k}{2}\right) = 0$ when $n - 1 - \frac{3k^2 - k}{2} < 0$, and the term $p\left(n - 1 - \frac{3k^2 + k}{2}\right)$ will vanish from the equation when $n - 1 - \frac{3k^2 + k}{2} < 0$.

By Equation (9) and Equation (15), we have

$$\begin{aligned} & p(n) - p(n-1) \\ = & (p(n-1) - p(n-2)) + (p(n-2) - p(n-3)) - \\ & (p(n-5) - p(n-6)) - (p(n-7) - p(n-8)) + \cdots + \\ & (-1)^{k-1}\left(p\left(n - \frac{3k^2 \pm k}{2}\right) - p\left(n - 1 - \frac{3k^2 \pm k}{2}\right)\right) + \cdots \end{aligned}$$

By Equation (14),

$$\begin{aligned} h(n) &= h(n-1) + h(n-2) - h(n-5) - h(n-7) + \cdots + \\ &\quad (-1)^{k-1} h\left(n - \frac{3k^2 \pm k}{2}\right) + \cdots\cdots \\ &= \sum_{k=1}^{k_1} (-1)^{k-1} h\left(n - \frac{3k^2 + k}{2}\right) + \sum_{k=1}^{k_2} (-1)^{k-1} h\left(n - \frac{3k^2 - k}{2}\right). \end{aligned} \tag{16}$$

[5] A year after this formula is obtained, the author found an identity

$$p(n+1) - p(n) = p(2, n+1), \ n \geqslant 1,$$

in reference [48], where $p(2, n+1)$ is the number of partitions (of $n+1$) with every part greater than 1, which is different from the notation here. This equation is essentially the same as Equation (14).

We can easily obtain the solutions of Equation (1) by hand when $n < 13$. By Equation (11), we can obtain the number $h(n)$ of solutions of Equation (1) without technical difficulties with the help of some Computer Algebra System (CAS) softwares such as "maple", "maxima", "axiom" or some other softwares likewise (be aware of that 0 is not a valid index value in some software such like maple).

The value of $h(n)$ when $n < 250$ are shown on Table 1 (on page 9) and Table 3 (on page 10). Some value of $H_q(n)$ are shown on Table 2 (on page 9).

Obviously, $h(n) < p(n)$ holds by definition (when $n > 1$). As $p(n)$ grows much more slowly than exponential functions, i.e., for any $r > 1$, $p(n) < r^n$ will hold when n is large enough, which means we can not estimate $p(n)$ and $h(n)$ by an exponential function. As $p(n)$ grows faster than any power of n, which means we can not estimate $p(n)$ by a polynomial function. (refer [36], page 53) So, $h(n)$ can not be estimated by a polynomial function, either; otherwise, if $h(n)$ can be estimated by a polynomial of order m, by Equation (14), $p(n) = \sum_{k=2}^{n} h(n) + p(2)$ $(n > 2)$ can be estimated by a polynomial of order $m + 1$. Contradiction.

3 The Estimation of $h(n)$

The recursion formula Equation (11) for $h(n)$ is not convenient in practical for a lot of people who do not want to write programs.

The figure of the data $(n, \ln(h(n)))$ ($n = 60+20k$, $k = 1, 2, \cdots, 397$) are shown on Figure 1 on page 14. The shape is the same as that of the data $(n, \ln(p(n)))$ and $(n, \ln(R_{\mathrm{h}}(n) - p(n)))$ in reference [34], at least we can not find the difference by our eyes. Here the data points are displayed by small hollow circles, and the circles are very crowded that we may believe that the circles themselves be a very thick curve if we notice only the right-hand part. In this figure, the data points in the lower left part are sparse (compared with the points in the right upper part), and we may find some hollow circles easily. If there is a curve passes through these hollow circles, we will notice it (as shown on Figure 3 on page 18). But later in Figure 2, the circles distribute uniformly on a curve, it will be difficult to distinguish the circles and a curve passes through the centers of the them.

The author has not found a practical estimation formula with good accuracy of the number $h(n)$ before. [6]

[6] In 2015, the author find that in [47] (or some related pages in *The On-Line Encyclopedia of Integer Sequences*, OEIS for short) the values of $h(n)$ when $1 \leqslant n \leqslant 50$, together with some programs to calculate $h(n)$ written by MAPLE or MATHEMATICA, and some "FORMULA "s of $h(n)$, but these formulae are not convenient in practical use for engineers who are not willing to write a program, either.

n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$
1	0	21	165	41	7245	61	155038	81	2207851
2	1	22	210	42	8591	62	178651	82	2501928
3	1	23	253	43	10087	63	205343	83	2832214
4	2	24	320	44	11914	64	236131	84	3205191
5	2	25	383	45	13959	65	270928	85	3623697
6	4	26	478	46	16424	66	310962	86	4095605
7	4	27	574	47	19196	67	356169	87	4624711
8	7	28	708	48	22519	68	408046	88	5220436
9	8	29	847	49	26252	69	466610	89	5887816
10	12	30	1039	50	30701	70	533623	90	6638248
11	14	31	1238	51	35717	71	609237	91	7478186
12	21	32	1507	52	41646	72	695578	92	8421448
13	24	33	1794	53	48342	73	792906	93	9476370
14	34	34	2167	54	56224	74	903811	94	10659543
15	41	35	2573	55	65121	75	1028764	95	11981699
16	55	36	3094	56	75547	76	1170827	96	13462885
17	66	37	3660	57	87331	77	1330772	97	15116626
18	88	38	4378	58	101066	78	1512301	98	16967206
19	105	39	5170	59	116600	79	1716486	99	19031739
20	137	40	6153	60	134647	80	1947826	100	21339417

Table 1: The value of $h(n)$ when $1 \leqslant n \leqslant 100$

n	$h(n)$	$H_1(n)$	$H_2(n)$	$H_3(n)$	$H_4(n)$	$H_5(n)$	$H_6(n)$	$H_7(n)$
4	2	1	1					
5	2	1	1					
6	4	1	2	1				
7	4	1	2	1				
8	7	1	3	2	1			
9	8	1	3	3	1			
10	12	1	4	4	2	1		
11	14	1	4	5	3	1		
12	21	1	5	7	5	2	1	
13	24	1	5	8	6	3	1	
14	34	1	6	10	9	5	2	1
15	41	1	6	12	11	7	3	1

Table 2: The number of solutions of Equation (1) for different q

n	$h(n)$	n	$h(n)$	n	$h(n)$	n	$h(n)$
101	23911834	116	124763797	131	593224104	146	2608194590
102	26784253	117	138801828	132	656291385	147	2871619379
103	29983571	118	154364067	133	725798623	148	3160747519
104	33552415	119	171594522	134	802411183	149	3477935703
105	37524344	120	190680895	135	886795381	150	3825880113
106	41950627	121	211798491	136	979745604	160	9775430911
107	46873053	122	235172861	137	1082063336	170	24329692015
108	52353455	123	261017329	138	1194696815	180	59110637816
109	58443396	124	289602259	139	1318608064	190	140453804468
110	65217506	125	321186852	140	1454928240	200	326926597263
111	72739457	126	356095340	141	1604811073	210	746521272980
112	81098953	127	394641603	142	1769604112	220	1674422848222
113	90374472	128	437214305	143	1950689437	230	3693304861665
114	100674037	129	484193270	144	2149671688	240	8019313019148
115	112093786	130	536043530	145	2368203564	250	17156634544056

Table 3: The value of $h(n)$ when $101 \leqslant n \leqslant 250$

Since we have several accurate estimation formula of $p(n)$ (refer [34]), such as

$$R'_{\mathrm{h2}}(n) = \left\lfloor \frac{\exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)}{4\sqrt{3}\left(n + a_2\sqrt{n+c_2} + b_2\right)} + \frac{1}{2} \right\rfloor, \quad (n \geqslant 80)$$

and

$$R'_{\mathrm{h0}}(n) = \left\lfloor \frac{\exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)}{4\sqrt{3}\left(n + C'_2(n)\right)} + \frac{1}{2} \right\rfloor, \quad 1 \leqslant n \leqslant 100,$$

where $a_2 = 0.4432884566$, $b_2 = 0.1325096085$, $c_2 = 0.274078$ and

$$C'_2(n) = \begin{cases} 0.4527092482 \times \sqrt{n+4.35278} - 0.05498719946, & n = 3, 5, 7, \cdots, 99; \\ 0.4412187317 \times \sqrt{n-2.01699} + 0.2102618735, & n = 4, 6, 8 \cdots, 100. \end{cases} \tag{17}$$

By Equation (14), we can obtain $h(n)$ by

$$h_1(n) = \begin{cases} R'_{\mathrm{h0}}(n) - R'_{\mathrm{h0}}(n-1), & 2 \leqslant n \leqslant 80; \\ R'_{\mathrm{h2}}(n) - R'_{\mathrm{h2}}(n-1), & n > 80. \end{cases} \tag{18}$$

and the error of this formula will not exceed twice of the error of $R'_{\mathrm{h2}}(n)$ or $R'_{\mathrm{h0}}(n)$. Of course, this formula will not be simple enough, but the accuracy is very good.

3.1 Asymptotic Formula

As $h(n) = p(n) - p(n-1)$, by *Hardy-Ramanujan's asymptotic formula*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$$

(refer [19], [14], [39], [40], [55], [4], [34]), we assume that, when $n \gg 1$, $h(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right) - \frac{1}{4\sqrt{3}(n-1)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n-1}\right)$. So,

$$h(n) \sim \frac{1}{4\sqrt{3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n-1}\right) \left(\frac{\exp\left(\pi\sqrt{\frac{2}{3}}\left(\sqrt{n}-\sqrt{n-1}\right)\right)}{n} - \frac{1}{(n-1)}\right)$$

$$= \frac{1}{4\sqrt{3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n-1}\right) \left(\frac{\exp\left(\frac{\pi\sqrt{2/3}}{\sqrt{n}+\sqrt{n-1}}\right)}{n} - \frac{1}{(n-1)}\right)$$

$$\sim \frac{1}{4\sqrt{3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right) \left(\frac{\exp\left(\frac{\pi\sqrt{2/3}}{2\sqrt{n}}\right)}{n} - \frac{1}{(n-1)}\right)$$

$$= \frac{1}{4\sqrt{3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right) \left(\frac{\exp\left(\frac{\pi}{\sqrt{6n}}\right)}{n} - \frac{1}{(n-1)}\right)$$

$$\sim \frac{1}{4\sqrt{3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right) \left(\frac{1+\frac{\pi}{\sqrt{6n}}}{n} - \frac{1}{(n-1)}\right)$$

($e^x \approx 1 + x$, when $x \ll 1$. when $n \gg 1$, $\frac{\pi}{\sqrt{6n}} \ll 1$.)

$$= \frac{1}{4\sqrt{3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right) \left(\frac{\frac{\pi\sqrt{n}}{\sqrt{6}} - 1 + \frac{\pi}{\sqrt{6n}}}{n(n-1)}\right)$$

$$\sim \frac{1}{4\sqrt{3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right) \left(\frac{\frac{\pi\sqrt{n}}{\sqrt{6}}}{n(n-1)}\right)$$

$$= \frac{\pi}{12\sqrt{2n}(n-1)} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right) \sim \frac{\pi}{12\sqrt{2n^3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right).$$

So,

$$h(n) \sim \frac{\pi}{12\sqrt{2n^3}} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right). \tag{19}$$

In coincidence, the author find an asymptotic formula

$$P_{a,b}(n) \sim \Gamma\left(\frac{b}{a}\right) \pi^{b/a-1} 2^{-(3/2)-(b/2a)} 3^{-(b/2a)} a^{-(1/2)+(b/2a)} n^{-\frac{a+b}{2a}} \exp\left(\pi\sqrt{\frac{2n}{3a}}\right), \tag{20}$$

in [24]. When $a = 1$, $b = 2$, we will have

$$P_{1,2}(n) \sim \frac{\pi}{12\sqrt{2n^3}} \exp\left(\pi\sqrt{\frac{2}{3}n}\right), \tag{21}$$

which coincides with the asymptotic formula obtained here.

The formula (19) will also be called the *Ingham-Meinardus asymptotic formula* in this thesis, since Daniel mentioned in [24] that the more general asymptotic formula (20) was first given by A. E. Ingham in [21] and the proof was refined by G. Meinardus later in another two papers written in German.

Later in this thesis $\frac{\pi}{12\sqrt{2n^3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n}\right)$ will be denoted by $I_{\mathrm{g}}(n)$ for short.

3.2 Estimation of $h(n)$ Method A: Fit the Denominator

It is not satisfying to estimate $h(n)$ by $I_{\mathrm{g}}(n)$ when n is small. The relative error of $I_{\mathrm{g}}(n)$ to $h(n)$ is shown on Table 4 (on page 13). The round approximation

$$I'_{\mathrm{g}}(n) = \left\lfloor I_{\mathrm{g}}(n) + \frac{1}{2} \right\rfloor$$

will not change the accuracy distinctly, as shown on Table 5 (on page 13).

So it is necessary to modify the asymptotic formula in order to obtain better accuracy. In reference [34], we found that the accuracy of the estimation formula to modify the exponent parts was not as good as that to modify the denominator part. [7]

[7] If we fit $h(n)$ by $I_{\mathrm{ga}} = \frac{\pi}{12\sqrt{2n^3}} \exp\left(\sqrt{\tfrac{2}{3}}\pi\sqrt{n + C_1(n)}\right)$, or fit $\left(n, \frac{3}{2\pi^2}\left(\ln\left(\frac{12\sqrt{2n^3}h(n)}{\pi}\right)\right)^2 - n\right)$ ($n = 60 + 20k$, $k = 1, 2, \cdots, 397$) by a function $C_1(n)$, the

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	50.30%	40	32.10%	220	13.10%	520	8.39%
2	146.24%	17	56.82%	50	28.60%	240	12.50%	540	8.23%
3	202.89%	18	46.69%	60	25.90%	260	12.00%	560	8.08%
4	95.59%	19	52.75%	70	23.90%	280	11.50%	580	7.93%
5	156.43%	20	44.94%	80	22.30%	300	11.10%	600	7.79%
6	68.62%	21	48.48%	90	20.90%	320	10.80%	640	7.54%
7	121.38%	22	43.47%	100	19.80%	340	10.40%	680	7.31%
8	65.43%	23	46.00%	110	18.80%	360	10.10%	720	7.10%
9	88.38%	24	41.09%	120	18.00%	380	9.86%	760	6.91%
10	62.58%	25	43.68%	130	17.20%	400	9.60%	800	6.73%
11	79.47%	26	39.93%	140	16.60%	420	9.36%	840	6.56%
12	53.29%	27	41.27%	150	16.00%	440	9.14%	880	6.41%
13	70.98%	28	38.50%	160	15.40%	460	8.93%	920	6.27%
14	53.12%	29	39.70%	180	14.50%	480	8.74%	960	6.13%
15	60.35%	30	37.00%	200	13.70%	500	8.56%	1000	6.01%

Table 4: The relative error of $I_{\mathrm{g}}(n)$ to $h(n)$ when $n \leqslant 1000$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	50.91%	40	32.10%	220	13.08%	520	8.39%
2	100%	17	57.58%	50	28.55%	240	12.50%	540	8.23%
3	200%	18	46.59%	60	25.92%	260	11.99%	560	8.08%
4	100%	19	52.38%	70	23.89%	280	11.54%	580	7.93%
5	150%	20	45.26%	80	22.25%	300	11.13%	600	7.79%
6	75%	21	48.48%	90	20.91%	320	10.77%	640	7.54%
7	125%	22	43.33%	100	19.77%	340	10.44%	680	7.31%
8	71.43%	23	45.85%	110	18.80%	360	10.13%	720	7.10%
9	87.50%	24	40.94%	120	17.96%	380	9.86%	760	6.91%
10	66.67%	25	43.60%	130	17.22%	400	9.60%	800	6.73%
11	78.57%	26	39.96%	140	16.56%	420	9.36%	840	6.56%
12	52.38%	27	41.29%	150	15.97%	440	9.14%	880	6.41%
13	70.83%	28	38.56%	160	15.44%	460	8.93%	920	6.27%
14	52.94%	29	39.67%	180	14.52%	480	8.74%	960	6.13%
15	60.98%	30	37.05%	200	13.74%	500	8.56%	1000	6.01%

Table 5: The relative error of $\left\lfloor I_{\mathrm{g}}(n) + \frac{1}{2} \right\rfloor$ to $h(n)$ when $n \leqslant 1000$.

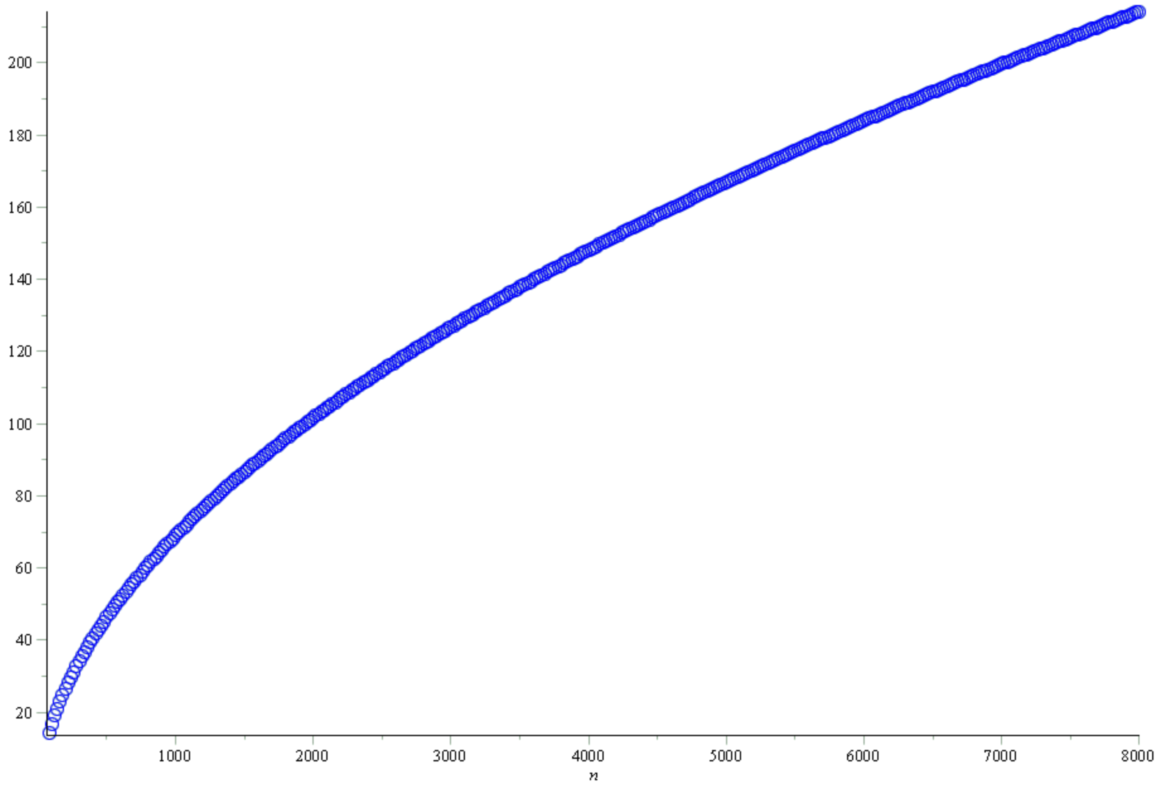

Figure 1: The graph of the data $(n, \ln h(n))$

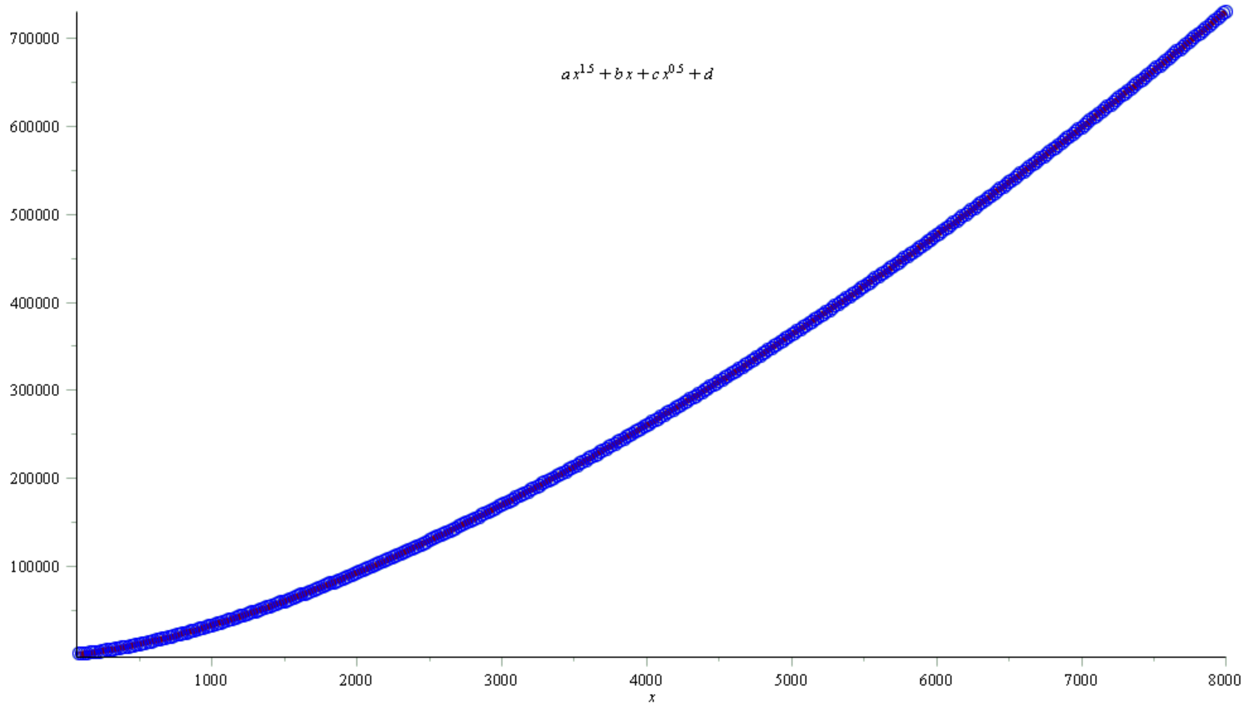

Figure 2: The graph of the data $\left(n, \dfrac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)}\right)$ and the fitting curve

Since $h(n) \sim \dfrac{\pi}{12\sqrt{2n^3}} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$, we first consider estimating $h(n)$ by $\dfrac{\pi}{12\sqrt{2C_3(n)}} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$, (i.e., fit $\dfrac{\pi^2 \exp\left(2\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{288h^2(n)}$ by a function $C_3(n)$), where $C_3(n)$ is a cubic function or a function like

$$ax^3 + bx^{2.5} + cx^2 + dx^{1.5} + ex + fx^{0.5} + g.$$

But the results are worse, as the relative errors are obviously much greater than the relative error of $I_{\mathrm{g}}(n)$ when $n < 350$.

Then we consider consider estimating $h(n)$ by $\dfrac{\pi}{12\sqrt{2}C_4(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$, or fit $\dfrac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)}$ by a function

$$C_4(n) = a_4x^{1.5} + b_4x + c_4x^{0.5} + d_4. \tag{22}$$

The result is very good. The figure of the data $\left(n, \dfrac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)}\right)$ and the fitting curve $C_4(n)$ are shown on Figure 2 on page 14. Here the fitting curve is displayed by a thick full curve, which lies in the middle of the area the circles occupied. Since the circles are too crowded, the circles themselves look like a very thick curve.

The values of the coefficients in the expression of $C_4(n)$ are as follow,

$$\begin{aligned} a_4 &= 1.000010809, \\ b_4 &= 1.862505234, \\ c_4 &= 1.169930087, \\ d_4 &= -0.7005460222. \end{aligned}$$

The value of a_4 is very close to 1, which means that this fitting function coincides with the Ingham-Meinardus asymptotic formula very well.

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result is

$$C_1(n) \doteq \frac{a_1}{\sqrt{x + c_1}} + b_1,$$

where $a_1 = 0.5145272581$, $b_1 = -1.453631562$, $c_1 = -0.555555$.

Here it is not valid to obtain the coefficients in $C_1(n)$ by iteration method described in reference [34].

The relative error of I_{ga} when $n < 1000$ is obviously greater than that of I_{g1} and I_{g2} obtained later in this section by modifying the denominator part ; when $4000 < n < 10000$, the relative error of I_{g0} is about 1000 times of that of I_{g2}.

Here the relative error of I_{ga} is not shown.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	-1.63%	40	-2.21E-05	220	7.23E-05	520	1.04E-06
2	-7.23%	17	3.82%	50	5.35E-04	240	5.74E-05	540	3.01E-07
3	29.97%	18	-1.87%	60	6.16E-04	260	4.59E-05	560	-3.53E-07
4	-8.44%	19	3.18%	70	6.15E-04	280	3.68E-05	580	-8.95E-07
5	27.94%	20	-1.21%	80	5.35E-04	300	2.97E-05	600	-1.37E-06
6	-11.61%	21	2.06%	90	4.56E-04	320	2.40E-05	640	-2.10E-06
7	20.76%	22	-0.61%	100	3.89E-04	340	1.93E-05	680	-2.64E-06
8	-6.74%	23	1.89%	110	3.30E-04	360	1.55E-05	720	-2.97E-06
9	9.21%	24	-0.85%	120	2.80E-04	380	1.24E-05	760	-3.20E-06
10	-3.44%	25	1.63%	130	2.40E-04	400	9.79E-06	800	-3.36E-06
11	8.85%	26	-0.40%	140	2.06E-04	420	7.63E-06	840	-3.43E-06
12	-5.28%	27	1.14%	150	1.78E-04	440	5.82E-06	880	-3.51E-06
13	7.42%	28	-0.29%	160	1.55E-04	460	4.32E-06	920	-3.49E-06
14	-2.35%	29	1.08%	180	1.18E-04	480	3.04E-06	960	-3.43E-06
15	3.66%	30	-0.32%	200	9.20E-05	500	1.97E-06	1000	-3.37E-06

Table 6: The relative error of $I_{\mathrm{g1}}(n)$ to $h(n)$ when $n \leqslant 1000$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	-1.82%	40	0	220	7.23E-05	520	1.06E-06
2	0	17	4.55%	50	5.21E-04	240	5.74E-05	540	2.95E-07
3	0	18	-2.27%	60	6.16E-04	260	4.59E-05	560	-3.55E-07
4	0	19	2.86%	70	6.15E-04	280	3.68E-05	580	-9.07E-07
5	50%	20	-1.46%	80	5.35E-04	300	2.97E-05	600	-1.38E-06
6	0	21	1.82%	90	4.56E-04	320	2.39E-05	640	-2.10E-06
7	25%	22	-0.48%	100	3.89E-04	340	1.93E-05	680	-2.62E-06
8	0	23	1.98%	110	3.30E-04	360	1.55E-05	720	-2.98E-06
9	12.5%	24	-0.94%	120	2.80E-04	380	1.24E-05	760	-3.22E-06
10	0	25	1.57%	130	2.40E-04	400	9.78E-06	800	-3.37E-06
11	7.14%	26	-0.42%	140	2.06E-04	420	7.63E-06	840	-3.45E-06
12	-4.76%	27	1.22%	150	1.78E-04	440	5.83E-06	880	-3.48E-06
13	8.33%	28	-0.28%	160	1.55E-04	460	4.32E-06	920	-3.48E-06
14	-2.94%	29	1.06%	180	1.18E-04	480	3.05E-06	960	-3.44E-06
15	4.88%	30	-0.29%	200	9.20E-05	500	1.97E-06	1000	-3.39E-06

Table 7: The relative error of $\left\lfloor I_{\mathrm{g1}}(n) + \frac{1}{2} \right\rfloor$ to $h(n)$ when $n \leqslant 1000$.

So we have an estimation formula

$$h(n) \sim I_{\mathrm{g1}}(n) = \frac{\pi}{12\sqrt{2}C_4(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right). \tag{23}$$

We may call it the *Ingham-Meinardus revised estimation formula 1.* The graph of $\ln\left(I_{\mathrm{g1}}(n)\right)$ is shown on Figure 3 on page 18, together with the data points of $(n, \ln h(n))$. This revised estimation formula is much more accurate than the asymptotic formula. The relative error is less than 1×10^{-6} when $n > 2000$ (as shown on Figure 4 on page 18), and less than 3‰ when $n \geqslant 30$ (as shown on Table 6 on page 16). The relative error of the round approximation $I'_{\mathrm{g1}}(n) = \left\lfloor I_{\mathrm{g1}}(n) + \frac{1}{2} \right\rfloor$ is shown on Table 7 on page 16.

But Equation (23) is not so satisfying when $n < 30$, especially when $n < 15$ as the relative error is not negligible for some value of n.

As we already know that $h(n) \sim \frac{\pi}{12\sqrt{2n^3}} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$, or

$n^{3/2} \sim \frac{\pi}{12\sqrt{2}h(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$, which means that when fitting $\frac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)}$ by a function $C_4(n)$ shown in Equation (22), the coefficient a_4 should be exactly 1, hence we should fit $\frac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)}$ by a function $C'_4(n) = x^{3/2} + b_5 x + c_5 x^{1/2} + d_5$, or fit $\frac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)} - n^{3/2}$ by a function

$$C_5(n) = b_5 x + c_5 x^{1/2} + d_5. \tag{24}$$

The figure of the data $\left(n, \frac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)} - n^{3/2}\right)$ is shown on Figure 5 on page 21 (together with the figure of the fitting function $C_5(n)$ generated by the least square method).

The values of the coefficients in Equation (24) are as follow

$$\begin{aligned} b_5 =& 1.864260743, \\ c_5 =& 1.084436400, \\ d_5 =& 0.4754177757. \end{aligned}$$

So we have another estimation formula for $h(n)$,

$$h(n) \sim I_{\mathrm{g2}}(n) = \frac{\pi}{12\sqrt{2}\left(n^{3/2} + C_5(n)\right)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right). \tag{25}$$

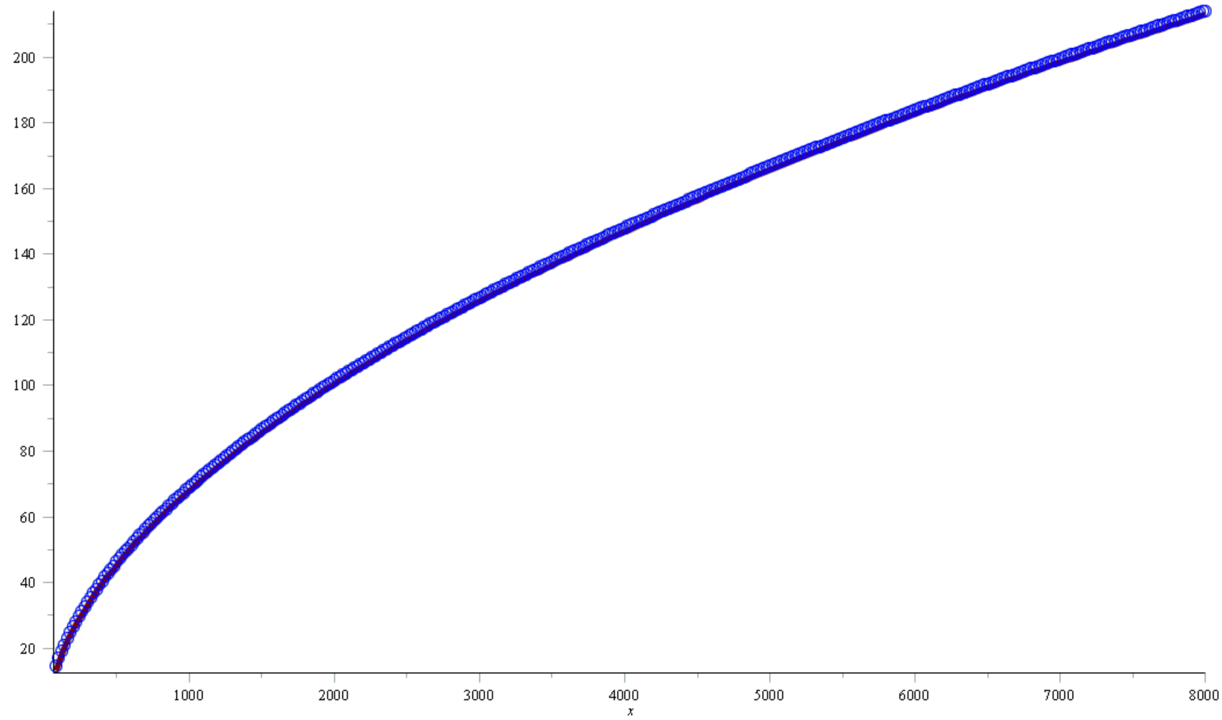

Figure 3: The graph of the data $(n, \ln h(n))$ and the fitting curve $\ln(I_{\mathrm{g}1}(n))$

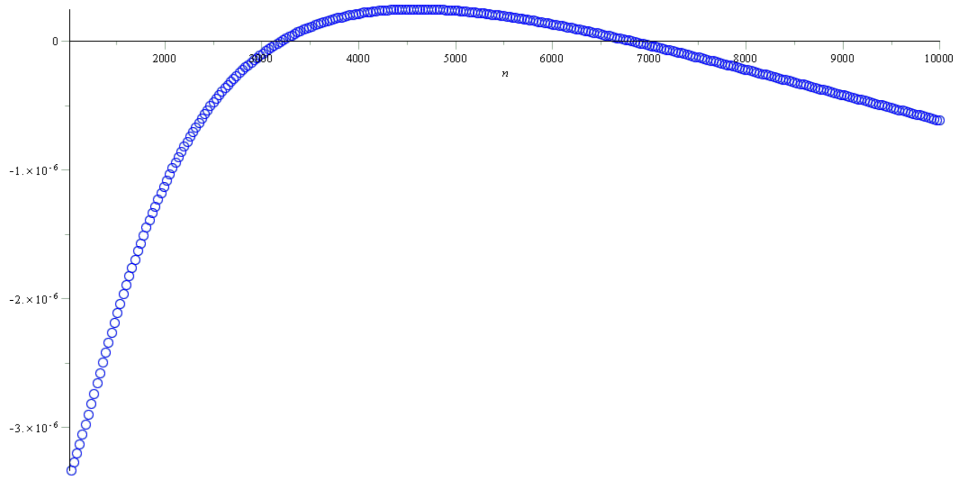

Figure 4: The Relative Error of $I_{\mathrm{g}1}(n)$when $1000 \leqslant n \leqslant 10000$

We may call it the *Ingham-Meinardus revised estimation formula 2.* The graph of $\ln(I_{g2}(n))$ is nearly the same as that of $\ln(I_{g1}(n))$ shown on Figure 2 on page 14. The second revised estimation formula is much more accurate than the first one. The relative error is less than 2×10^{-9} when $n > 3000$ (as shown on Figure 6 on page 21), about $\frac{1}{500}$ of the relative error of $I_{g1}(n)$. When $n < 10$, the relative error is also distinctly less than that of $I_{g1}(n)$ (as shown on Table 8 on page 20). The relative error of the round approximation $I'_{g2}(n) = \left\lfloor I_{g2}(n) + \frac{1}{2} \right\rfloor$ is shown on Table 9 (on page 20).

It should be mentioned that in Figure 5 on page 21, the graph of the data points lie in a line, so we might be willing to fit this line by a first order equation. The result is

$$C'_5(n) = 1.873818457 \times n + 27.08318017.$$

If we use this fitting function instead of $C_5(n)$ generated above, the relative error to fit $h(n)$ will be about 10000 times more, that is about 20 times more than that of $I_{g1}(n)$. So we do not use linear function to fit the data $\left(n, \frac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)} - n^{3/2} \right)$ before.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	-2.49%	40	-0.21%	220	2.15E-06	520	1.78E-07
2	-18.69%	17	2.99%	50	-9.06E-04	240	1.76E-06	540	1.83E-07
3	19.75%	18	-2.59%	60	-4.33E-04	260	1.46E-06	560	1.59E-07
4	-13.56%	19	2.48%	70	-1.80E-04	280	1.20E-06	580	1.51E-07
5	22.51%	20	-1.83%	80	-8.70E-05	300	9.94E-07	600	1.32E-07
6	-14.58%	21	1.46%	90	-4.13E-05	320	8.51E-07	640	1.03E-07
7	17.43%	22	-1.15%	100	-1.68E-05	340	7.18E-07	680	5.80E-08
8	-8.89%	23	1.37%	110	-5.98E-06	360	6.14E-07	720	6.80E-08
9	7.06%	24	-1.32%	120	-7.10E-07	380	5.23E-07	760	6.70E-08
10	-5.09%	25	1.18%	130	2.07E-06	400	4.61E-07	800	5.10E-08
11	7.23%	26	-0.82%	140	3.17E-06	420	3.90E-07	840	5.40E-08
12	-6.53%	27	0.74%	150	3.54E-06	440	3.34E-07	880	-4.30E-09
13	6.16%	28	-0.66%	160	3.59E-06	460	2.96E-07	920	7.00E-09
14	-3.38%	29	0.72%	180	3.16E-06	480	2.50E-07	960	2.80E-08
15	2.67%	30	-0.66%	200	2.64E-06	500	2.22E-07	1000	3.30E-08

Table 8: The relative error of $I_{\mathrm{g2}}(n)$ to $h(n)$ when $n \leqslant 1000$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	-1.82%	40	-0.21%	220	2.15E-06	520	2.00E-07
2	0	17	3.03%	50	-9.12E-04	240	1.75E-06	540	1.77E-07
3	0	18	-2.27%	60	-4.31E-04	260	1.44E-06	560	1.57E-07
4	0	19	2.86%	70	-1.80E-04	280	1.19E-06	580	1.39E-07
5	0	20	-2.19%	80	-8.68E-05	300	9.96E-07	600	1.24E-07
6	-25%	21	1.21%	90	-4.13E-05	320	8.38E-07	640	9.81E-08
7	25%	22	-0.95%	100	-1.69E-05	340	7.10E-07	680	7.91E-08
8	-14.29%	23	1.19%	110	-6.00E-06	360	6.05E-07	720	6.28E-08
9	12.5%	24	-1.25%	120	-7.08E-07	380	5.19E-07	760	5.04E-08
10	-8.33%	25	1.31%	130	2.08E-06	400	4.48E-07	800	4.07E-08
11	7.14%	26	-0.84%	140	3.17E-06	420	3.88E-07	840	3.22E-08
12	-4.76%	27	0.70%	150	3.54E-06	440	3.37E-07	880	2.59E-08
13	4.17%	28	-0.71%	160	3.57E-06	460	2.95E-07	920	2.01E-08
14	-2.94%	29	0.71%	180	3.16E-06	480	2.58E-07	960	1.52E-08
15	2.44%	30	-0.67%	200	2.63E-06	500	2.27E-07	1000	1.26E-08

Table 9: The relative error of $\left\lfloor I_{\mathrm{g2}}(n) + \frac{1}{2} \right\rfloor$ to $h(n)$ when $n \leqslant 1000$.

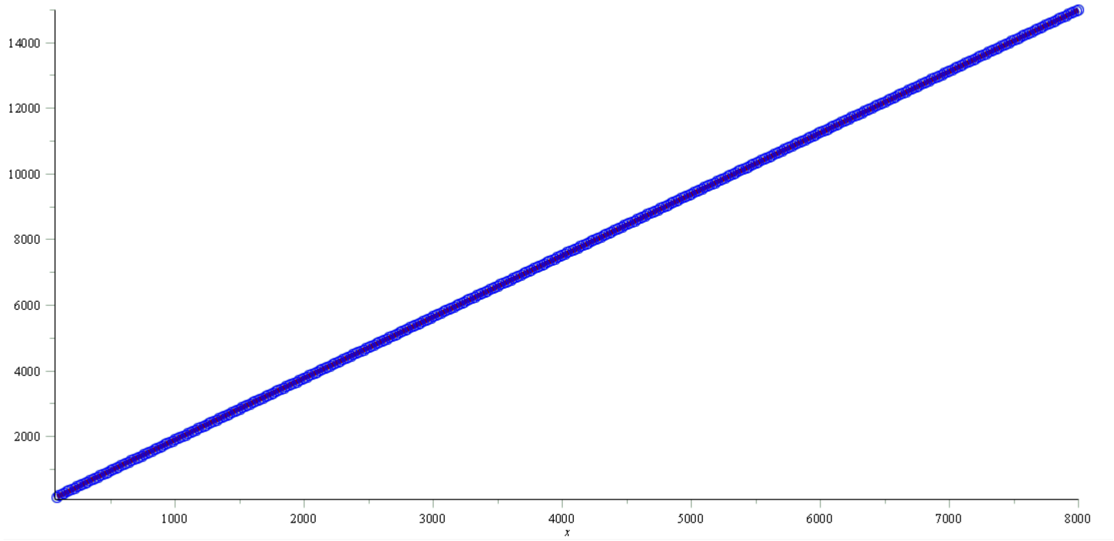

Figure 5: The graph of the data $\left(n,\ \dfrac{\pi\exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)}-n^{3/2}\right)$ and the fitting curve $C_5(n)$

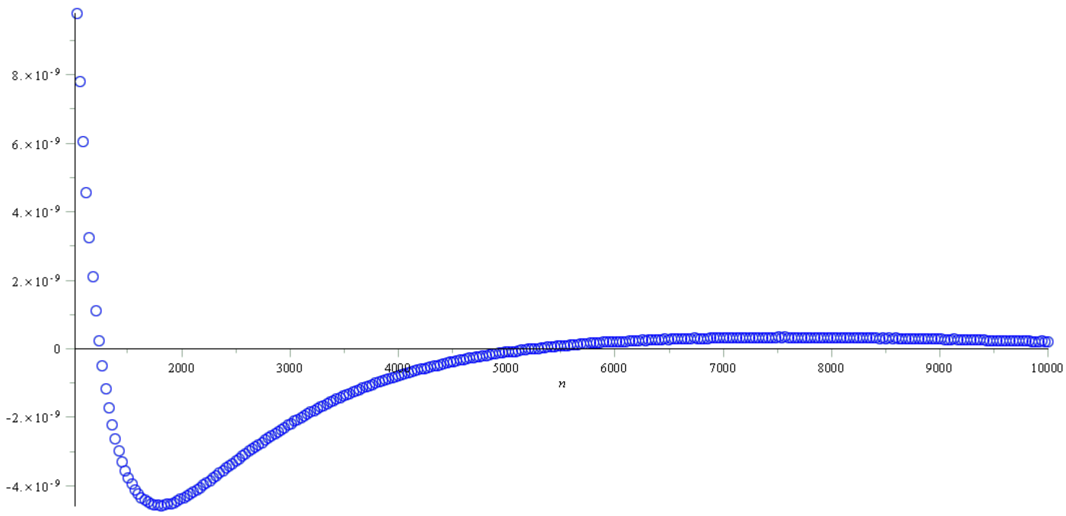

Figure 6: The Relative Error of $I_{\mathrm{g2}}(n)$when $1000 \leqslant n \leqslant 10000$

3.3 Estimation of $h(n)$ Method B: Fit $I_{\mathrm{g}}(n) - h(n)$

We wander whether we can fit $I_{\mathrm{g}}(n) - h(n)$ by a function $r(n)$, then estimate $h(n)$ by $I_{\mathrm{g}}(n) - r(n)$ which may be believed more accurate than $I_{\mathrm{g2}}(n)$ at the price of being more complicated.

By the same tricks used at the beginning of this subsection, we will have

$$I_{\mathrm{g}}(n) - I_{\mathrm{g}}(n-t) \sim \frac{t\pi^2}{24\sqrt{3}n^2} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right). \quad (t \ll n)$$

So we may fit $I_{\mathrm{g}}(n) - h(n)$ by $\dfrac{\pi^2}{24\sqrt{3}C_6(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$ where $C_6(n)$ is a quadratic function or a function like

$$ax^2 + bx^{1.5} + cx + dx^{0.5} + e.$$

That means, we can fit $\dfrac{\pi^2 \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)}{24\sqrt{3}\left(I_{\mathrm{g}}(n) - h(n)\right)}$ by a function $C_6(n)$. But the result is useless. Although $C_6(n)$ will fit the data $\dfrac{\pi^2 \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)}{24\sqrt{3}\left(I_{\mathrm{g}}(n) - h(n)\right)}$ very well, but the relative error of $I_{\mathrm{g}}(n) - \dfrac{\pi^2}{24\sqrt{3}C_6(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$ to $h(n)$ is much greater than that of $I_{\mathrm{g1}}(n)$ or $I_{\mathrm{g2}}(n)$, and the relative error differs very little with that of $I_{\mathrm{g}}(n)$ when n is small. Besides, the formula $I_{\mathrm{g}}(n) - \dfrac{\pi^2}{24\sqrt{3}C_6(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)$ are much more complicated than $I_{\mathrm{g1}}(n)$ and $I_{\mathrm{g2}}(n)$.

Then we consider fitting $\dfrac{\pi^2 \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right)}{24\sqrt{3}n^2\left(I_{\mathrm{g}}(n) - h(n)\right)}$ by a function $C_7(n)$. If $C_7(n)$ is in the form $\dfrac{a}{n} + b$ or $\dfrac{a}{n} + \dfrac{b}{n^2} + c$, the result is useless either. If $C_7(n)$ is in the form $\dfrac{a}{n^{0.5}} + b$, it will be barely satisfactory. If $C_7(n)$ is in the form $\dfrac{a}{n^{0.5}} + \dfrac{b}{n} + \dfrac{c}{n^{1.5}} + \dfrac{d}{n^2} + e$ or $\dfrac{a}{n^{0.5}} + \dfrac{b}{n} + \dfrac{c}{n^{1.5}} + e$, the result will be much better than the previous forms, but the accuracy (when estimating $h(n)$) is not as good as that of $I_{\mathrm{g1}}(n)$ and $I_{\mathrm{g2}}(n)$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	0	40	-8.13E-04	220	-1.74E-06	520	3.10E-07
2	100%	17	4.55%	50	-3.26E-04	240	-1.51E-06	540	3.29E-07
3	100%	18	-1.14%	60	-1.49E-04	260	-1.25E-06	560	3.42E-07
4	50%	19	3.81%	70	-2.81E-05	280	-9.91E-07	580	3.51E-07
5	50%	20	-0.73%	80	-4.62E-06	300	-7.56E-07	600	3.56E-07
6	0	21	2.42%	90	3.46E-06	320	-5.49E-07	640	3.57E-07
7	50%	22	-0.48%	100	6.70E-06	340	-3.72E-07	680	3.49E-07
8	0	23	1.98%	110	5.27E-06	360	-2.23E-07	720	3.36E-07
9	12.5%	24	-0.63%	120	3.37E-06	380	-9.87E-08	760	3.19E-07
10	0	25	1.83%	130	1.93E-06	400	3.54E-09	800	3.00E-07
11	14.29%	26	-0.21%	140	5.77E-07	420	8.70E-08	840	2.80E-07
12	0	27	1.22%	150	-4.01E-07	440	1.55E-07	880	2.59E-07
13	8.33%	28	-0.28%	160	-1.04E-06	460	2.09E-07	920	2.39E-07
14	0	29	1.06%	180	-1.72E-06	480	2.51E-07	960	2.19E-07
15	4.88%	30	-0.29%	200	-1.86E-06	500	2.85E-07	1000	1.99E-07

Table 10: The relative error of $\lfloor F_{7a}(n) + \frac{1}{2} \rfloor$ to $h(n)$ when $n \leqslant 1000$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1		16	-1.82%	40	-0.16%	220	-1.21E-06	520	-1.11E-06
2	0	17	3.03%	50	-6.19E-04	240	-1.77E-06	540	-1.01E-06
3	0	18	-2.27%	60	-2.60E-04	260	-2.08E-06	560	-9.24E-07
4	0	19	2.86%	70	-7.50E-05	280	-2.21E-06	580	-8.41E-07
5	50%	20	-1.46%	80	-1.85E-05	300	-2.24E-06	600	-7.64E-07
6	0	21	1.82%	90	3.62E-06	320	-2.20E-06	640	-6.25E-07
7	25%	22	-0.95%	100	1.30E-05	340	-2.12E-06	680	-5.05E-07
8	0	23	1.58%	110	1.37E-05	360	-2.02E-06	720	-4.00E-07
9	12.5%	24	-1.25%	120	1.21E-05	380	-1.90E-06	760	-3.10E-07
10	0	25	1.31%	130	1.01E-05	400	-1.78E-06	800	-2.31E-07
11	7.14%	26	-0.63%	140	7.74E-06	420	-1.66E-06	840	-1.63E-07
12	-4.76%	27	0.87%	150	5.68E-06	440	-1.54E-06	880	-1.04E-07
13	8.33%	28	-0.56%	160	3.97E-06	460	-1.42E-06	920	-5.24E-08
14	-2.94%	29	0.83%	180	1.40E-06	480	-1.31E-06	960	-7.83E-09
15	2.44%	30	-0.58%	200	-2.22E-07	500	-1.21E-06	1000	3.08E-08

Table 11: The relative error of $\lfloor F_{7b}(n) + \frac{1}{2} \rfloor$ to $h(n)$ when $n \leqslant 1000$.

The result of $C_7(n)$ is

$$C_{7a}(n) = \frac{0.8782296151}{n^{0.5}} + \frac{0.2567016063}{n} - \frac{3.580442785}{n^{1.5}} + \frac{21.28305831}{n^2} + 0.6879945549$$

or

$$C_{7b}(n) = \frac{0.8861039149}{n^{0.5}} - \frac{0.05719053203}{n} + \frac{0.9843423289}{n^{1.5}} + 0.6879343652.$$

The relative error of

$$F_{7a}(n) = I_{\mathrm{g}}(n) - \frac{\pi^2}{24\sqrt{3}n^2 C_{7a}(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right) \tag{26}$$

and

$$F_{7b}(n) = I_{\mathrm{g}}(n) - \frac{\pi^2}{24\sqrt{3}n^2 C_{7b}(n)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right) \tag{27}$$

to $h(n)$ when $1000 \leqslant n \leqslant 10000$ are shown on Figure 7 and Figure 8 (page 25), respectively. In this interval (1000, 10000), $F_{7a}(n)$ is obviously more accurate than $F_{7b}(n)$. When $n \leqslant 1000$ the relative error of $\lfloor F_{7a}(n) + \frac{1}{2} \rfloor$ and $\lfloor F_{7b}(n) + \frac{1}{2} \rfloor$ are shown on Table 10 (page 23) and Table 11 (page 23). In this case, $F_{7b}(n)$ is better than $F_{7a}(n)$. But neither of them is as good as $I_{\mathrm{g}1}(n)$ or $I_{\mathrm{g}2}(n)$, although they are more complicated than $I_{\mathrm{g}1}(n)$ and $I_{\mathrm{g}2}(n)$.

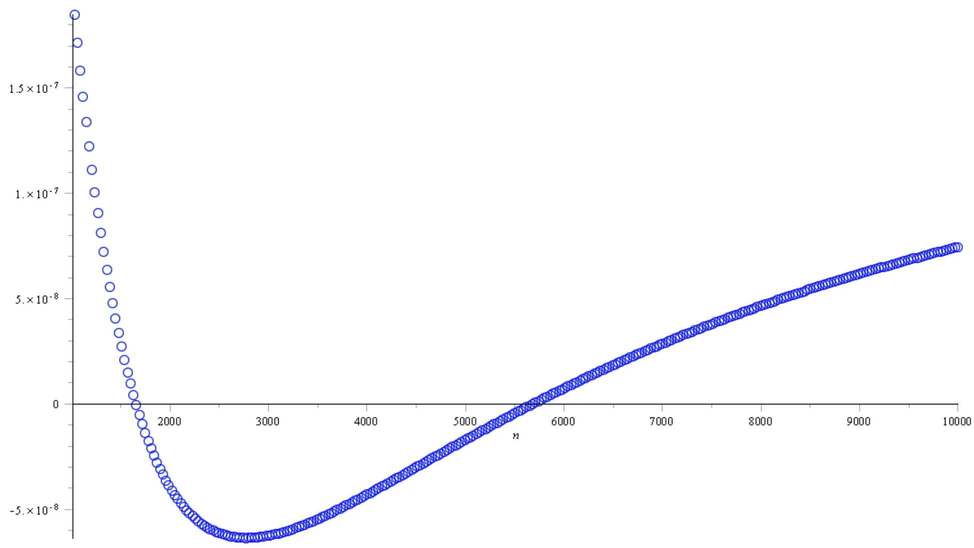

Figure 7: The Relative Error of $F_{7a}(n)$ when $1000 \leqslant n \leqslant 10000$

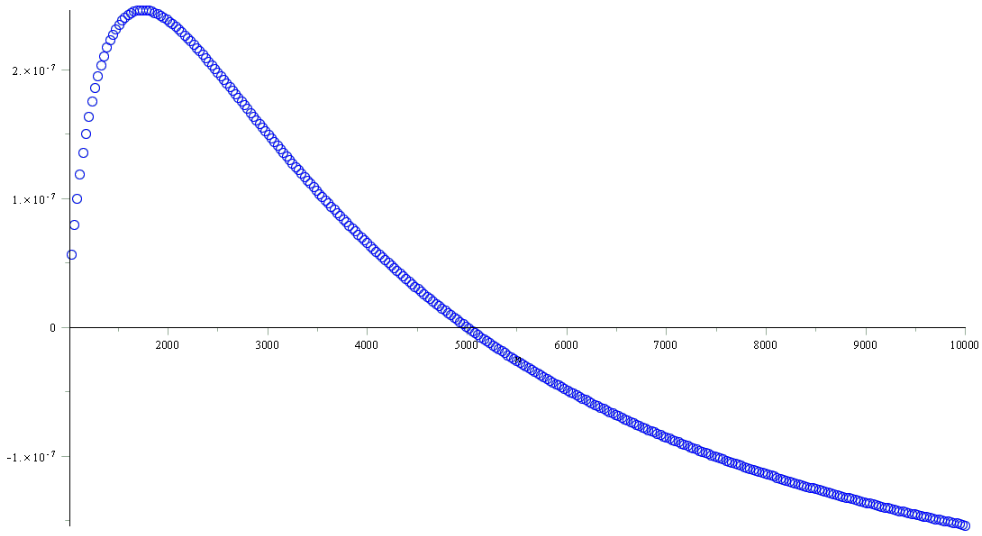

Figure 8: The Relative Error of $F_{7b}(n)$ when $1000 \leqslant n \leqslant 10000$

3.4 Estimate $h(n)$ When $n \leqslant 100$

All the estimation function for $h(n)$ found now are with very good accuracy when n is greater than 1000, but they are not so accurate when $n < 50$, especially when $n < 25$. Although $I'_{g1}(n)$ and $I'_{g2}(n)$ are better than others, the relative error are still greater than 1‰ when $n < 40$.

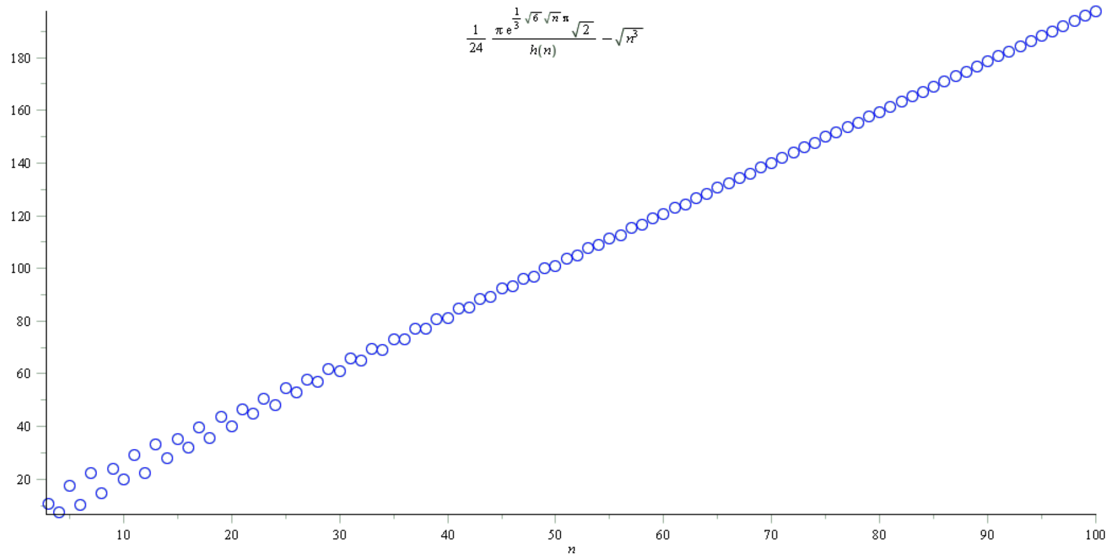

Figure 9: The graph of the data $(n,\, C_8(n))$

When $n < 40$, it is too difficult to fit $\dfrac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)} - n^{3/2}$ by a simple smooth function with high accuracy, as shown on Figure 9 (on page 26). The figure of the points $\left(n,\ \dfrac{\pi \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)}{12\sqrt{2}h(n)} - n^{3/2}\right)$ ($n = 3,\, 4,\, \cdots,\, 100$) is not so complicated (as shown on Figure 9). It seems that we can fit them by a simple piecewise function with 2 pieces, as the even points (where n is even) lie roughly on a smooth curve, so do the odd points. If we try to fit them respectively, we will have the fitting function below:

$$C_8(n) = \begin{cases} 1.942141112 \times x - 0.4796781366 \times \sqrt{x} + 8.291226268, & n = 3, 5, 7, \cdots, 99; \\ 1.803056782 \times x + 2.356539877 \times \sqrt{x} - 6.043824511, & n = 4, 6, 8 \cdots, 100. \end{cases} \tag{28}$$

Hence we can calculate $h(n)$ ($3 \leqslant n \leqslant 100$) by

$$h(n) \sim I_{\mathrm{g}0}(n) = \frac{\pi}{12\sqrt{2}\left(n^{3/2}+C_8(n)\right)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right), \quad 3 \leqslant n \leqslant 100. \tag{29}$$

Consider that $h(n)$ is an integer, we can take the round approximation of Equation (29),

$$I'_{\mathrm{g}0}(n) = \left\lfloor \frac{\pi}{12\sqrt{2}\left(n^{3/2}+C_8(n)\right)} \exp\left(\sqrt{\frac{2}{3}}\pi\sqrt{n}\right) + \frac{1}{2} \right\rfloor, \quad 3 \leqslant n \leqslant 100. \tag{30}$$

Here n begins from 3, not 1 or 2, because $\dfrac{I'_{\mathrm{g}0}(1)-h(1)}{h(1)}$ is meaningless since $h(1)=0$, and $I'_{\mathrm{g}0}(2)$ differs from $h(2)$ a lot. Besides, the value of $h(1)$ and $h(2)$ are clear by definition, so there is no need to use a complicated formula to estimate them.

The relative error of $I_{\mathrm{g}0}(n)$ (or $I'_{\mathrm{g}0}(n)$) to $h(n)$ are shown on Table 12 (or Table 13) on page 28. Compared them with Table 9 on page 20, we will find that when $n \geqslant 80$, $I'_{\mathrm{g}2}(n)$ is more accurate than $I'_{\mathrm{g}0}(n)$; when $n < 80$, $I'_{\mathrm{g}0}(n)$ is better.

4 Summary

In this paper, we have presented a recursion formula and several practical estimation formulae with high accuracy to calculated $h(n)$.

If we want to obtain the accurate value of $h(n)$, we can use the recursion formula (11) and write a program based on it, while sometimes (not always) we need to know the estimation value in the program for technique reason especially when we use a general programming language.

If we want to obtain the approximation value of $h(n)$ with high accuracy, we can use the formulae (25), (30), (23), etc.

When $2 \leqslant n \leqslant 80$, we can use $I'_{\mathrm{g}0}(n)$ (Equation (30)) , with a relative error less than 0.11% (while $32 \leqslant n \leqslant 80$) or mainly 0 with very few exceptions (while $2 \leqslant n \leqslant 31$); when $n > 80$, we can use $I'_{\mathrm{g}2}(n)$ (Equation (25)).

When $n \geqslant 100$, formulae $I'_{\mathrm{g}1}(n)$ (Equation (23)), $F_{7a}(n)$ (Equation (26)) and $F_{7b}(n)$ (Equation (27)) are also very accurate although they are not as good as Equations (25).

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	-	21	-0.16%	41	-5.72E-04	61	-1.04E-04	81	7.19E-05
2	-	22	0.12%	42	3.75E-04	62	1.43E-04	82	-5.18E-05
3	-14.85%	23	0.05%	43	-4.68E-04	63	-1.04E-04	83	8.12E-05
4	12.72%	24	-0.28%	44	4.91E-04	64	1.18E-04	84	-6.50E-05
5	1.99%	25	8.37E-04	45	-6.29E-04	65	-5.63E-05	85	8.69E-05
6	-1.83%	26	4.75E-04	46	5.45E-04	66	7.10E-05	86	-6.90E-05
7	4.76%	27	-0.17%	47	-4.45E-04	67	-2.40E-05	87	8.65E-05
8	-0.64%	28	6.69E-04	48	3.17E-04	68	5.74E-05	88	-7.40E-05
9	-0.92%	29	-3.78E-04	49	-3.42E-04	69	-1.67E-05	89	9.01E-05
10	0.69%	30	-4.43E-04	50	3.79E-04	70	4.05E-05	90	-8.07E-05
11	1.44%	31	-1.98E-04	51	-3.98E-04	71	1.30E-05	91	9.09E-05
12	-2.46%	32	4.21E-04	52	3.55E-04	72	6.81E-06	92	-8.24E-05
13	1.86%	33	-0.12%	53	-2.81E-04	73	3.41E-05	93	8.80E-05
14	-0.24%	34	9.27E-04	54	2.47E-04	74	-1.74E-06	94	-8.37E-05
15	-0.54%	35	-7.86E-04	55	-2.21E-04	75	3.84E-05	95	8.70E-05
16	-0.04%	36	1.82E-04	56	2.44E-04	76	-1.56E-05	96	-8.65E-05
17	0.46%	37	-4.80E-04	57	-2.25E-04	77	5.70E-05	97	8.44E-05
18	-0.67%	38	6.53E-04	58	2.29E-04	78	-3.58E-05	98	-8.56E-05
19	0.47%	39	-9.11E-04	59	-1.55E-04	79	6.90E-05	99	7.92E-05
20	-0.28%	40	6.34E-04	60	1.44E-04	80	-4.41E-05	100	-8.48E-05

Table 12: The relative error of $I_{\mathrm{g}0}(n)$ to $h(n)$ when $n \leqslant 100$.

n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err	n	Rel-Err
1	-	21	0	41	-5.52E-04	61	-1.03E-04	81	7.20E-05
2	-	22	0	42	3.49E-04	62	1.46E-04	82	-5.20E-05
3	0	23	0	43	-4.96E-04	63	-1.02E-04	83	8.12E-05
4	0	24	-0.31%	44	5.04E-04	64	1.19E-04	84	-6.49E-05
5	0	25	0	45	-6.45E-04	65	-5.54E-05	85	8.69E-05
6	0	26	0	46	5.48E-04	66	7.07E-05	86	-6.89E-05
7	0	27	-0.17%	47	-4.69E-04	67	-2.53E-05	87	8.65E-05
8	0	28	0	48	3.11E-04	68	5.64E-05	88	-7.39E-05
9	0	29	0	49	-3.43E-04	69	-1.71E-05	89	9.02E-05
10	0	30	0	50	3.91E-04	70	4.12E-05	90	-8.07E-05
11	0	31	0	51	-3.92E-04	71	1.31E-05	91	9.09E-05
12	-4.76%	32	0.07%	52	3.60E-04	72	7.19E-06	92	-8.24E-05
13	0	33	-0.11%	53	-2.90E-04	73	3.41E-05	93	8.80E-05
14	0	34	9.23E-04	54	2.49E-04	74	-2.21E-06	94	-8.38E-05
15	0	35	-7.77E-04	55	-2.15E-04	75	3.89E-05	95	8.70E-05
16	0	36	3.23E-04	56	2.38E-04	76	-1.54E-05	96	-8.65E-05
17	0	37	-5.46E-04	57	-2.29E-04	77	5.71E-05	97	8.44E-05
18	-1.14%	38	6.85E-04	58	2.28E-04	78	-3.57E-05	98	-8.56E-05
19	0	39	-9.67E-04	59	-1.54E-04	79	6.87E-05	99	7.92E-05
20	0	40	6.50E-04	60	1.41E-04	80	-4.42E-05	100	-8.48E-05

Table 13: The relative error of $I'_{\mathrm{g}0}(n)$ to $h(n)$ when $n \leqslant 100$.

Acknowledgements

The author would like to express the gratitude to Prof. *Ian M. Wanless* from Monash University in Australia and Prof. Jack H. Koolen from USTC for their valuable advice.

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