LATTICE PATHS INSIDE A TABLE, I

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Abstract. A lattice path L in $\mathbb{Z}^d$ of length k with steps in a given set $\mathbf{S} \subseteq \mathbb{Z}^d$, or $\mathbf{S}$-path for short, is a sequence $\nu_1, \nu_2, \ldots, \nu_k \in \mathbb{Z}^d$ such that the steps $\nu_i - \nu_{i-1}$ lie in $\mathbf{S}$ for all $i = 2, \ldots, k$. Let $T_{m,n}$ be the $m \times n$ table in the first area of xy-axis and put $\mathbf{S} = \{(1,0),(1,1),(1,-1)\}$. Accordingly, let $\mathcal{I}_m(n)$ denote the number of $\mathbf{S}$-paths starting from the first column and ending at the last column of T. We will study the numbers $\mathcal{I}_m(n)$ and give explicit formulas for special values of m and n. As a result, we prove a conjecture of *Alexander R. Povolotsky*. We conclude the paper with some applications to Fibonacci and Pell-Lucas numbers and posing an open problem.

1. Introduction

A *lattice path* L in $\mathbb{Z}^d$ is a path in the d-dimensional integer lattice $\mathbb{Z}^d$, which uses only points of the lattice; that is a sequence $\nu_1, \nu_2, \ldots, \nu_k$, where $\nu_i \in \mathbb{Z}^d$ for all i (see [9, 10]). The vectors $\nu_2 - \nu_1, \nu_3 - \nu_2, \ldots, \nu_k - \nu_{k-1}$ are called the *steps* of L. Recall that a *Dyck path* is a lattice path in $\mathbb{Z}^2$ starting from $(0,0)$ and ending at a point $(2n,0)$ (for some $n \geqslant 0$) consisting of up-steps $(1,1)$ and down-steps $(1,-1)$, which never passes below the x-axis. It is well known that Dyck paths of length $2n$ are counted by the n^{th}-*Catalan number* $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers arise in many combinatorial problems, see Stanley [12] for an extensive study of these numbers.

Let $T_{m,n}$ be the $m \times n$ table in the first quadrant composed of mn unit squares, whose (x,y)-blank is located in the x^{th}-column from the left and the y^{th}-row from the bottom hand side of $T_{m,n}$. For a set $\mathbf{S} \subseteq \mathbb{Z}^d$ of steps, let $L(i,j;s,t:\mathbf{S})$ denote the set of all lattice paths in $T_{m,n}$ starting form (i,j)-blank and ending at (s,t)-blank with steps in $\mathbf{S}$, where $1 \leqslant i,s \leqslant m$ and $1 \leqslant j,t \leqslant n$. The number of such lattice paths is denoted by $l(i,j;s,t:\mathbf{S})$. For example, assuming $\mathbf{S} := \{(1,0),(1,1),(1,-1)\}$, the set of all lattice paths in the table T starting

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from $(1,1)$ and ending at $(n,1)$, where allowed to move only to the right (up, down or straight) is shown by $L(1,1;n,1:\mathbf{S})$ and the number of such lattice paths, namely $l(1,1;n,1:\mathbf{S})$, is the n^{th}-*Motzkin number*.

Lattice paths starting from the first column and ending at the n column of $T_{m,n}$ with steps in $\mathbf{S}$, $\mathbf{S}$ being as above, are called *perfect lattice paths*, and the number of all perfect lattice paths is denoted by $\mathcal{I}_m(n)$. Indeed,

$$\mathcal{I}_m(n) = \sum_{i,j=1}^{m} l(1,i;n,j;\mathbf{S}).$$

Figure 1 shows the number of all perfect lattice paths for $m=2$ and $n=3$. Clearly, $l(1,i;n,j:\mathbf{S}) = l(1,i';n,j':\mathbf{S})$ when $i+i'=m+1$ and $j+j'=m+1$.

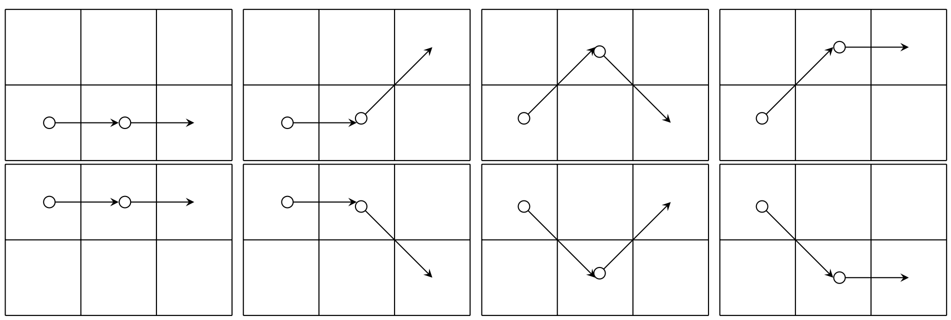

FIGURE 1. All perfect lattice paths in $T_{2,3}$.

We intend to evaluate $\mathcal{I}_m(n)$ for special cases of (m,n). In section 2, we obtain $\mathcal{I}_m(n)$ when $m \geqslant n$. Also, we prove a conjecture of *Alexander R. Povolotsky* posed in [8]. In section 3, we shall compute $\mathcal{I}_m(n)$ for small values of m, namely $m=1,2,3,4$. Finally, we present some results for $\mathcal{I}_5(n)$ and use Fibonacci and Pell-Lucas numbers to prove some relations concerning perfect lattice paths.

2. $\mathcal{I}_n(n)$ VS ALEXANDER R. POVOLOTSKY'S CONJECTURE

Let $T = T_{m,n}$ be the $m \times n$ table. For positive integers $1 \leqslant i,t \leqslant m$ and $1 \leqslant s \leqslant n$, the number of all perfect lattice paths from $(1,i)$ to (s,t) in T is denoted by $\mathcal{D}^i(s,t)$, that is, $\mathcal{D}^i(s,t) = l(1,i;s,t:\mathbf{S})$. Also, we put

$$\mathcal{D}_{m,n}(s,t) = \sum_{i=1}^{m} \mathcal{D}^i(s,t).$$

In case we are working in a single table, say T as above, we may simple use $\mathcal{D}(s,t)$ for $\mathcal{D}_{m,n}(s,t)$. Also, we put $\mathcal{D}_n(s,t) := \mathcal{D}_{n,n}(s,t)$. Clearly,

$\mathcal{D}(s,t)$ is the number of all perfect lattice paths from first column to the (s,t)-blank of T. The values of $\mathcal{D}(s,n)$ is computed in [4] in the cases where $m=n$ and T is a square table. By symmetry, $\mathcal{D}(s,t)=\mathcal{D}(s,t')$ when $t+t'=m+1$. Table 1 illustrates the values of $\mathcal{D}(6,t)$, for all $1 \leqslant t \leqslant 6$, where the number in (s,t)-blank of T determines the number $\mathcal{D}(s,t)$.

					$\mathcal{D}(6,t)$
1	2	5	13	35	96
1	3	8	22	61	170
1	3	9	26	74	209
1	3	9	26	74	209
1	3	8	22	61	170
1	2	5	13	35	96

TABLE 1. Values of $\mathcal{D}(6,t)$

Theorem 2.1. *For any positive integer n we have*

$$\mathcal{I}_n(n) = 3\mathcal{I}_{n-1}(n-1) + 3^{n-1} - 2\mathcal{D}_{n-1}(n-1,n-1).$$

Proof. Let $T := T_{n,n}$ and $T' := T_{n-1,n-1}$ with T' in the left-bottom side of T. Clearly, the number of perfect lattice paths of T which never meet the n^{th} row of T is

$$\mathcal{I}_{n-1}(n) = 3\mathcal{I}_{n-1}(n-1) - 2\mathcal{D}_{n-1}(n-1,n-1).$$

To obtain the number of all perfect lattice paths we must count those who meet the n^{th}-row of T, that is equal to 3^{n-1}. Thus $\mathcal{I}_n(n) - \mathcal{I}_{n-1}(n) = 3^{n-1}$, from which the result follows. □

Michael Somos [4] gives the following recurrence relation for $\mathcal{D}(n,n)$.

Theorem 2.2. *Inside the square $n \times n$ table we have*

$$n\mathcal{D}(n,n) = 2n\mathcal{D}(n-1,n-1) + 3(n-2)\mathcal{D}(n-2,n-2).$$

Utilizing Theorems 2.1 and 2.2 for $\mathcal{D}_n(n,n)$, we can prove a conjecture of Alexander R. Povolotsky posed in [8] as follows:

Conjecture 2.3. The following identity holds for the numbers $\mathcal{I}_n(n)$.

$$\begin{aligned}(n+3)\mathcal{I}_{n+4}(n+4) &= 27n\mathcal{I}_n(n) + 27\mathcal{I}_{n+1}(n+1)\\ &\quad - 9(2n+5)\mathcal{I}_{n+2}(n+2) + (8n+2)\mathcal{I}_{n+3}(n+3).\end{aligned}$$

Proof. Put

$$\begin{aligned}A &= (n+3)\mathcal{I}_{n+4}(n+4),\\ B &= (8n+21)\mathcal{I}_{n+3}(n+3),\\ C &= 9(2n+5)\mathcal{I}_{n+2}(n+2),\\ D &= 27\mathcal{I}_{n+1}(n+1),\\ E &= 27n\mathcal{I}_{n}(n).\end{aligned}$$

Using Theorem 2.1, we can write

$$\begin{aligned}A =&(3n+9)\mathcal{I}_{n+3}(n+3) + (n+3)3^{n+3} - (2n+6)\mathcal{D}(n+3,n+3)\\ =&(8n+21)\mathcal{I}_{n+3}(n+3) - (5n+12)\mathcal{I}_{n+3}(n+3) + (n+3)3^{n+3}\\ &- (2n+6)\mathcal{D}(n+3,n+3)\\ =&B + (n+3)3^{n+3} - (5n+12)\mathcal{I}_{n+3}(n+3)\\ &- (2n+6)\mathcal{D}(n+3,n+3).\end{aligned} \tag{2.1}$$

Utilizing Theorem 2.1 once more for $\mathcal{I}_{n+3}(n+3)$ and $\mathcal{I}_{n+2}(n+2)$ yields

$$\begin{aligned}A =&B + (n+3)3^{n+3} - (5n+12)3^{n+2}\\ &- (18n+45)\mathcal{I}_{n+2}(n+2) - (2n+6)\mathcal{D}(n+3,n+3)\\ &+ (10n+24)\mathcal{D}(n+2,n+2) + (3n+9)\mathcal{I}_{n+2}(n+2) + (n+3)3^{n+3}\\ =&B - C - (5n+12)3^{n+2} - (2n+6)\mathcal{D}(n+3,n+3)\\ &+ (10n+24)\mathcal{D}(n+2,n+2) + 9n\mathcal{I}_{n+1}(n+1)\\ &+ 27\mathcal{I}_{n+1}(n+1) + (3n+9)3^{n+1} - (6n+18)\mathcal{D}(n+1,n+1).\end{aligned}$$

It can be easily shown that

$$\begin{aligned}A =&B - C + D\\ &+ (n+3)3^{n+3} - (2n+6)\mathcal{D}(n+3,n+3) - (5n+12)3^{n+2}\\ &+ (10n+24)\mathcal{D}(n+2,n+2) + 9n\mathcal{I}_{n+1}(n+1)\\ &+ (3n+9)3^{n+1} - (6n+18)\mathcal{D}(n+1,n+1).\end{aligned} \tag{2.2}$$

Replacing $9n\mathcal{I}_{n+1}(n+1)$ by $27n\mathcal{I}_{n}(n) + n3^{n+2} - 18n\mathcal{I}_{n}(n)$ in 2.2 gives

$$\begin{aligned}A = &B - C + D + E\\ &- (2n+6)\mathcal{D}(n+3,n+3) + (10n+24)\mathcal{D}(n+2,n+2)\\ &- 18n\mathcal{D}(n,n) - (6n+18)\mathcal{D}(n+1,n+1).\end{aligned}$$

Since the coefficient of $\mathcal{D}(n+3, n+3)$ is $2(n+3)$, it follow from Theorem 2.2 that

$$\begin{aligned} A =& B - C + D + E - (4n+12)\mathcal{D}(n+2, n+2) - 18n\mathcal{D}(n, n) \\ &+ (10n+24)\mathcal{D}(n+2, n+2) - (6n+6)\mathcal{D}(n+1, n+1) \\ &- (6n+18)\mathcal{D}(n+1, n+1) \\ =& B - C + D + E - (4n+12)\mathcal{D}(n+2, n+2) \\ &- (6n+6)\mathcal{D}(n+1, n+1) + 18n\mathcal{D}(n, n) - 18n\mathcal{D}(n, n) \\ &- (12n+24)\mathcal{D}(n+1, n+1) + (6n+18)\mathcal{D}(n+1, n+1) \\ =& B - C + D + E, \end{aligned}$$

as required. □

Theorem 2.4. *Inside the $m \times n$ table we have*

$$\mathcal{I}_m(n) = m3^{n-1} - 2\sum_{s=1}^{n-1} 3^{n-s-1}\mathcal{D}(s, 1). \tag{2.3}$$

Proof. Let $T := T_{m,n}$. The number of all lattice paths from the first column to the last column is simply $n3^{n-1}$ if they are allowed to get out of T. Now we count all lattice paths that go out of T in some step. First observe that the number of lattice paths that leave T from the bottom row equals to those leave T from the the top row in the first times. Suppose a lattice path goes out of T from the bottom in column s for the first times. The number of all partial lattice paths from the first column to the $(s-1, 1)$-blank is simply $\mathcal{D}(s-1, 1)$, and every such path continues in 3^{n-s} ways until it reaches the last column of T. Hence we have $3^{n-s}\mathcal{D}(s-1, 1)$ paths leave the table T from the bottom in column s for any $s = 2, \ldots, n$. Hence, the number of perfect lattice paths is simply

$$\begin{aligned} \mathcal{I}_m(n) &= m3^{n-1} - 2\sum_{i=2}^{n} 3^{n-s}\mathcal{D}(s-1, 1) \\ &= m3^{n-1} - 2\sum_{s=1}^{n-1} 3^{n-s-1}\mathcal{D}(s, 1), \end{aligned}$$

as required. □

Example 2.5. Let T be the square 6×6 table. In Table 1, every blank represents the number of all perfect lattice paths from first column to that blank. Summing up the last column yields

$$\mathcal{I}_6(6) = 96 + 170 + 209 + 209 + 170 + 96 = 950.$$

Now, utilizing Theorem 2.4, we calculate $\mathcal{I}_6(6)$ in another way, as follows:

$$\begin{aligned}\mathcal{I}_6(6) =&6\cdot 3^{6-1} - 2\left(3^{6-1-1}\mathcal{D}(1,1) + 3^{6-2-1}\mathcal{D}(2,1) + 3^{6-3-1}\mathcal{D}(3,1)\right.\\ &\left.+3^{6-4-1}\mathcal{D}(4,1) + 3^{6-5-1}\mathcal{D}(5,1)\right)\\ =&1458 - 2\left(3^4\cdot 1 + 3^3\cdot 2 + 3^2\cdot 5 + 3^1\cdot 13 + 3^0\cdot 35\right) = 950.\end{aligned}$$

Remind that the number of lattice paths $L(1,1;n+1,1:\mathbf{S})$ in $\mathbb{Z}^2$ that never slides below the x-axis, is the n^{th}-*Motzkin number* $(n\geqslant 0)$, denoted by $\mathcal{M}_n$. Motzkin numbers begin with $1,1,2,4,9,21,\ldots$ (see [2]) and can be expressed in terms of binomial coefficients and Catalan numbers via

$$\mathcal{M}_n = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}\mathcal{C}_k.$$

Trinomial triangles are defined by the same steps $(1,1)$, $(1,-1)$ and $(1,0)$ (in our notation) with no restriction by starting from a fixed blank. The number of ways to reach a blank is simply the sum of three numbers in the adjacent previous column. The k^{th}-entry of the n^{th} column is denoted by $\binom{n}{k}_2$, where columns start by 0. The middle entries of the Trinomial triangle, namely $1,1,3,7,19,\ldots$ (see [6]) are studied by Euler. Analogously, Motzkin triangle are defined by recurrence sequence

$$\mathcal{T}(n,k) = \mathcal{T}(n-1,k-2) + \mathcal{T}(n-1,k-1) + \mathcal{T}(n-1,k),$$

for all $1\leqslant k\leqslant n-1$ and satisfy

$$\mathcal{T}(n,n) = \mathcal{T}(n-1,n-2) + \mathcal{T}(n-1,n-1)$$

for all $n\geqslant 1$ (see [5]).

Table 2 illustrates initial parts of the above triangles with Motzkin triangle in the left and trinomial triangle in the right. For a positive integer $1\leqslant s\leqslant n$, each entry of the column $\mathcal{D}_s(s,1)$ is the sum of all entries in the s^{th}-row in the rotated Motzkin triangle, that is, $\mathcal{D}_s(s,1) = \sum_{i=1}^{s}\mathcal{T}(s,i)$. For example,

$$\mathcal{D}(4,1) = \mathcal{T}(4,1) + \mathcal{T}(4,2) + \mathcal{T}(4,3) + \mathcal{T}(4,4) = 4+5+3+1 = 13.$$

The entries in the first column of rotated Motzkin triangle are indeed the Motzkin numbers.

Lemma 2.6. *Inside the square $n\times n$ table we have*

$$\mathcal{D}(s,1) = 3\mathcal{D}(s-1,1) - \mathcal{M}_{s-2},$$

for all $1\leqslant s\leqslant n$.

$\mathcal{D}_s(s,1)$						
1	1					
2	1	1				
5	2	2	1			
13	4	5	3	1		
35	9	12	9	4	1	
96	21	30	25	14	5	1

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

TABLE 2. Motzkin triangle (left) and trinomial triangle (right) rotates 90° clockwise

Proof. Let $T := T_{n,n}$. By the definition, $D(s,1)$ is the number of all lattice paths from the first column to $(s,1)$-blank. This number equals the number of lattice paths from $(s,1)$-blank to the first column with reverse steps that lie inside the table T, which is equal to 3^{s-1} minus those paths that leave T at some point. Consider all those lattice paths staring from $(s,1)$-blank with reverse steps that leaves T at $(i,0)$ for the first time, where $1 \leqslant i \leqslant s-1$. Clearly, the number of such paths are $3^{i-1}\mathcal{M}_{s-i-1}$. Thus

$$\mathcal{D}(s,1) = 3^{s-1} - \sum_{i=1}^{s-1} 3^{i-1}\mathcal{M}_{s-i-1},$$

from which it follows that $\mathcal{D}(s,1) = 3\mathcal{D}(s-1,1) - \mathcal{M}_{s-2}$, as required. □

Example 2.7. Consider the Table 2. Using Lemma 2.6 we can calculate $\mathcal{D}(6,1)$ as

$$\mathcal{D}(6,1) = 3\mathcal{D}(5,1) - \mathcal{M}_4 = 3 \cdot 35 - 9 = 96.$$

Corollary 2.8. *Inside the $n \times n$ table we have*

$$\mathcal{I}_n(n) = (n+2)3^{n-2} + 2\sum_{k=0}^{n-3}(n-k-2)3^{n-k-3}\mathcal{M}_k.$$

Proof. The result follows from Theorem 2.4 and Lemma 2.6. □

The next result shows that the number of perfect lattice paths in $T_{m,n}$ is independent of the number m or rows provided that m is big enough.

Theorem 2.9. *Inside the $m \times n$ table ($m \geqslant n$) we have*

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=0}^{n-1} \mathcal{D}(i,1)\mathcal{D}(n-i,1).$$

Proof. Consider the table $T := T_{m,n}$. We construct the table T' by adding a new row $m+1$ at the top of T. Now to count the number of all perfect lattice paths in T', it is sufficient to consider perfect lattice paths that reach to the new row $m+1$ for the first time. Assume a perfect lattice path reach to row $m+1$ at column i for the first time. Then its initial part from column 1 to column $i-1$ is a lattice path from the first column of T to $(i-1,m)$-blank. Also, its terminal part from column i to column n is a lattice path from $(i,m+1)$-blank of T' to its last column, which is in one to one correspondence with a lattice path from (i,m)-blank of T to its last column as $m \geqslant n$. Hence, the number of such paths is simply $\mathcal{D}(i-1,m)\mathcal{D}(n-i+1,m)$, which is equal to $\mathcal{D}(i-1,1)\mathcal{D}(n-i+1,1)$ by symmetry. Therefore

$$\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=1}^{n} \mathcal{D}(i-1,1)\mathcal{D}(n-i+1,1)$$

and the result follows. □

Corollary 2.10. *For $m \geqslant n$ we have*

$$\mathcal{I}_m(n) = (n+2)3^{n-2} + (m-n)\sum_{i=0}^{n-1} \mathcal{D}(i,1)\mathcal{D}(n-i,1) + 2\sum_{k=0}^{n-3}(n-k-2)3^{n-k-3}\mathcal{M}_k.$$

Proof. Let $m = n+k$, where k is a positive integer. Then

$$\begin{aligned}\mathcal{I}_m(n) - \mathcal{I}_n(n) &= (\mathcal{I}_m(n) - \mathcal{I}_{m-1}(n)) + \cdots + (\mathcal{I}_{n+1}(m) - \mathcal{I}_n(m)) \\ &= (m-n)\sum_{i=0}^{n-1} \mathcal{D}(i,1)\mathcal{D}(n-i,1).\end{aligned}$$

Now the result follows from Corollary 2.8. □

Theorem 2.11. *Inside the $m \times n$ table with $m \geqslant 2n-2$ we have*

(i) $\sum_{i=0}^{n-1} \mathcal{D}(i,n)\mathcal{D}(n-i,n) = 3^{n-1}$;
(ii) $\sum_{i=1}^{n-1} \mathcal{D}(i,n)\mathcal{D}(n-i,n) = \sum_{i=0}^{n-2} 3^{n-i-1}\mathcal{M}_i$;
(iii) $\mathcal{I}_m(n) = (3m-2n+2)3^{n-2} + 2\sum_{k=0}^{n-3}(n-k-2)3^{n-k-3}\mathcal{M}_k$.

Proof. (i) Let $T := T_{m,n}$ with $m = 2n-2$ and T' be the table obtained by adding a new row in the middle of T. It is sufficient to obtain $\mathcal{I}_{m+1}(n) - \mathcal{I}_m(n)$. Clearly, the number of perfect lattice paths reaching to any (i,n)-blank of T or T' is the same for all $i = 1, \ldots, n-1$. On the other hand, the number of all perfect lattice paths of T' reaching at (n,n)-blank is 3^{n-1} since we may begin the paths form the last (n,n)-blank and apply reverse steps with no limitation until to reach the first column. Thus

$$3^{n-1} = \mathcal{I}_{m+1}(n) - \mathcal{I}_m(n) = \sum_{i=0}^{n-1} \mathcal{D}(i,1)\mathcal{D}(n-i,1).$$

(ii) Put $\mathcal{D}(0,1) = 1$. Then

$$\mathcal{D}(n,1) = 3^{n-1} - \sum_{i=1}^{n-1} \mathcal{D}(i,1)\mathcal{D}(n-i,1).$$

On the other hand, by Lemma 2.6, we have

$$\mathcal{D}(n,1) = 3^{n-1} - \sum_{i=0}^{n-2} 3^{n-i-2}\mathcal{M}_i,$$

from which the result follows.

(iii) It follows from (i) and Corollary 2.10. □

Lemma 2.12. *Inside the $n \times n$ table we have*

$$\mathcal{D}(n,k+2) - \mathcal{D}(n,k) = \sum_{i=1}^{n-1} \left(\mathcal{D}(i,k+3) - \mathcal{D}(i,k-1)\right)$$

for all $1 \leqslant k \leqslant n$.

Proof. For $n = 2$, the result is trivially true. For any $\ell < n$ we have

$$\mathcal{D}(\ell+1,k+2) = \mathcal{D}(\ell,k+3) + \mathcal{D}(\ell,k+2) + \mathcal{D}(\ell,k+1)$$
$$\mathcal{D}(\ell+1,k) = \mathcal{D}(\ell,k+1) + \mathcal{D}(\ell,k) + \mathcal{D}(\ell,k-1),$$

which imply that

$$\mathcal{D}(\ell+1,k+2) - \mathcal{D}(\ell+1,k) \\ = \mathcal{D}(\ell,k+3) - \mathcal{D}(\ell,k-1) + \left(\mathcal{D}(\ell,k+2) - \mathcal{D}(\ell,k)\right).$$

Thus

$$\mathcal{D}(n,k+2) - \mathcal{D}(n,k) = \sum_{i=1}^{n-1} \left(\mathcal{D}(i,k+3) - \mathcal{D}(i,k-1)\right)$$

as $\mathcal{D}(1,k+2) - \mathcal{D}(1,k) = 0$. This completed the proof. □

3. Concluding some remarks and one conjecture

In this section, we shall compute $\mathcal{I}_m(n)$ for $m = 1, 2, 3, 4$ and arbitrary positive integers n. Some values of the $\mathcal{I}_3(n)$ and $\mathcal{I}_4(n)$ are given in [3] and [7], respectively.

Lemma 3.1. *$\mathcal{I}_1(n) = 1$ and $\mathcal{I}_2(n) = 2^n$ for all $n \geqslant 1$.*

Let x and y be arbitrary real numbers. By the binomial theorem, we have the following identity,

$$x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k},$$

where $n \geqslant 1$. This identity also can rewritten as

$$x^n + y^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k}, \quad (3.1)$$

where $\binom{r}{-1} = 0$. Pell-Lucas sequence [11] is defined as $\mathcal{Q}_1 = 1$, $\mathcal{Q}_2 = 3$, and $\mathcal{Q}_n = 2\mathcal{Q}_{n-1} + \mathcal{Q}_{n-2}$ for all $n \geqslant 3$. It can also be defined by the so called *Binet formula* as $\mathcal{Q}_n = (\alpha^n + \beta^n)/2$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are solutions of the quadratic equation $x^2 = 2x + 1$.

Lemma 3.2. *For all $n \geqslant 1$ we have $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$.*

Proof. The number of lattice paths to entries in columns $n-2$, $n-1$ and n of $T_{3,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$3x+2y$
y	$2x+y$	$4x+3y$
x	$x+y$	$3x+2y$

which imply that $\mathcal{I}_3(n-2) = 2x + y$, $\mathcal{I}_3(n-1) = 4x + 3y$ and $\mathcal{I}_3(n) = 10x + 7y$. Thus the following linear recurrence exists for $\mathcal{I}_3$.

$$\mathcal{I}_3(n) = 2\mathcal{I}_3(n-1) + \mathcal{I}_3(n-2). \quad (3.2)$$

Since $\mathcal{I}_3(1) = \mathcal{Q}_2 = 3$ and $\mathcal{I}_3(2) = \mathcal{Q}_3 = 7$, it follows that $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$ for all $n \geqslant 1$, as required. □

Corollary 3.3. *Let n be a positive integer. Then*

$$\mathcal{I}_3(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n-k+1}{k} + \binom{n-k}{k-1} \right] 2^{n-2k}.$$

Proof. It is sufficient to put $x = \alpha$ and $y = \beta$ in 3.1. □

The Fibonacci sequence [1] starts with the integers 0 and 1, and every other term is the sum of the two preceding ones, that is, $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$, and $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ for all $n \geqslant 2$. This recursion gives the Binet's formula $\mathcal{F}_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Lemma 3.4. *For all $n \geqslant 1$ we have $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$.*

Proof. The number of lattice paths to entries in columns $n-2$, $n-1$ and n of $T_{4,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$2x+3y$
y	$x+2y$	$3x+5y$
y	$x+2y$	$3x+5y$
x	$x+y$	$2x+3y$

which imply that $\mathcal{I}_4(n-2) = 2x + 2y$, $\mathcal{I}_4(n-1) = 4x + 6y$ and $\mathcal{I}_4(n) = 10x + 16y$. Hence we get the following linear recurrence for $\mathcal{I}_4$.

$$\mathcal{I}_4(n) = 3\mathcal{I}_4(n-1) - \mathcal{I}_4(n-2). \tag{3.3}$$

On the other hand,

$$\begin{aligned} \mathcal{F}_{2n+1} &= \mathcal{F}_{2n} + \mathcal{F}_{2n-1} \\ &= 2\mathcal{F}_{2n-1} + \mathcal{F}_{2n-2} \\ &= 3\mathcal{F}_{2n-1} - \mathcal{F}_{2n-3} \\ &= 3\mathcal{F}_{2(n-1)+1} - \mathcal{F}_{2(n-2)+1}. \end{aligned}$$

Now since $\mathcal{I}_4(1) = 2\mathcal{F}_3$ and $\mathcal{I}_4(2) = 2\mathcal{F}_5$, it follows that $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$ for all $n \geqslant 1$. The proof is complete. □

Corollary 3.5. *For all $n \geqslant 1$ we have*

$$\mathcal{I}_4(n) = \sum_{k=0}^{n} (-1)^k \left[\frac{2n+1}{k} \binom{2n-k}{k-1} \right] 5^{n-k}. \tag{3.4}$$

Proof. It is sufficient to put $x = \varphi$ and $y = \psi$ in 3.1. □

Consider the $m \times n$ table T with $2n \geqslant m$. For positive integers $\ell_1, \ell_2, \ldots, \ell_{\lceil \frac{m}{2} \rceil}$, we can write $\mathcal{I}_m(n)$ as

$$\mathcal{I}_m(n) = \ell_1 \mathcal{I}_m(n-1) + \ell_2 \mathcal{I}_m(n-2) + \cdots + \ell_{\lceil \frac{m}{2} \rceil} \mathcal{I}_m(n - \lceil \frac{m}{2} \rceil).$$

Also, for positive integers $0 \leqslant s \leqslant \lceil \frac{m}{2} \rceil$ and $k_{1,s}, k_{2,s}, \ldots, k_{\lceil \frac{m}{2} \rceil, s}$, we put

$$\mathcal{I}_m(n-s) = k_{1,s}x_1 + k_{2,s}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, s} x_{\lceil \frac{m}{2} \rceil},$$

where $x_t = \mathcal{D}(n - \lceil \frac{m}{2} \rceil, t) = \sum_{i=1}^{m} \mathcal{D}^i(n - \lceil \frac{m}{2} \rceil, t)$ is the number of all perfect lattice paths from the first column to the $(n - \lceil \frac{m}{2} \rceil, t)$-blank of T, for each $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant \lceil \frac{m}{2} \rceil$. Utilizing the above notations, we can can write

$$\begin{aligned} \mathcal{I}_m(n) =& k_{1,0}x_1 + k_{2,0}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 0} x_{\lceil \frac{m}{2} \rceil} \\ =& \ell_1 \mathcal{I}_{n-1} + \ell_2 \mathcal{I}_{n-2} + \cdots + \ell_{\lceil \frac{m}{2} \rceil} \mathcal{I}_{n - \lceil \frac{m}{2} \rceil} \\ =& \ell_1 (k_{1,1}x_1 + k_{2,1}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 1} x_{\lceil \frac{m}{2} \rceil}) \\ & + \ell_2 (k_{1,2}x_1 + k_{2,2}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 2} x_{\lceil \frac{m}{2} \rceil}) \\ & \vdots \\ & + \ell_{\lceil \frac{m}{2} \rceil} (k_{1,\lceil \frac{m}{2} \rceil} x_1 + k_{2,\lceil \frac{m}{2} \rfloor} x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil} x_{\lceil \frac{m}{2} \rceil}). \end{aligned} \tag{3.5}$$

From 3.5, we obtain the following system of linear equations

$$\left\{ \begin{array}{ccccccc} k_{1,1}\ell_1 & + & \cdots & + & k_{1,\lceil \frac{m}{2} \rceil} \ell_{\lceil \frac{m}{2} \rceil} & = & k_{1,0}, \\ k_{2,1}\ell_1 & + & \cdots & + & k_{2,\lceil \frac{m}{2} \rceil} \ell_{\lceil \frac{m}{2} \rceil} & = & k_{2,0}, \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ k_{\lceil \frac{m}{2} \rceil,1}\ell_1 & + & \cdots & + & k_{\lceil \frac{m}{2} \rceil,\lceil \frac{m}{2} \rceil} \ell_{\lceil \frac{m}{2} \rceil} & = & k_{\lceil \frac{m}{2} \rceil,0}. \end{array} \right. \tag{3.6}$$

Now consider the following coefficient matrix A of the system 3.6

$$A = \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,\lceil \frac{m}{2} \rceil} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,\lceil \frac{m}{2} \rceil} \\ \vdots & \vdots & \ddots & \vdots \\ k_{\lceil \frac{m}{2} \rceil,1} & k_{\lceil \frac{m}{2} \rceil,2} & \cdots & k_{\lceil \frac{m}{2} \rceil,\lceil \frac{m}{2} \rceil} \end{bmatrix},$$

which we call the *coefficient matrix* of the table T and denote it by $\mathcal{C}(T)$. We concluded this section by the following conjecture.

Conjecture 3.6. For a given $m \times n$ table T $(2n \geqslant m)$, we have $\det(\mathcal{C}(T)) = -2^{\lfloor \frac{m}{2} \rfloor}$.

Example 3.7. Let T be a $5 \times n$ table. The columns $n-3$, $n-2$, $n-1$, and n of T are given by

$n-3$	$n-2$	$n-1$	n
x_1	x_1+x_2	$2x_1+2x_2+x_3$	$4x_1+6x_2+3x_3$
x_2	$x_1+x_2+x_3$	$2x_1+4x_2+2x_3$	$6x_1+10x_2+6x_3$
x_3	$2x_2+x_3$	$2x_1+4x_2+3x_3$	$6x_1+12x_2+7x_3$
x_2	$x_1+x_2+x_3$	$2x_1+4x_2+2x_3$	$6x_1+10x_2+6x_3$
x_1	x_1+x_2	$2x_1+2x_2+x_3$	$4x_1+6x_2+3x_3$

from which it follows that

$$\begin{aligned}\mathcal{I}_5(n-3) &= 2x_1+2x_2+x_3,\\ \mathcal{I}_5(n-2) &= 4x_1+6x_2+3x_3,\\ \mathcal{I}_5(n-1) &= 10x_1+16x_2+9x_3,\\ \mathcal{I}_5(n) &= 28x_1+44x_2+25x_3\end{aligned}$$

Clearly,

$$\mathcal{I}_5(n) = \ell_1\mathcal{I}_5(n-1)+\ell_2\mathcal{I}_5(n-2)+\ell_3\mathcal{I}_5(n-3)$$

for some ℓ_1, ℓ_2, ℓ_3, and that the coefficient matrix of the table T is $\mathcal{C}(T) = \begin{bmatrix} 10 & 4 & 2\\ 16 & 6 & 2\\ 9 & 3 & 1\end{bmatrix}$. It is obvious that $\det(\mathcal{C}(T)) = -2^{\lfloor \frac{5}{2}\rfloor} = -4$.

4. Some result about perfect lattice paths by using of Fibonacci and Pell-Lucas numbers

In this section, we obtain some results about perfect lattice paths in the $5\times n$ table . Also, we get some relations and properties about Fibonacci and Pell-Lucas sequences by the aid of perfect lattice paths.

Proposition 4.1. *Inside the $5\times n$ table we have*

$$\mathcal{D}(s+2,1) = \mathcal{I}_5(s) \quad \textit{and} \quad \mathcal{D}(s+2,3) = 2\mathcal{I}_5(s)-1$$

for all $1\leqslant s\leqslant n$.

Proof. From the table in Example 3.7, it follows simply that $\mathcal{I}_5(s) = \mathcal{D}(s+2,1)$ for all $s\geqslant 1$. Also, from the table, it follows that

$$2\mathcal{D}(s+1,1)-\mathcal{D}(s+1,3) = 2\mathcal{D}(s,1)-\mathcal{D}(s,3)$$

for all $s\geqslant 1$, that is, $2\mathcal{D}(s,1)-\mathcal{D}(s,3)$ is constant. Since $2\mathcal{D}(1,1)-\mathcal{D}(1,3) = 1$, we get $2\mathcal{D}(s+2,1)-\mathcal{D}(s+2,3) = 1$, from which the result follows. □

Proposition 4.2. *Inside the $5\times n$ table we have*

$$\mathcal{D}(s,1)\times\mathcal{D}(s+t,3)-\mathcal{D}(s,3)\times\mathcal{D}(s+t,1) = \sum_{i=s}^{s+t-1}\mathcal{D}(i,2)$$

for all $1 \leqslant s, t \leqslant n$.

Proof. From Proposition 4.1, we know that $\mathcal{D}(s,3) = 2\mathcal{D}(s,1) - 1$ for all $1 \leqslant s \leqslant n$. Then

$$\begin{aligned}
&\mathcal{D}(s,1)\mathcal{D}(s+t,3) - \mathcal{D}(s,3)\mathcal{D}(s+t,1)\\
=&\mathcal{D}(s,1)(2\mathcal{D}(s+t,1)-1) - (2\mathcal{D}(s,1)-1)\mathcal{D}(s+t,1)\\
=&2\mathcal{D}(s,1)\mathcal{D}(s+t,1) - \mathcal{D}(s,1) - 2\mathcal{D}(s,1)\mathcal{D}(s+t,1) + \mathcal{D}(s+t,1)\\
=&\mathcal{D}(s+t,1) - \mathcal{D}(s,1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathcal{D}(s+t,1) - \mathcal{D}(s,1)\\
=&\mathcal{D}(s+t-1,1) + \mathcal{D}(s+t-1,2) - \mathcal{D}(s,1)\\
=&\mathcal{D}(s+t-2,1) + \mathcal{D}(s+t-2,2) + \mathcal{D}(s+t-1,2) - \mathcal{D}(s,1)\\
&\vdots\\
=&\sum_{i=s}^{s+t-1}\mathcal{D}(i,2) + \mathcal{D}(s,1) - \mathcal{D}(s,1)\\
=&\sum_{i=s}^{s+t-1}\mathcal{D}(i,2),
\end{aligned}$$

from which the result follows. □

Proposition 4.3. *Inside the* $4 \times n$ *table we have*

$$\mathcal{D}(s,1) = \mathcal{F}_{2s-1} \quad \textit{and} \quad \mathcal{D}(s,2) = \mathcal{F}_{2s}$$

for all $s \geqslant 1$. *As a result,*

$$\mathcal{D}(s,1) \times \mathcal{D}(s+t,2) - \mathcal{D}(s,2) \times \mathcal{D}(s+t,1) = \mathcal{D}(s,2).$$

for all $s, t \geqslant 1$.

Proof. Clearly $\mathcal{D}(1,1) = \mathcal{D}(1,2) = \mathcal{F}_1 = \mathcal{F}_2 = 1$. Now since

$$\begin{aligned}
\mathcal{D}(s,1) &= \mathcal{D}(s-1,1) + \mathcal{D}(s-1,2),\\
\mathcal{D}(s,2) &= 2\mathcal{D}(s-1,2) + \mathcal{D}(s-1,1).
\end{aligned}$$

we may prove, by using induction that, $\mathcal{D}(s,1) = \mathcal{F}_{2s-1}$ and $\mathcal{D}(s,2) = \mathcal{F}_{2s}$ for all $s \geqslant 1$. The second claim follows from the fact that

$$\mathcal{F}_{2s-1}\mathcal{F}_{2s+2t} - \mathcal{F}_{2s}\mathcal{F}_{2s+2t-1} = \mathcal{F}_{2s}.$$

The proof is complete. □

Proposition 4.4. *Inside the $4 \times n$ table we have*

$$\mathcal{I}_4(2s+1) = \frac{1}{4}\mathcal{I}_4(s+1)^2 + \mathcal{D}(s,2)^2$$

for all $1 \leqslant s \leqslant n$.

Proof. Following Lemma 3.4 and Proposition 4.3, it is enough to show that

$$2\mathcal{F}_{4s+3} = \mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s}^2.$$

First observe that the equation $\mathcal{F}_{2n-1} = \mathcal{F}_n^2 + \mathcal{F}_{n-1}^2$ yields $\mathcal{F}_{4s+1} = \mathcal{F}_{2s+1}^2 + \mathcal{F}_{2s+2}^2$ and $\mathcal{F}_{4s+5} = \mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s+2}^2$. Now, by combining these two formulas, we obtain

$$\begin{aligned}
\mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s}^2 &= \mathcal{F}_{4s+5} + \mathcal{F}_{4s+1} - (\mathcal{F}_{2s+1}^2 + \mathcal{F}_{2s+2}^2) \\
&= \mathcal{F}_{4s+4} + \mathcal{F}_{4s+3} + \mathcal{F}_{4s+1} - \mathcal{F}_{4s+3} \\
&= \mathcal{F}_{4s+3} + \mathcal{F}_{4s+2} + \mathcal{F}_{4s+1} \\
&= 2\mathcal{F}_{4s+3},
\end{aligned}$$

as required. ☐

Pell numbers $\mathcal{P}_n$ are defined recursively as $\mathcal{P}_1 = 1$, $\mathcal{P}_2 = 2$, and $\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$ for all $n \geqslant 3$. The Binet's formula corresponding to $\mathcal{P}_n$ is $\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Proposition 4.5. *Inside the $3 \times n$ table we have*

$$\mathcal{D}(s,1) = \mathcal{P}_s \quad \textit{and} \quad \mathcal{D}(s,2) = \mathcal{Q}_s$$

for all $s \geqslant 1$. As a result,

$$\mathcal{D}(s,1) \times \mathcal{D}(s+t,2) - \mathcal{D}(s,2) \times \mathcal{D}(s+t,1) = (-1)^{s+1}\mathcal{D}(t,1).$$

for all $s, t \geqslant 1$.

Proof. From the table in Lemma 3.2, we observe that

$$\begin{aligned}
\mathcal{D}(s,1) &= 2\mathcal{D}(s-1,1) + \mathcal{D}(s-2,1), \\
\mathcal{D}(s,2) &= 2\mathcal{D}(s-1,2) + \mathcal{D}(s-2,2)
\end{aligned}$$

for all $s \geqslant 3$. Now since $\mathcal{D}(1,1) = \mathcal{P}_1 = 1$, $\mathcal{D}(2,1) = \mathcal{P}_2 = 2$, $\mathcal{D}(1,2) = \mathcal{Q}_1 = 1$, and $\mathcal{D}(2,2) = \mathcal{Q}_2 = 3$ one can show, by using induction, that $\mathcal{D}(s,1) = \mathcal{P}_s$ and $\mathcal{D}(s,2) = \mathcal{Q}_s$ for all s. To prove the second claim, we use the following formula

$$\mathcal{P}_s\mathcal{Q}_{s+t} - \mathcal{Q}_s\mathcal{P}_{s+t} = (-1)^{s+1}\mathcal{P}_t$$

that can be proves simply using Binet's formulas. ☐

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