Parabolic Catalan numbers count efficient inputs for Gessel-Viennot flagged Schur function determinant

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February 13, 2017

Abstract

Let λ be a partition with no more than n parts. Let β be a weakly increasing n-tuple with entries from $\{1, ..., n\}$. The flagged Schur function in the variables $x_1, ..., x_n$ that is indexed by λ and β has been defined to be the sum of the content weight monomials for the semistandard Young tableaux of shape λ whose values are row-wise bounded by the entries of β. Gessel and Viennot gave a determinant expression for the flagged Schur function indexed by λ and β; this could be done since the pair (λ, β) satisfied their "nonpermutable" condition for the sequence of terminals of an n-tuple of certain lattice paths that they used to model the tableaux. We generalize the notion of flagged Schur function by dropping the requirement that β be weakly increasing. Then we give a condition on the entries of λ and β for the pair (λ, β) to be nonpermutable that is both necessary and sufficient. When the parts of λ are not distinct there will be multiple row bound n-tuples that will produce the same polynomial via the sum of tableau weights construction on λ. We accordingly group the bounding n-tuples into equivalence classes and identify the most efficient n-tuple in each class for the determinant computation. We have recently shown that many other sets of objects that are indexed by n and λ are enumerated by the number of these efficient n-tuples. It is noted that the $GL(n)$ Demazure characters (key polynomials) indexed by 312-avoiding permutations can also be expressed with these determinants.

Keywords. flagged Schur function, Gessel-Viennot method, sign reversing involution, nonintersecting lattice paths, Jacobi-Trudi identity

MSC Codes. 05E05, 05A19

1 Introduction

No particular background is needed to read this largely self-contained paper. Several prior results that are needed in Sections 2 and 8 and for Corollaries 5.3 and 5.4 were obtained in the predecessor paper [PW].

Fix $n \geq 1$ throughout the paper. Also fix a partition λ that has n nonnegative parts; this is a list of n weakly decreasing nonnegative integers. Flagged Schur functions are polynomials in $x_1, x_2, ..., x_n$ that were introduced by Lascoux and Schützenberger in 1982 as they studied Schubert polynomials. Given an n-tuple β such that $1 \leq \beta_1 \leq \beta_2 \leq ... \leq \beta_n \leq n$, the flagged Schur function indexed by λ and β is defined to be the sum of the content weight monomial $x^{\Theta(T)}$ over the semistandard tableaux T on the shape of λ whose values are row-wise bounded by the respective entries of β. Sometimes $\beta_i \geq i$ for $1 \leq i \leq n$ is also required to ensure nonvanishing. However, in this paper the notation $s_\lambda(\beta; x)$ will more generally denote this sum when β is only required to satisfy $\beta_i \geq i$ for $1 \leq i \leq n$.

Ira Gessel and X.G. Viennot were able [GV] to express a flagged Schur function with a determinant by modelling its tableaux with nonintersecting n-tuples of lattice paths: Their initial set-up fixed a sequence of n lattice points to serve as sources for the respective paths, to which were assigned sinks from a set of n fixed lattice points in any of the $n!$ possible ways. These "terminal" lattice points were specified in terms of the entries of λ and β. Initially $\beta_i \leq \beta_{i+1}$ was not required. Most of the resulting n-tuples of lattice paths contained intersections, and the desired tableaux corresponded only to the nonintersecting n-tuples of lattice paths for which the sinks were assigned from the set of terminals in their "native" order. The terms in the signed sum expansion of the proposed determinant gave the weights that they assigned to the n-tuples of paths. Then they introduced a sign reversing involution that paired up the intersecting n-tuples of paths so that the weights for these cancelled each other out from the expansion, leaving only the signed sum of the weights for the nonintersecting n-tuples of paths. For this method to give the tableau weight sum $s_\lambda(\beta; x)$, they needed to require that the set of terminals specified by the pair (λ, β) satisfied their "nonpermutable" property: This required that any n-tuple of lattice paths that had a sequence of sinks coming from a nontrivially permuted assignment of the terminals had to contain an intersection. As Stanley parenthetically noted in his presentation of their work in Theorem 2.7.1 of [St1], for any λ it can be seen that requiring $\beta_1 \leq \beta_2 \leq ... \leq \beta_n$ will guarantee that (λ, β) is nonpermutable. Directly in terms of the entries of λ and β, our main result gives a condition for a pair (λ, β) to be nonpermutable that is necessary as well as being sufficient. Although the references [GV], [St1], and [PS] each provide determinants for skew flagged Schur functions, we limit our considerations to the general sums $s_\lambda(\beta; x)$ on nonskew shapes.

Demazure characters were introduced by Demazure in 1974 when he studied singularities of

Schubert varieties. Coincidences between the Demazure characters for $GL(n)$ (key polynomials) and flagged Schur functions were studied by Reiner and Shimozono [RS] and then by Postnikov and Stanley [PS]. When the parts of λ are not distinct, there are multiple row bound n-tuples $\beta, \beta', ...$ that will produce the same polynomial $s_\lambda(\beta; x)$ via the sum of tableau weights construction on the shape of λ. The predecessor paper to this paper sharpened, deepened, and extended the results of [RS] and [PS]. Much machinery was introduced and several special kinds of n-tuples were defined. The foremost kinds were the "λ-312-avoiding permutations" and the "gapless λ-tuples". The crucial information for an n-tuple β was distilled into its "critical list", as its "λ-core" $\Delta_\lambda(\beta)$ was being computed.

It turns out that the machinery and notions that were introduced in [PW] for the purposes of that paper are surprisingly well-suited to solving the problem of characterizing the nonpermutable pairs (λ, β) that was implicitly raised by Stanley's Theorem 2.7.1 parenthetical remark. In addition to re-using the notion of gapless λ-tuple and the closely related notion of "gapless core λ-tuple", here we also need to extend the λ-ceiling flag map Ξ_λ of [PW] so that we can introduce a new condition that requires $\beta \leq \Xi_\lambda(\beta)$. Our main result, Theorem 5.1, presents our characterization of the nonpermutable pairs (λ, β). Its two halves are proved with Proposition 6.3 and 7.2. Corollary 5.2 gives the determinant expression for $s_\lambda(\beta; x)$ when β satisfies the characterization with respect to λ. Corollary 5.3 indicates how results of [PW] can be used to extend the realm of Corollary 5.2. Corollary 5.4 describes how Corollary 5.2 can be used to give a determinant expression for certain $GL(n)$ Demazure characters; this improves upon Corollary 14.6 of [PS].

In the last section, as in [PW], we define two row bound n-tuples β, β' to be equivalent if the sets of tableaux on the shape λ that satisfy these bounds are the same. Proposition 8.2 describes the equivalence classes of this relation within the set of row bound n-tuples that meet the criteria required to use the determinant expression. Within an equivalence class, one can seek the n-tuple for which the total number of monomials appearing in the corresponding determinant is as small as possible. Proposition 8.3 identifies these "maximum efficiency" n-tuples as being the gapless λ-tuples that appeared in Corollary 5.3. Corollary 8.4 then notes that the number of gapless λ-tuples was shown in [PW] to be the number of λ-312-avoiding permutations; there this number was taken to be the definition of the "parabolic Catalan number" indexed by n and λ.

When one sets all $\beta_i := n$, no special row bounds are imposed upon the tableaux and the resulting polynomial is the ordinary Schur function $s_\lambda(x)$. The Gessel-Viennot method made the Jacobi-Trudi determinant expression of Theorem 7.16.1 of [St2] for $s_\lambda(x)$ more efficient by reducing the number of variables that appeared in most of its entries. When the parts of λ are not distinct, Proposition 8.3 says that Corollary 5.3 provides a determinant for $s_\lambda(x)$ that is even more efficient in this regard.

One of the central themes of the predecessor paper [PW] is continued into this paper. Given a set $R \subseteq \{1, 2, ..., n-1\}$, an "$R$-tuple" is an n-tuple with entries from $\{1, 2, ..., n\}$ that is equipped with "dividers" between some of its entries. In these two papers the study of any one of the interrelated phenomena concerning sets of tableaux on the shape λ begins with the determination of the set $R_\lambda \subseteq \{1, 2, ..., n-1\}$ of the lengths of the columns in λ that are less than n. Much of the machinery needed to study these phenomena is formulated in terms of R_λ-tuples without reference to any other aspects of λ: Five preliminary sections of [PW] take place in the world of R-tuples, before shapes and tableaux are introduced. Continuing a notation convention of [PW], after λ has been introduced we replace 'R_λ' in prefixes and subscripts with 'λ'. This reduces clutter while explicitly retaining the dependence upon λ, which setting $R := R_\lambda$ would lose.

2 Definitions for n-tuples

Let i and k be nonnegative integers. Define $(i, k] := \{i+1, i+2, ..., k\}$ and $[k] := \{1, 2, ..., k\}$. Except for ζ, lower case Greek letters indicate n-tuples of non-negative integers; their entries are denoted with the same letter. An *nn-tuple* ν consists of n *entries* $\nu_i \in [n]$ indexed by *indices* $i \in [1, n]$, which together form n *pairs* (i, ν_i). Let $P(n)$ denote the poset of nn-tuples ordered by entrywise comparison. Fix an nn-tuple ν. A *subsequence* of ν is a sequence of the form $(\nu_i, \nu_{i+1}, ..., \nu_j)$ for some $i, j \in [n]$. A *staircase of ν within a subinterval* $[i, j]$ for some $i, j \in [n]$ is a maximal subsequence of $(\nu_i, \nu_{i+1}, ..., \nu_j)$ whose entries increase by 1. A *plateau* in ν is a maximal constant nonempty subsequence of ν. An nn-tuple φ is a *flag* if $\varphi_1 \leq \ldots \leq \varphi_n$. An *upper tuple* is an nn-tuple β such that $\beta_i \geq i$ for $i \in [n]$.

Fix $R \subseteq [n-1]$. Denote the elements of R by $q_1 < \ldots < q_r$ for some $r \geq 0$. Set $q_0 := 0$ and $q_{r+1} := n$. We use the q_h for $h \in [r+1]$ to specify the locations of $r+1$ "dividers" within nn-tuples: Let ν be an nn-tuple. On the graph of ν in the first quadrant draw vertical lines at $x = q_h + \epsilon$ for $h \in [r+1]$ and some small $\epsilon > 0$. These $r+1$ lines indicate the right ends of the $r+1$ *carrels* $(q_{h-1}, q_h]$ *of* ν for $h \in [r+1]$. An *R-tuple* is an nn-tuple that has been equipped with these $r+1$ dividers. Fix an R-tuple ν; we portray it by $(\nu_1, ..., \nu_{q_1}; \nu_{q_1+1}, ..., \nu_{q_2}; ...; \nu_{q_r+1}, ..., \nu_n)$. Let $U_R(n)$ denote the subposet of $P(n)$ consisting of upper R-tuples. Let $UF_R(n)$ denote the subposet of $U_R(n)$ consisting of upper flags. Fix $h \in [r+1]$. The h^{th} carrel has $p_h := q_h - q_{h-1}$ indices. An *R-increasing tuple* is an R-tuple α such that $\alpha_{q_{h-1}+1} < ... < \alpha_{q_h}$ for $h \in [r+1]$. Let $UI_R(n)$ denote the subset of $U_R(n)$ consisting of R-increasing upper tuples.

We distill the crucial information from an upper R-tuple into a skeletal substructure called its "critical list", and at the same time define two functions from $U_R(n)$ to $U_R(n)$. Fix $\beta \in U_R(n)$. To launch a running example, take $n := 9, R := \{3, 8\}$, and $\beta := (2, 7, 5; 8, 6, 6, 9, 9; 9)$. We will be constructing the images δ and ξ of β under *R-core* and *R-platform* maps Δ_R and Ξ_R. Fix $h \in [r+1]$. Working within the h^{th} carrel $(q_{h-1}, q_h]$ from the right we recursively find for $u = 1, 2, ...$: At $u = 1$

the *rightmost critical pair of* β *in the* h^{th} *carrel* is (q_h, β_{q_h}). Set $x_1 := q_h$. Recursively attempt to increase u by 1: If it exists, the *next critical pair to the left* is (x_u, β_{x_u}), where $q_{h-1} < x_u < x_{u-1}$ is maximal such that $\beta_{x_{u-1}} - \beta_{x_u} > x_{u-1} - x_u$. For $x_u < i \leq x_{u-1}$, write $x_{u-1} =: x$ and set $\delta_i := \beta_x - (x - i)$ and $\xi_i := \beta_x$. Otherwise, let $f_h \geq 1$ be the last value of u attained. For $q_{h-1} < i \leq x_{f_h}$, write $x_{f_h} =: x$ and again set $\delta_i := \beta_x - (x-i)$ and $\xi_i := \beta_x$. The *set of critical pairs of* β *for the* h^{th} *carrel* is $\{(x_u, \beta_{x_u}) : u \in [f_h]\} =: \mathcal{C}_h$. Equivalently, here f_h is maximal such that there exists indices $x_1, x_2, ..., x_{f_h}$ such that $q_{h-1} < x_{f_h} < ... < x_1 = q_h$ and $\beta_{x_{u-1}} - \beta_{x_u} > x_{u-1} - x_u$ for $u \in (1, f_h]$. The *R-critical list for* β is the sequence $(\mathcal{C}_1, ..., \mathcal{C}_{r+1}) =: \mathcal{C}$ of its $r+1$ sets of critical pairs. In our example $\mathcal{C} = (\{(1,2),(3,5)\};\{(6,6),(8,9)\};\{(9,9)\})$ and $\delta = (2,4,5;4,5,6,8,9;9)$ and $\xi = (2,5,5;6,6,6,9,9;9)$. It can be seen that the R-core $\Delta_R(\beta) = \delta$ of β and the R-platform $\Xi_R(\beta) = \xi$ of β have the same critical list as β. It can also be seen that $\Delta_R(\beta) \leq \beta$ and that $\Delta_R(\alpha) = \alpha$ for $\alpha \in UI_R(n)$. If (x, y_x) is a critical pair, we call x a *critical index* and y_x a *critical entry.* We say that an R-critical list is a *flag R-critical list* if whenever $h \in [r]$ we have $y_{q_h} \leq y_k$, where $k := x_{f_{h+1}}$. The example critical list is a flag critical list. If $\beta \in UF_R(n)$, then its R-critical list is a flag R-critical list.

A *gapless core R-tuple* is an upper R-tuple η whose critical list is a flag critical list. Let $UGC_R(n)$ denote the set of gapless core R-tuples. The example β above is a gapless core R-tuple. A *gapless R-tuple* is an R-increasing upper tuple γ whose critical list is a flag critical list. Let $UG_R(n) \subseteq UI_R(n)$ denote the set of gapless R-tuples. The example δ above is a gapless R-tuple. Originally a gapless R-tuple was defined in Section 3 of [PW] to be an R-increasing upper tuple γ such that whenever there exists $h \in [r]$ with $\gamma_{q_h} > \gamma_{q_h+1}$, then $\gamma_{q_h} - \gamma_{q_h+1} + 1 =: s \leq p_{h+1}$ and the first s entries of the $(h+1)^{st}$ carrel $(q_h, q_{h+1}]$ are $\gamma_{q_h} - s + 1, \gamma_{q_h} - s + 2, ..., \gamma_{q_h}$. Originally a gapless core R-tuple was defined in Section 3 of [PW] to be an upper R-tuple η whose R-core $\Delta_R(\eta)$ is a gapless R-tuple. Those original definitions were shown there to be equivalent to these definitions in Proposition 4.2. An upper R-tuple β is *bounded by its platform* if $\beta \leq \Xi_R(\beta)$. Let $UBP_R(n)$ denote the set of such upper R-tuples. The example β above is not bounded by its platform. Clearly $UG_R(n) \subseteq UGC_R(n)$ and $UF_R(n) \subseteq UGC_R(n)$. From the definition of Ξ_R, it is clear that $UF_R(n) \subseteq UBP_R(n)$ and $UI_R(n) \subseteq UBP_R(n)$. Since $UG_R(n) \subseteq UI_R(n)$ by definition, we have $UG_R(n) \subseteq UBP_R(n)$.

We illustrate some recent definitions. First consider an R-increasing upper tuple $\alpha \in UI_R(n)$: Each carrel subsequence of α is a concatenation of the staircases within the carrel in which the largest entries are the critical entries for the carrel. Now consider the definition of a gapless R-tuple, which begins by considering a $\gamma \in UI_R(n)$: This definition is equivalent to requiring for all $h \in [r]$ that if $\gamma_{q_h} > \gamma_{q_h+1}$, then the leftmost staircase within the $(h+1)^{st}$ carrel must contain an entry γ_{q_h}.

An *R-ceiling flag* ξ is an upper flag that is a concatenation of plateaus whose rightmost pairs are the R-critical pairs of ξ. Let $UCeil_R(n)$ denote the set of R-ceiling flags. It can be seen that the restriction of the R-platform map from $U_R(n)$ to $UG_R(n)$ is the R-ceiling map $\Xi_R : UG_R(n) \to UCeil_R(n)$ defined near the end of Section 5 of [PW]. So by that Proposition 5.4(ii) this restriction of Ξ_R is a bijection from $UG_R(n)$ to $UCeil_R(n)$ with inverse Δ_R, and for $\gamma \in UG_R(n)$ the upper flag $\xi := \Xi_R(\gamma)$ is the unique R-ceiling flag that has the same flag R-critical list as γ.

3 Definitions of shapes, tableaux, polynomials

A *partition* is an n-tuple $\lambda \in \mathbb{Z}^n$ such that $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$. The *shape* of λ, also denoted λ, consists of n left justified rows with $\lambda_1, \ldots, \lambda_n$ boxes. We denote its column lengths by $\zeta_1 \geq \ldots \geq \zeta_{\lambda_1}$. Since the columns were more important than the rows in [PW], the boxes of λ are transpose-indexed by pairs (j, i) such that $1 \leq j \leq \lambda_1$ and $1 \leq i \leq \zeta_j$. Define $R_\lambda \subseteq [n-1]$ to be the set of distinct column lengths of λ that are less than n. Using the language of Section 2 with $R := R_\lambda$, note that for $h \in [r+1]$ one has $\lambda_i = \lambda_{i'}$ for $i, i' \in (q_{h-1}, q_h]$. For $h \in [r+1]$ the coordinates of the p_h boxes in the h^{th} *cliff* form the set $\{(\lambda_i, i) : i \in (q_{h-1}, q_h]\}$. We will replace '$R_\lambda$' by '$\lambda$' in subscripts and in prefixes when using concepts from Section 2 via $R := R_\lambda$.

A *(semistandard) tableau of shape* λ is a filling of λ with values from $[n]$ that strictly increase from north to south and weakly increase from west to east. Let $\mathcal{T}_\lambda$ denote the set of tableaux of shape λ. Fix $T \in \mathcal{T}_\lambda$. For $j \in [\lambda_1]$, we denote the one column "subtableau" on the boxes in the j^{th} column by T_j. Here for $i \in [\zeta_j]$ the tableau value in the i^{th} row is denoted $T_j(i)$. To define the *content* $\Theta(T) := \theta$ *of* T, for $i \in [n]$ take θ_i to be the number of values in T equal to i. Let $x_1, \ldots, x_n$ be indeterminants. The *monomial* $x^{\Theta(T)}$ of T is $x_1^{\theta_1} \ldots x_n^{\theta_n}$, where θ is the content $\Theta(T)$.

Let β be a λ-tuple. We define the *row bound set of tableaux* to be $\mathcal{S}_\lambda(\beta) := \{T \in \mathcal{T}_\lambda : T_j(i) \leq \beta_i$ for $j \in [0, \lambda_1]$ and $i \in [\zeta_j]\}$. As in Section 12 of [PW], it can be seen that $\mathcal{S}_\lambda(\beta)$ is nonempty if and only if $\beta \in U_\lambda(n)$. Fix $\beta \in U_\lambda(n)$. As noted in Section 12 of [PW], it can be seen that $\mathcal{S}_\lambda(\beta)$ has a unique maximal element. In [PW] we introduced the *row bound sum* $s_\lambda(\beta; x) := \sum x^{\Theta(T)}$, sum over $T \in \mathcal{S}_\lambda(\beta)$. To connect to the literature, for $\varphi \in UF_\lambda(n)$ we also give the names *flag bound set* and *flag Schur polynomial* to $\mathcal{S}_\lambda(\varphi)$ and the flagged Schur function $s_\lambda(\varphi; x)$ respectively. As in [PW], for $\eta \in UGC_\lambda(n)$ it is also useful to give the names *gapless core bound set* and *gapless core Schur polynomial* to $\mathcal{S}_\lambda(\eta)$ and $s_\lambda(\eta; x)$ respectively.

Proposition 12.1 of [PW] stated that the collection of sets $\mathcal{S}_\lambda(\varphi)$ and of $\mathcal{S}_\lambda(\eta)$ are the same. Thus the gapless core Schur polynomials are already available as flag Schur polynomials. However, the additional indexing λ-tuples from $UGC_\lambda(n) \backslash UF_\lambda(n)$ are useful. The following theme from [PW] will be continued: Here we will prove that the row bound sums $s_\lambda(\beta; x)$ for $\beta \in U_\lambda(n) \backslash UGC_\lambda(n)$ are not "good" for the consideration at hand.

4 Lattice paths and Gessel-Viennot determinant

We introduce n-tuples of weighted lattice paths to model the tableaux in the row bound tableau set $\mathcal{S}_\lambda(\beta)$. To obtain a close visual correspondence we first flip the x-y plane containing the paths vertically so that its first quadrant is to the lower right (southeast) of the origin on the page. Re-use our indexing of boxes in shapes with transposed matrix coordinates to coordinatize the points in this first quadrant of $\mathbb{Z} \times \mathbb{Z}$: Let $l \geq j \geq 0$ and $k \geq i \geq 1$. The lattice point (j, i) is j units to the east of $(0, 0)$ and i units to the south of $(0, 0)$. For $j \geq 1$, the directed line segment from $(j-1, i)$ to (j, i) is an *easterly step of depth i*. A *(lattice) path with source (j, i) and sink (l, k)* is a connected set incident to (j, i) and (l, k) that is the union of $l-j$ easterly steps and $k-i$ *southerly steps*. The notation $... \rightarrow (j, i) \downarrow (j, k) \rightarrow (l, k) \downarrow ...$ indicates that an eastbound path arrives at (j, i), turns right and proceeds south to (j, k), turns left and proceeds east to (l, k), and then turns right and proceeds south. An *n-path* is an n-tuple $(\Lambda_1, ..., \Lambda_n) =: \Lambda$ of paths such that the component path Λ_m has source $(n-m, m)$ for $m \in [n]$.

Let $\beta \in P(n)$. The n points $(\lambda_1+n-1, \beta_1), (\lambda_2+n-2, \beta_2), ..., (\lambda_n, \beta_n)$ are *terminals* and (λ, β) is a *terminal pair*. This "strictification" of λ ensures that the longitudes of the terminals are distinct. Initially our n-paths $(\Lambda_1, ..., \Lambda_n)$ will use the terminals $(\lambda_1+n-1, \beta_1), (\lambda_2+n-2, \beta_2), ..., (\lambda_n, \beta_n)$ in this order as sinks for their respective components. Given such an n-path Λ, as in the proof of Theorem 7.16.1 of [St2] we attempt to create a corresponding tableau $T \in \mathcal{S}_\lambda(\beta)$. For each $m \in [n]$ we record the weakly increasing depths of the successive easterly steps in the path Λ_m from left to right in the boxes of the m^{th} row of the shape λ: Here the easterly step in Λ_m from $(n-m+j-1, p)$ to $(n-m+j, p)$ is recorded as the value p in the box (j, m) for T. The last value in the m^{th} row cannot exceed β_m. It can be seen that these values strictly increase down each column of λ if and only if there are no intersections among the Λ_m for $m \in [n]$. Let $\mathcal{LD}_\lambda(\beta)$ denote the set of such *disjoint* n-paths. There is at least one such disjoint n-path if and only if β is upper: To confirm this, with the correspondence above re-use the observations made near the beginning of Section 12 of [PW] that addressed the questions of when the set $\mathcal{S}_\lambda(\beta)$ is empty and nonempty. When β is upper, it can be seen that the recording process is bijective to the set $\mathcal{S}_\lambda(\beta)$. Since it will be seen that the cliffs of λ play a crucial role, we now determine R_λ and regard β as being a λ-tuple. Summarizing:

Fact 4.1. *We have $\mathcal{LD}_\lambda(\beta) \neq \emptyset$ if and only if $\beta \in U_\lambda(n)$. For $\beta \in U_\lambda(n)$, the recording process is a bijection from the set of disjoint n-paths $\mathcal{LD}_\lambda(\beta)$ to the row bound tableau set $\mathcal{S}_\lambda(\beta)$.*

Fix $\beta \in U_\lambda(n)$. To obtain the determinant expression for $s_\lambda(\beta; x)$ we will need to consider more general n-paths and introduce weights. Let Λ be an n-path with any sinks. Assigning a weight monomial $x^{\Theta(\Lambda)}$ to Λ in the following fashion emulates our assignment of the weight $x^{\Theta(T)}$

to a tableau $T \in \mathcal{T}_\lambda$ when $\Lambda \in \mathcal{LD}_\lambda(\beta)$, and it also extends the weight rule to all n-paths. For $m \in [n]$ assign the weight x_i to each easterly step of depth i in the path Λ_m, and then multiply these weights over its easterly steps. Multiply the weights of the n component paths to produce a monomial we denote $x^{\Theta(\Lambda)}$. When the sinks of Λ are the terminals from (λ, β) in their usual order, it can be seen that the multivariate generating function $\sum_{\Lambda \in \mathcal{LD}_\lambda(\beta)} x^{\Theta(\Lambda)}$ is our row bound sum $s_\lambda(\beta; x)$. Let $j \geq 0, i \geq 1, l \geq 0, k \geq 1$, and set $u := l - j$. If we sum the weights that are assigned to just one path as it varies over all paths from (j, i) to (l, k), we produce the *complete homogeneous symmetric function* $h_u(i, k; x)$ in the variables $x_i, x_{i+1}, ..., x_k$: Here $h_u(i, k; x) := 0$ for $u < 0$, and otherwise $h_u(i, k; x) := \sum x_{t_1} \cdots x_{t_u}$, sum over $i \leq t_1 \leq ... \leq t_u \leq k$.

We next consider n-paths that use the same terminals, but in a permuted order, for their list of sinks. Let π be a permutation of $[n]$. Let $\pi.(\lambda, \beta)$ denote the list of terminals $(\lambda_{\pi_1} + n - \pi_1, \beta_{\pi_1})$, $(\lambda_{\pi_2} + n - \pi_2, \beta_{\pi_2}), ..., (\lambda_{\pi_n} + n - \pi_n, \beta_{\pi_n})$. Let $\mathcal{LD}_\lambda(\beta; \pi)$ denote the set of disjoint n-paths $(\Lambda_1, ..., \Lambda_n)$ with respective sinks $\pi.(\lambda, \beta)$. The terminal pair (λ, β) is *nonpermutable* [GV] if $\mathcal{LD}_\lambda(\beta; \pi) = \emptyset$ when $\pi \neq (1, 2, ..., n)$.

Here is our non-skew version of Theorem 2.7.1 of [St1]; as in Theorem 7.16.1 of [St2] we have replaced the disjoint n-paths with the corresponding tableaux:

Proposition 4.2. *Let $\beta \in U_\lambda(n)$. If the terminal pair (λ, β) is nonpermutable, then the row bound sum $s_\lambda(\beta; x)$ is given by the $n \times n$ determinant $|h_{\lambda_j - j + i}(i, \beta_j; x)|$.*

To produce this expression with Theorem 2.7.1 of [St1], use the remark above that expressed $s_\lambda(\beta; x)$ as the $\mathcal{LD}_\lambda(\beta)$ generating function and note that $(\lambda_j + n - j) - (n - i) = \lambda_j - j + i$. Theorem 2.7.1 was proved with a signed involution pairing cancellation argument, as in [GV].

5 Main results

Our main result combines the forthcoming Propositions 6.3 and 7.2:

Theorem 5.1. *Let λ be a partition and let β be an upper λ-tuple. The terminal pair (λ, β) is nonpermutable if and only if β is a gapless core λ-tuple that is bounded by its platform.*

So under these circumstances we can employ the Gessel-Viennot method, as noted in Proposition 4.2:

Corollary 5.2. *Let $\beta \in U_\lambda(n)$. If $\beta \in UGC_\lambda \cap UBP_\lambda(n)$ then $s_\lambda(\beta; x) = |h_{\lambda_j - j + i}(i, \beta_j; x)|$.*

Although this determinant is not guaranteed to "work" when $\beta \in UGC_\lambda(n) \backslash UBP_\lambda(n)$, given our quotes in Section 8 of facts from [PW] the polynomial $s_\lambda(\beta; x)$ for such a β can be computed with the determinant using $\delta := \Delta_\lambda(\beta)$ instead of β itself. An example of the failure of the determinant for such a β is given before Lemma 6.1.

Corollary 5.3. *Let $\beta \in U_\lambda(n)$. Set $\delta := \Delta_\lambda(\beta)$. If $\beta \in UGC_\lambda(n)$ then $s_\lambda(\beta;x) = |h_{\lambda_j-j+i}(i,\delta_j;x)|$.*

At the end of Section 14 of [PW] we promised to give a determinant expression for certain $GL(n)$ Demazure characters (key polynomials) here. General Demazure characters $d_\lambda(\pi;x)$ for $GL(n)$ can be recursively defined with divided differences as noted in Section 1 of [PW] or defined as a sum of $x^{\Theta(T)}$ over a certain set of semistandard tableaux as in Section 14 of [PW]. Given that $UG_\lambda(n) \subseteq UBP_\lambda(n)$, the next statement is implied by Corollary 5.2 and Theorem 14.2(ii) of [PW]. For this result that theorem gives $d_\lambda(\pi;x) = s_\lambda(\gamma;x)$. Consult Section 3 of [PW] for the definitions of the λ-permutations and the map Ψ_λ.

Corollary 5.4. *Let λ be a partition and let π be a λ-permutation. If π is λ-312-avoiding, then $\Psi_\lambda(\pi) =: \gamma$ is a gapless λ-tuple and $d_\lambda(\pi;x) = |h_{\lambda_j-j+i}(i,\gamma_j;x)|$.*

A "less efficient" (in the sense of our Section 8) version of this expression appeared in the proof of Corollary 14.6 of [PS] when Postnikov and Stanley applied their skew flagged Schur function determinant identity Equation 13.1 to their $\mathrm{ch}_{\lambda,w}$.

6 Necessary condition for nonpermutability

Let $\beta \in U_\lambda(n)$. We prepare for two proofs by constructing an n-path Λ for each $d \in [q_r]$. To see that each $\Lambda \in \mathcal{LD}_\lambda(\beta)$, we also describe its corresponding (clearly semistandard) tableau T. Launching a running example, take $n = 16$ and $\lambda = (7^3; 5^8; 3^2; 1^2; 0^1)$ and $\beta = (5,5,8;5,12,13,9,11,11,15,15;16,16;14,16;16)$. Set $\delta := \Delta_\lambda(\beta)$. Here $\delta = (4,5,8;5,7,8,9,10,11,14,15;15,16;14,16;16)$. Let $d \in [q_r]$. For example, take $d = 9$. For $i \in (0, d-1]$ set $T_j(i) := i$ for $j \in (0,\lambda_i]$. The corresponding paths Λ_i are $(n-i,i) \to (\lambda_i+n-i,i) \downarrow (\lambda_i+n-i,\delta_i) \downarrow (\lambda_i+n-i,\beta_i)$. Six of these eight paths are shown with dots in Figure 6.1. Let $i \in (d-1,q_r]$. Let $h \in [r]$ be such that $i \in (q_{h-1},q_h]$. For $j \in (0,\lambda_{q_{h+1}}]$ set $T_j(i) := i$. For $j \in (\lambda_{q_{h+1}},\lambda_{q_h}]$ set $T_j(i) := \delta_i$. The corresponding paths Λ_i are $(n-i,i) \to (\lambda_{q_{h+1}}+n-i,i) \downarrow (\lambda_{q_{h+1}}+n-i,\delta_i) \to (\lambda_i+n-i,\delta_i) \downarrow (\lambda_i+n-i,\beta_i)$. For $i \in (q_r,n]$ set $T_j(i) := \delta_i$ $(= i)$ for $j \in (0,\lambda_i]$. The corresponding paths Λ_i are $(n-i,i) \to (\lambda_i+n-i,\delta_i) \downarrow (\lambda_i+n-i,\beta_i)$. The dots indicate the depths δ_i on the ending longitudes of the paths.

For a determinant example pertinent to the following lemma, take $n := 3, \lambda := (1,1,0)$, and $\beta := (3,2,3)$. Note that $\beta \in UGC_\lambda(n) \backslash UBP_\lambda(n)$, and so this lemma will imply that (λ,β) is not nonpermutable. Here $s_\lambda(\beta;x,y,z) = xy$, but the determinant of Proposition 4.2 evaluates to $xy - z^2$.

Lemma 6.1. *If $\beta \notin UBP_\lambda(n)$, then (λ,β) fails to be nonpermutable.*

Proof. Set $\Delta_\lambda(\beta) =: \delta \in UI_\lambda(n)$ and $\xi = \Xi_\lambda(\beta)$. In the example we have $\xi = (5,5,8;5,11,11,11,11,11,15,15;16,16;14,16;16)$. Since β is a λ-tuple and $\xi_i = n$ for $i \in (q_r,n]$, the failure of

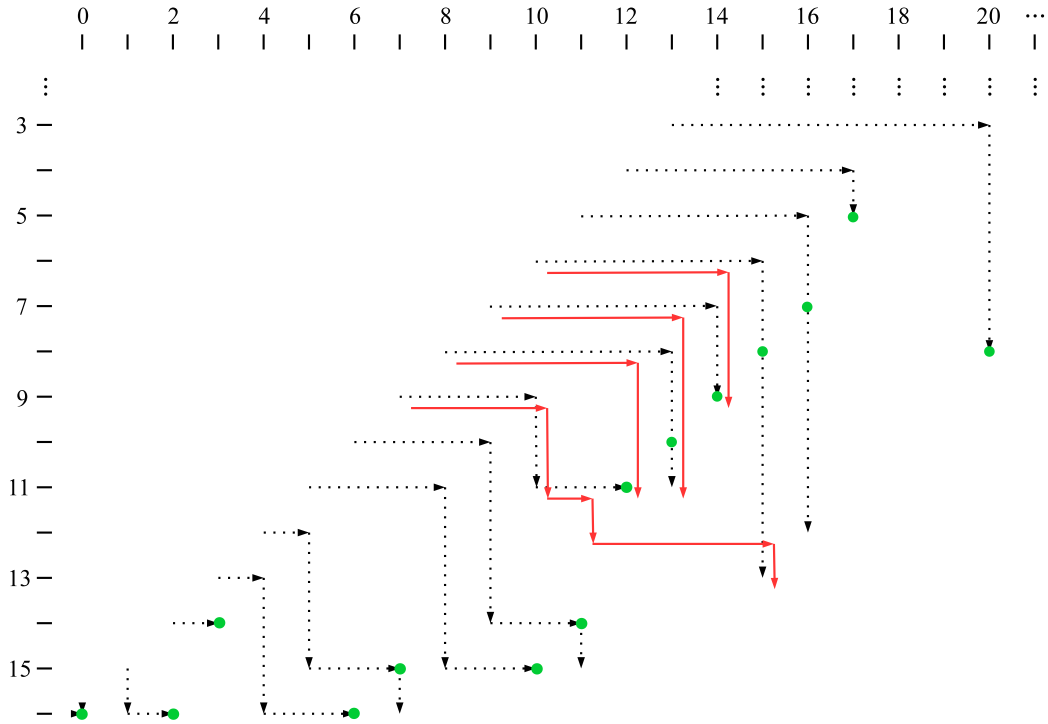

Figure 6.1. Rewiring four component paths produces a nonpermutability violation.

boundedness for β cannot occur in this last carrel. Let $h \in [r]$ be such that there exists $t \in (q_{h-1}, q_h]$ such that $\beta_t > \xi_t$, and then let $c \in (q_{h-1}, q_h]$ be maximal such that $\beta_c > \xi_c$. So c is not a critical index, since $\beta_c \neq \xi_c$. Let d be the leftmost critical index in $(q_{h-1}, q_h]$ such that $d > c$. Here we have $h = 2, c = 6$, and $d = 9$. Here $\beta_d = \delta_d = \xi_d = \xi_c < \beta_c$ implies $\delta_d + 1 \leq \beta_c$. Since $d \leq q_r$ we have $\lambda_d \geq 1$, which implies $\lambda_d + n - d - 1 \geq 0$. Now refer to the n-path Λ constructed above for this $d \in [q_r]$. We rewire the last part of its Λ_d to produce a new path Λ'_d as follows: Rather than finishing with ... $\to (\lambda_d + n - d, \delta_d) = (\lambda_d + n - d, \beta_d)$, the new path Λ'_d finishes with ... $(\lambda_d + n - d - 1, \delta_d) \downarrow (\lambda_d + n - d - 1, \delta_d + 1) \to (\lambda_d + n - c, \delta_d + 1) \downarrow (\lambda_c + n - c, \beta_c)$. Four rewirings are shown with solid paths. Here Λ'_d reaches $(\lambda_d + n - d - 1, \delta_d)$, goes one unit to the south, then turns left onto the latitude $\delta_d + 1$ and goes $d - c + 1$ units to the east, and then turns right to go straight south until it ends at $(\lambda_c + n - c, \beta_c)$. This new southerly edge $(\lambda_d + n - d - 1, \delta_d) \downarrow (\lambda_d + n - d - 1, \delta_d + 1)$ is not in use by Λ_{d+1} (or a later path): If $d = q_h$, then the longitude at $(\lambda_d + n - d) - 1$ is not used by any component of Λ since $\lambda_d > \lambda_{d+1}$ here implies that this longitude is strictly to the east of the longitude $\lambda_{d+1} + n - d - 1$ on which Λ_{d+1} finishes. If $d < q_h$, note that $\delta_d + 1 < \delta_{d+1}$ because d is a critical index. So here the southernmost point reached by Λ'_d on its new briefly used longitude at $\lambda_d + n - d - 1$ is strictly to the north of the northernmost point on this longitude used by Λ_{d+1}, which descended to the depth δ_{d+1} on the longitude $\lambda_{q_{h+1}} + n - d - 1$ to the west. Either way, for $m = d - 1, d - 2, ..., c$, next successively rewire the finishes of $\Lambda_{d-1}, \Lambda_{d-2}, ..., \Lambda_c$

to respectively produce finishes for the paths $\Lambda'_{d-1}, \Lambda'_{d-2}, ..., \Lambda'_c$ as follows: Rather than travelling $(n-m,m) \to (\lambda_m+n-m,m) \downarrow (\lambda_m+n-m,\delta_m) \downarrow (\lambda_m+n-m,\beta_m)$, the new path Λ'_m travels $(n-m,m) \to (\lambda_m+n-m-1,m) \downarrow (\lambda_m+n-m-1,\delta_{m+1}) \downarrow (\lambda_m+n-m-1,\beta_{m+1})$. Here Λ'_m is finishing by turning right one step early, using one (or more) new southerly step(s), and then using the final (possibly empty) "southerly stilt" that Λ_{m+1} had been using to finish. It can be seen that the "further" new southerly steps that could be used by Λ'_{d-1} are not used by Λ'_d. No intersections among these $d-c$ paths occur since the right turns that are each being executed one easterly step early are being coordinated along a staircase where $\lambda_m = \lambda_{q_h}$. Given the choices of c and d, for $i \in (c,d]$ we have $\beta_i \leq \xi_i = \xi_d = \delta_d$. So $\beta_i < \delta_d + 1$ for $i \in (c,d]$. Hence Λ'_m does not intersect Λ'_d for $m \in [c, d-1]$. When $m \notin [c,d]$ set $\Lambda'_m := \Lambda_m$. It can be seen that none of the rewired paths intersect any of these original paths. We have constructed a disjoint n-path $\Lambda' := (\Lambda'_1, ..., \Lambda'_n)$ whose respective sinks form a nontrivial permutation π of the original ordered terminals. Therefore $\mathcal{LD}_\lambda(\beta;\pi) \neq \emptyset$. □

For an example pertinent to the following lemma, take $n := 3, \lambda := (2,1,0)$, and $\beta := (3,2,3)$. Note that $\beta \in UBP_\lambda(n) \backslash UGC_\lambda(n)$, and so this lemma will imply that (λ, β) is not nonpermutable. Here $s_\lambda(\beta; x,y,z) = x^2y + xy^2 + xyz$, but the determinant of Proposition 4.2 evaluates to $x^2y + xy^2 + xyz - z^3$.

Lemma 6.2. *If $\beta \notin UGC_\lambda(n)$, then (λ, β) fails to be nonpermutable.*

Proof. If $\beta \notin UBP_\lambda(n)$ apply Lemma 6.1; otherwise $\beta \in UBP_\lambda(n)$. Set $\Delta_\lambda(\beta) =: \delta \in UI_\lambda(n)$ and $\xi := \Xi_\lambda(\beta)$. Having β failing to be a gapless core λ-tuple is equivalent to having δ failing to be a gapless λ-tuple. The only critical entry in the last carrel $(q_r, n]$ is n. So there cannot be a failure of λ-gapless based upon having $\delta_{q_r} > n$. Let $h \in (1, r]$ be such that δ fails to be λ-gapless based upon having $\delta_{q_{h-1}} > \delta_d$, where d is the leftmost critical index in the h^{th} carrel $(q_{h-1}, q_h]$. Set $c := q_{h-1}$. In each of the two cases below we refer to the n-path Λ for this d constructed above. Note that $\delta_d + 1 \leq \delta_c$. Since $d \leq q_r$ in each case we have $\lambda_d \geq 1$, which implies $\lambda_d + n - d - 1 \geq 0$. These facts will allow us to rewire the path Λ_d to produce the path Λ'_d in nearly the same fashion as in the previous proof. The only difference is that the new path Λ'_d now has to make $\lambda_{q_{h-1}} - \lambda_{q_h}$ additional easterly steps just before reaching its finishing longitude of $\lambda_{q_{h-1}} + n - q_{h-1}$. If $d = q_h$, then the reasoning used in the '$d = q_h$' case in the preceding proof to see that the southerly edge on the longitude $(\lambda_d + n - d) - 1$ from depth δ_d to depth $\delta_d + 1$ is not in use by Λ_{d+1} can be re-used here. Here d is the only critical index for the carrel $(q_{h-1}, q_h]$. If $d < q_h$, the reasoning used in the '$d < q_h$' case in the preceding proof to see that the early "jog" to the right is acceptable can be re-used here. Here d is the smallest critical index greater than q_{h-1}. Either way, for $m = d-1, d-2, ..., c+1$, next successively rewire $\Lambda_{d-1}, \Lambda_{d-2}, ..., \Lambda_{c+1}$ to respectively produce paths $\Lambda'_{d-1}, \Lambda'_{d-2}, ..., \Lambda'_{c+1}$ as in the previous proof. Then rewire the path Λ_c to produce the path Λ_c' in nearly the same fashion as in the previous proof. The only difference is that the new path Λ_c' now makes $\lambda_{q_{h-1}} - \lambda_{q_h}$ fewer easterly steps just before reaching its finishing longitude of $\lambda_{q_h} + n - q_{h-1} - 1$. The observation in the previous proof concerning the coordination of the right turns among the shifted $d-c$ modified paths needs a tiny modification to account for this. In each case the fact that d is the smallest critical index larger than c implies $\xi_i = \xi_d = \delta_d$ for $i \in (c, d]$. Since $\beta \in UBP_\lambda(n)$, we have $\beta_i \leq \xi_i = \xi_d = \delta_d < \delta_d + 1$ for $i \in (c, d]$. The rest of this proof is the same as the end of the previous proof. $\square$

Combine the contrapositives of these two lemmas:

Proposition 6.3. *Let $\beta \in U_\lambda(n)$. If (λ, β) is nonpermutable, then $\beta \in UGC_\lambda(n) \cap UBP_\lambda(n)$.*

7 Sufficient condition for nonpermutability

To prove the converse of Proposition 6.3, we will need:

Lemma 7.1. *Let $\beta \in UBP_\lambda(n)$. Set $\delta := \Delta_\lambda(\beta)$. Let π be a permutation of $[n]$. Let $\Lambda \in \mathcal{LD}_\lambda(\beta; \pi)$. For each $m \in [n]$, the component Λ_m of Λ must end with $(\lambda_{\pi_m} + n - \pi_m, \delta_{\pi_m}) \downarrow (\lambda_{\pi_m} + n - \pi_m, \beta_{\pi_m})$.*

Proof. To avoid forming the inverse of π and using double subscripts, we sidestep π by refering to the original indices for the terminals. Let x be a critical index for β. Let x' be the largest critical index that is less than x; if x is the leftmost critical index then take $x' := 0$. Here $\lambda_i = \lambda_x$ for $i \in (x', x]$. For such i, let M_i denote the component of Λ that sinks at $(\lambda_x + n - i, \beta_i)$. The claim is true for M_x since $\delta_x = \beta_x$. Let i decrement from x to $x' + 1$ and assume the claim is true for $i < i' \leq x$. So each $M_{i'}$ ends with $(\lambda_x + n - i', \delta_{i'}) \downarrow (\lambda_x + n - i', \beta_{i'})$. Set $\xi := \Xi_\lambda(\delta)$. Note that $\xi_i = \xi_x = \delta_x = \beta_x$. If $\beta_i = \delta_i$ there is nothing to show. Otherwise $\delta_i = \delta_{i+1} - 1$ and $\delta_i \leq \beta_i \leq \xi_i$ imply that $\delta_{i+1} \leq \beta_i \leq \xi_i$. By the induction we see that $(\lambda_x + n - i', \delta_{i'})$ is unavailable to M_i for $i < i' \leq x$. So this path M_i must pass through $(\lambda_x + n - i, \delta_i)$. Then it must finish with $(\lambda_x + n - i, \delta_i) \downarrow (\lambda_x + n - i, \beta_i)$. $\square$

Stanley remarked in Theorem 2.7.1 of [St1] that (λ, β) is nonpermutable when β is a flag. Since $UF_\lambda(n) \subseteq UGC_\lambda(n) \cap UBP_\lambda(n)$, the following proposition extends that remark. His remark can be justified with either of the arguments that we describe within Case (i) of this proof, but referring to β rather than to δ.

Proposition 7.2. *Let $\beta \in U_\lambda(n)$. If $\beta \in UGC_\lambda(n) \cap UBP_\lambda(n)$, then (λ, β) is nonpermutable.*

Proof. Let π be a permutation of $[n]$ such that $\pi \neq (1, 2, ..., n)$. For the sake of contradiction suppose $\mathcal{LD}_\lambda(\beta; \pi) \neq \emptyset$. Find a descent in π^{-1} and let $1 \leq i < k \leq n$ be such that $\pi_i = \pi_k + 1$.

Set $m := \pi_k$. Take $\Lambda \in \mathcal{LD}_\lambda(\beta;\pi)$. Set $\delta := \Delta_\lambda(\beta) \in UG_\lambda(n)$. By the lemma, without loss of generality we may revamp Λ by replacing (with respect to their original indexing) the sequence β of depths of its terminals with the sequence of shallower depths δ. This truncates its original paths by deleting their final stilts. We consider the components Λ_i and Λ_k of Λ. Here Λ_i arises at $(n-i,i)$ and sinks at $(\lambda_{m+1}+n-m-1, \delta_{m+1})$. Later Λ_k arises at $(n-k,k)$ and sinks at $(\lambda_m+n-m, \delta_m)$. Comparing the starting and finishing longitudes for Λ_k to those for Λ_i, we have $n-k < n-i$ and $\lambda_m + n - m > \lambda_{m+1}+n-m-1$. So every longitude that is visited by Λ_i is later visited by the longer Λ_k. Set $v := \lambda_{m+1}+n-m-1$; the earlier path Λ_i finishes on the longitude at v. Let's say that the later path Λ_k first reaches the longitude at v on the latitude at z, for some $z \geq 1$.

(i) First suppose that $z \leq \delta_{m+1}$, which is the finishing depth of Λ_i on the longitude at v. It is topologically evident that the path Λ_k must intersect the path Λ_i; this contradicts $\Lambda \in \mathcal{LD}_\lambda(\delta;\pi)$. (For an explicit discrete proof, consider the minimum and maximum depths used on each of the $\lambda_{m+1}-m+i$ longitudes visited by both Λ_i and Λ_k. Inequalities and equalities among these $4(\lambda_{m+1}-m+i)$ depths can be used to find a longitude on which Λ_i and Λ_k intersect.)

(ii) Otherwise we have $z > \delta_{m+1}$. See Figure 7.1. Since z cannot exceed the finishing depth δ_m for Λ_k, we have $z \leq \delta_m$. Hence $\delta_m > \delta_{m+1}$. But $\delta \in UG_\lambda(n)$ is λ-increasing. This forces $m = q_h$ for some $h \in [r]$. Set $s := \delta_m - \delta_{m+1}+1$. Since δ is λ-gapless we have $s \leq p_{h+1}$ and $\delta_{m+1} = \delta_m - s + 1$, $\delta_{m+2} = \delta_m - s + 2, ..., \delta_{m+s} = \delta_m$. Starting at the sink (v, δ_{m+1}) of Λ_i and moving exactly to the

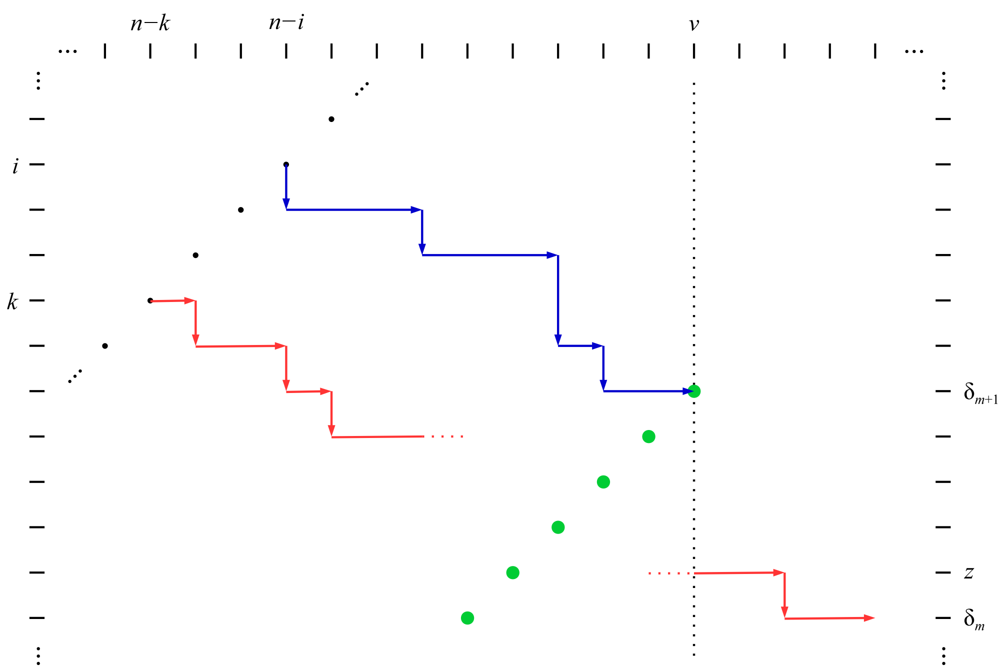

Figure 7.1. Paths Λ_i and Λ_k successively sink at terminals $(\lambda_{m+1}+n-m-1, \delta_{m+1})$ and $(\lambda_m+n-m, \delta_m)$.

southwest, we note that the s points $(v, \delta_{m+1}), (v-1, \delta_{m+1}+1), ..., (v-s+1, \delta_m)$ forming a staircase are terminals that are serving as sinks for some paths other than Λ_k. Since Λ_i and Λ_k are paths, we have $i \leq \delta_{m+1}$ and $k \leq \delta_m$. The source of Λ_k is exactly to the southwest of the source of Λ_i by $k-i$ diagonal steps. Since the source of Λ_i is weakly to the west of the longitude at v, if the source of Λ_k is on one of the latitudes appearing in the staircase it must be weakly to the west of the point of the staircase on that latitude. This implies that the source of Λ_k is not on the same side of this staircase as (v, z). This is also clear if the source of Λ_k is on a shallower latitude. Since the path Λ_k originates on the longitude at $n-k < v$ and reaches (v, z) with $z \in (\delta_{m+1}, \delta_m]$, it must intersect this staircase. This contradicts $\Lambda \in \mathcal{LD}_\lambda(\delta; \pi)$. Hence $\mathcal{LD}_\lambda(\delta; \pi) \neq \emptyset$ is impossible when $\pi \neq (1, 2, ..., n)$. $\square$

8 Equivalence and efficiency

We group the valid λ-tuple inputs for computing row bound sums using the Gessel-Viennot method into equivalence classes, and identify the most efficient λ-tuple within each class.

When λ has distinct parts, the row ending values for the unique maximal element of $\mathcal{S}_\lambda(\beta)$ are the entries of β. Hence the sets $\mathcal{S}_\lambda(\beta)$ for $\beta \in U_\lambda(n)$ are distinct in this case. For general λ, as in Section 12 of [PW], for $\beta, \beta' \in U_\lambda(n)$ define $\beta \approx_\lambda \beta'$ when $\mathcal{S}_\lambda(\beta) = \mathcal{S}_\lambda(\beta')$. Proposition 12.3(i) of [PW] stated that the sets $\mathcal{S}_\lambda(\beta)$ could be precisely labelled by requiring $\beta \in UI_\lambda(n)$, and that these λ-increasing upper tuples are the minimal elements of the equivalence classes in $U_\lambda(n)$ for $\approx_\lambda$. Proposition 12.2 said that the results in Sections 4 and 5 of [PW] for $\sim_R$ could be used for $\approx_\lambda$ by taking $R := R_\lambda$. Lemma 5.1(i) there said that $\beta, \beta' \in U_\lambda(n)$ are equivalent exactly when $\Delta_\lambda(\beta) = \Delta_\lambda(\beta')$ or when they have the same critical list. Since the $\beta \in U_\lambda(n) \backslash UGC_\lambda(n)$ are not valid n-tuple Gessel-Viennot inputs, the next statement considers only $UGC_\lambda(n)$ and $UF_\lambda(n)$. Its two parts follow from Lemma 5.1(i), Proposition 4.2, and Proposition 5.2(ii)(iii) of [PW].

Fact 8.1. *When $\approx_\lambda$ is restricted to $UGC_\lambda(n)$ and to $UF_\lambda(n)$, in each case the equivalence classes are the subsets consisting of λ-tuples that share a flag critical list. More specifically:*
(i) In $UGC_\lambda(n)$ these subsets are the nonempty intervals in $U_\lambda(n)$ of the form $[\gamma, \kappa]$, where $\gamma \in UG_\lambda(n)$ and κ is a "λ-canopy tuple".
(ii) In $UF_\lambda(n)$ these subsets are the nonempty intervals in $UF_\lambda(n)$ of the form $[\tau, \xi]$, where τ is a "λ-floor flag" and $\xi \in UCeil_\lambda(n)$.

To describe the equivalence classes of valid λ-tuple inputs as intervals, we "borrow" the minimum element of Part (i) above and the maximum element of Part (ii) above:

Proposition 8.2. *The equivalence classes for the restriction of $\approx_\lambda$ to $UGC_\lambda(n) \cap UBP_\lambda(n)$ are the subsets of $UGC_\lambda(n) \cap UBP_\lambda(n)$ consisting of λ-tuples that share a flag critical list. These subsets* *are the nonempty intervals in $U_\lambda(n)$ of the form $[\gamma, \xi]$, where $\gamma \in UG_\lambda(n)$ and $\xi \in UCeil_\lambda(n)$. The equivalence class for a particular $\eta \in UGC_\lambda(n) \cap UBP_\lambda(n)$ has $\gamma = \Delta_\lambda(\eta)$ and $\xi = \Xi_\lambda(\gamma)$.*

Since it was noted that $UF_\lambda(n) \subseteq UBP_\lambda(n)$ in Section 2, there is no need here to consider how the equivalence classes for $\approx_\lambda$ restrict to $UF_\lambda(n) \cap UBP_\lambda(n) = UF_\lambda(n)$.

Proof. Two upper λ-tuples are equivalent exactly when they share a critical list. And by Proposition 4.2(iii) of [PW] every gapless core λ-tuple has a flag critical list. Let $\eta \in UGC_\lambda(n) \cap UBP_\lambda(n)$, and denote its equivalence class in this set by $\langle \eta \rangle$. By Proposition 5.2(ii)(i) of [PW] and Fact 8.1(i), the minimum element of its equivalence class in $UGC_\lambda(n)$ is the gapless λ-tuple $\gamma := \Delta_\lambda(\eta)$. In Section 2 it was noted that $UG_\lambda(n) \subseteq UBP_\lambda(n)$. So $\gamma \in UGC_\lambda(n) \cap UBP_\lambda(n)$, and it must be the minimum element of $\langle \eta \rangle$. Set $\xi := \Xi_\lambda(\gamma)$; in Section 2 it was noted that ξ has the same flag critical list as is shared by η and γ. Let $\eta' \in \langle \eta \rangle$. Since it has the same critical list as γ, by the definition of Ξ_λ we have $\Xi_\lambda(\eta') = \xi$. By the definition of $UBP_\lambda(n)$ we have $\eta' \leq \xi$. Hence ξ is the maximum element of $\langle \eta \rangle$ and $\eta' \in [\gamma, \xi]$. Suppose $\eta'' \in [\gamma, \xi]$. By Lemma 5.1(i) and Proposition 5.2(i) of [PW], the critical list of η'' is the flag critical list shared by γ and ξ. So $\eta'' \in UGC_\lambda(n)$. And $\eta'' \leq \xi = \Xi_\lambda(\eta'')$ implies $\eta'' \in UBP_\lambda(n)$. Hence $\eta'' \in \langle \eta \rangle$. $\square$

So to compute $s_\lambda(\eta; x)$ for a given $\eta \in UGC_\lambda(n)$ we may apply the Gessel-Viennot method to any $\eta' \in [\gamma, \xi]$, where γ and ξ are respectively the unique gapless λ-tuple and the unique λ-ceiling flag that have the same flag critical list as η. If one does not care about efficiency and wishes to use an upper flag, then at least the λ-ceiling flag ξ will be available. In his Theorem 2.7.1 [St1], Stanley noted that flags were valid inputs for the Gessel-Viennot method. Via Proposition 8.2, our Theorem 5.1 implies that the Gessel-Viennot method cannot be used to compute a row bound sum $s_\lambda(\beta; x)$ for any upper λ-tuple β that is not equivalent to an upper flag. So Corollary 5.2 does not provide determinant expressions for any new row bound sum polynomials.

We say $\eta \in UGC_\lambda(n) \cap UBP_\lambda(n)$ attains *maximum efficiency* if $|h_{\lambda_j - j + i}(i, \eta_j; x)|$ has fewer total monomials among its entries than does the Gessel-Viennot determinant for any other $\eta' \in UGC_\lambda(n) \cap UBP_\lambda(n)$ that produces $s_\lambda(\eta; x)$. Fix one $\eta \in UGC_\lambda(n) \cap UBP_\lambda(n)$ and set $\Delta_\lambda(\eta) =: \gamma \in UG_\lambda(n)$. By Proposition 5.2(ii) of [PW] this is the minimum element of $U_\lambda(n)$ that is equivalent to η. Knowing $\gamma \leq \eta$ leads to:

Proposition 8.3. *Let $\eta \in UGC_\lambda(n)$. The gapless λ-tuple $\Delta_\lambda(\eta)$ attains maximum efficiency.*

Proof. To complete the proof, note that $\Delta_\lambda(\eta) \in UG_\lambda(n) \subseteq UBP_\lambda(n)$. So Proposition 4.2 can be applied. Corollary 14.4(i) of [PW] rules out an "accidental" polynomial equality between $s_\lambda(\eta; x)$ and any $s_\lambda(\beta; x)$ for which β is not equivalent to η. The (i,j) entry of $|h_{\lambda_j - j + i}(i, \eta_j; x)|$ has $\binom{\lambda_j - j + \eta_j}{\lambda_j - j + i}$ monomials. The sentences before the statement complete this proof. $\square$

So the $\gamma \in UG_\lambda(n)$ are the λ-tuples in $UGC_\lambda(n) \cap UBP_\lambda(n)$ that attain maximum efficiency. If β is replaced by γ, for each $j \in [n]$ the number of terms in the (i,j) entry of the determinant will be reduced by a factor of $[(\lambda_j - j + \gamma_j)_{(\lambda_j - j+i)}]/[(\lambda_j - j + \beta_j)_{(\lambda_j-j+i)}]$; this is a ratio of falling factorials. We have not been able to obtain this conversion with naive row and column operations. In the $\beta \in UGC_\lambda(n) \backslash UBP_\lambda(n)$ example given before Lemma 6.1, the "attempted" incorrect determinant expression for $s_\lambda(\beta; x)$ that uses β cannot be converted with row and column operations to the correct determinant expression for $s_\lambda(\beta; x)$ that uses $\gamma := \Delta_\lambda(\beta)$. So any row and column conversion that is proposed here must refer to the assumption $\beta \in UBP_\lambda(n)$. If $\lambda_n > 0$, one can also factor out $(x_1 x_2 \cdots x_n)^{\lambda_n}$ and work with $\lambda' := (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, ..., 0)$. Going further, when there are only $p := \zeta_1 < n$ nonempty rows in the shape λ, the determinant is equal to its upper left $p \times p$ minor because the last $n-p$ terminals coincide with the respective sources: There are no paths from the first p sources to these terminals, and the only path from one of the last $n-p$ sources to one of these last $n-p$ terminals is the null path at each source.

What does the equivalence class $[\gamma, \xi]$ for $\approx_\lambda$ look like in the path model? Fix $\gamma \in UG_\lambda(n)$ and $h \in [r+1]$. Since $UG_\lambda(n) \subseteq UI_\lambda(n)$, the graph of γ above the portion $(q_{h-1}, q_h]$ of the x-axis can be decomposed into "staircases" whose rightmost indices are the critical indices. When Ξ_λ is applied to γ to produce ξ, these staircases are converted to "plateaus" at the heights of the critical entries for γ in this carrel. Let $\eta \in [\gamma, \xi]$. The graph of η over this carrel lies between these graph portions for γ and ξ. To view the portions of these three gapless core λ-tuples as subsequences of the corresponding overall sequences of terminals, rotate this picture by 180°. The partition λ is constant on each of its carrels. Lemma 7.1 said that the the $q_h - q_{h-1}$ lattice paths that arrive at these terminals for η within a non-intersecting n-tuple of paths must pass through "staircases" of terminals specified by this portion of γ, and that the ending segments of these paths must then drop down in "stilts" to arrive at their terminals. As the lengths of each of these stilts is varied from γ_i to ξ_i for $i \in (q_{h-1}, q_h]$, the weight of the n-tuple of paths is unaffected since no horizontal steps are present.

In [PW] we defined the parabolic Catalan number C_n^λ to be the number of "λ-312-avoiding permutations". There in Theorem 18.1(ii) we noted that this is also the number of gapless λ-tuples. Given this, the following result is a consequence of the two propositions in this section. It was previewed as Part (xi) of Theorem 18.1 of that paper:

Corollary 8.4. *The number of valid upper λ-tuple inputs to the Gessel-Viennot determinant expression for flagged Schur polynomials on the shape λ that attain maximum efficiency is C_n^λ.*

For a sequence of examples, let $m \geq 1$. Suppose λ is a partition whose shape's set of column lengths that are less than $2m$ is $R_\lambda = \{2, 4, ..., 2m-2\}$. Then the number of maximum efficiency inputs here is given by the member of the sequences A220097 of the OEIS [Slo] that is indexed by m.

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