FINITISTIC AUSLANDER ALGEBRAS

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Abstract. Recently, Chen and Koenig in [CheKoe] and Iyama and Solberg in [IyaSol] independently introduced and characterised algebras with dominant dimension coinciding with the Gorenstein dimension and both dimensions being larger than or equal to two. In [IyaSol], such algebras are named Auslander-Gorenstein algebras. Those classes of algebras clearly generalise the well known class of higher Auslander algebras, where the dominant dimension additionally coincides with the global dimension. In this short article we generalise Auslander-Gorenstein algebras further to algebras having the property that the dominant dimension coincides with the finitistic dimension and both dimension are at least two. We call such algebras finitistic Auslander algebras. As an application we can specialise to reobtain known results about Auslander-Gorenstein algebras and higher Auslander algebras such as the higher Auslander correspondence, which now has a very short proof. We furthermore state the new homological conjecture that in fact all nonselfinjective algebras with high enough dominant dimension have automatically their finitistic dimension equal to the dominant dimension. The last section motivates this conjecture by examples. In particular, we show how to associate finitistic Auslander algbras to arbitrary local selfinjective algebras in different ways, which also indicates that the class of finitistic Auslander algebras is much larger than the class of Auslander-Gorenstein algebras. We give several related conjectures for Nakayama algebras with high dominant dimension. We verify those conjectures for $n \leq 13$ using the computer algebra system QPA, which is a package of GAP.

Introduction

Let A always be a finite dimensional connected algebra over a field K, which is not semi-simple. All modules are finite dimensional right modules if nothing is stated otherwise. In this article we generalise Auslander-Gorenstein algebras introduced in [IyaSol] as algebras having dominant dimension equal to the Gorenstein dimension and both dimensions being larger than or equal to two. Those algebras contain the important class of higher Auslander algebras, introduced in [Iya]. Note that in case an algebra is Gorenstein, the Gorenstein dimension equals the finitistic dimension (see for example [Che]). Thus algebras with finitistic dimension equal to the dominant dimension, which is larger than or equal to two, generalise Auslander-Gorenstein algebras. We call such algebras *finitistic Auslander algebras* and deduce some of their properties, including a generalisation of the celebrated higher Auslander correspondence for finite dimensional algebras first proven in [Iya]. We characterise finitistic Auslander algebras in terms of Gorenstein homological algebra and the category of modules having a certain dominant dimension. Let Dom_d denote the full subcategory of modules having dominant dimension at least d, $Gp(A)$ the subcategory of Gorenstein projective modules and $Gp_{\infty}(A)$ the full subcategory of modules having infinite Gorenstein projective dimension. We refer to the preliminaries for more information and definitions.

Theorem. (see 2.4) Let $A \cong End_B(M)$ be an algebra of dominant dimension $d \geq 2$, where M is a generator-cogenerator of $mod-B$. The following are equivalent:

(1) A is a finitistic Auslander algebra.
(2) $Dom_d \subseteq proj \cup proj_{\infty}$.
(3) $Dom_d \subseteq Gp(A) \cup Gp_{\infty}(A)$.
(4) $add(M) - resdim(X) = \infty$ for all $X \in M^{\perp d-2} \setminus add(M)$.

Thus generator-cogenerators M with $add(M)-resdim(X) = \infty$ for all $X \in M^{\perp d-2} \setminus add(M)$ generalise the classical cluster tilting objects introduced in [Iya] and the precluster tilting objects introduced in [IyaSol].

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In particular, specialising our results to finite Gorenstein or finite global dimension, we obtain quick proofs of some known facts such as the higher Auslander correspondence relating higher Auslander algebras and cluster tilting objects.

The final section gives examples and conjectures. We state the following new homological conjecture:

Conjecture. *There exists a polynomial function $f(n)$ depending only on n such that the following holds: Let A be a nonselfinjective algebra with n simple modules and dominant dimension at least $f(n)$. Then the finitistic dimension of A is equal to the dominant dimension.*

Our results suggest that something between $f(n) = n$ and $f(n) = 2n$ might do the job. At the moment no counterexample to the conjecture with $f(n) = n$ seems to be known. We verify this conjecture for the following classes of algebras and refer to the last section for details.

(1) Algebras B isomorphic to $End_A(A \oplus M)$, where A is a local selfinjective algebra with indecomposable module M such that $Ext^1(M, M) \neq 0$. Here B has two simples and $f(2) = 2$ suffices.
(2) Standardly stratified algebras. Here $f(n) = 2n - 2$ works (we do not know whether $f(n) = n$ is enough).
(3) Representation-finite gendo-symmetric biserial algebras. Here $f(n) = n$ is enough.
(4) Nakayama algebras with n simples for $n \leq 13$. Also $f(n) = n$ is enough here.
(5) Acyclic quiver algebras. This is in fact trivial since the global dimension is bounded by $n - 1$ for the class of such algebras with n simples. Thus $f(n) = n - 1$ is ok for this class of algebras.

We show the following, which illustrates that the class of of finitistic Auslander algebras is much larger than the class of Auslander-Gorenstein algebras in general:

Theorem. (see 3.10)

(1) Let A be a commutative selfinjective algebra with enveloping algebra $A^e = A \otimes_K A$. Then $B := End_{A^e}(A^e \oplus A)$ is a finitistic Auslander algebra of finitistic dimension equal to two. It is an Auslander-Gorenstein algebra iff A is a 2-periodic algebra iff $A \cong K[x]/(x^n)$ for some $n >= 2$. It is never a higher Auslander algebra.
(2) Let A be a selfinjective local algebra with simple module S. Then $End_A(A \oplus S)$ is a finitistic Auslander algebra with finitistic dimension equal to two. It is a higher Auslander-Gorenstein algebra iff $A \cong K[x]/(x^n)$ for some $n \geq 2$ and it is a higher Auslander algebra iff $A \cong K[x]/(x^2)$.

We end with conjectures related to finitistic Auslander algebras inside the class of Nakayama algebras. The conjectures are proven for algebras with $n \leq 13$ simples modules using the computer algebra system QPA.

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1. Preliminaries

Throughout A is a finite dimensional and connected algebra over a field K. Furthermore, we assume that A is not semisimple. We always work with finite dimensional right modules, if not stated otherwise. By $mod - A$, we denote the category of finite dimensional right A-modules. For background on representation theory of finite dimensional algebras and their homological algebra, we refer to [ASS]. For a module M, $add(M)$ denotes the full subcategory of $mod - A$ consisting of direct summands of M^n for some $n \geq 1$. A module M is called *basic* in case $M \cong M_1 \oplus M_2 \oplus ... \oplus M_n$, where every M_i is indecomposable and M_i is not isomorphic to M_j for $i \neq j$. The *basic version* of a module N is the unique (up to isomorphim) module M such that $add(M) = add(N)$ and such that M is basic. We denote by $S_i = e_iA/e_iJ$, $P_i = e_iA$ and $I_i = D(Ae_i)$ the simple, indecomposable projective and indecomposable injective module, respectively, corresponding to the primitive idempotent e_i.

The *dominant dimension* domdim(M) of a module M with a minimal injective resolution
$(I_i) : 0 \to M \to I_0 \to I_1 \to ...$ is defined as:
domdim(M):=sup$\{n | I_i$ is projective for $i = 0, 1, ..., n\}$+1, if I_0 is projective, and
domdim(M):=0, if I_0 is not projective.

The *codominant dimension* of a module M is defined as the dominant dimension of the A^{op}-module

$D(M)$. The dominant dimension of a finite dimensional algebra is defined as the dominant dimension of the regular module. It can be shown that the dominant dimension of an algebra always equals the dominant dimension of the opposite algebra, see for example [Ta]. So domdim$(A) \geq 1$ means that the injective hull of the regular module A is projective or equivalently, that there exists an idempotent e such that eA is a minimal faithful projective-injective module. Algebras with dominant dimension larger than or equal to 1 are called QF-3 algebras. For more information on dominant dimensions and QF-3 algebras, we refer to [Ta]. An algebra is called *Gorenstein* in case $Gdim(A) := injdim(A)$ equals $projdim(D(A)) < \infty$. In this case $Gdim(A)$ is called the *Gorenstein dimension* of A and we say that A has infinite Gorenstein dimension if $injdim(A) = \infty$. Note that $Gdim(A) = max\{injdim(e_i A) | e_i \text{ a primitive idempotent}\}$ and $domdim(A) = min\{domdim(e_i A) | e_i \text{ a primitive idempotent }\}$. We denote by $proj$ the full subcategory of projective modules and by $proj_\infty$ the full subcategory of modules of infinite projective dimension. The Morita-Tachikawa correspondence (see for example [Ta]) says that an algebra A has dominant dimension at least two iff $A \cong End_B(M)$ for some generator-cogenerator M of $mod - B$. Mueller's theorem says that in this case the dominant dimension of A equals $\inf\{i \geq 1 | Ext^i(M, M) \neq 0\} + 1$, see [Mue]. We will need the following results that can be viewed as refinements of results of Mueller, which can be found in [Mar] as theorem 2.2. with detailed references and which can be viewed as a special case of results in [APT].

Theorem 1.1. *Let A be an algebra of dominant dimension at least two with minimal faithful projective-injective left module P and minimal faithful projective-injective right module I. Let $B = End_A(P)$.*

(1) $F := Hom_A(P, -) : Dom_2 \to mod - B$ is an equivalence of categories. F restricts to an equivalence between add(I) and the category of injective B-modules.
(2) The functor $G := Hom_B(P, -) : mod - B \to Dom_2$ is inverse to F.
(3) For $i \geq 3$, F restricts to an equivalence $F : Dom_i \to (P)^{\perp i-2}$, where P is viewed as a B-module.

An algebra A is called *higher Auslander algebra* in case $\infty > domdim(A) = gldim(A) \geq 2$, see [Iya] and A is called *Auslander-Gorenstein algebra* in case $\infty > domdim(A) = Gdim(A)$, see [IyaSol]. A module M is called *Gorenstein projective* in case $Ext^i(D(A), \tau(M)) \cong Ext^i(M, A) \cong 0$ for all $i \geq 1$. Every non-projective Gorenstein projective modules has infinite projective dimension. As in the case of usual minimal projective resolutions, every module M has a resolution by Gorenstein projective modules and a corresponding *Gorenstein projective dimension $Gpd(M)$*, see [Che] for more details. $\Omega^i(A-mod)$ denotes the full subcategory of all projective modules and modules which are i-th syzygies, Dom_d denotes the full subcategory of modules having dominant dimension at least d, $Gp(A)$ denotes the full subcategory of Gorenstein projective modules and $Gp_\infty(A)$ denotes the full subcategory of modules having infinite Gorenstein projective dimension. We will need the following proposition:

Proposition 1.2. *Let A be an algebra of dominant dimension $d \geq 1$, then $\Omega^i(A - mod) = Dom_i$ for every $i \leq d$.*

Proof. See [MarVil], proposition 4. □

For a given subcategory C of $mod - A$, a *minimal right C-approximation* of a module X is a right minimal map $f : N \to X$ with $N \in C$ such that $Hom(L, f)$ is surjective for every $L \in C$. Such minimal right approximations always exist and are unique up to isomorphism in case $C = add(M)$ for some module M. In case $C = add(M)$, one defines $\Omega^0_M(X) := X$, $\Omega^1_M(X)$ as the kernel of such an f and inductively $\Omega^n_M(X) := \Omega^{n-1}_M(\Omega^1_M(X))$. One then defines $add(M) - resdim(X) := inf\{n \geq 0 | \Omega^n_M(X) \in add(M)\}$. Given an algebra A, which is isomorphic to $End_B(M)$ for some algebra B with generator-cogenerator M, one can show that minimal $add(M)$-resolutions in $mod - B$ of a module X corresponds to minimal projective resolutions of the module $Hom_B(M, X)$ in $mod - A$. See [CheKoe] section 2.1. for more information on this. A subcategory C of $mod - A$ is called *contravariantly finite* in case every module $X \in mod - A$ has a minimal right C-approximation. A subcategory C is called *resolving* in case it contains the projective modules, is closed under extensions and closed under kernels of surjections. We will need the following result, that can be found in [AR], 3.9.:

Proposition 1.3. *Let C be a resolving contravariantly finite subcategory of $mod - A$. Then every module in C has finite projective dimension bounded by t in case all of the modules X_i have finite projective dimension bounded by t, where $f_i : X_i \to S_i$ are minimal C-approximations of the simple modules S_i.*

For a module M, we define $M^{\perp n} := \{X \in mod-A | Ext^i(M,X)=0 \text{ for all } i=1,...,n\}$. The *finitistic dimension* of an algebra is defined as $findim(A) = \sup\{pd(N) | pd(N) < \infty\}$. The *global Gorenstein projective dimension* of an algebra is defined as the supremum of all Gorenstein projective dimensions of modules. It is known that the global Gorenstein projective dimension is finite iff the algbra is Gorenstein, see [Che] corollary 3.2.6. The *finitistic Gorenstein projective dimension* is defined as $Gfindim(A) = \sup\{Gpd(N) | Gpd(N) < \infty\}$ and in [Che] one finds a quick proof that this always equals the usual finitistic dimension in theorem 3.2.7. We call a module M d-rigid, in case $Ext^i(M,M)=0$ for $i=1,2,...,d$. The *finitistic dominant dimension*, first introduced in [Mar2], is defined as the suprumum of all dominant dimension of modules having finite dominant dimension. An algebra is called *CM-free* in case every Gorenstein projective module is projective. We will also need the following theorem, which can be found as theorem 3.2.5. in [Che].

Theorem 1.4. *Let M be a module. M has finite Gorenstein projective dimension at most n iff in every exact sequence of the form $0 \rightarrow K \rightarrow G^{n-1} \rightarrow ... \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ with Gorenstein projective modules G_i, also the module K is Gorenstein projective.*

We remark that we omit to state obvious dual results, which would involve dual concepts such as codominant dimenion or Gorenstein injective modules.

2. Finitistic Auslander algebras

This section introduces finitistic Auslander algebras and gives new relations between dominant dimension and the finitistic dimension.

Lemma 2.1. *Let A be an algebra of dominant dimension $d \geq 1$. $findim(A) = d + \sup\{pd(N) | domdim(N) \geq d, pd(N) < \infty\}$.*

Proof. First note that the finitistic dimension is larger than or equal to the dominant dimension: The exact sequence coming from a minimal injective coresolution of the regular module: $0 \rightarrow A \rightarrow I_0 \rightarrow \cdots \rightarrow I_{d-1} \rightarrow \Omega^{-d}(A) \rightarrow 0$ shows that the module $\Omega^{-d}(A)$ has finite projective dimension d. Thus the finitistic dimension is at least d. Assume the projective dimension $s \geq d$ is attained at the module X: $pd(X) = s$. Looking at the minimal projective resolution of X: $0 \rightarrow P_s \rightarrow \cdots P_0 \rightarrow X \rightarrow 0$ and using $pd(X) = pd(\Omega^d(X)) + s - d$ one immediatly obtains the lemma, since $\Omega^d(X)$ has dominant dimension at least d and its projective dimension equals $s-d$. □

Proposition 2.2. *Let A be an algebra of positive dominant dimension d, then A has finite finitistic dimension in case the subcategory $Dom_d \cap proj_{<\infty}$ is contravariantly finite.*

Proof. Let $C := Dom_d \cap proj_{<\infty}$. We want to use 1.3. First note that the intersection of two resolving subcategories is resolving and that Dom_d is resolving (see for example [MarVil]), while the property that $proj_{<\infty}$ is resolving is well known. Thus C is a contravariantly finite resolving subcategory and the X_i (defined as the modules, such that $f_i : X_i \rightarrow S_i$ are minimal right C-approximations of the simples) have finite projective dimension bounded by some number t, since they are contained in $proj_{<\infty}$. Thus all modules in C have finite projective dimension bounded by t and the result follows from 2.1. □

Now we come to the generalisation of Auslander-Gorenstein algebras:

Definition 2.3. An algebra with finite dominant dimension $d \geq 2$ is called a *finitistic Auslander algebra* in case its finitistic dimension equals its dominant dimension.

Note that by the Morita-Tachikawa correspondence, every finitistic Auslander algebra A is isomorphic to an algebra of the form $End_B(M)$ for some algebra B with generator-cogenerator M, since by assumption A has dominant dimension at least two. We remark that every Auslander-Gorenstein algebra and thus every higher Auslander algebra is a finitistic Auslander algebra, since the finitistic dimension equals the Gorenstein dimension in case the Gorenstein dimension is finite. We will later see an example of a finitistic Auslander algebra of infinite Gorenstein dimension, showing that the class of finitistic Auslander algebras is really bigger than the class of Auslander-Gorenstein algebras.

The next theorem gives another characterisation of finitistic Auslander algebras using the subcategory of modules having dominant dimension at least d.

Theorem 2.4. *Let $A \cong End_B(M)$ be an algebra of finite dominant dimension $d \geq 2$, where M is a generator-cogenerator. The following are equivalent:*

(1) A is a finitistic Auslander algebra.
(2) $Dom_d \subseteq proj \cup \mathrm{proj}_{<\infty}$.
(3) $Dom_d \subseteq Gp(A) \cup Gp_\infty(A)$.
(4) $add(M) - resdim(X) = \infty$ for all $X \in M^{d-2} \setminus add(M)$

Proof. First we show that (1) and (2) are equivalent: Just note that by 2.1, A is a finitistic Auslander algebra iff every module of dominant dimension at least d has infinite projective dimension or is projective. Assume now (1), that is the finitistic dimension equals the dominant dimension. Assume $X \in Dom_d$ and X having finite and non-zero Gorenstein projective dimension s. Then there exists the following exact sequence, where the left side comes from a minimal Gorenstein projective resolution and the right side comes from a minimal injective coresolution: $0 \to G_s \to \cdots \to G_0 \to X \to I_0 \to \cdots \to \Omega^{-d}(X) \to 0$. This shows that the module $\Omega^{-d}(X)$ has finite Gorenstein projective dimension $s + d > s$ using that X is not Gorenstein projective, by 1.4.

This contradicts the fact that the finitistic Gorenstein projective dimension equals the finitistic dimension which is equal to s. This shows that (1) implies (3).
Now assume (3), that is $Dom_d \subseteq Gp(A) \cup Gp_\infty(A)$. We use 2.1 and show that $\sup\{pd(N) | domdim(N) \geq d, pd(N) < \infty\} = 0$. But this is obvious since every non-projective module in $Dom_d \subseteq Gp(A) \cup Gp_\infty(A)$ has infinite projective dimension (recall that Gorenstein projective modules are projective or have infinite projective dimension). This shows that (3) implies (1).
Now we show that (4) is equivalent to (1):
Assume A has dominant dimension $d \geq 2$. By 2.1, the finitistic dimension equals the dominant dimension iff every non-projective module of dominant dimension at least d has infinite projective dimension. This translates into the condition $add(M) - resdim(X) = \infty$ for all $X \in M^{d-2} \setminus add(M)$ since $add(M)$ resolutions correspond to minimal projective resolutions in A and the subcategory Dom_d without the projectives corresponds to $M^{d-2} \setminus add(M)$ by (3) of 1.1. □

The next lemma was also noted in [Mar].

Lemma 2.5. *Let A be an algebra of dominant dimension $d \geq 1$, then every Gorenstein projective module has dominant dimension at least d.*

Proof. By definition every Gorenstein projective module is in $\Omega^i(A - mod)$ for every $i \geq 1$. Now $\Omega^d(A - mod) = Dom_d$ by 1.2 and thus $Gp(A) \subseteq \Omega^d(A - mod) = Dom_d$. □

Note that in the next proposition, (2) contains the higher Auslander correspondence from [Iya], where generator-cogenerators with the condition $add(M) = M^{d-2}$ are called cluster tilting objects. We give a very quick proof of the higher Auslander correspondence in (2) but refer to [CheKoe] or [IyaSol] for the second equivalence in (1).

Proposition 2.6. *Let B an algebra with generator-cogenerator M and $A = End_B(M)$. Assume A has finite dominant dimension $d \geq 2$, which by Mueller's theorem is equivalent to M being $d-2$ rigid and not $d-1$ rigid.*

(1) A is an Auslander-Gorenstein algebra iff $Dom_d = Gp(A)$ iff $add(M) = add(\tau(\Omega^{d-2}(M \oplus D(A))))$.
(2) A is a higher Auslander algebra iff $Dom_d = proj$ iff $add(M) = M^{\perp d-2}$.

Proof.
(1) Being an Auslander-Gorenstein algebra is equivalent to being a finitistic Auslander algebra and additionally having finite Gorenstein dimension. An algebra is Gorenstein iff it has finite global Gorenstein dimension and thus iff $Gp_\infty(A)$ is empty. But by 2.5 $Gp(A) \subseteq Dom_d$ and thus it is an Auslander-Gorenstein algebra iff $Dom_d = Gp(A)$, using (4) of 2.4. For the second equivalence, see [CheKoe] corollary 3.18.
(2) Recall that an algebra has finite global dimension iff it is Gorenstein and every Gorenstein projective module is projective, see for example [Che]. Thus the first equivalence follows by the first equivalence in (1). Now let Af be the minimal faithful projective-injective left A-module. Then the functor $(-)f$ is an equivalence between $proj$ and $add(M)$ and between Dom_d and $M^{\perp d-2}$ by 1.1 and this shows the second equivalence.

□

We explicitly state the case $d = 2$ since here finitistic Auslander algebras generalise the well known Auslander algebras.

Corollary 2.7. *Let B be an algebra with generator-cogenerator M and $A = End_B(M)$. Then A is a finitistic Auslander algebra with finitistic dimension two iff $Ext^1(M,M) \neq 0$ and $add(M) - resdim(X) = \infty$ for all $X \in mod - B \setminus add(M)$.*

Proof. By Mueller's theorem $Ext^1(M,M)$ implies that A has dominant dimension $d = 2$ and by 1. of the previous proposition the result follows by noting that $M^{\perp 0} = mod - B$. □

3. Finitistic Auslander algebras inside standardly stratified algebras and Nakayama algebras

In this section we give several examples and conjectures related to finitistic Auslander algebras. The main homological conjecture is as follows:

Conjecture. *There exists a polynomial function $f(n)$ depending only on n such that the following holds: Let A be a nonselfinjective algebra with n simple modules and dominant dimension at least $f(n)$. Then the finitistic dimension of A is automatically equal to the dominant dimension.*

The results of this section suggest that $f(n)$ might be something between $f(n) = n$ and $f(n) = 2n$.

Remark 3.1. In [ChMar] the class of representation-finite gendo-symmetric biserial algebras was classified (generalising the classical Brauer tree algebras). All those algebras were Gorenstein and thus their finitistic dimension coincides with the Gorenstein dimension. Explicit values for the dominant and Gorenstein dimension are obtained and one can easily show that the above conjecture is true for this class of algebras with $f(n) = n$.

3.1. Finitistic Auslander algebras from selfinjective local algebras.

The next lemma is due to Jeremy Rickard.

Lemma 3.2. *Let A be a local selfinjective algebra and M an indecomposable module and $\alpha : M^m \to M^n$ with $n, m > 0$ a map between direct sums of M all of whose components are radical maps. Let F be an additive functor such that $F(\alpha)$ is injective, then $F(M) = 0$.*

Proof. We can assume that n is a multiple of m by possibly adding extra summands to M^n. Write $n = dm$ for some integer d. We then have maps $M^{d^s m} \to M^{d^{s+1} m}$ for $s \geq 0$ by taking direct sums of the map α. This gives a sequence of maps $M^m \to M^{dm} \to M^{d^2 m} \to ... \to M^{d^k m}$, all of which become injective when applying F, since F is additive. But choosing k greater than the Loewy length of $End_A(M)$, the composition of this sequence of maps is zero and thus $F(M^m) = 0$, giving also $F(M) = 0$ using that F is addtive. □

The next theorem is due to Jeremy Rickard.

Theorem 3.3. *Let A be a local selfinjective algebra and let M be an indecomposable nonprojective module with $Ext^1(M,M) \neq 0$. Then $B := End_A(A \oplus M)$ is a finitistic Auslander algebra of finitistic dimension 2.*

Proof. The condition $Ext^1(M,M) \neq 0$ gives us that B has dominant dimension equal to two. We have to show that every non-projective module of dominant dimension at least two has infinite projective dimension. Let $N := A \oplus M$. This translates into the condition that every A-module not in $add(N)$ has infinite $add(N)$-resolution dimension. Assume there is an indecomposable module with finite $add(N)$-resolution. Then there is a short exact sequence as follows, where the maps are minimal right $add(N)$-approximations:

$$0 \to N_1 \to N_0 \to U \to 0,$$

with $N_0, N_1 \in add(N)$ and N_1 being a direct sum of copies of M because of the minimality. Now applying the functor $Hom(M,-)$ to this short exact sequence the map $Ext^1(M,N_1) \to Ext^1(M,N_0)$ has to be injective, since the right map in the short exact sequence is assumed to be a minimal $add(N)$-approximation. Now after removing free summands of the left map in the short exact sequence, we obtain a map $\alpha : N_1 \to N_0'$ between direct sums of copies of M with the property that all components of this map are radical maps. Now $Ext^1(M,-)$ is a functor sending α to an injection. By the previous lemma this is only possible if $Ext^1(M,M)=0$. This contradicts our assumptions and thus there is no module with finite $add(N)$-resolution. □

Corollary 3.4. *Let A be a local Hopf algebra (for example a groupalgebra of a p-group over a field of characteristic p) with a nonprojective indecomposable module M. Then $B := End_A(A \oplus M)$ is a finitistic Auslander algebra of finitistic dimension 2.*

Proof. In [Mar4], theorem 3.8. it was shown that $Ext^1(M,M) \neq 0$ for such M and thus the previous theorem 3.3 applies to give the result.

□

Remark 3.5. No example of a nonprojective module M over a local selfinjective algebra A with $Ext^1(M,M)=0$ seems to be known. See also [Mar4] for more on this.

3.2. **Standardly stratified algebras.** For the basics on standardly stratified algebras, we refer to [Rei]. Recall that a quasi-hereditary algebra is a standardly stratified algebra with finite global dimension. Some of the results in this subsection can also be otained using 3.3, but we give an alternative proof using theorems for standardly stratified algebras. Recall the following result:

Theorem 3.6. *(see* [AHLU]*) Let A be a standardly stratified algebra with n simple modules. Then the finitistic dimension of A is bounded by $2n-2$.*

Remark 3.7. The previous theorem trivially implies that our conjecture in the beginning of this section is true for standardly stratified algebras when choosing $f(n)=2n-2$. We do not know if $f(n)=n$ is also ok for the class of standardly stratified algebras.

Using this theorem, we can give several examples of finitistic Auslander algebras inside the class of standardly stratified algebras. We need the following result, which is the main result of [CheDl]:

Theorem 3.8. *Let A be a local, commutative selfinjective algebra over an algebraically closed field. Let $\mathcal{X} = (A = X(1), X(2), ..., X(n))$ be a sequence of local-colocal modules (meaning that all modules have simple socle and top and therefore can be viewed as ideals of A) with $X(i) \subseteq X(j)$ implying $j \leq i$. Let $X = \bigoplus_{i=1}^{n} X(i)$ and $B = End_A(X)$. Then B is properly stratified with a duality iff the following two conditions are satisfied:*
1. $X(i) \cap X(j)$ is generated by suitable $X(t)$ of $\mathcal{X}$ for any $1 \leq i,j \leq n$
2. $X(j) \cap \sum_{t=j+1}^{n} X(t) = \sum_{t=j+1}^{n} X(j) \cap X(t)$ for any $1 \leq j \leq n$.

Lemma 3.9. *Let A be a commutative selfinjective algebra with an ideal I with $D(I) \cong I$. Then $B := End_A(A \oplus I)$ is a finitistic Auslander algebra with finitistic dimension equal to 2.*

Proof. First note that I being an ideal has simple socle. Now $top(I) \cong top(D(I)) \cong D(soc(I))$ is again simple. Thus 3.8 applies to give that B is standardly stratified. Since B has two simple modules, the finitistic dimension of B is bounded by 2. But since B is an endomorphism ring of a generator-cogenerator, its dominant dimension is at least two. Since the dominant dimension is bounded by the finitsitic dimension for non-selfinjective algebras, B is a finitistic Auslander algebra with finitistic dimension equal to two. □

The next corollary illustrates that being an Auslander-Gorenstein algebra might be extremely rare compared to the more general concept of being a finitistic Auslander algebra. Recall that the eveloping algebra A^e for an arbitrary algebra A is defined as $A^e := A \otimes_K A^{op}$ and an algebra is called m-periodic in case the A^e-module A has Ω-period m.

Theorem 3.10. *Let A be an K-algebra.*

(1) Let A be a commutative selfinjective algebra with enveloping algebra $A^e = A \otimes_K A$. Then $B := End_{A^e}(A^e \oplus A)$ is a finitistic Auslander algebra of finitistic dimension equal to two. It is an Auslander-Gorenstein algebra iff A is a 2-periodic algebra iff $A \cong K[x]/(x^n)$ for some $n \geq 2$. It is never a higher Auslander algebra.

(2) *Let A be a selfinjective local algebra with simple module S. Then $End_A(A \oplus S)$ is a finitistic Auslander algebra with finitistic dimension equal to two. It is an Auslander-Gorenstein algebra iff $A \cong K[x]/(x^n)$ for some $n \geq 2$ and it is a higher Auslander algebra iff $A \cong K[x]/(x^2)$.*

Proof. (1) Note that being commutative selfinjective implies that A is even symmetric and thus $D(A) \cong A$ has A^e-bimodules. Then 3.9 applies to show that B is a finitistic Auslander algebra with finitistic dimension equal to two. Now by (1) of 2.6, B is an Auslander-Gorenstein algebra iff $\tau(A) \cong A$ as A^e-bimodules. Now since A^e is symmetric: $\tau \cong \Omega^2$. And thus B is an Auslander-Gorenstein algebra iff A is 2-periodic iff $A \cong K[x]/(x^n)$ for some $n \geq 2$ by corollary 2.10. of [Sko]. What is left to do is calculate when the algebra B has finite global dimension in case $A \cong K[x]/(x^n)$. But B can have only global dimension equal to the dominant dimension equal to two iff it is a Auslander algebra iff A^e has A^e and A as its only indecomposable modules. This is certainly never the case, since A^e has at least two loops in its quiver and thus is never representation-finite.

(2) In [We] theorem 1.1., it was proven that $End_A(A \oplus S)$ is always standardly stratified in that situation. One has $Ext^1(S, S) \neq 0$ since the algebra is local. Thus the dominant and finitistic dimension are equal to two. Again by [Sko] corollary 2.10. the module S is 2-periodic iff $A \cong K[x]/(x^n)$ and the global dimension is equal to two iff it is finite iff $End_A(A \oplus S)$ is an Auslander algebra iff $A \cong K[x]/(x^2)$.

□

We give another example, where high dominant dimension automatically leads to being a finitistic Auslander algebra and the bound $2n - 2$ for the finitistic dimension of standardly stratified algebras is attained for an arbitrary $n \geq 2$.

Example 3.11. Let A be a representation-finite block of a Schur algebra with n simple modules. Then A has dominant dimension equal to $2n - 2$. This was noted and proven in [ChMar] and [Mar]. By 3.6 it is a finitistic Auslander algebra and even a higher Auslander algebra since it has finite global dimension, being a block of a Schur algebra.

3.3. **Finitistic Auslander algebras in the class of Nakayama algebras.** In this section we present computer experiments with the GAP-package QPA to motivate several conjectures on homological dimensions of Nakayama algebras. For this section we call an algebra an n-Nakayama algebra in case it is a Nakayama algebra with n simple modules and we always assume furthermore that our Nakayama algebras are not selfinjective in this section. Recall that an n-Nakayama algebra is uniquely determined by its Kupisch series $[c_0, c_1, ..., c_{n-1}]$, where c_i is the length of the indecomposable projective module $e_i A$. We assume here that all Nakayama algebras are given by quiver and relations, since the calculations of homological dimension for those algebras is independent of Morita equivalence and field extensions. We are mainly interested in n-Nakayama algebras having dominant dimension at least n and thus their quiver is a circle, since n-Nakayama algebras with a line as a quiver have their global, and thus dominant, dimension bounded above by $n - 1$. Thus in the following we assume that Nakayama algebras have a cyclic quiver, where the indices are number from 0 to $n - 1$, when the algebra has n simples. In this case the ordering of the simples can be shifted such that $c_{n-1} = c_0 + 1$ and we assume this condition in the following without loss of generality. There are infinitely many Nakayama algebras with n simple modules up to isomorphism and thus we restrict to difference classes of Nakayama algebras. Here we say that two Nakayama algebras with Kupisch series $[c_0, c_1, ..., c_{n-1}]$ and $[e_0, e_1, ..., e_{m-1}]$ are in the same difference class in case $n = m$ and $c_i = e_i \mod n$ for all $i = 0, 1, .., n - 1$. Then there are only finitely many difference classes and we always choose a representative in this difference class of smallest possible Loewy length. We then just write the Kupisch series of this representative to show with which difference class we deal. Two Nakayama algebras with Kupisch series in the same difference class have the same dominant and finitistic dimension, see [Mar4] theorem 1.1.4. for the case of dominant dimension, while a proof for the finitistic dimension case works with the same arguments as there. Motivated by this, we just count representatives of the difference classes with the smallest Loewy length inside the difference class. This is also motivated by the fact that Nakayama algebras with n simples and Loewy length larger than or equal to $2n$ have infinite global dimension by a result of Gustafson, see [Gus]. We remark that the global or Gorenstein dimension may differ for two Nakayama algebras in the same difference class.

The following conjectures are verified for $n \leq 13$ with QPA and the relevant sequences have been found via the sequence database http://oeis.org/.

Conjecture. *(1) An n-Nakayama algebra with dominant dimension at least n has Kupisch series of the form $[a, a, a, ..., a, a+1, a+1, ..., a+1]$ for some $a \geq 2$.*

(2) An n-Nakayama algebra with dominant dimension at least n has also finitistic dimension equal to n.

(3) The number of representatives of n-Nakayama algebras, which are finitistic Auslander algebras with dominant dimension at least n is equal to $\frac{2n^2-1+(-1)^n}{8}$ for $n \geq 2$ (the sequence starts as follows: 1,2,4,6,9,12,16,20,25,30,36,42,49,.... and also counts other algebraic configurations such as the order dimension of the (strong) Bruhat order on the Coxeter group A_{n-1} according to the OEIS database).

(4) The number of representatives of n-Nakayama algebras, which are Auslander-Gorenstein algebras with dominant dimension at least n is equal to a_{n-2}, when $a_n := (\frac{1}{16})(2n^2+18n+15+(2n+1)(-1)^n)$ for $n \geq 4$ (the sequence starts as follows: 1, 2, 4, 5, 8, 9, 13, 14, 19, 20, 26, 27, 34, ... and the description is "Write two numbers, skip one, write two, skip two, write two, skip three... and so on.").

(5) For every m with $1 \leq m \leq n-1$, there is exactly one n-Nakayama algebra which is a higher Auslander algebra with dominant dimension $n+m-1$.

(6) A n-Nakayama algebra with dominant dimension at least n is an Auslander-Gorenstein algebra of infinite global dimension iff its finitistic dominant dimension is not equal to the dominant dimension.

We list all n-Nakayama algebras with dominant dimension at least n and their homological dimensions for $n \leq 13$. The first entry is the Kupisch series, the second entry the dominant dimension, the third entry the global dimension, the fourth entry the Gorenstein dimension, the fifth entry the finitistic dimension and the last entry is the finitistic dominant dimension. "false" here means that the relevant dimension is infinite. This verifies the conjectures for $n \leq 13$. The program just filtered the Kupisch series of every difference class with n simples for those with dominant dimension at least n and then calculated the relevant homological dimensions using QPA.

3.3.1. *n=2:* [[[2, 3], 2, 2, 2, 2, 2]]

3.3.2. *n=3:* [[[2, 2, 3], 3, 3, 3, 3, 3],
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3.3.3. *n=4:* [[[2, 2, 2, 3], 4, 4, 4, 4, 4],
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3.3.4. *n=5:* [[[2, 2, 2, 2, 3], 5, 5, 5, 5, 5],
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3.3.5. *n=6:* [[[2, 2, 2, 2, 2, 3], 6, 6, 6, 6, 6],
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3.3.6. *n=7:* [[[2, 2, 2, 2, 2, 2, 3], 7, 7, 7, 7, 7],
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3.3.7. *n=8:* [[[2, 2, 2, 2, 2, 2, 2, 3], 8, 8, 8, 8, 8],
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3.3.8. *n=9:* [[[2, 2, 2, 2, 2, 2, 2, 2, 3], 9, 9, 9, 9, 9],
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3.3.9. *n=10:* [[[2, 2, 2, 2, 2, 2, 2, 2, 2, 3], 10, 10, 10, 10, 10],
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3.3.10. *n=11:* [[[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3], 11, 11, 11, 11, 11],
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3.3.11. *n=12:* [[[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3], 12, 12, 12, 12, 12],
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3.3.12. *n=13:* [[[2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3], 13, 13, 13, 13, 13],
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Example 3.12. Many of the above n-Nakayama algebras with infinite Gorenstein dimension and dominant dimension at least n are CM-free, but this is not in general true: The smallest counterexample is the algebra A with Kupisch series $[6,6,6,6,7,7,7]$, which has dominant dimension equal to $d = 7$. Note that in a Nakayama algebra every indecomposable Gorenstein projective non-projective module is periodic, since there are only finitely many indecomposable Gorenstein projective modules and with M also $\Omega^1(M)$ is Gorenstein projective. Here are the periodic modules of A with their period:

(1) $S_5 = e_5A/e_5J^1$ with period 2.
(2) e_4A/e_4J^2 with period 2.
(3) $S_4 = e_4A/e_4J^1$ with period 2.
(4) e_6A/e_6J^5 with period 2.
(5) e_6A/e_6J^6 with period 2.
(6) e_5A/e_5J^6 with period 2.

Since a module in a representation-finite algebra A is Gorenstein projective iff $Ext^i(M,A) = 0$ for all $i \geq 1$, the above modules are Gorenstein-projective iff $Ext^i(M,A) = 0$ for $i = 1,2$ since they are 2-periodic. Only the modules S_4 and $\Omega^1(S_4) = e_5A/e_5J^6$ have $Ext^i(M,A) = 0$ for $i = 1,2$ and thus are the only non-projective Gorenstein projective modules. Note also that S_1 is in $Gp(A)_\infty$ but has only dominant dimension 4. Thus there is in general no equality in $Dom_d \subseteq Gp(A) \cup Gp(A)_\infty$ in 2.4.

3.4. **Questions.** We end this article with several questions:

(1) Is there a bound for the global dimension of non-semisimple higher Auslander algebras with n simple modules? To give a more concrete questions: Is the global dimension of higher Auslander algebras with n simple modules bounded by $2n-2$? In fact we know no example where the global dimension is higher than $2n-2$, where this number is attained for example for representation-finite blocks of Schur algebras. Of course one can ask similar questions for the class of Auslander-Gorenstein or finitistic Auslander algebras, by replacing global dimension by Gorenstein or finitistic dimension.
(2) Can one classify the quasi-hereditary (or even standardly stratified) algebras with n simple modules and having dominant dimension equal to $2n-2$?
(3) Can one generalise other theorem of [IyaSol] from Auslander-Gorenstein algebras to the more general finitistic Auslander algebras?
(4) Can one describe the category of Gorenstein projective modules for a general finitistic Auslander algebra?

(5) Is there an example of an algebra A with positive dominant dimension d such that $Dom_d \cap proj_{<\infty}$ is contravariantly finite but $proj_{<\infty}$ is not?
(6) Is there an example of an algebra A with positive dominant dimension d such that $Dom_d \cap proj_{<\infty}$ is not contravariantly finite?
(7) It was shown in [Hap], that the finitistic dimension of algebras with two simple modules can get arbitrary large. Can this also happen if we restrict to algebras with two simples and dominant dimension at least two?

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