

ABELIAN-SQUARE-RICH WORDS

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ABSTRACT. An abelian square is the concatenation of two words that are anagrams of one another. A word of length n can contain at most $\Theta(n^2)$ distinct factors, and there exist words of length n containing $\Theta(n^2)$ distinct abelian-square factors, that is, distinct factors that are abelian squares. This motivates us to study infinite words such that the number of distinct abelian-square factors of length n grows quadratically with n . More precisely, we say that an infinite word w is *abelian-square-rich* if, for every n , every factor of w of length n contains, on average, a number of distinct abelian-square factors that is quadratic in n ; and *uniformly abelian-square-rich* if every factor of w contains a number of distinct abelian-square factors that is proportional to the square of its length. Of course, if a word is uniformly abelian-square-rich, then it is abelian-square-rich, but we show that the converse is not true in general. We prove that the Thue-Morse word is uniformly abelian-square-rich and that the function counting the number of distinct abelian-square factors of length $2n$ of the Thue-Morse word is 2-regular. As for Sturmian words, we prove that a Sturmian word s_α of angle α is uniformly abelian-square-rich if and only if the irrational α has bounded partial quotients, that is, if and only if s_α has bounded exponent.

1. INTRODUCTION

A fundamental topic in combinatorics on words is the study of repetitions. A *repetition* in a word is a factor that is formed by the concatenation of two or more identical blocks. The simplest kind of repetition is a *square*, that is, the concatenation of two copies of the same block, such as the English word **hotshots**. A famous conjecture of Fraenkel and Simpson [20] states that a word of length n contains fewer than n distinct square factors. Experiments strongly suggest that the conjecture is true, but a theoretical proof of the conjecture seems difficult. In [20], the authors proved a bound of $2n$. In [25], Ilie improved this bound to $2n - \Theta(\log n)$, and recently Deza et al. showed the current best bound of $\frac{11}{6}n$ [12], but the conjectured bound is still out of reach.

Other variations on counting squares include counting squares in partial words (e.g., [5]) and pseudo-repetitions (e.g., [22]).

Among the different generalizations of the notion of repetition, a prominent one is that of an abelian repetition. An *abelian repetition* in a word is a factor that is formed by the concatenation of two or more blocks that have the same number of occurrences of each letter in the alphabet. Of course, the simplest kind of abelian repetition is an *abelian square*, that is, the concatenation of a word with an anagram of itself, such as the English word **intestines**. Abelian squares were considered in 1961 by Erdős [16], who conjectured that there exist infinite words avoiding abelian squares. This conjecture was later confirmed, and the smallest possible size of an alphabet for which it holds is known to be 4 [26].

Some of the results contained in this paper were presented (without the third author) at the 10th International Conference on Words, WORDS 2015 [19].

We focus on the maximum number of distinct abelian squares that a word can contain. In contrast to the case of ordinary squares, a word of length n can contain $\Theta(n^2)$ distinct abelian-square factors (see [27]). Since the total number of factors in a word of length n is quadratic in n , this means that there exist words in which a constant fraction of all factors are abelian squares. So we turn our attention to infinite words, and we ask whether there exist infinite words such that for every n the factors of length n contain, on average, a number of distinct abelian-square factors that is quadratic in n . We call such an infinite word *abelian-square-rich*. Since a random binary word of length n contains $\Theta(n\sqrt{n})$ distinct abelian-square factors [10], the existence of abelian-square-rich words is not immediate. We also introduce *uniformly abelian-square-rich* words; these are infinite words such that for every n , every factor of length n contains a quadratic number of distinct abelian squares. Of course, if a word is uniformly abelian-square-rich, then it is abelian-square-rich, but the converse is not true in general — we provide in this paper an example of a word that is abelian-square-rich but not uniformly abelian-square-rich. However, we show that for linearly recurrent words the two definitions are equivalent. Moreover, we prove that if an infinite word w is uniformly abelian-square-rich, then w has bounded exponent (that is, there exists an integer $k \geq 2$ such that w does not contain any repetition of order k as a factor).

We then prove that the famous Thue-Morse word is uniformly abelian-square-rich. Furthermore, we look at the function that counts the number of distinct abelian squares of length $2n$ in the Thue-Morse word and prove that this function is 2-regular.

Then we look at the class of Sturmian words; these are aperiodic infinite words with the lowest possible factor complexity. In this case, we prove that a Sturmian word has bounded exponent if and only if it is uniformly abelian-square-rich, and leave open the question of determining whether a Sturmian word is not abelian-square-rich in the case when it does not have bounded exponent.

2. NOTATION AND BACKGROUND

Let $\Sigma = \{a_1, a_2, \dots, a_\sigma\}$ be an ordered σ -letter alphabet. Let Σ^* stand for the free monoid generated by Σ , whose elements are called *words* over Σ . The *length* of a word w is denoted by $|w|$. The *empty word*, denoted by ε , is the unique word of length zero and is the neutral element of Σ^* . We also define $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$.

A *prefix* (respectively, a *suffix*) of a word w is a word u such that $w = uz$ (respectively, $w = zu$) for some word z . A *factor* of w is a prefix of a suffix (or, equivalently, a suffix of a prefix) of w . The set of prefixes, suffixes and factors of the word w are denoted, respectively, by $\text{Pref}(w)$, $\text{Suff}(w)$ and $\text{Fact}(w)$. From the definitions, we have that ε is a prefix, a suffix and a factor of every word.

A word w is a *k-power* (also called a *repetition of order k*), for an integer $k \geq 2$, if there exists a nonempty word u such that $w = u^k$. A 2-power is called a *square*. The *period* of a word $w = w_1w_2 \cdots w_{|w|}$ is the minimal integer p such that $w_{i+p} = w_i$ for every $1 \leq i \leq |w| - p$. The *exponent* $e(w)$ of a word w is the ratio between its length $|w|$ and its period p . For example, the period of $w = abaab$ is $p = 3$, hence $e(w) = 5/3$. Of course, if a word w avoids k -powers (that is, no factor of w is a k -power), then the supremum of the exponents of factors of w is smaller than k .

For a word w and a letter $a_i \in \Sigma$, we let $|w|_{a_i}$ denote the number of occurrences of a_i in w . The *Parikh vector* (sometimes called the *composition vector*) of a word w over $\Sigma = \{a_1, a_2, \dots, a_\sigma\}$ is the vector $P(w) = (|w|_{a_1}, |w|_{a_2}, \dots, |w|_{a_\sigma})$. An *abelian k-power* is a nonempty word of the form $v_1v_2 \cdots v_k$ where all the v_i have the same Parikh vector (and

therefore in particular the same length). An abelian 2-power is called an *abelian square*; an example in English is the word **reappear**.

An *infinite word* w over Σ is an infinite sequence of letters from Σ , that is, a function $w : \mathbb{N} \mapsto \Sigma$. An infinite word is *recurrent* if each of its factors occurs infinitely often. Given an infinite word w , the *recurrence index* $R_w(n)$ of w is defined to be the least integer m such that every factor of w of length m contains all factors of w of length n , or $+\infty$ if such an integer does not exist. If the recurrence index is finite for every n , the infinite word w is called *uniformly recurrent* and the function $R_w(n)$ the *recurrence function* of w . A uniformly recurrent word is of course recurrent, but the converse is not always true. For example, the *Champernowne word* $w = 011011100101 \dots$, obtained by concatenating the base-2 representations of the natural numbers, is recurrent but not uniformly recurrent (to see this, it is sufficient to observe that it contains arbitrarily large consecutive blocks of the same letter). A uniformly recurrent word w is called *linearly recurrent* if the ratio $R_w(n)/n$ is bounded by a constant. Given a linearly recurrent word w , the real number $r_w = \limsup_{n \rightarrow \infty} R_w(n)/n$ is called the *recurrence quotient* of w . The *factor complexity function* (sometimes called *subword complexity*) of an infinite word w is the integer function $p_w(n)$ defined by $p_w(n) = |\text{Fact}(w) \cap \Sigma^n|$. An infinite word w has *linear complexity* if $p_w(n) = O(n)$. In particular, if a word is linearly recurrent, then it has linear complexity (see, for example, [15]).

A *substitution* over the alphabet Σ is a map $\tau : \Sigma \mapsto \Sigma^+$. A substitution τ over Σ can be naturally extended to a (non-erasing) morphism from Σ^* to Σ^* . A substitution can be iterated: for every substitution τ and every $n > 0$, using the extension to a morphism, one can define the substitution τ^n . A substitution τ is *r-uniform* if there exists an integer $r \geq 1$ such that for all $a \in \Sigma$, $|\tau(a)| = r$. A substitution is called *uniform* if it is *r-uniform* for some $r \geq 1$. A substitution τ is *primitive* if there exists an integer $n \geq 1$ such that for every $a \in \Sigma$, the word $\tau^n(a)$ contains every letter of Σ at least once. In this paper, we will only consider primitive substitutions such that $\tau(a_1) = a_1v$ for a letter a_1 and some nonempty word v . These substitutions always have a fixed point, which is the infinite word $w = \lim_{n \rightarrow \infty} \tau^n(a_1)$. Moreover, this fixed point is linearly recurrent (see, for example, [11]) and therefore has linear complexity.

For an integer $k \geq 2$, we say that an infinite word w is *k-power-free* if no factor of w is a *k*-power. If an infinite word w is *k*-power-free for some k , we say that w has *bounded exponent*. For example, if a word is linearly recurrent, then it has bounded exponent (see, for example, [15]).

3. ABELIAN-SQUARE-RICH WORDS

Kociumaka et al. [27] showed that a word of length n can contain a number of distinct abelian-square factors that is quadratic in n . For the sake of completeness, we give a proof of this fact here.

Proposition 1. *A word of length n can contain $\Theta(n^2)$ distinct abelian-square factors.*

Proof. Consider the word $w_n = a^n b a^n b a^n$, of length $3n + 2$. For every $0 \leq i, j \leq n$ such that $i + j + n$ is even, the factor $a^i b a^n b a^j$ of w is an abelian square. Since the number of possible choices for the pair (i, j) is quadratic in n , we are done. \square

Motivated by the previous result, we might ask whether there exist infinite words such that all their factors contain a number of distinct abelian squares that is quadratic in their length. But first we relax this condition and consider words in which, for every n , each factor of length n contains, on average, a number of distinct abelian-square factors that is quadratic in n .

Let us define some notation. Given a finite or infinite word w , we let $\text{AS}_n(w)$ denote the number of distinct abelian-square factors of w of length n . Of course, $\text{AS}_n(w) = 0$ if n is odd, so this quantity is significant only for even values of n . Furthermore, for a finite word w of length n , we let $\text{AS}(w) = \sum_{m \leq n} \text{AS}_m(w)$ denote the total number of distinct abelian-square factors, of all lengths, in w .

Definition 2. An infinite word w is *abelian-square-rich* if there exists a positive constant C such that for every n one has

$$\frac{1}{p_w(n)} \sum_{v \in \text{Fact}(w) \cap \Sigma^n} \text{AS}(v) \geq Cn^2.$$

Notice that Christodoulakis et al. [10] proved that a binary word of length n contains $\Theta(n\sqrt{n})$ distinct abelian-square factors on average; hence a random infinite binary word is almost surely not abelian-square-rich.

In an abelian-square-rich word the number of distinct abelian squares contained in any factor is, on average, quadratic in the length of the factor. A stronger condition is that *every* factor contains a quadratic number of distinct abelian squares. We thus introduce the concept of uniformly abelian-square-rich words.

Definition 3. An infinite word w is *uniformly abelian-square-rich* if there exists a positive constant C such that $\text{AS}(v) \geq C|v|^2$ for all $v \in \text{Fact}(w)$.

Clearly, if a word is uniformly abelian-square-rich, then it is also abelian-square-rich, but the converse is not always true (we will provide an example of a word that is abelian-square-rich but not uniformly abelian-square-rich at the end of this section). However, in the case of linearly recurrent words, the two definitions are equivalent, as shown in the next lemma.

Lemma 4. *Let w be an infinite word. If w is abelian-square-rich and linearly recurrent, then it is uniformly abelian-square-rich.*

Proof. If w is linearly recurrent, then there exists a positive integer K such that, for every n , every factor of w of length Kn contains all the factors of w of length n . Let v be a factor of w of length n containing the largest number of distinct abelian squares among the factors of w of length n . Hence the number of distinct abelian squares in v is at least the average number of distinct abelian squares in a factor of w of length n . Since w is abelian-square-rich, the number of distinct abelian squares in v is greater than or equal to $C'n^2$, for a positive constant C' . Since v is contained in every factor of w of length Kn , the number of distinct abelian squares in every factor of w of length Kn is greater than or equal to $C'n^2$, whence w is uniformly abelian-square-rich. \square

The following lemma will be useful in the next sections.

Lemma 5. *Let w be a linearly recurrent infinite word. If there exists a positive constant C such that for every n one has $\sum_{m \leq n} \text{AS}_m(w) \geq Cn^2$, then w is (uniformly) abelian-square-rich.*

Proof. Since w is linearly recurrent, there exists a constant K such that, for every n , every factor of w of length Kn contains all the factors of w of length n . Take any factor u of w of length Kn . Since u contains all the factors of w of length n , it contains a number of distinct abelian-square factors that is larger than a constant times n^2 . Therefore, every factor of length n (and hence in particular, on average) contains a number of distinct abelian-square factors that is proportional to the square of its length. \square

In the next proposition we show that a necessary condition for a word to be uniformly abelian-square-rich is that it does not contain factors with arbitrarily large exponent.

Proposition 6. *Let w be an infinite word. If w is uniformly abelian-square-rich, then w has bounded exponent.*

Proof. Let u^k be a nonempty factor of w of length $n = km$, $m = |u|$. Every abelian square in u^k has an occurrence starting at a position smaller than m . We separate the abelian squares of u^k in two disjoint sets: those whose first occurrence ends at, or before, position m (i.e., those occurring in u), and those whose first occurrence ends after position m . Since there are no more than m^2 distinct abelian squares of the first kind and no more than $m \cdot (k-1)m = (k-1)m^2$ distinct abelian squares of the second kind, we have that u^k contains no more than $km^2 = n^2/k$ distinct abelian-square factors.

If w does not have bounded exponent then, for every $k \geq 2$, w contains a nonempty factor v of the form $v = u^k$, for some u . Hence, for every positive constant C , taking k such that $1/k < C$, the word w contains a factor v such that $\text{AS}(v) < C|v|^2$. \square

To conclude this section, we exhibit an example of a word that is abelian-square-rich but not uniformly abelian-square-rich.

Consider the sequence of words $(w_k)_{k \geq 1}$ defined by: $w_1 = aabaabaab$, and for every $k > 1$

$$w_k = w_{k-1}a^{2^k}ba^{2^k}ba^{2^k}b. \quad (3.1)$$

Proposition 7. *The infinite word $w = \lim_{k \rightarrow \infty} w_k$ is abelian-square-rich but not uniformly abelian-square-rich.*

Proof. Let us first prove that w is abelian-square-rich. We first observe that w has linear complexity. Indeed, this is an immediate consequence of the fact that for every n there is a constant number of distinct factors of length n that can be extended to the right both by the letter a or by the letter b to factors of w of length $n+1$ (these factors are usually called *right special factors*). Actually, for every n the number of right special factors of w of length n is bounded by 4, since these can only be of the following kinds:

- (1) a^n ;
- (2) $a^i b a^j$, for some $i \geq 0$ and $j > 0$ such that $j = 2^r$ for some r and $i + j + 1 = n$;
- (3) $a^i b a^j b a^j$, for some $i \geq 0$ and $j > 0$ such that $j = 2^r$ for some r and $i + 2j + 2 = n$;
- (4) $a^i b a^j b a^j b a^j$, for some $i \geq 0$ and $j > 0$ such that $j = 2^r$ for some r and $i + 3j + 3 = n$.

It is readily verified that if a factor of w contains more than 3 b 's or does not end in a^{2^r} for some $r > 0$, then it cannot be a right special factor of w .

So, it is sufficient to prove that for every n there is a linear (in n) number of factors of w of length n that contain a number of distinct abelian-square factors proportional to n^2 . This follows from the fact that for every n there is a linear number of distinct factors that contain $a^{2^k} b a^{2^k} b a^{2^k}$ as factor, for some value of 2^k proportional to n (for example, take the largest k such that $2^k \leq n/100$, and consider the factors of length n in which $a^{2^k} b a^{2^k} b a^{2^k}$ appears at different positions), and these factors contain a number of distinct abelian-square factors proportional to the square of their length (see the proof of Proposition 1), hence proportional to n^2 .

Finally, w is not uniformly abelian-square-rich by Proposition 6, since it contains arbitrarily large powers of the letter a . \square

It remains to prove that a recurrent (or even a uniformly recurrent) word exists that is abelian-square-rich but not uniformly abelian-square-rich, but such an example is probably more technical and involved.

4. THE THUE-MORSE WORD

Let

$$\mathbf{t} = 011010011001011010010110 \dots$$

be the Thue-Morse word, i.e., the fixed point starting with 0 of the uniform substitution $\mu : 0 \mapsto 01, 1 \mapsto 10$. It is well known that \mathbf{t} is linearly recurrent and that \mathbf{t} does not contain any factor with exponent larger than 2. In particular, \mathbf{t} does not contain *overlaps*, i.e., factors of the form $avava$, with $a \in \{0, 1\}$ and $v \in \{0, 1\}^*$.

For every $n \geq 4$, the factors of length n of \mathbf{t} belong to two disjoint sets: those that start only at even positions in \mathbf{t} , and those that start only at odd positions in \mathbf{t} . This is a consequence of two facts: first, that \mathbf{t} is overlap-free (and so 0101 cannot be preceded by 1 nor followed by 0) and second, that 00 and 11 are not images under μ of letters, so they cannot appear at even positions.

Let $p(n)$ be the factor complexity function of \mathbf{t} . It is known [6, Proposition 4.3] that for every $n \geq 1$ one has

$$p(2n) = p(n) + p(n+1), \quad p(2n+1) = 2p(n+1). \quad (4.1)$$

We define $f_{aa}(n)$ (respectively, $f_{ab}(n)$) to be the number of factors of \mathbf{t} of length n that begin and end with the same letter (respectively, with different letters). The next lemma (proved in [8]) shows that the Thue-Morse word has the property that for every n , at least one-third of the length- n factors begin and end with the same letter, and at least one-third of the length- n factors begin and end with different letters.

Lemma 8 ([8]). *For every $n \geq 2$, one has $f_{aa}(n) \geq p(n)/3$ and $f_{ab}(n) \geq p(n)/3$.*

Since from (4.1) we have $p(n) \geq 3(n-1)$ for every n , we get the following result.

Corollary 9. *For every $n \geq 2$, one has $f_{aa}(n) \geq n-1$ and $f_{ab}(n) \geq n-1$.*

We are now ready to prove that the Thue-Morse word is uniformly abelian-square-rich.

Proposition 10. *The Thue-Morse word \mathbf{t} is uniformly abelian-square-rich.*

Proof. Let u be a factor of length $n > 1$ of \mathbf{t} that begins and ends with the same letter. Since the image of every even-length word under μ is an abelian square, we have that $\mu^2(u)$ is an abelian-square factor of \mathbf{t} of length $4n$ that begins and ends with the same letter. Moreover, the word obtained from $\mu^2(u)$ by removing the first and the last letter is an abelian-square factor of \mathbf{t} of length $4n-2$. So, by Corollary 9, \mathbf{t} contains at least $n-1$ distinct abelian-square factors of length $4n$ and at least $n-1$ distinct abelian-square factors of length $4n-2$. This implies that for every even n the number of distinct abelian-square factors of \mathbf{t} of length n is linear in n . Hence, for every n the number of distinct abelian-square factors of \mathbf{t} of length at most n is quadratic in n . The statement then follows from Lemmas 5 and 4. \square

4.1. More detailed analysis for the Thue-Morse word. Let $f(n)$ denote the number of distinct abelian squares of length $2n$ (or, equivalently, of order n) in the Thue-Morse sequence \mathbf{t} . Table 1 gives the first few terms of this sequence.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f(n)$	2	4	4	10	8	24	10	22	12	36	20	52	24	54	20	46	24	72	32	76

TABLE 1. First few values of the number of distinct abelian-square factors of order n in the Thue-Morse word.

We can use the decision procedure discussed in [1, 9, 23, 24, 36] to analyze the function f in more detail. Although in general abelian questions about arbitrary automatic sequences are *not* decidable using this method, the Thue-Morse sequence has symmetries that make it amenable. One nice feature of this approach is that it can be almost entirely automated, using the freely-available `Walnut` package [34].

Define

$$d(n) := |\mathbf{t}[0..n-1]|_0 - \frac{n}{2}$$

to be the *prefix defect function* for the Thue-Morse sequence, that is, the function that counts the number of 0's in a prefix of length n over the amount “expected to occur”. (Since $\mathbf{t} \in (01+10)^\omega$, we expect $\frac{n}{2}$ 0's in a prefix of length n .)

Recall that we say that an infinite word $\mathbf{a} = (a_n)_{n \geq 0} \in \Delta^\omega$ is a k -*automatic sequence* if there exists a deterministic finite automaton with output (DFAO) $M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$ that computes \mathbf{a} in the following sense: if the input to M is $(n)_k$, the base- k representation of n , then the output $\tau(\delta(q_0, x))$ is equal to a_n . Here $\tau : Q \rightarrow \Delta$. Without loss of generality, we assume all automata discussed in this section take, as input, the base-2 representations of numbers starting with the most significant digit. See, for example, [3].

Now it is not hard to see that $\mathbf{d} = (d_n)_{n \geq 0}$ is a 2-automatic sequence. In fact, the four-state automaton in Figure 1 computes it. Here the notation a/b in a state indicates that the state name is a and its output is b . Thus, for example, starting in state 0, and reading the base-2 expansion of 23 (namely, 10111), we reach the state 3 with output $-\frac{1}{2}$. And indeed, $\mathbf{t}[0..22]$ has 11 zeroes, so its defect is $11 - \frac{23}{2} = -\frac{1}{2}$.

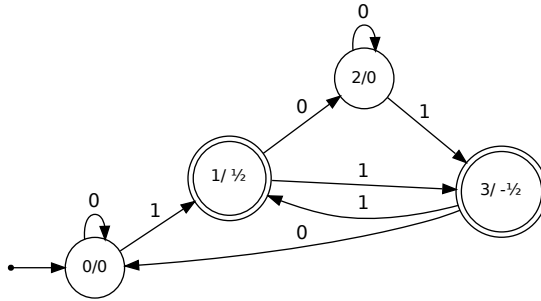


FIGURE 1. Automaton calculating prefix defect function for \mathbf{t} .

Now that we have the sequence \mathbf{d} , we define the function $D(i, n)$ be the defect associated with the factor $\mathbf{t}[i..i+n-1]$, that is,

$$D(i, n) := |\mathbf{t}[i..i+n-1]|_0 - \frac{n}{2}.$$

Then we have $D(i, n) = d(i + n) - d(i) \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$.

We can view D as a *two-dimensional automatic sequence* (see, e.g., [3, Chapter 14]). Here the input alphabet is $\{0, 1\}^2$, where the first components of the input spell out the base-2 representation of i and the second components spell out the base-2 representation of n . We write this input as $(i, n)_2$. (The shorter of the two representations is, if necessary, padded with leading zeroes so that the two representations can be read in parallel.) From the DFAO for \mathbf{d} we can easily generate an automaton for the two-dimensional infinite array $\mathbf{D} = (D(i, n))_{i, n \geq 0}$. More usefully, we can produce 5 different automata A_1, A_2, A_3, A_4, A_5 accepting those inputs $(i, n)_2$ for which $D(i, n) = -1$ (respectively, $-\frac{1}{2}, 0, \frac{1}{2}, 1$). We can think of each automaton A_m , $1 \leq m \leq 5$, as computing the function $\mathcal{A}_m(i, n)$ that is **true** when $(i, n)_2$ is accepted and **false** otherwise. When we compute these using **Walnut** we discover that they have 10, 13, 7, 13, 10 states respectively.

Once we have the automata A_1, A_2, A_3, A_4, A_5 , we can create an automaton T accepting those $(i, n)_2$ such that $\mathbf{t}[i..i+2n-1]$ is an abelian square. This automaton is created by taking the disjunction of the assertions $\mathcal{A}_m(i, n) \wedge \mathcal{A}_m(i+n, n)$ for $1 \leq m \leq 5$, and is implemented by an automaton with 36 states.

Finally, using T we can create an automaton U accepting those $(i, n)_2$ such that $\mathbf{t}[i..i+2n-1]$ is a novel abelian square; that is, an abelian square of length $2n$ that has never appeared previously in \mathbf{t} :

$$U(i, n) := T(i, n) \wedge (\forall j (j < i) \implies (\exists k (k < 2n) \wedge (\mathbf{t}[i+k] \neq \mathbf{t}[j+k]))).$$

The automaton U has 64 states, and can be computed using the techniques described in [24]. Again, we can view U as computing the function $\mathcal{U}(i, n)$ that is **true** when $(i, n)_2$ is accepted and **false** otherwise.

Now define $f(n) = |\{i : \mathcal{U}(i, n)\}|$. As a moment's reflection will reveal, this is the number of distinct abelian squares in \mathbf{t} of order n (and length $2n$). As the techniques in [9] show, from the transition diagram of the automaton U we can immediately deduce a so-called *linear representation* for the function f , that is, square matrices M_0, M_1 and vectors v, w such that

$$f(n) = vM_{a_1} \cdots M_{a_n}w$$

if $(n)_2 = a_1 \cdots a_n$. We can minimize the representation using the algorithm in [4, Section 2.3], obtaining the following linear representation of rank 11:

$$v := [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$$

$$M_0 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{64}{13} & 1 & -\frac{171}{26} & -\frac{69}{26} & 0 & 0 & \frac{43}{26} & \frac{9}{13} & \frac{51}{26} \\ 0 & 0 & \frac{90}{13} & -1 & -\frac{275}{26} & \frac{9}{26} & 0 & 2 & \frac{69}{26} & -\frac{17}{13} & \frac{51}{26} \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{68}{13} & 0 & -\frac{88}{13} & -\frac{33}{13} & 0 & 0 & \frac{20}{13} & \frac{12}{13} & \frac{34}{13} \\ 0 & 0 & \frac{54}{13} & 0 & -\frac{89}{13} & -\frac{22}{13} & 0 & 0 & \frac{35}{13} & \frac{8}{13} & \frac{27}{13} \end{bmatrix}$$

$$M_1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{8}{13} & 0 & -\frac{8}{13} & \frac{10}{13} & 0 & 0 & \frac{16}{13} & \frac{7}{13} & -\frac{4}{13} \\ 0 & 0 & -\frac{87}{13} & -1 & \frac{203}{26} & \frac{3}{26} & 2 & 0 & -\frac{3}{26} & \frac{16}{13} & -\frac{61}{26} \\ 0 & 0 & -\frac{7}{13} & 1 & \frac{25}{26} & -\frac{93}{26} & 0 & 0 & \frac{15}{26} & \frac{37}{13} & -\frac{7}{26} \\ 0 & 0 & \frac{20}{13} & 0 & -\frac{45}{13} & -\frac{12}{13} & 0 & 0 & \frac{25}{13} & \frac{28}{13} & -\frac{3}{13} \\ 0 & 0 & -\frac{40}{13} & 0 & \frac{25}{13} & -\frac{2}{13} & 0 & 0 & \frac{15}{13} & \frac{22}{13} & -\frac{7}{13} \\ 0 & 0 & -\frac{36}{13} & 0 & \frac{29}{13} & -\frac{20}{13} & 0 & 0 & \frac{7}{13} & \frac{38}{13} & -\frac{5}{13} \end{bmatrix}$$

$$w := [1, 2, 4, 4, 10, 8, 24, 10, 22, 12, 36]^T$$

We have therefore proved

Theorem 11. *The function f counting the number of distinct abelian squares of order n of \mathbf{t} is given by $f(n) = vM_{a_1} \cdots M_{a_i}w$ if $(n)_2 = a_1 \cdots a_i$, for the vectors v, w and matrices M_0, M_1 given above.*

A sequence that can be computed in this manner is called 2-regular; see [2].

From the matrices above we can, using the method described in [23], obtain defining recursive relations for f :

$$\begin{aligned}
f(8n) &= -2f(2n) + 3f(4n) \\
f(8n + 6) &= 4f(2n + 1) - f(4n + 1) + 4f(4n + 2) + f(4n + 3) + f(8n + 1) - f(8n + 2) + f(8n + 3) \\
&\quad - 2f(8n + 4) + 2f(8n + 5) \\
f(16n + 3) &= f(2n) - \frac{3}{2}f(2n + 1) - f(4n) - \frac{15}{4}f(4n + 1) + 5f(4n + 2) + \frac{5}{2}f(8n + 1) \\
&\quad + \frac{13}{4}f(8n + 3) - \frac{9}{4}f(8n + 4) + \frac{1}{2}f(16n + 1) - \frac{1}{2}f(16n + 2) \\
f(16n + 4) &= f(2n) - \frac{3}{2}f(2n + 1) - f(4n) - \frac{7}{4}f(4n + 1) + f(4n + 2) + \frac{3}{2}f(8n + 1) \\
&\quad + 2f(8n + 2) + \frac{5}{4}f(8n + 3) - \frac{1}{4}f(8n + 4) - \frac{1}{2}f(16n + 1) - \frac{1}{2}f(16n + 2) \\
f(16n + 5) &= f(2n) + \frac{1}{2}f(2n + 1) - f(4n) - \frac{11}{4}f(4n + 1) + 3f(4n + 2) + \frac{5}{2}f(8n + 1) \\
&\quad + \frac{9}{4}f(8n + 3) - \frac{5}{4}f(8n + 4) + \frac{1}{2}f(16n + 1) - \frac{1}{2}f(16n + 2) \\
f(16n + 7) &= 2f(4n + 1) - 2f(4n + 2) + f(8n + 3) \\
f(16n + 9) &= f(4n + 1) - 2f(4n + 2) + f(4n + 3) + f(8n + 3) + f(8n + 4) \\
f(16n + 11) &= 4f(2n + 1) - f(4n + 1) + 2f(4n + 2) + f(4n + 3) - f(8n + 3) - f(8n + 4) + 2f(8n + 5) \\
f(16n + 12) &= 8f(2n + 1) - 3f(4n + 1) + 10f(4n + 2) + 3f(4n + 3) + 2f(8n + 1) - 2f(8n + 2) \\
&\quad + f(8n + 3) - 5f(8n + 4) + 5f(8n + 5) \\
f(16n + 13) &= 2f(2n + 1) - 2f(4n + 1) + 4f(4n + 2) + f(4n + 3) - 2f(8n + 3) \\
&\quad - 2f(8n + 4) + 4f(8n + 5) \\
f(16n + 15) &= -2f(2n + 1) + f(4n + 3) + 2f(8n + 7) \\
f(32n + 1) &= -4f(2n) + 2f(2n + 1) + 4f(4n) + 7f(4n + 1) - 12f(4n + 2) - f(8n + 1) \\
&\quad - 5f(8n + 3) + 5f(8n + 4) - 2f(16n + 1) + 2f(16n + 2) \\
f(32n + 2) &= -4f(2n) + 4f(2n + 1) + 4f(4n) + 8f(4n + 1) - 14f(4n + 2) - 4f(8n + 1) \\
&\quad + f(8n + 2) - 5f(8n + 3) + 5f(8n + 4) - 3f(16n + 1) + 4f(16n + 2) \\
f(32n + 10) &= 3f(2n) - \frac{5}{2}f(2n + 1) - 3f(4n) - \frac{17}{4}f(4n + 1) + 13f(4n + 2) + \frac{17}{2}f(8n + 1) \\
&\quad - f(8n + 2) + \frac{35}{4}f(8n + 3) - \frac{19}{4}f(8n + 4) + \frac{3}{2}f(16n + 1) - \frac{3}{2}f(16n + 2) \\
f(32n + 17) &= 2f(4n + 1) - 4f(4n + 2) + 2f(8n + 3) + 2f(8n + 4) + f(8n + 5) \\
f(32n + 18) &= 14f(2n + 1) + 4f(4n + 1) - 4f(4n + 2) + f(4n + 3) + f(8n + 1) - f(8n + 2) \\
&\quad + 6f(8n + 3) + f(8n + 4) + 2f(8n + 5) \\
f(32n + 26) &= -10f(2n + 1) - 6f(4n + 1) + 8f(4n + 2) + 3f(4n + 3) - f(8n + 1) + f(8n + 2) \\
&\quad - 8f(8n + 3) - 3f(8n + 4) + 5f(8n + 5) + 3f(8n + 7) + 2f(16n + 10)
\end{aligned}$$

Corollary 12. $f(4^n - 1) = (4^{n+1} - 4)/3$ and $f(3 \cdot 2^n) = 14 \cdot 2^n - 4$ for $n \geq 1$.

We suspect the values in Corollary 12 are, respectively, the local minima (maxima) of $f(n)/n$, but we do not have a proof yet.

5. STURMIAN WORDS

In this section we fix the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.

Recall that a (finite or infinite) word w over Σ is *balanced* if and only if for every pair of factors u, v of w of the same length, one has $\|u\|_{\mathbf{a}} - \|v\|_{\mathbf{a}}\| \leq 1$.

We start with a simple lemma.

Lemma 13. *Let w be a finite balanced word over Σ . Then for every $k > 0$, $P(w) \equiv (0, 0) \pmod{k}$ if and only if w is an abelian k -power.*

Proof. Let w be balanced and $P(w) = (ks, kt)$, for a positive integer k and some $s, t \geq 0$. Then we can write $w = v_1 v_2 \cdots v_k$ where each v_i has length $s + t$. Now each v_i must have Parikh vector equal to (s, t) , otherwise w would not be balanced, whence the ‘only if’ part of the statement follows. The ‘if’ part is straightforward. \square

A binary infinite word is *Sturmian* if and only if it is balanced and aperiodic. Sturmian words are precisely the infinite words having $n + 1$ distinct factors of length n for every $n \geq 0$. There are many other equivalent definitions of Sturmian words. A classical reference on Sturmian words is [31, Chapter 2]. Let us recall the definition of Sturmian words as codings of a rotation.

We fix the torus $I = \mathbb{R}/\mathbb{Z} = [0, 1)$. Given δ, γ in I , if $\delta > \gamma$, we use the notation $[\delta, \gamma)$ for the interval $[\delta, 1) \cup [0, \gamma)$. Recall that given a real number α , $\lfloor \alpha \rfloor$ is the greatest integer smaller than or equal to α , $\lceil \alpha \rceil$ is the least integer greater than or equal to α , and $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ is the fractional part of α . Notice that $\{-\alpha\} = 1 - \{\alpha\}$ for non-integer α .

Let $\alpha \in I$ be irrational, and $\rho \in I$. The Sturmian word $s_{\alpha, \rho}$ (respectively, $s'_{\alpha, \rho}$) of *angle* α and *initial point* ρ is the infinite word $a_0 a_1 a_2 \cdots$ defined by

$$a_n = \begin{cases} \mathbf{b}, & \text{if } \{\rho + n\alpha\} \in I_{\mathbf{b}}; \\ \mathbf{a}, & \text{if } \{\rho + n\alpha\} \in I_{\mathbf{a}}. \end{cases}$$

where $I_{\mathbf{b}} = [0, 1 - \alpha)$ and $I_{\mathbf{a}} = [1 - \alpha, 1)$ (respectively, $I_{\mathbf{b}} = (0, 1 - \alpha]$ and $I_{\mathbf{a}} = (1 - \alpha, 1]$).

In other words, take the unit circle and consider a point initially in position ρ . Then start rotating this point on the circle (clockwise) by an angle $\alpha, 2\alpha, 3\alpha$, etc. For each rotation, take the letter \mathbf{a} or \mathbf{b} associated with the interval within which the point falls. The infinite sequence obtained in this way is the Sturmian word $s_{\alpha, \rho}$ (or $s'_{\alpha, \rho}$, depending on the choice of the two intervals). See Figure 2 for an illustration.

For example, if $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ is the golden ratio, the Sturmian word

$$F = s_{\varphi-1, \varphi-1} = \mathbf{abaababaabaabaabaabaabaabaabaabaab} \cdots$$

is called the *Fibonacci word*.

A Sturmian word for which $\rho = \alpha$, such as the Fibonacci word, is called *characteristic*. Note that for every α one has $s_{\alpha, 0} = \mathbf{b}s_{\alpha, \alpha}$ and $s'_{\alpha, 0} = \mathbf{a}s_{\alpha, \alpha}$.

An equivalent way to visualize the coding of a rotation consists of fixing the point and rotating the intervals. In this representation, the interval $I_{\mathbf{b}} = I_{\mathbf{b}}^0$ is rotated at each step, so that after i rotations it is transformed into the interval $I_{\mathbf{b}}^{-i} = [\{-i\alpha\}, \{-(i+1)\alpha\})$, while $I_{\mathbf{a}}^{-i} = I \setminus I_{\mathbf{b}}^{-i}$.

This representation is convenient, since one can read within it not only a Sturmian word, but also all of its factors. More precisely, for every positive integer n , the factor of length n of $s_{\alpha, \rho}$ starting at position $j \geq 0$ is determined by the value of $\{\rho + j\alpha\}$ only. Indeed, for every

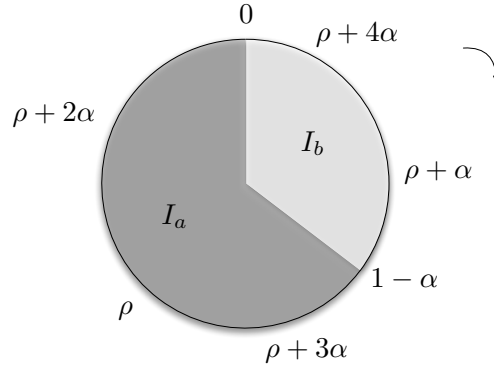


FIGURE 2. The rotation of angle $\alpha = \varphi - 1$ (where $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ is the golden ratio) and initial point $\rho = \alpha$ generating the Fibonacci word $F = s_{\varphi-1, \varphi-1} = \mathbf{abaababaabaabab} \dots$.

j and i , we have

$$a_{j+i} = \begin{cases} \mathbf{b}, & \text{if } \{\rho + j\alpha\} \in I_{\mathbf{b}}^{-i}; \\ \mathbf{a}, & \text{if } \{\rho + j\alpha\} \in I_{\mathbf{a}}^{-i}. \end{cases}$$

As a consequence, we have that given a Sturmian word $s_{\alpha, \rho}$ and a positive integer n , the $n + 1$ different factors of $s_{\alpha, \rho}$ of length n are completely determined by the intervals $I_{\mathbf{b}}^0, I_{\mathbf{b}}^{-1}, \dots, I_{\mathbf{b}}^{-(n-1)}$, that is, only by the points $\{-i\alpha\}$ for $0 \leq i < n$. In particular, they do not depend on ρ , so that the set of factors of $s_{\alpha, \rho}$ is the same as the set of factors of $s_{\alpha, \rho'}$ for every ρ and ρ' . Hence, from now on, we let s_{α} denote a Sturmian word of angle α .

If we arrange the $n+2$ points $0, 1, \{-\alpha\}, \{-2\alpha\}, \dots, \{-n\alpha\}$ in increasing order, we determine a partition of I in $n + 1$ subintervals, $L_0(n), L_1(n), \dots, L_n(n)$. Each of these subintervals is in bijection with a different factor of length n of s_{α} (see Figure 3).

Recall that a factor of length n of a Sturmian word s_{α} has a Parikh vector equal either to $(\lfloor n\alpha \rfloor, n - \lfloor n\alpha \rfloor)$ (in which case it is called *light*) or to $(\lceil n\alpha \rceil, n - \lceil n\alpha \rceil)$ (in which case it is called *heavy*). The following proposition relates the intervals $L_i(n)$ to the Parikh vectors of the factors of length n (see [17, 18, 35]).

Proposition 14. *Let s_{α} be a Sturmian word of angle α , and n a positive integer. Let t_i be the factor of length n associated with the interval $L_i(n)$. Then t_i is heavy if $L_i(n) \subset \{-n\alpha, 1\}$, while it is light if $L_i(n) \subset [0, \{-n\alpha\})$.*

Example 15. Let $\alpha = \varphi - 1 \approx 0.618$ and $n = 6$. We have $6\alpha \approx 3.708$, so that $\{-6\alpha\} \approx 0.292$. The reader can see in Figure 3 that the factors of length 6 corresponding to intervals above (respectively, below) $\{-6\alpha\} \approx 0.292$ all have Parikh vector $(4, 2)$ (respectively, $(3, 3)$). That is, the intervals L_0 and L_1 are associated with light factors (**babaab**, **baabab**), while the intervals L_2 to L_6 are associated with heavy factors (**baabaa**, **ababaa**, **abaaba**, **aababa**, **aabaab**).

Observe that, by Lemma 13, every factor of a Sturmian word having even length and containing an even number of **a**'s (or, equivalently, of **b**'s) is an abelian square. The following proposition relates the abelian-square factors of a Sturmian word of angle α with the arithmetic properties of α .

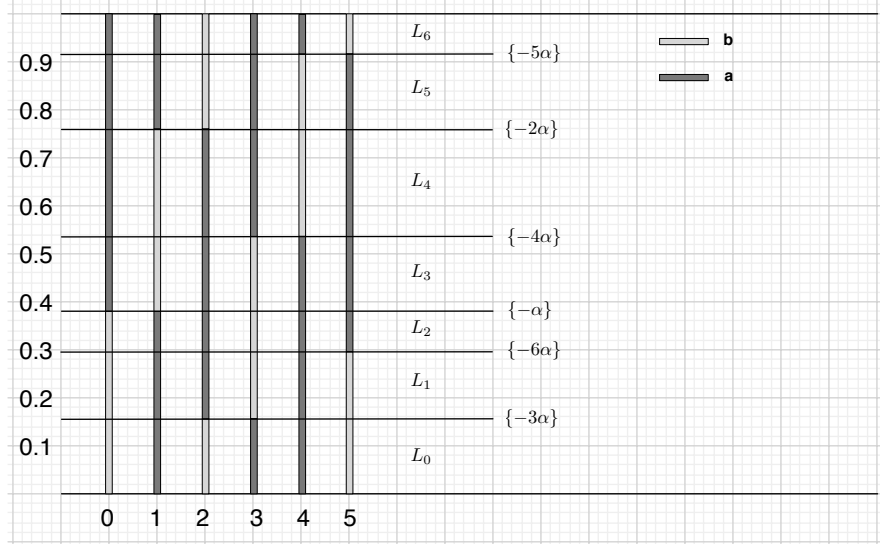


FIGURE 3. The points 0, 1 and $\{-\alpha\}, \{-2\alpha\}, \{-3\alpha\}, \{-4\alpha\}, \{-5\alpha\}, \{-6\alpha\}$ ($\alpha = \varphi - 1$), arranged in increasing order, define the intervals $L_0(6) \approx [0, 0.146)$, $L_1(6) \approx [0.146, 0.292)$, $L_2(6) \approx [0.292, 0.382)$, $L_3(6) \approx [0.382, 0.528)$, $L_4(6) \approx [0.528, 0.764)$, $L_5(6) \approx [0.764, 0.910)$, $L_6(6) \approx [0.910, 1)$. Each interval is associated with one of the factors of length 6 of the Fibonacci word, respectively **babaab**, **baabab**, **baabaa**, **ababaa**, **abaaba**, **aababa**, **aabaab**.

Proposition 16. *Let s_α be a Sturmian word of angle α , and n a positive even integer. Let t_i be the factor of length n associated with the interval $L_i(n)$. Then t_i is an abelian square if and only if $L_i(n) \subset \{\{-n\alpha\}, 1\}$ if $\lfloor n\alpha \rfloor$ is even, or $L_i(n) \subset [0, \{-n\alpha\})$ if $\lfloor n\alpha \rfloor$ is odd.*

Proof. By Proposition 14, t_i is heavy if $L_i(n) \subset \{\{-n\alpha\}, 1\}$, while it is light if $L_i(n) \subset [0, \{-n\alpha\})$. If $\lfloor n\alpha \rfloor$ is even, then every light factor of length n contains an even number of **a**'s and hence is an abelian square, while if $\lfloor n\alpha \rfloor$ is odd, then every heavy factor of length n contains an even number of **a**'s and hence is an abelian square, whence the statement follows. \square

Recall that given a finite or infinite word w , $AS_n(w)$ denotes the number of distinct abelian-square factors of w of length n .

Corollary 17. *Let s_α be a Sturmian word of angle α . For every positive even n , let $I_n = \{\{-i\alpha\} \mid 1 \leq i \leq n\}$. Then*

$$AS_n(s_\alpha) = \begin{cases} \#\{x \in I_n \mid x \leq \{-n\alpha\}\}, & \text{if } \lfloor n\alpha \rfloor \text{ is even;} \\ \#\{x \in I_n \mid x \geq \{-n\alpha\}\}, & \text{if } \lfloor n\alpha \rfloor \text{ is odd.} \end{cases}$$

Example 18. The factors of length 6 of the Fibonacci word F are, lexicographically ordered: **aabaab**, **aababa**, **abaaba**, **ababaa**, **baabaa** (heavy factors), **baabab**, **babaab** (light factors). The light factors, whose number of **a**'s is $\lfloor 6\alpha \rfloor = 3$, are not abelian squares; the heavy factors, whose number of **a**'s is $\lceil 6\alpha \rceil = 4$, are all abelian squares.

We have $I_6 = \{0.382, 0.764, 0.146, 0.528, 0.910, 0.292\}$ (values are approximated) and $6\alpha \simeq 3.708$, so $\lfloor 6\alpha \rfloor$ is odd. Thus, there are 5 elements in I_6 that are $\geq \{-6\alpha\}$, so by Corollary 17 there are 5 distinct abelian-square factors of length 6.

n	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
$AS_n(F)$	0	1	3	5	1	9	5	5	15	3	13	13	5	25	9	15	25	1	27

TABLE 2. The first few values of the sequence $AS_n(F)$ of the number of distinct abelian-square factors of length n in the Fibonacci word $F = s_{\varphi-1, \varphi-1}$. See OEIS sequence A241674.

The factors of length 8 of the Fibonacci word are, lexicographically ordered: **aabaabab**, **aababaab**, **abaabaab**, **abaababa**, **ababaaba**, **baabaaba**, **baababaa**, **babaabaa** (heavy factors), **babaabab** (light factor). The light factor, whose number of **a**'s is $\lfloor 8\alpha \rfloor = 4$, is an abelian square; the heavy factors, whose number of **a**'s is $\lfloor 8\alpha \rfloor = 5$, are not abelian squares. We have $I_8 = \{0.382, 0.764, 0.146, 0.528, 0.910, 0.292, 0.674, 0.056\}$ (values are approximated) and $8\alpha \simeq 4.944$, so $\lfloor 8\alpha \rfloor$ is even. Thus, there is only one element in I_8 that is $\leq \{8\alpha\}$, so by Corollary 17 there is only one abelian-square factor of length 8.

In Table 2 we report the first few values of the sequence $AS_n(F)$ for the Fibonacci word F . More detailed analysis of $AS_n(F)$ can be found, using the decision method we used for Thue-Morse, in [13, 14].

Recall that every irrational number α can be uniquely written as a (simple) continued fraction as follows:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} \quad (5.1)$$

where $a_0 = \lfloor \alpha \rfloor$, and the infinite sequence $(a_i)_{i \geq 0}$ is called the sequence of *partial quotients* of α . The continued fraction expansion of α is usually denoted by its sequence of partial quotients as follows: $\alpha = [a_0; a_1, a_2, \dots]$, and each of its finite truncations $[a_0; a_1, a_2, \dots, a_k]$ is a rational number n_k/m_k called the k th convergent to α . We say that an irrational $\alpha = [a_0; a_1, a_2, \dots]$ has bounded partial quotients if and only if the sequence $(a_i)_{i \geq 0}$ is bounded.

The continued fraction expansion of α is deeply related to the exponent of the factors of the Sturmian word s_α . The second author [32] proved that a Sturmian word of angle α has bounded exponent if and only if α has bounded partial quotients.

Since the golden ratio φ is defined by the equation $\varphi = 1 + 1/\varphi$, we have from Equation 5.1 that $\varphi = [1; 1, 1, 1, \dots]$ and therefore $\varphi - 1 = [0; 1, 1, 1, \dots]$, so the Fibonacci word is an example of a Sturmian word with bounded exponent (actually, one can prove that the superior limit of the exponent of a factor of the Fibonacci word is $(2 + \varphi)$ [33]).

Now we prove that if α has bounded partial quotients, then the Sturmian word s_α is (uniformly) abelian-square-rich. For this, we will use a result on the *discrepancy* of uniformly distributed modulo 1 sequences from [29]. To the best of our knowledge, this is the first application of this result to the theory of Sturmian words, and we think that this correspondence might be useful for deriving other results on Sturmian words.

Let $\omega = (x_n)_{n \geq 0}$ be a sequence of real numbers. For a positive integer N and a subset E of the torus I , we define $A(E; N; \omega)$ as the number of terms x_n , $0 \leq n \leq N$, for which $\{x_n\} \in E$. If there is no risk of confusion, we will write $A(E; N)$ instead of $A(E; N; \omega)$.

Definition 19. The sequence $\omega = (x_n)_{n \geq 0}$ of real numbers is said to be *uniformly distributed modulo 1* if and only if for every pair γ, δ of real numbers with $0 \leq \gamma < \delta \leq 1$ we have

$$\lim_{N \rightarrow \infty} \frac{A([\gamma, \delta); N; \omega)}{N} = \delta - \gamma.$$

Definition 20. Let x_0, x_1, \dots, x_N be a finite sequence of real numbers. The number

$$D_N = D_N(x_0, x_1, \dots, x_N) = \sup_{0 \leq \gamma < \delta \leq 1} \left| \frac{A([\gamma, \delta); N)}{N} - (\delta - \gamma) \right|$$

is called the *discrepancy* of the given sequence. For an infinite sequence ω of real numbers the discrepancy $D_N(\omega)$ is the discrepancy of the initial segment formed by the first $N + 1$ terms of ω .

The two previous definitions are related by the following result.

Theorem 21 ([29]). *The sequence ω is uniformly distributed modulo 1 if and only if $\lim_{N \rightarrow \infty} D_N(\omega) = 0$.*

An important class of uniformly distributed modulo 1 sequences is given by the sequence $(n\alpha)_{n \geq 0}$ with α an irrational number. The discrepancy of the sequence $(n\alpha)$ will depend on the finer arithmetical properties of α . In particular, we have the following theorem, stating that if α has bounded partial quotients, then its discrepancy has the least order of magnitude possible.

Theorem 22 ([29]). *Suppose the irrational $\alpha = [a_0; a_1, \dots]$ has partial quotients bounded by K . Then the discrepancy $D_N(\omega)$ of $\omega = (n\alpha)$ satisfies $ND_N(\omega) = O(\log N)$. More exactly, we have*

$$ND_N(\omega) \leq 3 + \left(\frac{1}{\log \varphi} + \frac{K}{\log(K+1)} \right) \log N \tag{5.2}$$

where φ is the golden ratio.

As a consequence, we obtain the following corollary:

Corollary 23. *Suppose the irrational $\alpha = [a_0; a_1, \dots]$ has bounded partial quotients. Then there exists a positive constant C such that for every pair γ, δ of real numbers with $0 \leq \gamma < \delta \leq 1$ and for every n , we have*

$$A([\gamma, \delta); n; (n\alpha)) \geq n(\delta - \gamma) - C \log n.$$

Now we use the results above to prove that Sturmian words with bounded exponent are abelian-square-rich.

Theorem 24. *Let s_α be a Sturmian word of angle α such that α has bounded partial quotients. Then there exists a positive constant C such that for every n one has $\sum_{m \leq n} \text{AS}_m(s_\alpha) \geq Cn^2$.*

Proof. For every even n , let $I'_n = \{\{i\alpha\} \mid 1 \leq i \leq n\}$. By Corollary 17 and basic arithmetical properties of the fractional part, we have

$$\text{AS}_n(s_\alpha) = \begin{cases} \#\{x \in I'_n \mid x \geq \{n\alpha\}\}, & \text{if } \lfloor n\alpha \rfloor \text{ is even;} \\ \#\{x \in I'_n \mid x \leq \{n\alpha\}\}, & \text{if } \lfloor n\alpha \rfloor \text{ is odd.} \end{cases}$$

So,

$$\sum_{m \leq n} AS_m(s_\alpha) \quad (5.3)$$

$$\geq \sum_{m \leq n, \lfloor m\alpha \rfloor \text{ odd}} \#\{\{i\alpha\} \mid \{i\alpha\} \leq \{m\alpha\}, 1 \leq i \leq m\} \quad (5.4)$$

$$\geq \sum_{m \leq n, \lfloor m\alpha \rfloor \text{ odd}, \{m\alpha\} \geq 1/2} \#\{\{i\alpha\} \mid \{i\alpha\} \leq 1/2, 1 \leq i \leq m\} \quad (5.5)$$

$$= \sum_{m \leq n, \{m\alpha/2\} \geq 1/2, \{m\alpha\} \geq 1/2} \#\{\{i\alpha\} \mid \{i\alpha\} \leq 1/2, 1 \leq i \leq m\} \quad (5.6)$$

$$\geq \sum_{m \leq n, \{m\alpha/2\} \geq 3/4} \#\{\{i\alpha\} \mid \{i\alpha\} \leq 1/2, 1 \leq i \leq m\} \quad (5.7)$$

$$\geq \sum_{m \leq n, \{m\alpha/2\} \geq 3/4} \#\{\{i\alpha/2\} \mid \{i\alpha/2\} \leq 1/4, 1 \leq i \leq m\} \quad (5.8)$$

$$\geq \sum_{n/2 \leq m \leq n, \{m\alpha/2\} \geq 3/4} \#\{\{i\alpha/2\} \mid \{i\alpha/2\} \leq 1/4, 1 \leq i \leq n/2\} \quad (5.9)$$

$$= \#\{\{i\alpha/2\} \mid \{i\alpha/2\} < 1/4, 1 \leq i \leq n/2\} \cdot \sum_{n/2 \leq m \leq n} \{m \mid \{m\alpha/2\} \geq 3/4\}, \quad (5.10)$$

where: (5.6) follows from (5.5) because for every integer m one has that $\lfloor m\alpha \rfloor$ is odd if and only if $\{m\alpha/2\} \geq 1/2$; (5.7) follows from (5.6) because for every integer m , $\{m\alpha/2\} \geq 3/4$ implies $\{m\alpha\} \geq 1/2$; (5.8) follows from (5.7) because for every integer i , $\{i\alpha/2\} \leq 1/4$ implies $\{i\alpha\} \leq 1/2$; finally (5.10) follows from (5.9) because the cardinality of the first set is independent from the sum. The other (in-)equalities are obvious.

Since α has bounded partial quotients, so does $\alpha/2$ (see, e.g., [30]), and we can apply Corollary 23 to evaluate the two factors of (5.10). Therefore, we have

$$\begin{aligned} & \#\{\{i\alpha/2\} \mid \{i\alpha/2\} < 1/4, 1 \leq i \leq n/2\} \\ &= A([0, 1/4]; n/2; (n\alpha/2)) \\ &\geq (1/4)(n/2) - C_1 \log n \\ &= n/8 - C_1 \log n, \end{aligned}$$

for a positive constant C_1 , and

$$\begin{aligned} & \sum_{n/2 \leq m \leq n} \{m \mid \{m\alpha/2\} \geq 3/4\} \\ &= A([3/4, 1]; n; (n\alpha/2)) - A([3/4, 1]; n/2; (n\alpha/2)) \\ &\geq (1/4)n - C_2 \log n - (1/4)(n/2) + C_3 \log n \\ &= n/8 - C_4 \log n, \end{aligned}$$

for constants C_2, C_3, C_4 . From this, we can conclude that the product of the two factors of (5.10) is greater than a constant times n^2 , as required. \square

Next, we recall a result of J. Cassaigne:

Lemma 25. [7, Prop. 5] *The recurrence quotient r_α of a Sturmian word of angle $\alpha = [0; a_1, a_2, \dots]$ is finite if and only if α has bounded partial quotients and, in this latter case, one has $2 + \limsup a_i < r_\alpha < 3 + \limsup a_i$.*

Thus, we have the following:

Corollary 26. *Let s_α be a Sturmian word of angle α . Then s_α has bounded exponent if and only if s_α is uniformly abelian-square-rich.*

Proof. We know that s_α has bounded exponent if and only if α has bounded partial quotients, if and only if s_α is linearly recurrent. The statement then follows from Theorem 24, Lemmas 5 and 4 and Proposition 6. \square

6. CONCLUSIONS

We proved that the Thue-Morse word is uniformly abelian-square-rich. We suspect that the technique we used for the proof can be generalized to some extent, and could be used, for example, to prove that a subclass of fixed points of uniform substitutions are uniformly abelian-square-rich.

We also proved that Sturmian words of bounded exponent are uniformly abelian-square-rich (and the converse also holds). The proof we gave is based on a classical result on the discrepancy of the uniformly distributed modulo 1 sequence $(n\alpha)_{n \geq 0}$, where α is the angle of the Sturmian word. To the best of our knowledge, this is the first application of this result to the theory of Sturmian words, and we believe that the correspondence we have shown might be useful for deriving other results on Sturmian words.

We leave open the question of determining whether s_α is not abelian-square-rich in the case when α has unbounded partial quotients. Notice that, by Proposition 6, such s_α cannot be uniformly abelian-square-rich, since a Sturmian word has bounded partial quotients if and only if it has bounded exponent.

We mostly investigated binary words in this paper. We conjecture that binary words have the largest number of distinct abelian-square factors. More precisely, we propose the following conjecture.

Conjecture 27. *If a word of length n contains k distinct abelian-square factors, then there exists a binary word of length n containing at least k distinct abelian-square factors.*

A slightly different point of view from the one we considered in this paper consists in identifying two abelian squares if they have the same Parikh vector. Two abelian squares are therefore called *inequivalent* if they have different Parikh vectors [21]. Sturmian words only have a linear number of inequivalent abelian squares. Nevertheless, a word of length n can contain $\Omega(n\sqrt{n})$ inequivalent abelian squares [28]. Computations support the following conjecture:

Conjecture 28 (see [27, 28]). *Every word of length n contains at most $\Theta(n\sqrt{n})$ inequivalent abelian squares.*

7. ACKNOWLEDGMENTS

The authors warmly thank Julien Cassaigne for useful discussions about the number of abelian-square factors in the Thue-Morse word, and the anonymous referees for their valuable comments, which allowed us to improve the presentation of the paper.

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