Bar code for monomial ideals

Michela Ceria
Department of Mathematics
University of Trento
Via Sommarive 14, 38123, Trento
michela.ceria@unitn.it

Abstract

Aim of this paper is to count 0-dimensional stable and strongly stable ideals in 2 and 3 variables, given their (constant) affine Hilbert polynomial.

To do so, we define the *Bar Code*, a bidimensional structure representing any finite set of terms M and allowing to desume many properties of the corresponding monomial ideal I, if M is an order ideal. Then, we use it to give a connection between (strongly) stable monomial ideals and integer partitions, thus allowing to count them via known determinantal formulas.

1 Introduction

Strongly stable ideals play a special role in the study of Hilbert scheme, introduced first by Grothendieck [22], since their escalier allows to study the Hilbert function of any homogeneous ideal, exploiting the theory of Groebner bases, as pointed out by Bayer [5] and Eisenbud [18].

The notion of generic initial ideal was introduced by Galligo [21] with the name of *Grauert invariant*. Galligo proved that the generic initial ideal of any homogeneous ideal is closed w.r.t the action of the Borel group and gave a combinatorial characterization of such ideals, provided that they are defined on a field of characteristic zero. Also Eisenbud and Peeva [18, 42], focused on that monomial ideals, labelling them 0-*Borel-fixed ideals*. Later, Aramova-Herzog [2, 3] renamed them *strongly stable ideals*.

A combinatorial description of the ideals closed w.r.t the action of the Borel group over a polynomial ring on a field of characteristic $p > 0$ has been provided by Pardue in his Thesis [41] and Galligo's result has been extended to that setting by Bayer-Stillman [6].

The notion of *stable ideal* has been introduced by Eliahou-Kervaire [19] as a generalization of 0-Borel-fixed ideals. They were able to give a minimal resolution for stable ideals.

Such minimal resolution was used by Bigatti [10] and Hulett [26] to extend Macaulay's result [37]; they proved that the lex-segment ideal has maximal Betti numbers, among all ideals sharing the same Hilbert function.

In connection with the study of Hilbert schemes [8, 9, 14, 33, 38, 45] it has been considered relevant to list all the stable ideals [7] and strongly stable ideals [15, 34] with a fixed Hilbert polynomial.

Aim of this paper is to count zerodimensional stable and strongly stable ideals in 2 and 3 variables, given their (constant) affine Hilbert polynomial.

To do so, we first introduce a bidimensional structure, called *Bar Code* which allows, a priori, to represent any (finite[1]) set of terms M and, if M is an order ideal, to authomatically desume many properties of the corresponding monomial ideal I. For example, a Pommaret basis [48, 12] of I can be easily desumed.

The Bar Code is strictly connected to Felzeghy-Rath-Ronyay's Lex Trie [20, 35], even if our goal and methods are completely different from theirs.

Using the Bar Code, we provide a connection between stable and strongly stable monomial ideals and integer partitions.

For the case of two variables, we see that there is a biunivocal correspondence between (strongly) stable ideals with affine Hilbert polynomial p and partitions of p with distinct parts.

The case of three variables is more complicated and some more technology is required. Thanks to the Bar Code, we provide a bijection between (strongly) stable ideals and some special plane partitions of their constant affine Hilbert polynomial p.

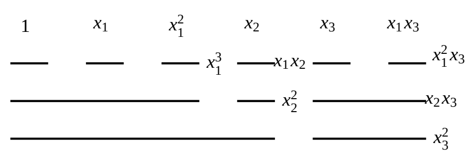
These plane partitions have been studied by Krattenthaler [31, 32], who proved determinantal formulas to find their norm generating functions and - finally - to count them.

As an example, we consider the stable monomial ideal

$$I_1 = (x_1^3, x_1x_2, x_2^2, x_1^2x_3, x_2x_3, x_3^2) \triangleleft \mathbf{k}[x_1, x_2, x_3],$$

whose Groebner escalier is $\mathsf{N}(I_1) = \{1, x_1, x_1^2, x_2, x_3, x_1x_3\}$.

It can be represented by the Bar Code below



and it corresponds to the plane partition

$$\begin{matrix} 3 & 1 \\ 2 & \end{matrix}$$

The correspondence can be seen observing the rows of the Bar Code above: since the bottom row is composed by two segments, the plane partition has exactly two rows. The number of entries in the i-th row of the partition, $i = 1, 2$ (*i.e.* 2 and 1 resp.), is given by the number of segments in the middle-row, lying over the i-th segment of the bottom row. Finally, the entries are represented by the number of segments in the top row, lying over the segments representing the corresponding entry.

[1]There is also the possibility to have *infinite* Bar Codes for infinite sets of terms, but it is out of the purpose of this paper, so we will only see an example for completeness' sake.

Exploiting this bijection and the determinantal formulas by Krattenthaler, we are finally able to count stable and strongly stable ideals in three variables.

Even if the Bar Code can easily represent finite sets of terms in any number of variables, the generalization of our results to the case of 4 or more variables would require the introduction of n-dimensional partitions, for which, in my knowledge, it does not exist a complete study from the point of view of counting them[2], so, in this paper, we do not extensively deal with them.

2 Some algebraic notation

Throughout this paper, in connection with monomial ideals, we mainly follow the notation of [39].
We denote by $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ the graded ring of polynomials in n variables with coefficients in the field $\mathbf{k}$, assuming, once for all, that $char(\mathbf{k}) = 0$.
The *semigroup of terms*, generated by the set $\{x_1, ..., x_n\}$ is:

$$\mathcal{T} := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \mid \gamma := (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\}.$$

If $\tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, then $\deg(\tau) = \sum_{i=1}^n \gamma_i$ is the *degree* of τ and, for each $h \in \{1, ..., n\}$ $\deg_h(\tau) := \gamma_h$ is the *h-degree* of τ.

For each $d \in \mathbb{N}$, $\mathcal{T}_d$ is the d-degree part of $\mathcal{T}$, i.e. $\mathcal{T}_d := \{x^\gamma \in \mathcal{T} \mid \deg(x^\gamma) = d\}$ and it is well known that $|\mathcal{T}_d| = \binom{n+d-1}{d}$. For each subset $M \subseteq \mathcal{T}$ we set $M_d = M \cap \mathcal{T}_d$. The symbol $\mathcal{T}(d)$ denotes the degree $\leq d$ part of $\mathcal{T}$, namely $\mathcal{T}(d) = \{x^\gamma \in \mathcal{T} \mid \deg(x^\gamma) \leq d\}$. Analogously, $\mathcal{P}(d)$ denotes the degree $\leq d$ part of $\mathcal{P}$ and given an ideal I of $\mathcal{P}$, $I(d)$ is its degree $\leq d$ part, i.e. $I(d) = I \cap \mathcal{P}(d)$.
We notice that $\mathcal{P}(d)$ is the vector space generated by $\mathcal{T}(d)$ and we observe that $I(d)$ is a vector subspace of $\mathcal{P}(d)$.
A *semigroup ordering* $<$ on $\mathcal{T}$ is a total ordering such that $\tau_1 < \tau_2 \Rightarrow \tau\tau_1 < \tau\tau_2$, $\forall \tau, \tau_1, \tau_2 \in \mathcal{T}$. For each semigroup ordering $<$ on $\mathcal{T}$, we can represent a polynomial $f \in \mathcal{P}$ as a linear combination of terms arranged w.r.t. $<$, with coefficients in the base field $\mathbf{k}$:

$$f = \sum_{\tau \in \mathcal{T}} c(f, \tau)\tau = \sum_{i=1}^{s} c(f, \tau_i)\tau_i : \ c(f, \tau_i) \in \mathbf{k}^*, \ \tau_i \in \mathcal{T}, \ \tau_1 > ... > \tau_s,$$

with $\mathsf{T}(f) := \tau_1$ the *leading term* of f, $Lc(f) := c(f, \tau_1)$ the *leading coefficient* of f and $tail(f) := f - c(f, \mathsf{T}(f))\mathsf{T}(f)$ the *tail* of f.
A *term ordering* is a semigroup ordering such that 1 is lower than every variable or, equivalently, it is a *well ordering*.
Unless otherwise specified, we consider the *lexicographical ordering* induced by $x_1 < ... < x_n$, i.e:

$$x_1^{\gamma_1} \cdots x_n^{\gamma_n} <_{Lex} x_1^{\delta_1} \cdots x_n^{\delta_n} \Leftrightarrow \exists j \mid \gamma_j < \delta_j, \ \gamma_i = \delta_i, \ \forall i > j,$$

[2] In [1], Chapter 11, the author observes:

Surprisingly, there is much of interest when the dimension is 1 or 2, and very little when the dimension exceeds 2.

which is a term ordering.

Since in all the paper we will consider the lexicographical ordering, no confusion may arise and so we drop the subscript and denote it by $<$ instead of $<_{Lex}$.

For each term $\tau \in \mathcal{T}$ and $x_j | \tau$, the only $\upsilon \in \mathcal{T}$ such that $\tau = x_j \upsilon$ is called j-th *predecessor* of τ.
Given a term $\tau \in \mathcal{T}$, we denote by $\min(\tau)$ the smallest variable x_i, $i \in \{1, ..., n\}$, s.t. $x_i \mid \tau$.
For $M \subset \mathcal{T}$, we denote by $\overline{M}$ the list obtained by ordering the elements of M increasingly w.r.t. Lex. For example, if $M = \{x_2, x_1^2\} \subset \mathbf{k}[x_1, x_2]$, $x_1 < x_2$, $\overline{M} = \{x_1^2, x_2\}$.

A subset $J \subseteq \mathcal{T}$ is a *semigroup ideal* if $\tau \in J \Rightarrow \sigma\tau \in J$, $\forall \sigma \in \mathcal{T}$; a subset $\mathsf{N} \subseteq \mathcal{T}$ is an *order ideal* if $\tau \in \mathsf{N} \Rightarrow \sigma \in \mathsf{N}\ \forall \sigma | \tau$. We have that $\mathsf{N} \subseteq \mathcal{T}$ is an order ideal if and only if $\mathcal{T} \setminus \mathsf{N} = J$ is a semigroup ideal.

Given a semigroup ideal $J \subset \mathcal{T}$ we define $\mathsf{N}(J) := \mathcal{T} \setminus J$. The minimal set of generators $\mathsf{G}(J)$ of J, called the *monomial basis* of J, satisfies the conditions below

$$\begin{aligned} \mathsf{G}(J) &:= \{\tau \in J \,|\, \text{ each predecessor of } \tau \in \mathsf{N}(J)\} \\ &= \{\tau \in \mathcal{T} \,|\, \mathsf{N}(J) \cup \{\tau\} \text{ is an order ideal}, \ \tau \notin \mathsf{N}(J)\}. \end{aligned}$$

For all subsets $G \subset \mathcal{P}$, $\mathsf{T}\{G\} := \{\mathsf{T}(g),\ g \in G\}$ and $\mathsf{T}(G)$ is the semigroup ideal of leading terms defined as $\mathsf{T}(G) := \{\tau\mathsf{T}(g),\ \tau \in \mathcal{T}, g \in G\}$.
Fixed a term order $<$, for any ideal $I \triangleleft \mathcal{P}$ the monomial basis of the semigroup ideal $\mathsf{T}(I) = \mathsf{T}\{I\}$ is called *monomial basis* of I and denoted again by $\mathsf{G}(I)$, whereas the ideal $In(I) := (\mathsf{T}(I))$ is called *initial ideal* and the order ideal $\mathsf{N}(I) := \mathcal{T} \setminus \mathsf{T}(I)$ is called *Groebner escalier* of I. The *border set* of I is defined as:

$$\begin{aligned} \mathsf{B}(I) &:= \{x_h\tau,\ 1 \le h \le n,\ \tau \in \mathsf{N}(I)\} \setminus \mathsf{N}(I) \\ &= \mathsf{T}(I) \cap (\{1\} \cup \{x_h\tau,\ 1 \le h \le n,\ \tau \in \mathsf{N}(I)\}). \end{aligned}$$

If $I \triangleleft \mathcal{P}$ is an ideal, we define its associated *variety* as

$$V(I) = \{P \in \overline{\mathbf{k}}^n,\ f(P) = 0,\ \forall f \in \mathcal{I}\},$$

where $\overline{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$.

Definition 1. *Let $I \triangleleft \mathcal{P}$ be an ideal. The* affine Hilbert function *of I is the function*

$$HF_I : \mathbb{N} \to \mathbb{N}$$

$$d \mapsto dim(\mathcal{P}(d)/I(d)).$$

For d sufficiently large, the affine Hilbert function of I can be written as:

$$HF_I(d) = \sum_{i=0}^{l} b_i \binom{d}{l-i},$$

where l is the Krull dimension of $V(I)$, b_i are integers called *Betti numbers* and b_0 is positive.

Definition 2. *The polynomial which is equal to $HF_I(d)$, for d sufficiently large, is called the* affine Hilbert polynomial *of I and denoted $H_I(d)$.*

3 On the Integer Partitions

In this section, we give some definitions and theorems from the theory of integer partitions that we will use as a tool for our study, mainly following [1, 31, 32, 49].
Let us start giving the definition of *integer partition*.

Definition 3 ([49])**.** *An* integer partition *of $p \in \mathbb{N}$ is a k-tuple $(\lambda_1, ..., \lambda_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^{k} \lambda_i = p$ and $\lambda_1 \geq ... \geq \lambda_k$.*

We regard two partitions as identical if they only differ in the number of terminal zeros. For example $(3, 2, 1) = (3, 2, 1, 0, 0)$.
The nonzero terms are called *parts* of λ and we say that λ has k parts if $k = |\{i,\ \lambda_i > 0\}|$.
We will mainly deal with the special case $\lambda_1 > ... > \lambda_k > 0$ i.e. with integer partitions of p into k non-zero *distinct parts*, denoting by $I_{(p,k)}$ the set containing them, i.e.

$$I_{(p,k)} := \{(\lambda_1, ..., \lambda_k) \in \mathbb{N}^k,\ \lambda_1 > ... > \lambda_k > 0 \text{ and } \sum_{j=1}^{k} \lambda_j = p\}.$$

The number $Q(p, i)$ of integer partitions of p into i distinct parts is well known in literature. For example, we can find in [16] the formulas allowing to compute it:

$$\forall p, i \in \mathbb{N},\ i \neq 1,\ Q(p, i) = P\left(p - \binom{i}{2}, i\right),\ Q(p, 1) = 1$$

where $P(n, k)$ denotes the number of integer partitions of n with largest part equal to k:

$$\forall n, k \in \mathbb{N},\ P(n, k) = P(n-1, k-1) + P(n-k, k),$$

with

$$\begin{cases} P(n,k) = 0 \text{ for } k > n \\ P(n,n) = 1 \\ P(n,0) = 0 \end{cases}$$

We define now the notion of *plane partition*.

Definition 4 ([31])**.** *A* plane partition *π of a positive integer $p \in \mathbb{N}$, is a partition of p in which the parts have been arranged in a 2-dimensional array, weakly decreasing across rows and down columns. If the inequality is strict across rows (resp. columns), we say that the partition is* row-strict *(resp* column-strict*).*
Different configurations are regarded as different plane partitions.
The norm *of π is the sum $n(\pi) := \sum_{i,j} \pi_{i,j}$ of all its parts.*

We point out that an integer partition (see Definition 3) is a simple and particular case of plane partition.

Example 5. An example of plane partition of $p = 6$ is

$$\begin{array}{ccc} 2 & 1 & 1 \\ & 1 & 1 \end{array}$$

which is different from the plane partition

$$\begin{array}{ccc} 2 & 1 & 1 \\ & 1 & \\ & 1 & \end{array}$$

◊

In sections 6, 7, we will be interested in some particular plane partitions, that we define in what follows.

Definition 6 ([31]). *Let D_r denote the set of all r-tuples $\lambda = (\lambda_1, ..., \lambda_r)$ of integers with $\lambda_1 \geq ... \geq \lambda_r$.*
For $\lambda, \mu \in D_r$, we write $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all $i = 1, 2, ..., r$. Let c, d arbitrary integers and $\lambda, \mu \in D_r$, with $\lambda \geq \mu$. We call an array ρ of integers of the form

$$\begin{array}{cccccccccc} & & & \rho_{1,\mu_1+1} & \rho_{1,\mu_1+2} & \cdots & \cdots & \cdots & \rho_{1,\lambda_1} \\ & \rho_{2,\mu_2+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{2,\lambda_2} & \\ & & & \cdots & \cdots & \cdots & \cdots & & \\ \rho_{r,\mu_r+1} & \cdots & \cdots & \rho_{r,\lambda_r} & & & & & \end{array}$$

a (c, d)-plane partition *of shape λ/μ if*

$$\rho_{i,j} \geq \rho_{i,j+1} + c \text{ for } 1 \leq i \leq r,\ \mu_i < j < \lambda_i,$$

$$\rho_{i,j} \geq \rho_{i+1,j} + d \text{ for } 1 \leq i \leq r-1,\ \mu_i < j \leq \lambda_{i+1}.$$

In the case $\mu = 0$, we shortly say that ρ is of shape λ.

We denote by $\mathcal{P}_\lambda(c, d)$ the set of (c, d)-plane partitions of shape λ.

A $(1, 1)$-plane partition containing only positive parts is a row and column-strict plane partition; these partitions will be useful while dealing with stable ideals (see section 6).

Definition 7 ([32]). *Let c, d be arbitrary integers and λ be a partition with $\lambda_r \geq r$. We call* "shifted (c, d)-plane partition *of* shape λ" *an array π of integers of the form*

$$\begin{array}{ccccccccc} \pi_{1,1} & \pi_{1,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \\ & \pi_{2,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_{2,\lambda_2} & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & \pi_{r,r} & \cdots & \cdots & \pi_{r,\lambda_r} & & \end{array}$$

and for which

$$\pi_{i,j} \geq \pi_{i,j+1} + c \text{ for } 1 \leq i \leq r,\ i \leq j < \lambda_i,$$

$$\pi_{i,j} \geq \pi_{i+1,j} + d \text{ for } 1 \leq i \leq r-1,\ i < j \leq \lambda_{i+1}.$$

We point out that, according to definition 7, there are $\lambda_i - i + 1$ integers in the i-th row.

We denote by $\mathcal{S}_\lambda(c,d)$ the set of shifted (c,d)-plane partitions of shape λ. These partitions will be useful in section 7, where we will count strongly stable ideals.

Example 8. The plane partition

$$\begin{matrix} 5 & 4 & 3 \\ 4 & 1 & \end{matrix}$$

is a $(1,1)$-plane partition with shape $\lambda = (3,2)$ and norm 17.
On the other hand, the plane partition

$$\begin{matrix} 5 & 4 & 3 \\ & 4 & 1 \end{matrix}$$

is a shifted $(1,0)$-plane partition of shape $\lambda = (3,3)$ and norm 17. It contains $\lambda_1 = 3$ elements in the first row and $\lambda_2 - 1 = 2$ elements in the second row. ◇

We introduce now the notion of *norm generating function*, for counting plane partitions.

Definition 9 ([31]). *The* norm generating function *for a class C of* (c,d)*-plane partitions is*

$$\sum_{\pi \in C} x^{n(\pi)}.$$

If x is an indeterminate, we introduce the x-notations (see [31]):

$$[n] = 1 - x^n$$

$$[n]! = [1][2]\cdots[n],\ [0]! = 1$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!},\ \text{if } n \geq k \neq 0.$$

If $k = 0$, $\begin{bmatrix} n \\ k \end{bmatrix} = 1$; if $k \neq 0$ and $n < k$, then we set $\begin{bmatrix} n \\ k \end{bmatrix} = 0$.

Theorems 10 and 12 give a way to compute the norm generating function for plane partitions of the forms introduced in Definitions 6 and 7, under some hypotheses on the size of their parts.
Let us start with the plane partitions of Definition 6.

Theorem 10 (Krattenthaler,[31]). *Let c, d be arbitrary integers, $\lambda, \mu \in D_r$ and let a, b be r-tuples of integers satisfying*

$$a_i - c(\mu_i - \mu_{i+1}) + (1-d) \geq a_{i+1}$$

$$b_i + c(\lambda_i - \lambda_{i+1}) + (1-d) \geq b_{i+1}$$

for $i = 1, 2, ..., r-1$.

Then, denoting $N_1(s,t) = b_s(\lambda_s - s - \mu_t + t) + (1-c-d)\left[\binom{\mu_t+s-t}{2} - \binom{\mu_t}{2}\right] + c\binom{\lambda_s-s-\mu_t+t}{2}$, the polynomial

$$det_{1\leq s,t\leq r}\left(x^{N_1(s,t)}\begin{bmatrix}(1-c)(\lambda_s-\mu_t)-d(s-t)+a_t-b_s+c\\ \lambda_s-s-\mu_t+t\end{bmatrix}\right),$$

is the norm generating function for (c, d)-plane partitions of shape λ/μ in which the first part in row i is at most a_i and the last part in row i is at least b_i.

Example 11. Let us consider the $(1, 1)$-plane partitions of shape $\lambda = (2, 1)$ (so $\mu = 0$), such that $a = (4, 3)$ and $b = (1, 1)$, i.e. row and column strict plane partitions of the form

$$\begin{pmatrix}\rho_{1,1} & \rho_{1,2}\\ \rho_{2,1} & 0\end{pmatrix}$$

with $\rho_{1,1} \leq 4$, $1 \leq \rho_{2,1} \leq 3$, $\rho_{1,2} \geq 1$, With the notation introduced above, we have $r = 2$.

Since

$$4 = a_1 - c(\mu_1 - \mu_2) + (1-d) \geq a_2 = 3$$

$$2 = b_1 + c(\lambda_1 - \lambda_2) + (1-d) \geq b_2 = 1,$$

we can apply the formula of Theorem 10, which, substituting our data, turns out to be significantly simplified:

$$det_{1\leq s,t\leq 2}\left(x^{N_1(s,t)}\begin{bmatrix}-(s-t)+a_t-b_s+1\\ \lambda_s-s+t\end{bmatrix}\right),$$

where $N_1(s,t) = b_s(\lambda_s - s + t) + (-1)\left[\binom{s-t}{2}\right] + \binom{\lambda_s-s+t}{2}$.
Now, we have $N(1,1) = (2-1+1) + \binom{2}{2} = 2$; $N(1,2) = (2-1+2) + \binom{3}{2} = 5$; $N(2,1) = 0$; $N(2,2) = (1-2+2) = 1$, so we have to compute $det\begin{pmatrix} x^3\begin{bmatrix}4\\2\end{bmatrix} & x^6\begin{bmatrix}4\\3\end{bmatrix}\\ \begin{bmatrix}3\\0\end{bmatrix} & x\begin{bmatrix}3\\1\end{bmatrix}\end{pmatrix} =$ $det\begin{pmatrix} x^3(1+x^2)(1+x+x^2) & x^5(1+x)(1+x^2)\\ 1 & x(1+x+x^2)\end{pmatrix} = x^{10}+2x^9+3x^8+3x^7+3x^6+x^5+x^4$
For example, there are exactly 3 partitions with norm 8, namely

$$\begin{pmatrix}\mathbf{4} & 1\\ \mathbf{3} & 0\end{pmatrix}, \begin{pmatrix}\mathbf{4} & 2\\ \mathbf{2} & 0\end{pmatrix}, \begin{pmatrix}\mathbf{4} & 3\\ \mathbf{1} & 0\end{pmatrix}$$

◇

We see now how to construct the norm generating function for the partitions of Definition 7.

Theorem 12 (Krattenthaler, [32]). *Let c, d be arbitrary integers, λ a partition with $\lambda_r \geq r$ and let a, b be r-tuples of integers satisfying*

$$a_i - c - d \geq a_{i+1}$$

$$b_i + c(\lambda_i - \lambda_{i+1}) + (1 - d) \geq b_{i+1}$$

for $i = 1, 2, ..., r-1$. Then, denoting $N_1 = \sum_{i=1}^{r}(b_i(\lambda_i - i) + a_i + c\binom{\lambda_i - i}{2})$, the polynomial

$$x^{N_1} det_{1\leq s,t\leq r}\left(\begin{bmatrix}(\lambda_s - s)(1-c) + (1-c-d)(s-t) + a_t - b_s \\ \lambda_s - s\end{bmatrix}\right),$$

is the norm generating function for shifted (c, d)-plane partitions of shape λ in which the first part in row i is equal to a_i and the last part in row i is at least b_i.

Example 13. Let us consider the shifted $(1, 0)$-plane partitions of shape $\lambda = (3, 3, 3)$, such that $a = (6, 3, 1)$ and $b = (1, 1, 1)$. By definition, they are matrices

$$\begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} \\ 0 & \pi_{2,2} & \pi_{2,3} \\ 0 & 0 & \pi_{3,3} \end{pmatrix}$$

with $\pi_{1,1} = 6, \pi_{2,2} = 3, \pi_{3,3} = 1$. Moreover, $\pi_{1,3}, \pi_{2,3} \geq 1$.
We compute the norm generating function for these partitions, via Theorem 12.
First of all $N_1 = \sum_{i=1}^{r}(b_i(\lambda_i - i) + a_i + c\binom{\lambda_i - i}{2}) = 14$.
Then we have to compute each $m_{s,t} = \begin{bmatrix}(\lambda_s - s)(1-c)+(1-c-d)(s-t)+a_t-b_s \\ \lambda_s - s\end{bmatrix}$, $1 \leq s, t \leq r$ and then the determinant of the matrix $M = (m_{s,t})_{1\leq s,t\leq r}$.
We have:
$m_{1,1} = \begin{bmatrix}5\\2\end{bmatrix} = \frac{\prod_{i=1}^{5}(1-x^i)}{\prod_{i=1}^{2}(1-x^i)\cdot\prod_{i=1}^{3}(1-x^i)} = (x^2 + 1)(x^4 + x^3 + x^2 + x + 1)$
$m_{1,2} = \begin{bmatrix}2\\2\end{bmatrix} = 1$
$m_{1,3} = \begin{bmatrix}0\\2\end{bmatrix} = 0$
$m_{2,1} = \begin{bmatrix}5\\1\end{bmatrix} = \frac{\prod_{i=1}^{5}(1-x^i)}{\prod_{i=1}^{1}(1-x^i)\cdot\prod_{i=1}^{4}(1-x^i)} = x^4 + x^3 + x^2 + x + 1$
$m_{2,2} = \begin{bmatrix}2\\1\end{bmatrix} = \frac{\prod_{i=1}^{2}(1-x^i)}{\prod_{i=1}^{1}(1-x^i)\cdot\prod_{i=1}^{1}(1-x^i)} = x + 1$
$m_{2,3} = \begin{bmatrix}0\\1\end{bmatrix} = 0$
$m_{3,1} = m_{3,2} = m_{3,3} = 1$.
This way

$$M = \begin{pmatrix} (x^2+1)(x^4+x^3+x^2+x+1) & 1 & 0 \\ x^4+x^3+x^2+x+1 & x+1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

so $det(M) = x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + x$. The generating function is then $x^{14}det(M) = x^{15} + 2x^{16} + 3x^{17} + 3x^{18} + 3x^{19} + 2x^{20} + x^{21}$.
If we consider, for example, $n(\pi) = 17$, the coefficient of x^{17} in the above polynomial is 3, so it tells us that there are exactly three shifted $(1, 0)$-plane partitions of shape

$\lambda = (3, 3, 3)$, such that $a = (6, 3, 1)$ and $b = (1, 1, 1)$.
We can write them down for completeness'sake:

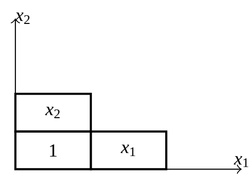
$$\begin{pmatrix} \mathbf{6} & 5 & 1 \\ 0 & \mathbf{3} & 1 \\ 0 & 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{6} & 4 & 2 \\ 0 & \mathbf{3} & 1 \\ 0 & 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{6} & 3 & 2 \\ 0 & \mathbf{3} & 2 \\ 0 & 0 & \mathbf{1} \end{pmatrix}$$

$\diamond$

4 Bar Code associated to a finite set of terms

In this section, we provide a language in order to represent zerodimensional monomial ideals, which are characterized by having a constant affine Hilbert polynomial.
In the case of two or three variables, this will allow us to establish a connection between (strongly) stable ideals $I \triangleleft \mathcal{P}$ with constant affine Hilbert polynomial $H_I(t) = p \in \mathbb{N}$ and some particular plane partitions of the integer number p. More precisely, we will give a combinatorial representation for the associated (finite) lexicographical Groebner escalier $\mathsf{N}(I)$.
First of all, we point out that, since $\mathcal{T} \cong \mathbb{N}^n$, a term $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ can be regarded as the point $(\gamma_1, ..., \gamma_n)$ in the n-dimensional space.
Using this convention, we can represent $\mathsf{N}(I)$ with a n-dimensional picture, called *tower structure* of I (for more details see [11] [39, II.33]).

Example 14. Consider the radical ideal $I = (x_1^2 - x_1, x_1x_2, x_2^2 - 2x_2) \triangleleft \mathbf{k}[x_1, x_2]$, defined by its lexicographical reduced Groebner basis. Since w.r.t. Lex[3], we have $\mathsf{T}(x_1^2 - x_1) = x_1^2$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$, we can conclude that the lexicographical Groebner escalier of I is $\mathsf{N}(I) = \{1, x_1, x_2\}$, so it can be represented by the following picture:



$\diamond$

For a radical ideal I, notice that if $|\mathsf{N}(I)| < \infty$ also $|V(I)| < \infty$ (and, more precisely, it holds $|\mathsf{N}(I)| = |V(I)|$), so the associated variety consists of a finite set of points.
It has been proved by Cerlienco-Mureddu ([13]) that, in this case, any ordering on the points in $V(I)$ gives a precise one-to-one correspondence between the terms in $\mathsf{N}(I)$ and the points in $V(I)$, so it is also possible to label the points in the tower structure with the corresponding point of the ordered $V(I)$.

[3] Since, in this paper, we are working with the lexicographical order, I precised here "w.r.t." Lex. Anyway, it can be easily observed that $\mathsf{T}(x_1^2 - x_1) = x_1^2$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$ trivially holds for each term order.

Example 15. Consider again the radical ideal $I = (x_1^2 - x_1, x_1x_2, x_2^2 - 2x_2) \triangleleft \mathbf{k}[x_1, x_2]$ of example 14. The corresponding variety can be easily computed and, actually, it is finite:

$$V(I) = \{(0,0), (0,2), (1,0)\}.$$

We can also note that, exactly as expected, $|\mathsf{N}(I)| = |V(I)| = 3$. The correspondence given by Cerlienco-Mureddu (see [13] for more details on how the correspondence is constructed) is displayed below; the corresponding reorderings of $V(I)$ are indicated in square brackets:

$$\begin{gathered} \Phi_1 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (0,0) \\ x_2 \mapsto (0,2) \\ x_1 \mapsto (1,0). \end{gathered}$$

$$[(0,0),(0,2),(1,0)];$$
$$[(0,0),(1,0),(0,2)].$$

$$\begin{gathered} \Phi_2 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (1,0) \\ x_2 \mapsto (0,2) \\ x_1 \mapsto (0,0). \end{gathered}$$

$$[(1,0),(0,0),(0,2)].$$

$$\begin{gathered} \Phi_3 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (1,0) \\ x_2 \mapsto (0,0) \\ x_1 \mapsto (0,2). \end{gathered}$$

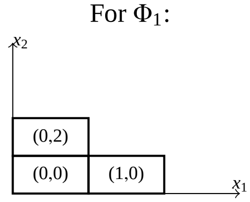
$$[(1,0),(0,2),(0,0)].$$

$$\begin{gathered} \Phi_4 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (0,2) \\ x_2 \mapsto (0,0) \\ x_1 \mapsto (1,0). \end{gathered}$$

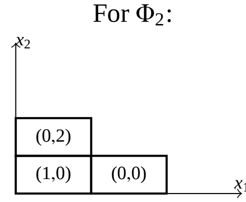
$$[(0,2),(0,0),(1,0)];$$
$$[(0,2),(1,0),(0,0)].$$

Now, we can label the points in the tower structure with the corresponding point of $V(I)$, as it can be seen in the pictures below.

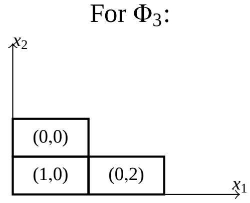
For Φ_1:



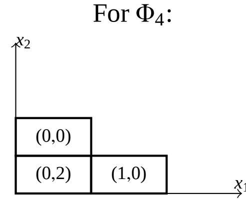
For Φ_2:



For Φ_3:



For Φ_4:



◇

The construction of Examples 14 and 15 is a sort of "inverse" of Macaulay's construction (see [37] p.548) in which from a finite order ideal N, a finite set of point $\mathbf{X}$ and a Groebner basis of $I(\mathbf{X})$ are produced so that the lexicographical Groebner escalier $\mathsf{N}(I(\mathbf{X}))$ is exactly N.

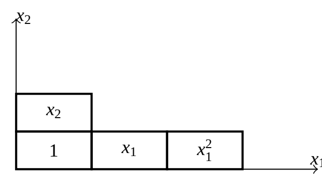
Example 16. For the case of two variables, the tower structure of a zerodimensional radical ideal I s.t. $V(I) = \{P_1, ..., P_s\}$ is represented by h towers, where h is the number of different values appearing as first coordinate of the points in $V(I)$, so that each tower corresponds to a "first coordinate". For each $1 \leq i \leq h$, the i-th tower contains as many elements as the number of occurrences of the associated first coordinate. Displaying these towers in nonincreasing order by height, one obtains a tower structure for I (see the one obtained in example 15 via the map Φ_1).

This is not the case for three or more variables, since some shifts in the towers' planes are needed. For example, given the zerodimensional radical ideal $I = (x_1^2 - x_1, x_1x_2, x_2^2 - x_2, x_1x_3 - x_3, x_2x_3, x_3^2 - x_3) \triangleleft \mathbf{k}[x_1, x_2, x_3]$, whose variety is

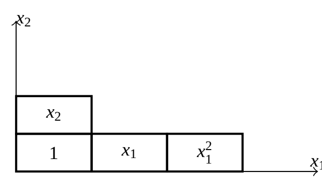
$$V(I) = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 0, 1)\},$$

we have $\mathsf{N}(I) = \{1, x_1, x_2, x_3\}$, which cannot be represented with a natural extension to three variables of the procedure explained above. In such an extension, the towers are in the $x(2)$ direction if the points have only the same first coordinate and in the $x(3)$ direction if both the first and the second coordinate are the same. ◇

Example 17. Let us consider the zerodimensional radical ideal $I = (x_1^3 - 3x_1^2 + 2x_1, x_1x_2, x_2^2 - 2x_2) \triangleleft \mathbf{k}[x_1, x_2]$, defined by its lexicographical reduced Groebner basis. Since, w.r.t. Lex, $\mathsf{T}(x_1^3 - 3x_1^2 + 2x_1) = x_1^3$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$, we can conclude that the lexicographical Groebner escalier of I is $\mathsf{N}(I) = \{1, x_1, x_1^2, x_2\}$, so it can be represented with the following picture:



Consider now the zerodimensional radical ideal $I' = (x_1^3 - x_1, x_1x_2, x_2^2 - 2x_2, x_3 + x_1^2 - x_1) \triangleleft \mathbf{k}[x_1, x_2, x_3]$, defined via its reduced lexicographical Groebner basis. Since w.r.t. Lex, we have $\mathsf{T}(x_1^3 - x_1) = x_1^3$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$, $\mathsf{T}(x_3 + x_1^2 - x_1) = x_3$, we can conclude that the lexicographical Groebner escalier of I' is $\mathsf{N}(I') = \{1, x_1, x_1^2, x_2\}$, so it can be represented with the following picture:



We point out that the tower structure above is exactly the same as for I, even if $I' \triangleleft \mathcal{P} = \mathbf{k}[x_1, x_2, x_3]$ and $I \triangleleft \mathbf{k}[x_1, x_2]$.

The reason of this fact is that $x_3 \notin \mathsf{N}(I')$; indeed, x_3 is the leading term of $x_3 + x_1^2 - x_1$. In general, the reason is that there is a polynomial $(x_3 - \sum_{t \in \mathsf{N}(I')} c_t t) \in I'$.

In a slightly different situation (i.e. in solving equations) the ability of detecting linear relations $\mod I'$ among the elements of $\{1, x_1, x_2, x_3\}$ and, equivalently, producing a basis of the vector space generated by $\{1, x_1, x_2, x_3\}$, $\mathbf{Span}(1, x_1, x_2, x_3) \mod I'$, is crucial (see [4, 36]).

This is the case, for instance of $I'' = (x_1^3 - x_1, x_1x_2, x_2^2 - 2x_2, x_3 - x_1) \triangleleft \mathbf{k}[x_1, x_2, x_3]$, where $\mathbf{Span}(1, x_1, x_2, x_3) = \mathbf{Span}(1, x_1, x_2) \mod I''$

◊

Unfortunately, as one can easily understand, the tower structure becomes rather complicated when we have an high number of terms in $\mathsf{N}(I)$ and/or of linearly *independent* variables in $\mathcal{P}$, i.e. when we deal with a large number of points, and/or we have really to draw the structure for high-dimensional spaces[4].

Moreover, as shown in example 17, from the tower structure it is impossible to understand the ring in which the Groebner escalier has been computed, since linearly dependent variables are discarded (see [36]).

For these reasons, we introduce now the *Bar Code diagram*, namely a (rather compact) *bidimensional picture* which keeps track of all the information contained in the tower structure, making them simple to be extracted.

We define now, in general, what is a Bar Code. After that, we see how to associate to a finite set of terms a Bar Code and, vice versa, how to associate a finite set of terms to a given Bar Code.

Definition 18. *A Bar Code* B *is a picture composed by segments, called* bars*, superimposed in horizontal rows, which satisfies conditions a., b. below. Denote by*

- $\mathsf{B}_j^{(i)}$ *the j-th bar (from left to right) of the i-th row (from top to bottom), i.e. the* j-th i-bar*;*
- $\mu(i)$ *the number of bars of the i-th row*
- $l_1(\mathsf{B}_j^{(1)}) := 1, \forall j \in \{1, 2, ..., \mu(1)\}$ *the* $(1-)$length *of the* 1*-bars;*
- $l_i(\mathsf{B}_j^{(k)}), 2 \leq k \leq n, 1 \leq i \leq k-1, 1 \leq j \leq \mu(k)$ *the* i-length *of* $\mathsf{B}_j^{(k)}$*, i.e. the number of i-bars lying over* $\mathsf{B}_j^{(k)}$

a. $\forall i, j, 1 \leq i \leq n-1, 1 \leq j \leq \mu(i), \exists! \overline{j} \in \{1, ..., \mu(i+1)\}$ *s.t.* $\mathsf{B}_{\overline{j}}^{(i+1)}$ *lies under* $\mathsf{B}_j^{(i)}$

b. $\forall i_1, i_2 \in \{1, ..., n\}, \sum_{j_1=1}^{\mu(i_1)} l_1(\mathsf{B}_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(\mathsf{B}_{j_2}^{(i_2)})$*; we will then say that* all the rows have the same length.

We denote by $\mathcal{B}_n$ the set of all Bar Codes composed by n rows.

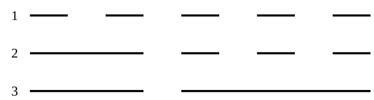
Note that if $1 \leq i_1 < i_2 \leq n$, $1 \leq j_1 \leq \mu(i_1)$, $1 \leq j_2 \leq \mu(i_2)$ and $\mathsf{B}_{j_2}^{(i_2)}$ lies below $\mathsf{B}_{j_1}^{(i_1)}$, then $l_1(\mathsf{B}_{j_2}^{(i_2)}) \geq l_1(\mathsf{B}_{j_1}^{(i_1)})$.

Definition 19. *We call* bar list *of a Bar Code* B*, composed by n rows, the list*

$$\mathsf{L}_{\mathsf{B}} := (\mu(1), ..., \mu(n)).$$

Example 20. An example of Bar Code B is

[4] Actually, in this context, "high-dimensional" means "of dimension greater than or equal to" 4.



The 1-bars have length 1. As regards the other rows, $l_1(\mathsf{B}_1^{(2)}) = 2$, $l_1(\mathsf{B}_2^{(2)}) = l_1(\mathsf{B}_3^{(2)}) = l_1(\mathsf{B}_4^{(2)}) = 1$, $l_2(\mathsf{B}_1^{(3)}) = 1$,$l_1(\mathsf{B}_1^{(3)}) = 2$ and $l_2(\mathsf{B}_2^{(3)}) = l_1(\mathsf{B}_2^{(3)}) = 3$, so

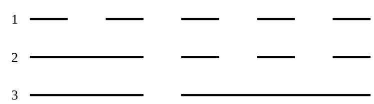
$$\sum_{j_1=1}^{\mu(1)} l_1(\mathsf{B}_{j_1}^{(1)}) = \sum_{j_2=1}^{\mu(2)} l_1(\mathsf{B}_{j_2}^{(2)}) = \sum_{j_3=1}^{\mu(3)} l_1(\mathsf{B}_{j_3}^{(3)}) = 5.$$

The bar list is $\mathsf{L}_\mathsf{B} := (5, 4, 2)$.

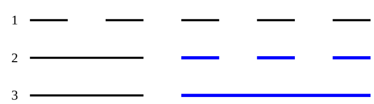
$\diamond$

Definition 21. *Given a Bar Code* B*, for each* $1 \leq l \leq n$*,* $l \leq i \leq n$*,* $1 \leq j \leq \mu(i)$*, an* l-block *associated to a bar* $B_j^{(i)}$ *of* B *is the set containing* $B_j^{(i)}$ *itself and all the bars of the* $(l-1)$ *rows lying immediately above* $B_j^{(i)}$*.*

Example 22. Take again the Bar Code B of example 20



Consider the bar $B_2^{(3)}$ (so $i = n = 3$, $j = 2 = \mu(3)$) and set $l = 2$. The 2-block associated to $B_2^{(3)}$ consists of $B_2^{(3)}$ itself and of the bars $B_2^{(2)}, B_3^{(2)}, B_4^{(2)}$, as shown by the thick blue lines in the picture below:



$\diamond$

We outline now the construction of the Bar Code associated to a finite set of terms. In order to do it, we need to introduce the operators P_{x_i}, $i = 1, ..., n$ on the terms.

First of all, we associate to each term $\tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$, n terms (one for each variable in $\mathcal{P}$). More precisely, for each $i \in \{1, ..., n\}$, we let

$$P_{x_i}(\tau) := x_i^{\gamma_i} \cdots x_n^{\gamma_n} \in \mathcal{T}, \text{ i.e. } P_{x_i}(\tau) = \frac{\tau}{x_1^{\gamma_1} \cdots x_{i-1}^{\gamma_{i-1}}}.$$

We can extend this procedure to a finite set of terms $M \subset \mathcal{T}$, defining, for each $i \in \{1, ..., n\}$,

$$M^{[i]} := P_{x_i}(M) := \{\sigma \in \mathcal{T}, \mid \exists \tau \in M, P_{x_i}(\tau) = \sigma\}.$$

The terms in $M^{[i]}$ will play a fundamental role for the construction of the Bar Code diagram.

Here we list some features of the operators P_{x_i}, that will be useful in what follows.

1. For each $\tau \in \mathcal{T}$, $P_{x_1}(\tau) = \tau$.

2. If $\tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, $\gamma_i = deg_i(\tau) = 0$ then $P_{x_i}(\tau) = x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n} = P_{x_{i+1}}(\tau)$.

3. It holds
$$\tau <_{Lex} \sigma \Rightarrow P_{x_i}(\tau) \leq_{Lex} P_{x_i}(\sigma),\ \forall i \in \{1, ..., n\}.$$

4. For each term τ and for any pair of indices i, j, say $1 \leq i < j \leq n$, we have that, since $x_i < x_j$,
$$P_{x_j}(P_{x_i}(\tau)) = P_{x_i}(P_{x_j}(\tau)) = P_{x_j}(\tau).$$

5. For each $\sigma, \tau \in \mathcal{T}$, $\forall 1 \leq i < n$, it holds
$$P_{x_i}(\tau) = P_{x_i}(\sigma) \Rightarrow P_{x_{i+1}}(\tau) = P_{x_{i+1}}(\sigma).$$

Example 23. Consider the term $\tau = x_1 x_2^3 x_3^4 \in \mathbf{k}[x_1, x_2, x_3]$.
Clearly $P_{x_1}(\tau) = x_1 x_2^3 x_3^4$, while $P_{x_2}(\tau) = x_2^3 x_3^4$ and $P_{x_3}(\tau) = x_3^4$. For $\sigma_1 := x_2 x_3^5 >_{Lex} \tau$, $P_{x_2}(\tau) = x_2^3 x_3^4 <_{Lex} P_{x_2}(\sigma_1) = x_2 x_3^5$ and $P_{x_3}(\tau) = x_3^4 <_{Lex} P_{x_3}(\sigma_1) = x_3^5$; for $\sigma_2 := x_1^5 x_2^3 x_3^4 >_{Lex} \tau$, $P_{x_2}(\tau) = x_2^3 x_3^4 = P_{x_2}(\sigma_2)$ and $P_{x_3}(\tau) = P_{x_3}(\sigma_2) = x_3^4$. Moreover, $P_{x_3}(P_{x_2}(\tau)) = P_{x_3}(x_2^3 x_3^4) = x_3^4 = P_{x_2}(P_{x_3}(\tau))$. ◇

Now we take $M \subseteq \mathcal{T}$, with $|M| = m < \infty$ and we order its elements increasingly w.r.t. Lex, getting the list $\overline{M} = [\tau_1, ..., \tau_m]$. Then, we construct the sets $M^{[i]}$, and the corresponding lexicographically ordered lists $\overline{M}^{[i]}$, for $i = 1, ..., n$. We notice that $\overline{M}$ cannot contain repeated terms, while the $\overline{M}^{[i]}$, for $1 < i \leq n$, can. In case some repeated terms occur in $\overline{M}^{[i]}$, $1 < i \leq n$, they clearly have to be adjacent in the list, due to the lexicographical ordering.
We can now define the $n \times m$ matrix of terms $\mathcal{M}$ as the matrix s.t. its i-th row is $\overline{M}^{[i]}$, $i = 1, ..., n$, i.e.

$$\mathcal{M} := \begin{pmatrix} P_{x_1}(\tau_1) & ... & P_{x_1}(\tau_m) \\ P_{x_2}(\tau_1) & ... & P_{x_2}(\tau_m) \\ \vdots & & \vdots \\ P_{x_n}(\tau_1) & ... & P_{x_n}(\tau_m) \end{pmatrix}$$

Definition 24. *The* Bar Code diagram B *associated to M (or, equivalently, to $\overline{M}$) is a $n \times m$ diagram, made by segments s.t. the i-th row of* B, $1 \leq i \leq n$ *is constructed as follows:*

1. *take the i-th row of $\mathcal{M}$, i.e. $\overline{M}^{[i]}$*

2. *consider all the sublists of repeated terms, i.e. $[P_{x_i}(\tau_{j_1}), P_{x_i}(\tau_{j_1+1}), ..., P_{x_i}(\tau_{j_1+h})]$ s.t. $P_{x_i}(\tau_{j_1}) = P_{x_i}(\tau_{j_1+1}) = ... = P_{x_i}(\tau_{j_1+h})$, noticing that*[5] $0 \leq h < m$

3. *underline each sublist with a segment*

4. *delete the terms of $\overline{M}^{[i]}$, leaving only the segments (i.e. the* i-bars*).*

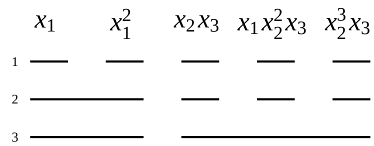
[5]Clearly if a term $P_{x_i}(\tau_{\overline{j}})$ is not repeated in $\overline{M}^{[i]}$, the sublist containing it will be only $[P_{x_i}(\tau_{\overline{j}})]$, i.e. $h = 0$.

We usually label each 1*-bar* $\mathsf{B}_j^{(1)}$*,* $j \in \{1, ..., \mu(1)\}$ *with the term* $\tau_j \in \overline{M}$*.*

By property 5. of the operators P_{x_i} and, since for each $1 \leq i \leq n$, $|\overline{M}^{[i]}| = \sum_{j=1}^{\mu(i)} l_1(\mathsf{B}_j^{(i)})$, a Bar Code diagram is a Bar Code in the sense of Definition 18.

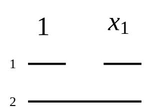
Example 25. Given $M = \{x_1, x_1^2, x_2x_3, x_1x_2^2x_3, x_2^3x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, we have:
$\overline{M}^{[1]} = [x_1, x_1^2, x_2x_3, x_1x_2^2x_3, x_2^3x_3]$
$\overline{M}^{[2]} = [1, 1, x_2x_3, x_2^2x_3, x_2^3x_3]$
$\overline{M}^{[3]} = [1, 1, x_3, x_3, x_3]$,
leading to the 3×5 table on the left and then to the Bar Code on the right:

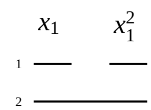
x_1	x_1^2	x_2x_3	$x_1x_2^2x_3$	$x_2^3x_3$
1	1	x_2x_3	$x_2^2x_3$	$x_2^3x_3$
1	1	x_3	x_3	x_3



◊

Remark 26. We can easily observe that Bar Codes associated to different sets of terms, *need not* to be different.
For example, if $M := \{1, x_1\}$, $M' := \{x_1, x_1^2\} \subset \mathbf{k}[x_1, x_2]$, both the Bar Code B associated to M and the Bar Code B' associated to M' are





We will see soon that this cannot happen for order ideals.

Now we explain how to associate a finite set of terms M_{B} to a given Bar Code B. In order to do it, we have to follow the steps below:

BC1 consider the n-th row, composed by the bars $B_1^{(n)}, ..., B_{\mu(n)}^{(n)}$. Let $l_1(B_j^{(n)}) = \ell_j^{(n)}$, for $j \in \{1, ..., \mu(n)\}$ and $a_1, ..., a_{\mu(n)} \in \mathbb{N}$, s.t. $a_k < a_h$ if $k < h$. Label each bar $B_j^{(n)}$ with $\ell_j^{(n)}$ copies of $x_n^{a_j}$.

BC2 For each $i = 1, ..., n-1$, $1 \leq j \leq \mu(n-i+1)$ consider the bar $B_j^{(n-i+1)}$ and suppose that it has been labelled by $\ell_j^{(n-i+1)}$ copies of a term τ. Construct the 2-block associated to $B_j^{(n-i+1)}$ which, by definition, is composed by $B_j^{(n-i+1)}$ and by all the $(n-i)$-bars $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$, lying immediately above $B_j^{(n-i+1)}$; note that h satisfies $0 \leq h \leq \mu(n-i) - \overline{j}$.
Denote the 1-lenghts of $B_{\overline{j}}^{(n-i)} ... \, B_{\overline{j}+h}^{(n-i)}$ by $l_1(B_{\overline{j}}^{(n-i)}) = \ell_{\overline{j}}^{(n-i)}, ..., l_1(B_{\overline{j}+h}^{(n-i)}) = \ell_{\overline{j}+h}^{(n-i)}$ and fix $h+1$ natural numbers $a_{\overline{j}} < a_{\overline{j}+1} < ... < a_{\overline{j}+h}$. For each $0 \leq k \leq h$, label $B_{\overline{j}+k}^{(n-i)}$ with $\ell_{\overline{j}+k}^{(n-i)}$ copies of $\tau x_{n-i}^{a_{\overline{j}+k}}$.

Clearly, if, given a Bar Code B, we apply BC1 and BC2 to get a set $M \subset \mathcal{T}$, and then we construct the Bar Code associated to M, we get back B. Indeed, BC1 and BC2 exactly construct the elements of the ordered lists $\overline{M}^{[i]}$, $i = 1, ..., n$.

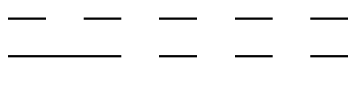
Given a Bar Code B, applying steps BC1 and BC2, we can generate an *infinite* number of sets $M \subset \mathcal{T}$.
We modify the steps BC1 and BC2 getting BbC1 and BbC2 so that, for each Bar Code B, the set of terms generated by applying them turns out to be *unique*:

BbC1 consider the n-th row, composed by the bars $B_1^{(n)}, ..., B_{\mu(n)}^{(n)}$. Let $l_1(B_j^{(n)}) = \ell_j^{(n)}$, for $j \in \{1, ..., \mu(n)\}$. Label each bar $B_j^{(n)}$ with $\ell_j^{(n)}$ copies of x_n^{j-1}.

BbC2 For each $i = 1, ..., n-1$, $1 \leq j \leq \mu(n-i+1)$ consider the bar $B_j^{(n-i+1)}$ and suppose that it has been labelled by $\ell_j^{(n-i+1)}$ copies of a term τ. Construct the 2-block associated to $B_j^{(n-i+1)}$ which, by definition, is composed by $B_j^{(n-i+1)}$ and by all the $(n-i)$-bars $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ lying immediately above $B_j^{(n-i+1)}$; note that h satisfies $0 \leq h \leq \mu(n-i) - \overline{j}$. Denote the 1-lenghts of $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ by $l_1(B_{\overline{j}}^{(n-i)}) = \ell_{\overline{j}}^{(n-i)}$,..., $l_1(B_{\overline{j}+h}^{(n-i)}) = \ell_{\overline{j}+h}^{(n-i)}$. For each $0 \leq k \leq h$, label $B_{\overline{j}+k}^{(n-i)}$ with $\ell_{\overline{j}+k}^{(n-i)}$ copies of τx_{n-i}^k.

It is important to notice that not all Bar Codes can be associated to order ideals, as easily shown by the example below.

Example 27. Consider the Bar Code B



We cannot associate any order ideal to it.
Indeed, using either BC1, BC2 or BbC1,BbC2, we obtain terms of the form

$$\begin{array}{ccccc} x_1^{\alpha_1}x_2^{\beta_1}x_3^{\gamma_1} & x_1^{\alpha_2}x_2^{\beta_1}x_3^{\gamma_1} & x_1^{\alpha_3}x_2^{\delta_1}x_3^{\gamma_2} & x_1^{\alpha_4}x_2^{\delta_2}x_3^{\gamma_2} & x_1^{\alpha_5}x_2^{\delta_3}x_3^{\gamma_2} \\ x_2^{\beta_1}x_3^{\gamma_1} & x_2^{\beta_1}x_3^{\gamma_1} & x_2^{\delta_1}x_3^{\gamma_2} & x_2^{\delta_2}x_3^{\gamma_2} & x_2^{\delta_3}x_3^{\gamma_2} \\ x_3^{\gamma_1} & x_3^{\gamma_1} & x_3^{\gamma_2} & x_3^{\gamma_2} & x_3^{\gamma_2} \end{array},$$

with $\gamma_1 < \gamma_2$, $\delta_1 < \delta_2 < \delta_3$, $\alpha_1 < \alpha_2$ and so the associated set of terms M turns out to be

$$M = \{x_1^{\alpha_1}x_2^{\beta_1}x_3^{\gamma_1}, x_1^{\alpha_2}x_2^{\beta_1}x_3^{\gamma_1}, x_1^{\alpha_3}x_2^{\delta_1}x_3^{\gamma_2}, x_1^{\alpha_4}x_2^{\delta_2}x_3^{\gamma_2}, x_1^{\alpha_5}x_2^{\delta_3}x_3^{\gamma_2}\}.$$

To be an order ideal, M must contain all the divisors of its elements:

$$\forall \tau \in M, \text{ if } \sigma \mid \tau \text{ then } \sigma \in M,$$

so we have to lay down some conditions on the exponents.
Let us start examining $x_1^{\alpha_1}x_2^{\beta_1}x_3^{\gamma_1}$ and $x_1^{\alpha_2}x_2^{\beta_1}x_3^{\gamma_1}$. Knowing that $\alpha_1 < \alpha_2$, we need to take $\alpha_1 = 0$ and $\alpha_2 = 1$. Indeed, otherwise, M should contain at least another term of the form $x_1^{\alpha_0}x_2^{\beta_1}x_3^{\gamma_1}$, $\alpha_0 \neq \alpha_1, \alpha_2$ and $\alpha_0 < \max(\alpha_1, \alpha_2)$. The exponent β_1 must be equal to

zero, otherwise at least $x_1^{\alpha_1}x_2^{\beta_1-1}x_3^{\gamma_1}$ and $x_1^{\alpha_2}x_2^{\beta_1-1}x_3^{\gamma_1}$ would belong to M. For analogous reasons, we have to choose $\gamma_1 = 0, \gamma_2 = 1$ and $\alpha_3 = \alpha_4 = \alpha_5 = 0$. We get

$$M = \{1, x_1, x_2^{\delta_1}x_3, x_2^{\delta_2}x_3, x_2^{\delta_3}x_3\}.$$

But let us examine $\delta_1 < \delta_2 < \delta_3$. Similarly to what said for the other exponents, we have only one possible choice for them, i.e. $\delta_1 = 0$, $\delta_2 = 1\,\delta_3 = 2$[6], but then also x_2 and x_2^2 should belong to M, and this is impossible: there is only one possible power of x_2 for $\gamma_1 = 0$ and this contradiction proves that B cannot be associated to any order ideal. ◇

Inspired by example 27, we define *admissible Bar Codes* as follows:

Definition 28. *A Bar Code* B *is* admissible *if the set M obtained by applying BbC*1 *and BbC*2 *to* B *is an order ideal.*

Remark 29. By definition of order ideal, using BbC1 and BbC2 is the only way an order ideal can be associated to an admissible Bar Code. Indeed, if we label two consecutive bars with two terms $\tau x_i^{a_i}$, $\tau x_i^{a_i+h}$, $h > 1$, then also the terms σ with $P_{x_i}(\sigma) = \tau x_i^{a_i+1}$ would belong to M and it would have to label a bar between those labelled by $\tau x_i^{a_i}$ and $\tau x_i^{a_i+h}$, giving a contradiction.

We need now an *admissibility criterion* for Bar Codes. In order to be able to state it, we start with the following trivial lemma.

Lemma 30. *Given a set $M \subset \mathcal{T}$, the following conditions are equivalent*

1. *M is an order ideal.*
2. *$\forall \tau \in M$, if $\sigma \mid \tau$, then $\sigma \in M$.*
3. *$\forall \tau \in M$ each predecessor of τ belongs to M.*

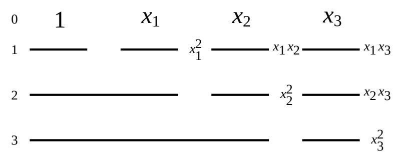
We give then the definition of *e-list*, associated to each 1-bar of a given Bar Code.

Definition 31. *Given a Bar Code* B*, let us consider a* 1*-bar* $B_{j_1}^{(1)}$*, with* $j_1 \in \{1, ..., \mu(1)\}$*. The* e-list *associated to* $B_{j_1}^{(1)}$ *is the n-tuple* $e(B_{j_1}^{(1)}) := (b_{j_1,1}, ..., b_{j_1,n})$*, defined as follows:*

- *consider the n-bar $B_{j_n}^{(n)}$, lying under $B_{j_1}^{(1)}$. The number of n-bars on the left of $B_{j_n}^{(n)}$ is $b_{j_1,n}$.*
- *for each $i = 1, ..., n-1$, let $B_{j_{n-i+1}}^{(n-i+1)}$ and $B_{j_{n-i}}^{(n-i)}$ be the $(n-i+1)$-bar and the $(n-i)$-bar lying under $B_{j_1}^{(1)}$. Consider the $(n-i+1)$-block associated to $B_{j_{n-i+1}}^{(n-i+1)}$. The number of $(n-i)$-bars of the block, which lie on the left of $B_{j_{n-i}}^{(n-i)}$ is $b_{j_1,n-i}$.*

Example 32. For the Bar Code B

[6]Notice that these assignments are those given by BbC1 and BbC2.



the e-lists are $e(B_1^{(1)}) := (0,0,0)$; $e(B_2^{(1)}) := (1,0,0)$; $e(B_3^{(1)}) := (0,1,0)$ and $e(B_4^{(1)}) := (0,0,1)$. ◊

Remark 33. Given a Bar Code B, fix a 1-bar $B_j^{(1)}$, with $j \in \{1, ..., \mu(1)\}$.
Comparing definition 31 and the steps BbC1 and BbC2 described above, we can observe that the values of the e-list $e(B_j^{(1)}) := (b_{j,1}, ..., b_{j,n})$ are exactly the exponents of the term labelling $B_j^{(1)}$, obtained applying BbC1 and BbC2 to B.

Proposition 34 (Admissibility criterion)**.** *A Bar Code* B *is admissible if and only if, for each* 1*-bar* $\mathsf{B}_j^{(1)}$, $j \in \{1, ..., \mu(1)\}$, *the e-list* $e(\mathsf{B}_j^{(1)}) = (b_{j,1}, ..., b_{j,n})$ *satisfies the following condition:* $\forall k \in \{1, ..., n\}$ *s.t.* $b_{j,k} > 0$, $\exists \overline{j} \in \{1, ..., \mu(1)\} \setminus \{j\}$ *s.t.*

$$e(\mathsf{B}_{\overline{j}}^{(1)}) = (b_{j,1}, ..., b_{j,k-1}, (b_{j,k}) - 1, b_{j,k+1}, ..., b_{j,n}).$$

Proof. It is a trivial consequence of Lemma 30 and Remark 33. □

Consider the following sets

$$\mathcal{A}_n := \{\mathsf{B} \in \mathcal{B}_n \text{ s.t. } \mathsf{B} \text{ admissible}\}$$

$$\mathcal{N}_n := \{\mathsf{N} \subset \mathcal{T},\ |\mathsf{N}| < \infty \text{ s.t. } \mathsf{N} \text{ order ideal}\}.$$

We can define the map

$$\eta : \mathcal{A}_n \to \mathcal{N}_n$$
$$\mathsf{B} \mapsto \mathsf{N},$$

where N is the order ideal obtained applying BbC1 and BbC2 to B.
By BbC1 and BbC2, η is a function; it is trivially surjective. Moreover, it is injective since, if $\mathsf{B}, \mathsf{B}' \in \mathcal{A}_n$ and $\mathsf{B} \neq \mathsf{B}'$ they have at least one pair of indices i, j s.t. $l_1(\mathsf{B}_j^{(i)}) \neq l_1(\mathsf{B}'^{(i)}_j)$ and this changes the result of the application of BbC1/BbC2.
From the arguments above, we can then deduce that there is a biunivocal correspondence between admissible n-Bar Codes and finite order ideals of $\mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$.

In the Lemma below we state some properties of admissible Bar Codes related to lengths.

Lemma 35. *If* B *is an admissible Bar Code, the following two conditions hold:*

a) $l_{n-1}(\mathsf{B}_1^{(n)}) \geq ... \geq l_{n-1}(\mathsf{B}_{\mu(n)}^{(n)})$

b) $\forall 1 \leq i \leq n-2, \forall 1 \leq j \leq \mu(i+2)$ *take the* $(i+2)$*-bar* $\mathsf{B}_j^{(i+2)}$ *and let* $\mathsf{B}_{j_1}^{(i+1)}, ..., \mathsf{B}_{j_1+h}^{(i+1)}$ *(where h satisfies* $h \in \{0, ..., \mu(i+1) - j_1\}$*) be the* $(i+1)$*-bars lying over* $\mathsf{B}_j^{(i+2)}$. *Then* $l_i(\mathsf{B}_{j_1}^{(i+1)}) \geq ... \geq l_i(\mathsf{B}_{j_1+h}^{(i+1)})$.

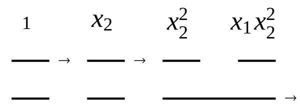
Proof. Let us start proving a). If for some $1 \leq l \leq \mu(n) - 1$ it holds $l_{n-1}(\mathsf{B}^{(n)}_l) < l_{n-1}(\mathsf{B}^{(n)}_{l+1})$ the Bar Code would be not admissible. Indeed, let $\mathsf{B}^{(1)}_k$ be the rightmost 1-bar over $\mathsf{B}^{(n)}_{l+1}$ and $e(\mathsf{B}^{(1)}_k) = (b_{k,1}, ..., b_{k,n})$ be its e-list. By construction (see Definition 31), $b_{k,n-1} = l_{n-1}(\mathsf{B}^{(n)}_{l+1}) - 1$. Now, this proves that there cannot exist a 1-bar labelling $(b_{k,1}, ..., b_{k,n-1}, b_{k,n} - 1)$, since $l_{n-1}(\mathsf{B}^{(n)}_l) < l_{n-1}(\mathsf{B}^{(n)}_{l+1})$ and so the 1-bars $\mathsf{B}^{(1)}_{\overline{k}}$ over $\mathsf{B}^{(n)}_l$ have $b_{\overline{k},n-1} \leq l_{n-1}(\mathsf{B}^{(n)}_l) - 1 < l_{n-1}(\mathsf{B}^{(n)}_{l+1}) - 1 = b_{k,n-1}$, contradicting the assumption of admissibility (see Proposition 34).

An analogous argument proves that if for some $\forall 1 \leq i \leq n-2$, $\forall 1 \leq j \leq \mu(i+2)$ we take the $(i+2)$-bar $\mathsf{B}^{(i+2)}_j$ and $\mathsf{B}^{(i+2)}_{j_1+h}$ s.t. h satisfies $h \in \{0, ..., \mu(i+1) - j_1\}$ is the $(i+1)$-bars lying over $\mathsf{B}^{(i+2)}_j$, it happens that for a fixed $l \in \{1, ..., \mu(i+1) - 1 - j_1\}$ $l_i(\mathsf{B}^{(i+1)}_{j_1+l}) < l_i(\mathsf{B}^{(i+1)}_{j_1+l+1})$, B is not admissible and so also b) is true. □

In what follows, unless differently specified, we always consider admissible Bar Codes, so, in general, we will omit the word "admissible".

Remark 36. In principle, it is possible to represent with a Bar Code also infinite order ideals, by means of a simple modification, i.e. the introduction of the symbol "$\rightarrow$" immediately after a l-bar for some $1 \leq l \leq n$, meaning that there should actually be infinitely many l-blocks equal to that containing that bar.

For example, the Bar Code of $I = (x_1^2 x_2^2) \triangleleft \mathbf{k}[x_1, x_2]$, whose lexicographical Groebner escalier is $\mathsf{N}(I) = \{x_1^{h_1} x_2^{h_2}, x_1^{h_3} x_2^{h_4},\ h_1, h_4 \in \mathbb{N}, h_2, h_3 \in \{0, 1\}\}$, turns out to be



In particular, the arrow on the right of 1 represents the terms of the form $x_1^{h_1}$, $h_1 \in \mathbb{N} \setminus \{0\}$, the one on the right of x_2 represents the terms of the form $x_1^{h_1} x_2$, $h_1 \in \mathbb{N} \setminus \{0\}$; finally the bottom arrow represents the terms of the form $x_2^{h_4}, x_1 x_2^{h_4}$, $h_4 \in \mathbb{N}$, $h_4 > 2$. Since infinite Bar Codes are out of the topics of this paper, we will not treat them in detail.

5 The star set

Up to this point, we have discussed the link between Bar Codes and order ideals, i.e. we focused on the link between Bar Codes and Groebner escaliers of monomial ideals.

In this section, we show that, given a Bar Code B and the order ideal $\mathsf{N} = \eta(\mathsf{B})$ it is possible to deduce a very specific generating set for the monomial ideal I s.t. $\mathsf{N}(I) = \mathsf{N}$.

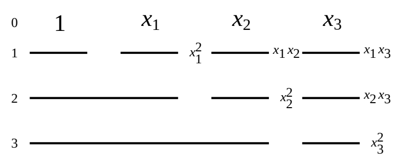
Definition 37. *The* star set *of an order ideal* N *and of its associated Bar Code* $\mathsf{B} = \eta^{-1}(\mathsf{N})$ *is a set* $\mathcal{F}_{\mathsf{N}}$ *constructed as follows:*

a) $\forall 1 \leq i \leq n$, *let* τ_i *be a term which labels a* 1*-bar lying over* $\mathsf{B}^{(i)}_{\mu(i)}$, *then* $x_i P_{x_i}(\tau_i) \in \mathcal{F}_{\mathsf{N}}$*;*

b) $\forall 1 \leq i \leq n-1, \forall 1 \leq j \leq \mu(i)-1$ *let* $\mathsf{B}_j^{(i)}$ *and* $\mathsf{B}_{j+1}^{(i)}$ *be two consecutive bars not lying over the same* $(i+1)$*-bar and let* $\tau_j^{(i)}$ *be a term which labels a* 1*-bar lying over* $\mathsf{B}_j^{(i)}$*, then* $x_i P_{x_i}(\tau_j^{(i)}) \in \mathcal{F}_{\mathsf{N}}$.

We usually represent $\mathcal{F}_{\mathsf{N}}$ within the associated Bar Code B, inserting each $\tau \in \mathcal{F}_{\mathsf{N}}$ on the right of the bar from which it is deduced. Reading the terms from left to right and from the top to the bottom, $\mathcal{F}_{\mathsf{N}}$ is ordered w.r.t. Lex.

Example 38. For $\mathsf{N} = \{1, x_1, x_2, x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, associated to the Bar Code of example 32, we have $\mathcal{F}_{\mathsf{N}} = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}$; looking at Definition 37, we can see that the terms x_1x_3, x_2x_3, x_3^2 come from a), whereas the terms x_1^2, x_1x_2, x_2^2 come from b).



◇

In [12], given a monomial ideal I, the authors define the following set, calling it *star set*:

$$\mathcal{F}(I) = \left\{ x^\gamma \in \mathcal{T} \setminus \mathsf{N}(I) \,\middle|\, \frac{x^\gamma}{\min(x^\gamma)} \in \mathsf{N}(I) \right\}.$$

We can prove the following proposition, which connects the definition above to our construction.

Proposition 39. *With the above notation* $\mathcal{F}_{\mathsf{N}} = \mathcal{F}(I)$.

Proof. We start proving $\mathcal{F}_{\mathsf{N}} \subseteq \mathcal{F}(I)$.
Consider $\sigma \in \mathcal{F}_{\mathsf{N}}$; by definition of $\mathcal{F}_{\mathsf{N}}$ there are two possibilities

a) $\sigma = x_i P_{x_i}(\tau_i)$, with $1 \leq i \leq n$ and τ_i a term which labels a 1-bar lying over $\mathsf{B}_{\mu(i)}^{(i)}$;

b) $\sigma = x_i P_{x_i}(\tau_j^{(i)})$, with $1 \leq i \leq n-1$, $1 \leq j \leq \mu(i)-1$ $\tau_j^{(i)}$ a term which labels a 1-bar lying over $\mathsf{B}_j^{(i)}$, under the condition that $\mathsf{B}_j^{(i)}$ $\mathsf{B}_{j+1}^{(i)}$ do not lie over the same $(i+1)$-bar.

Let us examine a) and b) separately.

a) By definition, $\sigma > \tau_i$; indeed $\deg_h(\sigma) = \deg_h(\tau_i)$ for $i+1 \leq h \leq n$ and $\deg_i(\sigma) > \deg_i(\tau_i)$. Clearly, $\sigma \notin \mathsf{N}$, because if it was in the Groebner escalier, applying the steps described in Definition 24, $P_{x_i}(\sigma) = \sigma = x_i P_{x_i}(\tau_i)$ would be put in a list that is subsequent to the one containing $P_{x_i}(\tau_i)$, but, in this case, there would be $\mu(i)+1$ i-bars instead of $\mu(i)$, contradicting the definition of $\mu(i)$. Since $\min(\sigma) = x_i$, $\frac{\sigma}{\min(\sigma)} = P_{x_i}(\tau_i) \mid \tau_i$, so $\frac{\sigma}{\min(\sigma)} \in \mathsf{N}$ and $\sigma \in \mathcal{F}(I)$.

b) Analogously to case a), $\sigma > \tau_j^{(i)}$. Let us prove that $\sigma \notin \mathsf{N}$. If $\sigma \in \mathsf{N}$ then σ would label a 1-bar lying over $\mathsf{B}_{j+1}^{(i)}$ but, since $P_{x_{i+1}}(\sigma) = P_{x_{i+1}}(\tau_j^{(i)})$, $\mathsf{B}_j^{(i)}$ $\mathsf{B}_{j+1}^{(i)}$ would lie over the same $(i+1)$-bar, contradicting the hypothesis. As above, since $\min(\sigma) = x_i$, $\frac{\sigma}{\min(\sigma)} = P_{x_i}(\tau_j^{(i)}) \mid \tau_j^{(i)}$, so $\frac{\sigma}{\min(\sigma)} \in \mathsf{N}$ and $\sigma \in \mathcal{F}(I)$.

We prove now that $\mathcal{F}_{\mathsf{N}} \supseteq \mathcal{F}(I)$.
Let us consider $\sigma \in \mathcal{F}(I)$ and let $\min(\sigma) = x_i$, $1 \leq i \leq n$. By definition of $\mathcal{F}(I)$, $\sigma \notin \mathsf{N}$ and $\widetilde{\sigma} := \frac{\sigma}{x_i} \in \mathsf{N}$, so it labels a 1-bar lying over some i-bar $\mathsf{B}_j^{(i)}$. Denote by $\mathsf{B}_{\overline{j}}^{(1)}, ..., \mathsf{B}_{\overline{j}+h}^{(1)}$ (where h satisfies $0 \leq h \leq \mu(i) - \overline{j}$) the 1-bars lying over $\mathsf{B}_j^{(i)}$. Two possibilities may occur:

a) $\overline{j} + h = \mu(i)$; in this case $x_i P_{x_i}(\widetilde{\sigma}) = \sigma \in \mathcal{F}_{\mathsf{N}}$ by Definition 37.

b) otherwise consider the term $\tau_{\overline{j}+h}$, which labels $\mathsf{B}_{\overline{j}+h}^{(1)}$, and the subsequent term $\tau_{\overline{j}+h+1}$, labelling $\mathsf{B}_{\overline{j}+h+1}^{(1)}$. Notice that $P_{x_i}(\tau_{\overline{j}+h}) = P_{x_i}(\widetilde{\sigma})$. By Definition 24, $\tau_{\overline{j}+h} <_{Lex} \tau_{\overline{j}+h+1}$. If $P_{x_i}(\tau_{\overline{j}+h}) = P_{x_i}(\tau_{\overline{j}+h+1})$ this would contradict the maximality of h, so, by property 3. of the operators P_{x_i}, it must be $P_{x_i}(\tau_{\overline{j}+h}) <_{Lex} P_{x_i}(\tau_{\overline{j}+h+1})$. But, if $P_{x_{i+1}}(\tau_{\overline{j}+h}) = P_{x_{i+1}}(\tau_{\overline{j}+h+1})$, then $\sigma \mid \tau_{\overline{j}+h+1}$ and so $\sigma \in \mathsf{N}$, that is impossible since $\sigma \in \mathcal{F}(I)$. This means then that $P_{x_{i+1}}(\tau_{\overline{j}+h}) <_{Lex} P_{x_{i+1}}(\tau_{\overline{j}+h+1})$, so we can deduce that $\mathsf{B}_{\overline{j}+h}^{(1)}$ and $\mathsf{B}_{\overline{j}+h+1}^{(1)}$ lie over two consecutive i-bars not lying over the same $(i+1)$-bar, so $\sigma = x_i P_{x_i}(\widetilde{\sigma}) = x_i P_{x_i}(\tau_{\overline{j}+h}) \in \mathcal{F}_{\mathsf{N}}$.

□

Remark 40. By Proposition 39, being $\mathcal{F}_{\mathsf{N}} = \mathcal{F}(I)$, it trivially holds $\mathsf{G}(I) \subseteq \mathcal{F}_{\mathsf{N}} \subseteq \mathsf{B}(I)$. In general, the inclusions may be strict; if $\mathcal{F}_{\mathsf{N}} = \mathsf{G}(I)$, we say that $\mathsf{B}_{\mathsf{N}} := \eta^{-1}(\mathsf{N})$ is a *full* Bar Code.

The star set $\mathcal{F}(I)$ of a monomial ideal I is strongly connected to Janet's theory [27, 28, 29, 30] and to the notion of Pommaret basis [43, 44, 48], as explicitly pointed out in [12]. For completeness sake, we recall it below.

Definition 41. *[27, ppg.75-9] Let $M \subset \mathcal{T}$ be a set of terms and $\tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ be an element of M. A variable x_j is called* multiplicative *for τ with respect to M if there is no term in M of the form $\tau' = x_1^{\delta_1} \cdots x_j^{\delta_j} x_{j+1}^{\gamma_{j+1}} \cdots x_n^{\gamma_n}$ with $\delta_j > \gamma_j$. We will denote by $mult_M(\tau)$ the set of multiplicative variables for τ with respect to M.*

Definition 42. *With the previous notation, the* cone *of τ with respect to M is the set*

$$C_M(\tau) := \{\tau x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid \textit{where } \lambda_j \neq 0 \textit{ only if } x_j \textit{ is multiplicative for } \tau \textit{ w.r.t. } M\}.$$

Definition 43. *[27, ppg.75-9] A set of terms $M \subset \mathcal{T}$ is called* complete *if for every $\tau \in M$ and $x_j \notin mult_M(\tau)$, there exists $\tau' \in M$ such that $x_j \tau \in C_M(\tau')$.*

Moreover, M is stably complete *[48, 12] if it is complete and for every $\tau \in M$ it holds $mult_M(\tau) = \{x_i \mid x_i \leq \min(\tau)\}$.*
If a set M is stably complete and finite, then it is the Pommaret basis *of $I = (M)$.*

Theorem 44. *For every monomial ideal I, the star set $\mathcal{F}(I)$ is the unique stably complete system of generators of I. Hence, if M is stably complete, $M = \mathcal{F}((M))$.*

By Proposition 39, the Bar Code gives a simple way to deduce the star set from the Groebner escalier of a zerodimensional monomial ideal.

6 Counting stable ideals

In this section, we connect the Bar Code associated to the Groebner escalier of a stable monomial ideal to the theory of integer and plane partitions, in order to find the number of stable ideals in two or three variables with constant affine Hilbert polynomial $H_{\_}(t) = p \in \mathbb{N}$.

We start recalling some definitions and known facts about stable and strongly stable ideals.

Definition 45. ([28][pg.41], [30]) (c.f.[39][IV.pg.673,679]) *A monomial ideal $J \triangleleft \mathcal{P} = \mathbf{k}[x_1, ..., x_n]$ is called* stable [19] *if it holds*

$$\tau \in J,\ x_j > \min(\tau) \Longrightarrow \frac{x_j \tau}{\min(\tau)} \in J.$$

Definition 46 ([46, 47, 23, 24, 21, 42])**.** *A monomial ideal $I \triangleleft \mathcal{P} = \mathbf{k}[x_1, ..., x_n]$ is called* strongly stable [3, 2] *if, for every term $\tau \in I$ and pair of variables $x_i,\ x_j$ such that $x_i | \tau$ and $x_i < x_j$, then also $\frac{\tau x_j}{x_i}$ belongs to I or, equivalently, for every $\sigma \in \mathsf{N}(I)$, and pair of variables $x_i,\ x_j$ such that $x_i | \sigma$ and $x_i > x_j$, then also $\frac{\sigma x_j}{x_i}$ belongs to $\mathsf{N}(I)$.*

It is well known that, in order to verify the (strong) stability of a monomial ideal, we can verify the conditions above for the terms in $\mathsf{G}(I)$.

Example 47 ([12]). In $k[x_1, x_2, x_3]$ with $x_1 < x_2 < x_3$:

- the ideal $I_1 = (x_1^3, x_1x_2, x_2^2, x_1^2x_3, x_2x_3, x_3^2)$ is stable.
  Indeed, we have:
  $\frac{(x_1^3)x_2}{x_1} = x_1^2x_2 \in I_1$,
  $\frac{(x_1^3)x_3}{x_1} = x_1^2x_3 \in I_1$,
  $\frac{(x_1x_2)x_2}{x_1} = x_2^2 \in I_1$,
  $\frac{(x_1x_2)x_3}{x_1} = x_2x_3 \in I_1$,
  $\frac{(x_2)^2x_3}{x_2} = x_2x_3 \in I_1$,
  $\frac{(x_1^2x_3)x_2}{x_1} = x_1x_2x_3 \in I_1$,
  $\frac{(x_1^2x_3)x_3}{x_1} = x_1x_3^2 \in I_1$,
  and $\frac{(x_2x_3)x_3}{x_2} = x_2x_3^2 \in I_1$.
  Anyway, it is not strongly stable, since $x_1x_2 \in I_1$, but $\frac{(x_1x_2)x_3}{x_2} = x_1x_3 \notin I_1$;

- the ideal $I_2 = (x_1^2, x_1x_2, x_2^2, x_3)$ is strongly stable, since
$\frac{(x_1^2)x_2}{x_1} = x_1x_2 \in I_2$,
$\frac{(x_1^2)x_3}{x_1} = x_1x_3 \in I_2$,
$\frac{(x_1x_2)x_2}{x_1} = x_1x_2^2 \in I_2$,
$\frac{(x_1x_2)x_3}{x_1} = x_2x_3 \in I_2$,
$\frac{(x_1x_2)x_3}{x_2} = x_1x_3 \in I_2$,
$\frac{(x_2^2)x_3}{x_2} = x_2x_3 \in I_2$

$\diamond$

Proposition 48 ([12])**.** *Let J be a monomial ideal. Then TFAE:*

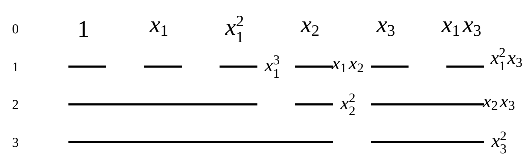
i) *J is stable*

ii) $\mathcal{F}(J) = \mathsf{G}(J)$

A simple property, useful for what follows, and trivially following from Remark 40 and Proposition 48, is that Bar Codes of (strongly) stable ideals are *full*.

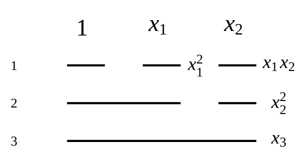
Example 49. In $\mathbf{k}[x_1, x_2, x_3]$ with $x_1 < x_2 < x_3$, consider again the ideals I_1, I_2 of example 47:

- the Bar Code B_1 associated to $I_1 = (x_1^3, x_1x_2, x_2^2, x_1^2x_3, x_2x_3, x_3^2)$ is



  and we have $\mathcal{F}(I_1) = \mathsf{G}(I_1) = \{x_1^3, x_1x_2, x_2^2, x_1^2x_3, x_2x_3, x_3^2\}$

- the Bar Code B_2 associated to $I_2 = (x_1^2, x_1x_2, x_2^2, x_3)$ is



  and we have $\mathcal{F}(I_2) = \mathsf{G}(I_2) = \{x_1^2, x_1x_2, x_2^2, x_3\}$

We see that, as expected, both their Bar Codes are full.

$\diamond$

Proposition 50. *Let $I \triangleleft \mathbf{k}[x_1, ..., x_n]$ be a stable zerodimensional monomial ideal and let B be its Bar Code. Then the following two conditions hold:*

a) $l_{n-1}(\mathsf{B}_1^{(n)}) > ... > l_{n-1}(\mathsf{B}_{\mu(n)}^{(n)})$

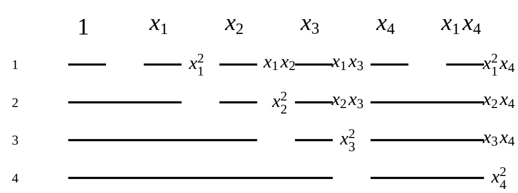
b) $\forall 1 \leq i \leq n-2, \forall 1 \leq j \leq \mu(i+2)$ *take the* $(i+2)$*-bar* $\mathsf{B}_j^{(i+2)}$ *and let* $\mathsf{B}_{j_1}^{(i+1)}, ..., \mathsf{B}_{j_1+h}^{(i+1)}$, *s.t.* h *satisfies* $h \in \{0, ..., \mu(i+1) - j_1\}$ *be the* $(i+1)$*-bars lying over* $\mathsf{B}_j^{(i+2)}$. *Then* $l_i(\mathsf{B}_{j_1}^{(i+1)}) > ... > l_i(\mathsf{B}_{j_1+h}^{(i+1)})$.

Proof. By lemma 35 the case $<$ cannot occur.
Suppose now that for some $1 \leq l \leq \mu(n) - 1$ it holds $l_{n-1}(\mathsf{B}_l^{(n)}) = l_{n-1}(\mathsf{B}_{l+1}^{(n)})$, let $\mathsf{B}_k^{(1)}$ be the rightmost 1-bar over $\mathsf{B}_l^{(n)}$ and call τ_k the term labelling $\mathsf{B}_k^{(1)}$. By definition of star set $x_{n-1}P_{x_{n-1}}(\tau_k) \in \mathcal{F}(I) \subset I$; moreover, clearly we know that $P_{x_{n-1}}(\tau_k) \in \mathsf{N}(I)$. But if $l_{n-1}(\mathsf{B}_l^{(n)}) = l_{n-1}(\mathsf{B}_{l+1}^{(n)})$, then $x_n P_{x_{n-1}}(\tau_k) = \frac{x_{n-1}P_{x_{n-1}}(\tau_k)}{x_{n-1}} x_n \notin I$ and this contradicts the stability of I.

If for some $1 \leq i \leq n-2$, $\forall 1 \leq j \leq \mu(i+2)$ we take the $(i+2)$-bar $\mathsf{B}_j^{(i+2)}$ and $\mathsf{B}_{j_1}^{(i+1)}..., \mathsf{B}_{j_1+h}^{(i+i)}$ (where h satisfies $h \in \{0, ..., \mu(i+1) - j_1\}$) are the $(i+1)$-bars lying over $\mathsf{B}_j^{(i+2)}$, it happens that for a fixed $l \in \{1, ..., \mu(i+1) - 1 - j_1\}$ $l_i(\mathsf{B}_{j_1+l}^{(i+1)}) = l_i(\mathsf{B}_{j_1+l+1}^{(i+1)})$, an analogous argument proves that I cannot be stable. ◻

In the example below, we show that there are also non-stable ideals satisfying conditions a) and b).

Example 51. For the ideal $I = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2, x_2x_4, x_3x_4, x_4^2) \triangleleft \mathbf{k}[x_1, x_2, x_3, x_4]$, we have $\mathsf{N}(I) = \{1, x_1, x_2, x_3, x_4, x_1x_4\}$ and the associated Bar Code B is



The star set is $\mathcal{F}(I) = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2, x_1^2x_4, x_2x_4, x_3x_4, x_4^2\}$ and we have $\mathcal{F}(I) \supsetneq \mathsf{G}(I)$, so I is not stable[7].
We can observe that B satisfies conditions a) b) of Proposition 50. Indeed:
a) $2 = l_3(\mathsf{B}_1^{(4)}) > 1 = l_3(\mathsf{B}_2^{(4)})$;
b) $2 = l_1(\mathsf{B}_1^{(2)}) > 1 = l_1(\mathsf{B}_2^{(2)})$; $2 = l_2(\mathsf{B}_1^{(3)}) > 1 = l_2(\mathsf{B}_2^{(3)})$. ◇

In the following two examples, we show that the result of Proposition 50 is only *local*, even if we consider strongly stable ideals, then strengthening the hypothesis of Proposition 50.

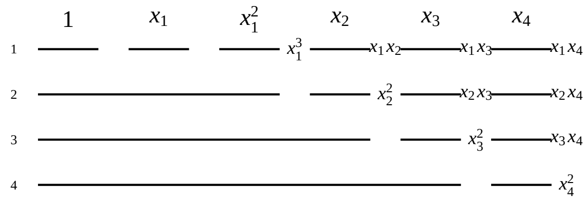
This means that in general, fixed a row $2 \leq i < n$ of the Bar Code B associated to a (even strongly) stable monomial ideal I, it does not hold

$$l_{(i-1)}(\mathsf{B}_1^{(i)}) > ... > l_{(i-1)}(\mathsf{B}_{\mu(i)}^{(i)}),$$

in particular, the $(i-1)$-length could even be completely unordered.

[7]We can also prove that I is not stable using the definition, indeed we have $x_1^2 \in I$ but $x_1x_4 \notin I$.

Example 52. The Bar Code B, associated to the (strongly) stable monomial ideal $I = (x_1^3, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2, x_1x_4, x_2x_4, x_3x_4, x_4^2) \triangleleft \mathbf{k}[x_1, x_2, x_3, x_4]$, is:

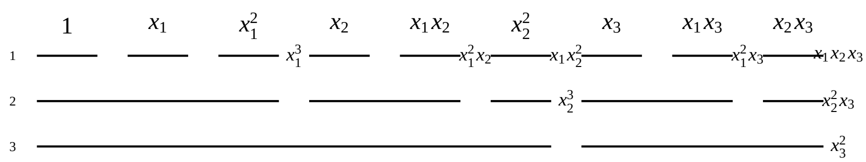


and it holds

$$2 = l_2(\mathsf{B}_1^{(3)}) > l_2(\mathsf{B}_2^{(3)}) = l_2(\mathsf{B}_3^{(3)}) = 1.$$

◊

Example 53. The (strongly) stable monomial ideal $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_1x_2x_3, x_2^2x_3, x_3^2) \triangleleft \mathbf{k}[x_1, x_2, x_3]$ is associated to the Bar Code displayed below



This monomial ideal is strongly stable, but

$$l_1(\mathsf{B}_1^{(2)}) = 3,\ l_1(\mathsf{B}^{(2)}) = 2,\ l_1(\mathsf{B}_3^{(2)}) = 1,\ l_1(\mathsf{B}_4^{(2)}) = 2 \text{ and } l_1(\mathsf{B}_5^{(2)}) = 1,$$

so in this case the 1-lengths are unordered. ◊

The proposition below gives a way to count zerodimensional stable ideals in two variables, once known their affine Hilbert polynomial.

Proposition 54. *The number of Bar Codes $\mathsf{B} \subset \mathcal{B}_2$ with bar list (p, h) and such that $\eta(B) = \mathsf{N} \subset \mathbf{k}[x_1, x_2]$ is the Groebner escalier of a stable ideal $J \triangleleft \mathbf{k}[x_1, x_2]$ equals the number of integer partitions of p into h distinct parts.*

Proof. Consider the set

$$\mathcal{B}_{(p,h)} := \{\mathsf{B} \in \mathcal{A}_2,\ \text{s.t. } \mathsf{L}_\mathsf{B} = (p, h) \text{ and } \eta(\mathsf{B}) = \mathsf{N}(J),\ J \text{ stable}\}$$

and the set of integer partitions of p into h distinct parts, i.e.

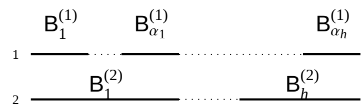
$$I_{(p,h)} = \left\{ (\alpha_1, ..., \alpha_h) \in \mathbb{N}^h,\ \alpha_1 > ... > \alpha_h \text{ and } \sum_{j=1}^{h} \alpha_j = p \right\}.$$

We define

$$\Xi : \mathcal{B}_{(p,h)} \longrightarrow \mathbb{N}^h$$
$$\mathsf{B} \mapsto (l_1(\mathsf{B}_1^{(2)}), ..., l_1(\mathsf{B}_h^{(2)}))$$

and we prove that Ξ defines a biunivocal correspondence between $\mathcal{B}_{(p,h)}$ and $I_{(p,h)} \subset \mathbb{N}^h$.

Let $\mathsf{B} \in \mathcal{B}_{p,h}$. We have $\eta(\mathsf{B}) = \mathsf{N}(J)$, $J \triangleleft \mathbf{k}[x_1, x_2]$ stable.
For each $1 \le j \le h$ set $\alpha_j = l_1(\mathsf{B}^{(2)}_j)$. By Proposition 50 a), we have $\alpha_1 > ... > \alpha_h$ and by definition of Bar Code (see Definition 18) $p = \sum_{i=1}^{p} l_1(\mathsf{B}^{(1)}_i) = \sum_{j=1}^{h} l_1(\mathsf{B}^{(2)}_j) = \sum_{j=1}^{h} \alpha_j$, so we can desume that $(l_1(\mathsf{B}^{(2)}_1), ..., l_1(\mathsf{B}^{(2)}_h)) = (\alpha_1, ..., \alpha_h) \in I_{(p,h)}$, so $\Xi(\mathcal{B}_{(p,h)}) \subseteq I_{(p,h)}$.
The map is injective by definition of 1-length of a bar.
Now, let us consider $(\alpha_1, ..., \alpha_h) \in I_{(p,h)}$ and construct a Bar Code $\mathsf{B} \subset \mathcal{B}_2$ with h 2-bars $\mathsf{B}^{(2)}_1, ..., \mathsf{B}^{(2)}_h$ and s.t. for each $1 \le j \le h$ there are α_j 1-bars lying over $\mathsf{B}^{(2)}_j$.



Clearly:

- B is univocally determined by $(\alpha_1, ..., \alpha_h) \in I_{(p,h)}$
- for each $1 \le j \le h$, $l_1(\mathsf{B}^{(2)}_j) = \alpha_j$.

We prove that $\mathsf{B} \in \mathcal{A}_2$, i.e. that B is admissible. Let $\mathsf{B}^{(1)}_i$ be a 1-bar, $1 \le i \le p$ and let $e(\mathsf{B}^{(1)}_i) = (b_{i,1}, b_{i,2})$ be its e-list. If $b_{i,1} = b_{i,2} = 0$ there is nothing to prove. If $b_{i,1} > 0$ trivially there is a 1-bar with e-list $(b_{i,1} - 1, b_{i,2})$; if $b_{i,2} > 0$, the assumption $\alpha_1 > ... > \alpha_h$ proves that there is a 1-bar with e-list $(b_{i,1}, b_{i,2} - 1)$.

Finally, we prove that the order ideal $\mathsf{N} = \eta(\mathsf{B})$ is the Groebner escalier $\mathsf{N} = \mathsf{N}(J)$ of a stable ideal J.
Let us take $\sigma \in \mathcal{F}(J)$; it can be constructed from a) or b) of Definition 37:

- If σ comes from a), $\sigma = x_i P_{x_i}(\tau_i)$, $i = 1, 2$. For $i = 2$, there is nothing to prove. We prove then the case $i = 1$, so we write $\sigma = x_1 P_{x_1}(\tau_1)$, where τ_1 labels $\mathsf{B}^{(1)}_{\mu(1)}$, and we prove that $\frac{\sigma x_2}{x_1} = x_2 P_{x_1}(\tau_1)$ belongs to J.
  Since $P_{x_2}(\tau_1) \mid P_{x_1}(\tau_1)$, $x_2 P_{x_2}(\tau_1) \mid x_2 P_{x_1}(\tau_1)$. Now, τ_1 labels a 1-bar over $\mathsf{B}^{(2)}_{\mu(2)}$, so $x_2 P_{x_2}(\tau_1) \in \mathcal{F}(J)$ and so we are done.

- Suppose now σ coming from b), so $\sigma = x_1 P_{x_1}(\tau^{(1)}_j)$, where $\tau^{(1)}_j$ is the term labelling a bar $\mathsf{B}^{(1)}_j$, $1 \le j \le \mu(1) - 1$, and $\mathsf{B}^{(1)}_j$ and $\mathsf{B}^{(1)}_{j+1}$ are two consecutive 1-bars not lying over the same 2-bar; in particular, we say that $\mathsf{B}^{(1)}_j$ lies over $\mathsf{B}^{(2)}_{j_1}$ and $\mathsf{B}^{(1)}_{j+1}$ lies over $\mathsf{B}^{(2)}_{j_1+1}$.
  We have to prove that $x_2 P_{x_1}(\tau^{(1)}_j)$ belongs to J.
  Denoted $\tau^{(1)}_{\bar{j}}$ the term labelling the rightmost 1-bar over $\mathsf{B}^{(2)}_{j_1+1}$, we have $\deg_2(\tau^{(1)}_{\bar{j}}) = \deg_2(\tau^{(1)}_j) + 1$ and $\deg_1(\tau^{(1)}_{\bar{j}}) < \deg_1(\tau^{(1)}_j)$, so $\deg_1(x_1 P_{x_1}(\tau^{(1)}_{\bar{j}})) \le \deg_1(x_2 P_{x_1}(\tau^{(1)}_j))$ and $\deg_2(x_1 P_{x_1}(\tau^{(1)}_{\bar{j}})) = \deg_2(x_2 P_{x_1}(\tau^{(1)}_j))$, whence $x_1 P_{x_1}(\tau^{(1)}_{\bar{j}}) \mid x_2 P_{x_1}(\tau^{(1)}_j)$ and since $x_1 P_{x_1}(\tau^{(1)}_{\bar{j}}) \in J$ we are done.

∎

With the Proposition below, we prove which is the maximal value that h can assume.

Proposition 55. *Denoting by* B *a Bar Code associated to a stable ideal* $I \triangleleft \mathbf{k}[x_1, x_2]$ *with affine Hilbert polynomial* $H_I(d) = p \in \mathbb{N}$ *and by* $\mathsf{L}_{\mathsf{B}} = (p, h)$ *its bar list, the maximal value that h can assume is*

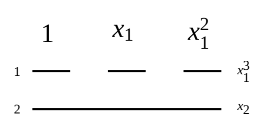
$$h := \left\lfloor \frac{-1 + \sqrt{1+8p}}{2} \right\rfloor$$

Proof. By Proposition 54, the Bar Codes associated to stable ideals s.t. the associated bar list is (p, i) are in bijection with the integer partitions of p with i distinct parts.
An integer partition of p with i distinct parts is a partition $(\alpha_1, ..., \alpha_i) \in \mathbb{N}^i$, $\alpha_1 > ... > \alpha_i$, $\sum_{j=1}^{i} \alpha_j = p$. Since the minimal value we can give to $\alpha_j, 1 \leq j \leq i$, so that $\alpha_1 > ... > \alpha_i$, is $\alpha_j = i - j + 1$ and $\sum_{j=1}^{i}(i - j + 1) = \frac{i(i+1)}{2}$, we have that $\frac{i(i+1)}{2}$ is the minimal sum of i positive distinct integer numbers. If $\frac{i(i+1)}{2} > p$, there cannot exist any partition of p with i distinct parts; if $\frac{i(i+1)}{2} = p$, the i-tuple $(\alpha_1, ..., \alpha_i) \in \mathbb{N}^i$ is such a partition and if $\frac{i(i+1)}{2} \leq p$, it is possible to find a partition of p with i distinct parts starting from $(\alpha_1, ..., \alpha_i) \in \mathbb{N}^i$, for example by increasing the value of α_1, until $\sum_{j=1}^{i} \alpha_j = p$.
Then, we have proved that the maximal number h of distinct parts in a partition of p is $h := \max_{i \in \mathbb{N}} \left\{ \frac{i(i+1)}{2} \leq p \right\}$. Since $\frac{i(i+1)}{2} \leq p$ for $\frac{-1-\sqrt{1+8p}}{2} \leq i \leq \frac{-1+\sqrt{1+8p}}{2}$, then

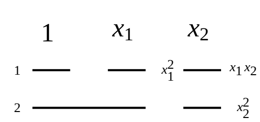
$$h := \left\lfloor \frac{-1 + \sqrt{1+8p}}{2} \right\rfloor$$

∎

Example 56. Applying proposition 55, we get that for $p = 1, 2$, we have $h = 1$, so the only (strongly) stable monomial ideals of $\mathbf{k}[x_1, x_2]$, with constant affine Hilbert polynomial $p = 1, 2$ are the ideals $I_1 = (x_1, x_2)$ and $I_2 = (x_1^2, x_2)$ (see Remark 59).
For the affine Hilbert polynomial $p = 3$ we have $h = 2$, so we have two (strongly) stable monomial ideals, $J_1 = (x_1^3, x_2)$ and $J_2 = (x_1^2, x_1x_2, x_2^2)$.
The Bar Code B_1 associated to J_1 is



whose bar list is $\mathsf{L}_{\mathsf{B}_1} = (3, 1)$.
The Bar Code associated B_2 to J_2 is



and its bar list is $\mathsf{L}_{\mathsf{B}_2} = (3, 2)$. ◊

In order to deal with stable ideals $J \triangleleft \mathbf{k}[x_1, ..., x_n]$ for $n > 2$, the following corollary will be rather useful.

Corollary 57. *The number of Bar Codes associated to stable ideals in* $\mathbf{k}[x_1, ..., x_n]$*,* $n > 2$*, whose bar list is* $(p, h, \underbrace{1, ..., 1}_{3,...,n})$*,* $p, h \in \mathbb{N}$*,* $p \geq h$ *equals the number of integer partitions of p in h distinct parts, namely*

$$p = \alpha_1 + ... + \alpha_h,\ \alpha_1 > ... > \alpha_h > 0.$$

Moreover, the maximal value that h can assume in the bar list $(p, h, 1, ..., 1)$ *is*

$$h := \left\lfloor \frac{-1 + \sqrt{1 + 8p}}{2} \right\rfloor.$$

Proof. It is a straightforward consequence of Propositions 54 and 55, noticing that, if $\mu(3) = ... = \mu(n) = 1$, $x_3, ..., x_n$ do not appear in any term of M_{B} with nonzero exponent. □

The following proposition is a consequence of 54 and 55 and completely solves the problem of counting stable monomial ideals in two variables.

Proposition 58. *The number of stable ideals* $J \triangleleft \mathbf{k}[x_1, x_2]$ *with* $H_{\_}(t, J) = p$ *is*

$$\sum_{i=1}^{h} Q(p, i),$$

where $h := \left\lfloor \frac{-1+\sqrt{1+8p}}{2} \right\rfloor$ *and* $Q(p, i)$ *is the number of integer partitions of p into i distinct parts.*

Remark 59. Let $I \triangleleft \mathbf{k}[x_1, x_2]$ be a strongly stable monomial ideal with affine Hilbert polynomial $H_I(t) = p$, B be the corresponding Bar Code and suppose that $\mathsf{L}_{\mathsf{B}} = (p, 1)$. In this case, we can easily deduce that $I = (x_1^p, x_2)$ so I is a *lex-segment ideal*, i.e., for each degree $i \in \mathbb{N}$, I is $\mathbf{k}$-spanned by the first $H_I(i)$ terms w.r.t. Lex.

By Remark 59, for each $p \in \mathbb{N}$, there exists a (strongly) stable monomial ideal $I \triangleleft \mathbf{k}[x_1, x_2]$ with affine Hilbert polynomial $H_I(t) = p$ and s.t. the corresponding Bar Code B has $\mathsf{L}_{\mathsf{B}} = (p, 1)$, so the minimal value that h can assume is 1.

We summarize in the following table the possible bar lists for stable ideals corresponding to some small values of p, together with the corresponding ideals.

$H_{\_}(t) = p$	Bar lists	Ideals
1	$(1, 1)$	(x_1, x_2)
2	$(2, 1)$	(x_1^2, x_2)
3	$(3, 1), (3, 2)$	$(x_1^3, x_2), (x_1^2, x_1x_2, x_2^2)$
4	$(4, 1), (4, 2)$	$(x_1^4, x_2), (x_1^3, x_1x_2, x_2^2)$
5	$(5, 1), (5, 2), (5, 2)$	$(x_1^5, x_2), (x_1^4, x_1x_2, x_2^2), (x_1^3, x_1^2x_2, x_2^2)$
6	$(6, 1), (6, 2), (6, 2), (6, 3)$	$(x_1^6, x_2), (x_1^5, x_1x_2, x_2^2), (x_1^4, x_1^2x_2, x_2^2), (x_1^3, x_1^2x_2, x_1x_2^2, x_2)$

We notice that the above ideals are also strongly stable.

Example 60. For the polynomial ring $\mathbf{k}[x_1, x_2]$, consider $H_{\_}(t) = p = 10$.
In this case, we have $h = 4$, so we have to compute the sum

$$Q(10, 1) + Q(10, 2) + Q(10, 3) + Q(10, 4).$$

We have:
$Q(10, 1) = 1$;
$Q(10, 2) = P(9, 2) = P(8, 1)+P(7, 2) = 1+P(7, 2) = 1+P(6, 1)+P(5, 2) = 2+P(5, 2) = 2 + P(4, 1) + P(3, 2) = 3 + P(2, 1) = 4$
$Q(10, 3) = P(7, 3) = P(6, 2) + P(4, 3) = 1 + P(4, 2) + P(3, 2) = 1 + P(3, 1) + P(2, 2) + P(2, 1) = 1 + 1 + 1 + 1 = 4$
$Q(10, 4) = P(4, 4) = 1$.
Then, we have exactly 10 strongly stable monomial ideals with $H_{\_}(t) = 10$.
More precisely, they are:

- ⋆ $J_1 = (x_1^{10}, x_2)$;
- ⋆ $J_2 = (x_1^9, x_1x_2, x_2^2)$;
- ⋆ $J_3 = (x_1^8, x_1^2x_2, x_2^2)$;
- ⋆ $J_4 = (x_1^7, x_1^3x_2, x_2^2)$;
- ⋆ $J_5 = (x_1^7, x_1x_2^2, x_2x_1^2, x_2^3)$;
- ⋆ $J_6 = (x_1^6, x_1^4x_2, x_2^2)$;
- ⋆ $J_7 = (x_1^6, x_1x_2^2, x_1^3x_2, x_2^3)$;
- ⋆ $J_8 = (x_1^5, x_2^2x_1, x_2x_1^4, x_2^3)$;
- ⋆ $J_9 = (x_1^5, x_2^2x_1^2, x_2x_1^3, x_2^3)$;
- ⋆ $J_{10} = (x_1^4, x_2^3x_1, x_2^2x_1^2, x_2x_1^3, x_2^4)$.

◊

Example 61. Employing the same formula (all the computation has been performed using Singular [17]), we can get that the strongly stable monomial ideals with $H_{\_}(t) = 100$ are exactly 444793. ◊

Now we start studying the case of three variables; in this case we need to consider the bar lists of the form (p, h, k). By Corollary 57, we can use the formulas for two variables in order to count the stable monomial ideals in three variables, associated to bar lists of the form $(p, h, 1)$. This means that we only have to deal with the bar lists of the form (p, h, k), such that $k > 1$.
In order to handle these new bar lists, we define the concept of *minimal sum* of a list of positive integers.

Definition 62. *The* minimal sum *of a given list of positive integers* $[\alpha_1, ..., \alpha_g]$ *is the integer*

$$Sm([\alpha_1, ..., \alpha_g]) := \sum_{i=1}^{g} \frac{\alpha_i(\alpha_i + 1)}{2}.$$

Lemma 63. *With the previous notation, it holds:*

1. $k \in \{1, ..., l\}$, *where* $l := \max_{i\in\mathbb{N}}\{i^3 + 3i^2 + 2i \leq 6p\}$;
2. $h \in \{\frac{k(k+1)}{2}, ..., m\}$, *where* $m = \max_{r\geq \frac{k(k+1)}{2}} \{r \mid \exists \lambda \in I_{(r,k)}, Sm(\lambda) \leq p\}$.

Proof. By Corollary 57 the minimal value for k is 1.
Now, in order to construct a Bar Code B associated to a stable ideal, we should at least meet the requirements of Proposition 50, so, given k, for each 3-bar $\mathsf{B}^{(3)}_j$ there should be at least $(k - j + 1)$ 2-bars lying over it, so that $h \geq \frac{k(k+1)}{2}$.
Now, select a 3-bar $\mathsf{B}^{(3)}_{\overline{j}}$, $1 \leq \overline{j} \leq k$ and let $\mathsf{B}^{(2)}_{j_1}, ..., \mathsf{B}^{(2)}_{j_1+t-1}$, $t \geq k - \overline{j}$ be the 2-bars over $\mathsf{B}^{(3)}_{\overline{j}}$. Now, with an analogous argument w.r.t. the one for 2-bars, we can say that for $\mathsf{B}^{(2)}_{j_1+j-1}$, $1 \leq j \leq t$, we must have at least $t - j + 1$ 1-bars, so that their total number will be $\mathrm{Sm}([1, 2, ..., k]) = \sum_{i=1}^{k} \frac{i(i+1)}{2}$. Since the number of elements in $\eta(\mathsf{B})$ equals the Hilbert polynomial p, we must have $\mathrm{Sm}([1, 2, ..., k]) = \sum_{i=1}^{k} \frac{i(i+1)}{2} \leq p$.
Now $\sum_{i=1}^{k} \frac{i(i+1)}{2} = \sum_{i=1}^{k} \binom{i+1}{2} = \binom{k+2}{3} \leq p$, so $k^3 + 3k^2 + 2k \leq 6p$ and we are done.
As regards the maximal value that h can assume, from anologous arguments, to meet the requirements of Proposition 50, it is enough to be able to find a partition $\lambda \in I_{(h,k)}$ with $\mathrm{Sm}(\lambda) \leq p$. ◻

Thanks to the previous Lemma 63, now we know which are the bar lists we have to take into account in order to count the stable ideals with affine Hilbert polynomial $H_{\_}(t) = p$.
Next step then, is to find out how many stable ideals with $H_{\_}(t) = p$ and such that their Bar Code B has bar list (p, h, k) are there.

Take then a bar list (p, h, k) and let $\overline{\beta} \in I_{(h,k)}$, so $\overline{\beta_1} > ... > \overline{\beta_k}$ and $\sum_{i=1}^{k} \overline{\beta_i} = h$.
We can construct plane partitions ρ of the form

$$\rho = (\rho_{i,j}) = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & ... & ... & ... & ... & ... & ... & \rho_{1,\overline{\beta_1}} \\ \rho_{2,1} & ... & ... & ... & ... & ... & \rho_{2,\overline{\beta_2}} & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ \rho_{k,1} & ... & ... & ... & ... & \rho_{k,\overline{\beta_k}} & 0 & ... & ... \end{pmatrix}$$

s.t.

1. $\rho_{i,j} > 0$, $1 \leq i \leq k$, $1 \leq j \leq \overline{\beta_i}$;
2. $\rho_{i,j} > \rho_{i,j+1}$, $1 \leq i \leq k$, $1 \leq j \leq \overline{\beta_i} - 1$;
3. $\rho_{i,j} > \rho_{i+1,j}$ $1 \leq i \leq k-1$, $1 \leq j \leq \overline{\beta_{i+1}}$;

4. $n(\rho) = \sum_{i=1}^{k} \sum_{j=1}^{\overline{\beta_i}} \rho_{i,j} = p$.

These plane partitions are exactly of the form defined in 6, with shape $\overline{\beta}$, $c = 1$ and $d = 1$, so they are row-strict and column-strict plane partitions of shape $\overline{\beta}$.
Fixed $\overline{\beta} \in I_{(h,k)}$, we denote by $\mathcal{P}_{(p,h,k),\overline{\beta}}$ the set of all partitions defined as above and $\mathcal{P}_{(p,h,k)} = \bigcup_{\overline{\beta} \in I_{(h,k)}} \mathcal{P}_{(p,h,k),\overline{\beta}}$. In other words,

$$\mathcal{P}_{(p,h,k),\overline{\beta}} = \{\rho \in \mathcal{P}_{\overline{\beta}}(1,1) \text{ s.t } n(\rho) = p\}$$

$$\mathcal{P}_{(p,h,k)} = \{\rho \in \mathcal{P}_{\overline{\beta}}(1,1) \text{ for some } \overline{\beta} \in I_{(h,k)} \text{ and s.t. } n(\rho) = p\}.$$

Each plane partition $\rho \in \mathcal{P}_{(p,h,k)}$ uniquely identifies a Bar Code B:

(a) each row i represents a 3-bar $\mathsf{B}_i^{(3)}$, $1 \le i \le k$;

(b) for each row i, $1 \le i \le k$, $l_2(\mathsf{B}_i^{(3)}) = \overline{\beta_i}$; the $\overline{\beta_i}$ nonzero entries represent the $\overline{\beta_i}$ 2-bars over $\mathsf{B}_i^{(3)}$, i.e the j-th entry of row i, $1 \le j \le \overline{\beta_i}$, represents the 2-bar $\mathsf{B}_t^{(2)}$, where $t = (\sum_{l=1}^{i-1} \overline{\beta_l}) + j$;

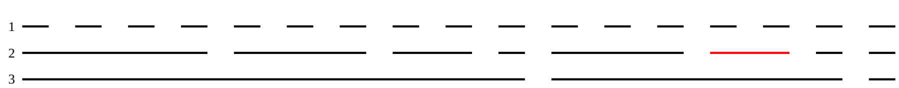
(c) for each $1 \le i \le k$, and each $1 \le j \le \overline{\beta_i}$, the number $\rho_{i,j}$ represents the number of 1-bars over $\mathsf{B}_t^{(2)}$, $t = (\sum_{l=1}^{i-1} \overline{\beta_l}) + j$, the j-th 2-bar lying over $\mathsf{B}_i^{(3)}$. In other words, $l_1(\mathsf{B}_t^{(2)}) = \rho_{i,j}$.

In conclusion, for each $1 \le i \le k$, and each $1 \le j \le \overline{\beta_i}$, the number $\rho_{i,j}$ means that in B there are 1-bars labelled by $(0, j-1, i-1), (1, j-1, i-1), ..., (\rho_{i,j}-1, j-1, i-1)$, but there is no 1-bar labelled by $(\rho_{i,j}, j-1, i-1)$, that is also equivalent to say that $x_1^0 x_2^{j-1} x_3^{i-1}, x_1 x_2^{j-1} x_3^{i-1}, ..., x_1^{\rho_{i,j}-1} x_2^{j-1} x_3^{i-1}$ belong to the set of terms associated to B via Bbc1 and Bbc2, but $x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-1}$ does not belong to the aforementioned set[8].

Example 64. Taken the plane partition

$$\rho = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & \mathbf{2} & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let us examine the position in bold, i.e. $\rho_{2,2} = 2$.
The Bar Code B associated to ρ is



We have $t = \overline{\beta_1} + 2 = 6$, so $2 = \rho_{2,2} = l_1(\mathsf{B}_6^{(2)})$ (we have marked $\mathsf{B}_6^{(2)}$ in red in the picture). Applying Bbc1 and Bbc2 we can see, absolutely in agreement, with the above comments, that $x_2 x_3, x_1 x_2 x_3$ are in the set of terms associated to B, whereas $x_1^2 x_2 x_3$ does not. ◊

[8] Actually, we will see that $x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-1}$ will belong to the star set associated to the Bar Code B, after proving that it is admissible.

Remark 65. The Bar Code B, uniquely identified by ρ, has bar list $L_B = (p, h, k)$. The relation $\mu(3) = k$ comes from (a), $\mu(2) = h$ comes from (b), since $\beta \in I_{(h,k)}$, so $\sum_{i=1}^{k} \beta_i = h$, whereas $\mu(1) = p$ is an easy consequence of (c).

In the following Lemma, we prove that a Bar Code B, defined as above, is admissible.

Lemma 66. *Fixed (p, h, k) and $\beta \in I_{(h,k)}$, let ρ be a partition in $\mathcal{P}_{(p,h,k),\beta}$. The Bar Code* B*, uniquely identified by ρ, is admissible.*

Proof. By Remark 65, $L_B = (p, h, k)$, so consider a 1-bar $B_l^{(1)}$, $1 \leq l \leq p$ and its e-list that we denote $e(B_l^{(1)}) = (b_{l,1}, b_{l,2}, b_{l,3})$. From the construction of B from ρ, we desume that $\rho_{b_{l,3}+1, b_{l,2}+1} \geq b_{l,1} + 1$; moreover $(m, b_{l,2}, b_{l,3})$, $0 \leq m \leq \rho_{b_{l,3}+1, b_{l,2}+1} - 1$ are e-lists for some bars of B, so, if $b_{l,1} \geq 1$, $(b_{l,1} - 1, b_{l,2}, b_{l,3})$ is an e-list labelling a 1-bar of B.
For B being admissible, we also need two other conditions:

a. if $b_{l,2} > 0$, then $(b_{l,1}, b_{l,2} - 1, b_{l,3})$ labels a 1-bar of B;

b. if $b_{l,3} > 0$, then $(b_{l,1}, b_{l,2}, b_{l,3} - 1)$ labels a 1-bar of B.

Let us prove them:

a. suppose $b_{l,2} > 0$; for $(b_{l,1}, b_{l,2} - 1, b_{l,3})$ labelling a 1-bar of B, we would need $\rho_{b_{l_3}+1, b_{l_2}} \geq b_{l_1} + 1$, but since $\rho_{b_{l_3}+1, b_{l_2}} > \rho_{b_{l_3}+1, b_{l_2}+1} \geq b_{l_1} + 1$ we are done

b. suppose $b_{l,3} > 0$; for $(b_{l,1}, b_{l,2}, b_{l,3} - 1)$ labelling a 1-bar of B, we would need $\rho_{b_{l_3}, b_{l_2}+1} \geq b_{l_1} + 1$, but since $\rho_{b_{l_3}, b_{l_2}+1} > \rho_{b_{l_3}+1, b_{l_2}+1} \geq b_{l_1} + 1$ we are done again and B turns out to be admissible.

□

Lemma 67. *Let $\rho \in \mathcal{P}_{(p,h,k)}$ be a strict plane partition and* B *be the Bar Code uniquely determined by ρ. Denoted by J the monomial ideal s.t. $\eta(B) = N(J)$ and by A the set*

$$A := \{x_3^k, x_2^{\beta_i} x_3^{i-1}, x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-1},\ 1 \leq i \leq k,\ 1 \leq j \leq \beta_i\},$$

then $\mathcal{F}(J) = A$.

Proof. Let us first prove $\mathcal{F}(J) \supseteq A$.
Neither x_3^k, nor $x_2^{\beta_i} x_3^{i-1}$, nor $x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-1}$ belong to $N(J)$ by the definition of η and by the construction of B from ρ.
Consider x_3^k; clearly, being $k > 0$, $\min(x_3^k) = x_3$, so we prove that $x_3^{k-1} \in N(J)$. Since $k = \mu(3)$, there are exactly k 3-bars. By BbC1, the k-th 3-bar of B is labelled by $l_1(B_k^{(3)})$ copies of x_3^{k-1}, so the 1-bars over $B_k^{(3)}$ are labelled by terms which are multiple of x_3^{k-1}. The Bar Code B is admissible, then also $x_3^{k-1} \in N(J)$[9].
As regards $x_2^{\beta_i} x_3^{i-1}$, $1 \leq i \leq k$, $\beta_i > 0$, whence $\min(x_2^{\beta_i} x_3^{i-1}) = x_2$, so we have to prove that $x_2^{\beta_i - 1} x_3^{i-1} \in N(J)$.

[9] Actually, by BbC1, x_3^{k-1} labels the first 1-bar over $B_k^{(3)}$.

We take the i-th 3-bar $\mathsf{B}_i^{(3)}$; it is labelled by $l_1(\mathsf{B}_i^{(3)})$ copies of x_3^{i-1}. Now, over $\mathsf{B}_i^{(3)}$ there are exactly β_i 2-bars and, by BbC2, the β_i-th 2-bar over $\mathsf{B}_i^{(3)}$ (i.e. $\mathsf{B}_t^{(2)}$, $t = \sum_{l=1}^{i}\beta_l$) is labelled by $l_1(\mathsf{B}_t^{(2)})$ copies of $x_2^{\beta_i-1}x_3^{i-1}$, so the 1-bars over $\mathsf{B}_i^{(3)}$ are labelled by terms which are multiple of $x_2^{\beta_i-1}x_3^{i-1}$; by the admissibility of B, we get $x_2^{\beta_i-1}x_3^{i-1} \in \mathsf{N}(J)$[10].
Take then $x_1^{\rho_{i,j}}x_2^{j-1}x_3^{i-1}$, $1 \le i \le k$, $1 \le j \le \beta_i$; since $\rho_{i,j} > 0$, $\min(x_1^{\rho_{i,j}}x_2^{j-1}x_3^{i-1}) = x_1$ and so we have to prove that $x_1^{\rho_{i,j}-1}x_2^{j-1}x_3^{i-1} \in \mathsf{N}(J)$, but this is trivial by the construction of B from ρ.

We prove now that $\mathcal{F}(J) \subseteq A$.
Let $\tau \in \mathcal{F}(J)$; we have to show that it belongs to A.
If $\min(\tau) = x_3$, then $\tau = x_3^{h_3}$ for some $h_3 \in \mathbb{N}$; we show that necessarily $h_3 = k$ and so $\tau = x_3^k \in A$.

By the construction of B from ρ we have $\mu(3) = k$, i.e. B has exactly k 3-bars; by Definition 37 a), with $i = n = 3$, $x_3P_{x_3}(\tau_3) \in \mathcal{F}(J)$, where τ_3 is a term labelling a 1-bar over $\mathsf{B}_k^{(3)}$. Now, by BbC1, each $\tau_3 \in \mathcal{T}$ labelling a 1-bar over $\mathsf{B}_k^{(3)}$ is s.t. $P_{x_3}(\tau_3) = x_3^{k-1}$, so $x_3P_{x_3}(\tau_3) = x_3^k \in \mathcal{F}(J)$.
No other pure powers of x_3 can occur in $\mathcal{F}(J)$ by Definition 37, indeed, x_3^k is the only term with minimal variable x_3 derived by a) and there cannot be terms derived by b), since each term σ coming from b) has $\min(\sigma) \le x_2$.
We can conclude that the only pure power of x_3 in $\mathcal{F}(J)$ is $\tau = x_3^k$, which is also an element of A.

Let now be $\min(\tau) = x_2$, so $\tau = x_2^{h_2}x_3^{h_3}$, for some $h_2, h_3 \in \mathbb{N}$. This term may be derived either from a) or from b) of Definition 37; we have to prove that, in any case, it belongs to A.

a) In this case, $\tau = x_2P_{x_2}(\tau_2)$, where τ_2 is a term labelling a 1-bar over $\mathsf{B}_{\mu(2)}^{(2)}$. But $\mu(2) = h$; since $\mathsf{B}_{\mu(2)}^{(2)} = \mathsf{B}_h^{(2)}$ is the rightmost 2-bar, it lies over $\mathsf{B}_k^{(3)}$, where $k = \mu(3)$ and, in particular it is the β_k-th bar over $\mathsf{B}_k^{(3)}$. Now, by BbC1 and BbC2, we can desume that $h_3 = k-1$ and $h_2 = \beta_k - 1$, so $\tau_2 = x_2^{\beta_k-1}x_3^{k-1}$ and so $\tau = x_2^{\beta_k}x_3^{k-1} \in A$.

b) In this case, for $1 \le l \le h-1$, we consider two consecutive 2-bars $\mathsf{B}_l^{(2)}$, $\mathsf{B}_{l+1}^{(2)}$ not lying over the same 3-bar, i.e. lying over two consecutive 3-bars $\mathsf{B}_{l_1}^{(3)}$, $\mathsf{B}_{l_1+1}^{(3)}$, $1 \le l_1 < k$; let $\tau_l^{(2)}$ a term labelling a 1-bar over $\mathsf{B}_l^{(2)}$.
Since $\tau_l^{(2)}$ labels a 2-bar lying over $\mathsf{B}_{l_1}^{(3)}$, $1 \le l_1 < k$, it holds $x_3^{l_1-1} \mid \tau_l^{(2)}$ and $x_3^{l_1} \nmid \tau_l^{(2)}$.
Now, over $\mathsf{B}_{l_1}^{(3)}$ there are β_{l_1} 2-bars and since $\mathsf{B}_{l+1}^{(2)}$ lies over $\mathsf{B}_{l_1+1}^{(3)}$, then $\mathsf{B}_l^{(2)}$ lies over the β_{l_1}-th 2-bar over $\mathsf{B}_{l_1}^{(3)}$, so $x_2^{\beta_{l_1}-1} \mid \tau_l^{(2)}$ and $x_2^{\beta_{l_1}} \nmid \tau_l^{(2)}$. This implies that $\tau = x_2P_{x_2}(\tau_l^{(2)}) = x_2^{\beta_{l_1}}x_3^{l_1-1} \in A$, $1 \le l_1 < k$.

Finally, let $\min(\tau) = x_1$; as for the above case, we have to examine a) and b) separately:

[10] Actually, by BbC1, $x_2^{\beta_i-1}x_3^{i-1}$ labels the first 1-bar over $\mathsf{B}_t^{(2)}$.

a) In this case, $\tau = x_1 P_{x_1}(\tau_1)$, where τ_1 labels $\mathsf{B}^{(1)}_{\mu(1)} = \mathsf{B}^{(1)}_p$. Now, $\mathsf{B}^{(1)}_p$ is the rightmost 1-bar, so it lies over $\mathsf{B}^{(2)}_h$, which, in turn, lies over $\mathsf{B}^{(3)}_k$. By BbC1 and BbC2, $x_3^{k-1} \mid \tau_1$, $x_3^k \nmid \tau_1$, $x_2^{\beta_k-1} \mid \tau_1$, $x_2^{\beta_k} \nmid \tau_1$ From $l_1(\mathsf{B}^{(2)}_h) = \rho_{k,\beta_k}$ we desume that $\tau = x_1 P_{x_1}(\tau_1) = x_1^{\rho_{k,\beta_k}} x_2^{\beta_k-1} x_3^{k-1} \in A$.

b) In this case, for $1 \le l_1 \le \mu(1) - 1 = p - 1$ we consider two consecutive 1-bars $\mathsf{B}^{(1)}_{l_1}$ and $\mathsf{B}^{(1)}_{l_1+1}$, lying over two consecutive 2-bars $\mathsf{B}^{(2)}_{l_2}$, $\mathsf{B}^{(2)}_{l_2+1}$, $1 \le l_2 < h$ and we denote $\mathsf{B}^{(3)}_{l_3}$, $1 \le l_3 \le k$, the 3-bar underlying[11] $\mathsf{B}^{(2)}_{l_2}$.
Let $\tau^{(1)}_{l_1}$ be the term labelling $\mathsf{B}^{(1)}_{l_1}$; by BbC1 and BbC2 $x_3^{l_3-1} \mid \tau^{(1)}_{l_1}$, $x_3^{l_3} \nmid \tau^{(1)}_{l_1}$, $x_2^{u-1} \mid \tau^{(1)}_{l_1}$, $x_2^{u} \nmid \tau^{(1)}_{l_1}$, $u = l_2 - \sum_{r=1}^{l_3-1} \beta_r \le \beta_{l_3}$ and $x_1^{\rho_{l_3,u}-1} \mid \tau^{(1)}_{l_1}$, $x_1^{\rho_{l_3,u}} \nmid \tau^{(1)}_{l_1}$, so we have $\tau = x_1 P_{x_1}(\tau^{(1)}_{l_1}) = x_1^{\rho_{l_3,u}} x_2^{u-1} x_3^{l_3-1} \in A$.

□

Theorem 68. *There is a biunivocal correspondence between $\mathcal{P}_{(p,h,k)}$ and the set* $\mathsf{B}^{(S)}_{(p,h,k)} = \{\mathsf{B} \in \mathcal{A}_3$ *s.t.* $\mathsf{L}_\mathsf{B} = (p,h,k)$, $\eta(\mathsf{B}) = \mathsf{N}(J)$, J *stable*$\}$.

Proof. Let $\mathsf{B} \in \mathsf{B}^{(S)}_{(p,h,k)}$; we construct a plane partition

$$\rho = (\rho_{i,j}) = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \dots & \dots & \dots & \dots & \dots & \dots & \rho_{1,\beta_1} \\ \rho_{2,1} & \dots & \dots & \dots & \dots & \dots & \rho_{2,\beta_2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \rho_{k,1} & \dots & \dots & \dots & \dots & \rho_{k,\beta_k} & 0\dots & \dots & \dots \end{pmatrix}$$

with k rows and $l_2(\mathsf{B}^{(3)}_1) = \beta_1$ columns.

Chosen $1 \le i \le k$ as row index and $1 \le j \le \beta_1$ as column index and set $\beta_i = l_2(\mathsf{B}^{(3)}_i)$, we define

$$\rho_{i,j} = \begin{cases} l_1(\mathsf{B}^{(2)}_t) & \text{with } t = (\sum_{l=1}^{i-1} \beta_l) + j, \text{ for } 1 \le i \le k,\ 1 \le j \le \beta_i, \\ 0 & \text{if } 1 \le i \le k,\ \beta_i < j \le \beta_1, \end{cases}$$

so β is the shape of ρ.
We notice that the partition ρ is uniquely determined by B and that $\beta \in I_{(h,k)}$; indeed $\sum_{i=1}^k \beta_i = h = \mu(2)$ and, by Proposition 50 a), $\beta_1 > \dots > \beta_n$.
Now, we prove that $\rho \in \mathcal{P}_{(p,h,k)}$.

The nonzero parts of ρ are positive by definition of length of a bar.

Clearly $\rho_{i,j} > \rho_{i,j+1}$, $1 \le i \le k$, $1 \le j < \beta_i$, indeed, this can be stated as $l_1(\mathsf{B}^{(2)}_t) > l_1(\mathsf{B}^{(2)}_{t+1})$, $t = (\sum_{l=1}^{i-1} \beta_l) + j$, with $\mathsf{B}^{(2)}_t$ and $\mathsf{B}^{(2)}_{t+1}$ lying over the same 3-bar $\mathsf{B}^{(3)}_i$. This statement follows from Proposition 50 b).

Moreover, $\rho_{i,j} > \rho_{i+1,j}$ $1 \le i \le k-1$, $1 \le j \le \beta_{i+1}$.

Indeed, for $1 \le i \le k-1$, $1 \le j \le \beta_{i+1}$, $\sigma := x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-1} \in J$; being $\rho_{i,j} > 0$, $\min(\sigma) = x_1 < x_3$, so $\frac{\sigma x_3}{x_1} = x_1^{\rho_{i,j}-1} x_2^{j-1} x_3^{i}$ should belong to the stable ideal J.

[11] We remark that $\mathsf{B}^{(2)}_{l_2+1}$ may lie over $\mathsf{B}^{(3)}_{l_3}$ or - if it exists - to its consecutive 2-bar, but we do not care about it, since it has no influence on τ. Remember also that, by construction, $l_2 = \sum_{r=1}^{l_3-1} \beta_r + \overline{j}$ with $1 \le \overline{j} \le \beta_{l_3}$.

But this implies $\rho_{i,j} > \rho_{i+1,j}$ since $\rho_{i,j} \leq \rho_{i+1,j}$ implies $\widetilde{\sigma} := x_1^{\rho_{i+1,j}-1} x_2^{j-1} x_3^{i} \in \mathsf{N}(J)$ and $\frac{\sigma x_3}{x_1} \mid \widetilde{\sigma}$, contradicting the stability of J.

Finally, $n(\rho) = p$ by definition of 1-length.

Then, we can define a map

$$\Xi : \mathcal{B}^{(S)}_{(p,h,k)} \to \mathcal{P}_{(p,h,k)}$$

$$\mathsf{B} \mapsto \rho,$$

where ρ is constructed from B as described above. We prove that Ξ is a bijection.

It is clearly an injection by definition of lenght of a bar: two different Bar Codes have at least one bar with different length.

Now, we have to prove the surjectivity of Ξ, so let us take $\rho \in \mathcal{P}_{(p,h,k)}$. We know that it uniquely identifies a Bar Code B and by Lemma 66 that B is admissible, so we only have to prove that $\mathsf{L}_{\mathsf{B}} = (p, h, k)$ and that $\eta(B) = \mathsf{N}(J)$, J stable.
The statement $\mathsf{L}_{\mathsf{B}} = (p, h, k)$ is trivial, since

1. there are k 3-bars,
2. for each $1 \leq i \leq k$, $l_2(\mathsf{B}^{(3)}_i) = \beta_i$ and $\sum_{i=1}^k \beta_i = h$,
3. for each $1 \leq i \leq k$, $1 \leq j \leq \beta_i$, $l_1(\mathsf{B}^{(2)}_t) = \rho_{i,j}$, $t = (\sum_{l=1}^{i-1} \beta_l) + j$ and $n(\rho) = p$.

A monomial ideal J is stable if and only if $\mathcal{F}(J) = \mathsf{G}(J)$; by Lemma 67 $\mathcal{F}(J) = A = \{x_3^k, x_2^{\beta_i} x_3^{i-1}, x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-1}, 1 \leq i \leq k, 1 \leq j \leq \beta_i\}$, so we only have to prove that $A \subset \mathsf{G}(J)$, i.e. that, for each element in the star set, all the predecessors belong to the Groebner escalier.
We have already proved that $x_3^{k-1} \in \mathsf{N}(J)$, since $\min(x_3^k) = x_3$ and $x_3^k \in \mathcal{F}(J)$.

Let us take $x_2^{\beta_i} x_3^{i-1}$, $1 \leq i \leq k$; since it belongs to the star set, $x_2^{\beta_i - 1} x_3^{i-1} \in \mathsf{N}(J)$, so we only have to prove that $x_2^{\beta_i} x_3^{i-2} \in \mathsf{N}(J)$, $2 \leq i \leq k$.
The bar $\mathsf{B}^{(3)}_{i-1}$ is labelled by x_3^{i-2} and, over $\mathsf{B}^{(3)}_{i-1}$, there are $\beta_{i-1} > \beta_i$ 2-bars. The $(\beta_i + 1)$-th 2-bar over $\mathsf{B}^{(3)}_{i-1}$, i.e. $\mathsf{B}^{(2)}_t$, $t = \sum_{l=1}^{i-2} \beta_l + (\beta_i + 1)$, is labelled by $x_2^{\beta_i} x_3^{i-2}$, so all the terms labelling the 1-bars over $\mathsf{B}^{(2)}_t$ are multiples of $x_2^{\beta_i} x_3^{i-2}$ and since the Bar Code is admissible, we can desume that $x_2^{\beta_i} x_3^{i-2} \in \mathsf{N}(J)$.
Let us finally take $x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-1}$, $1 \leq i \leq k$, $1 \leq j \leq \beta_i$; we need to prove that $x_1^{\rho_{i,j}} x_2^{j-2} x_3^{i-1}$ and $x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-2}$, when they are defined, belong to $\mathsf{N}(J)$.

- $x_1^{\rho_{i,j}} x_2^{j-2} x_3^{i-1} \in \mathsf{N}(J)$: we take $\mathsf{B}^{(2)}_t$, $t = \sum_{l=1}^{i-1} \beta_l + (j-1)$, i.e. the $(j-1)$-th 2-bar over $\mathsf{B}^{(3)}_i$; since $\rho_{i,j-1} > \rho_{i,j}$ the $(\rho_{i,j} + 1)$-th 1-bar over $\mathsf{B}^{(2)}_t$ is labelled by $x_1^{\rho_{i,j}} x_2^{j-2} x_3^{i-1}$, so belonging to $\mathsf{N}(J)$;
- $x_1^{\rho_{i,j}} x_2^{j-1} x_3^{i-2} \in \mathsf{N}(J)$: analogously as above, it comes from the inequality $\rho_{i-1,j} > \rho_{i,j}$.

This proves the stability of J, concluding our proof. □

Now, by Theorem 68, counting stable ideals in three variables becomes an application of Theorem 10 (see [31]).

Fix a constant Hilbert polynomial p. Lemma 63 allows to enumerate all bar lists. Fix then a bar list (p, h, k) and construct the plane partitions ρ as explained above, denoting by $(\beta_1, ..., \beta_k)$ their shape. Finally, denote by $b = (1, ..., 1)$ and $a = (a_1, ..., a_k)$ such that

$$\begin{cases} a_1 = p - \frac{\beta_1(\beta_1-1)}{2} - \sum_{i=2}^{k} \frac{\beta_i(\beta_i+1)}{2} \\ a_i = a_{i-1} - 1,\ 2 \le i \le k \end{cases} \tag{1}$$

the vectors of Theorem 10. We can compute the number of stable ideals by exploiting the formula in the aforementioned Theorem (see appendix A.1).

We remark that our choice for a and b meets the required inequalities of Theorem 10, remembering that $\mu = 0$ and $\lambda_i > \lambda_{i+1}$ for each $i = 1, ..., k-1$. Indeed, $a_i = a_{i+1} + 1$ so $a_i \ge a_{i+1}$ and $b_i + (\lambda_i - \lambda_{i+1}) = 1 + (\lambda_i - \lambda_{i+1}) \ge 1 = b_{i+1}$.

7 Counting strongly stable ideals

In this section, we extensively deal with strongly stable ideals (see Definition 46).

An asymptotical estimation of the number of strongly stable ideals with a fixed constant Hilbert polynomial has been given by Onn-Sturmfels in [50]; in the aforementioned paper, $\binom{\mathbb{N}^2}{n}_{\text{stair}}$ denotes the size-n subsets of $\mathbb{N}^2$ that are also staircases.

Proposition 69. *The number of Borel-fixed staircases in* $\binom{\mathbb{N}^2}{n}_{stair}$ *is* $2^{\Omega(\sqrt{n})}$.

The following Lemma is enough to deal with the case of two variables.

Lemma 70. *An ideal* $I \triangleleft \mathbf{k}[x_1, x_2]$ *is stable if and only if it is strongly stable.*

Proof. A strongly stable ideal is trivially stable, so we only need to prove the converse, namely, given a stable ideal I, we have to show that for each for every term $\tau \in I$ and pair of variables $x_i,\ x_j$ such that $x_i | \tau$ and $x_i < x_j$, then also $\frac{\tau x_j}{x_i}$ belongs to I. The only pair of variables of the above type is $x_1 < x_2$ and x_1 is the smallest variable in the polynomial ring $\mathbf{k}[x_1, x_2]$ so, if $x_1 \mid \tau \in I$, then $x_1 = \min(\tau)$ and $\frac{\tau x_2}{x_1} \in I$ by definition of stable ideal, whereas if $x_1 \nmid \tau$ there is nothing to do. This proves the claimed equivalence. ◻

By the above Lemma and by Proposition 58, we can conclude that the number of strongly stable ideals $J \triangleleft \mathbf{k}[x_1, x_2]$ with $H_{\_}(t, J) = p$ is $\sum_{i=1}^{h} Q(p, i)$, where $h := \left\lfloor \frac{-1+\sqrt{1+8p}}{2} \right\rfloor$ and $Q(p, i)$ is the number of integer partitions of p into i distinct parts.

Let us examine now the case of strongly ideals in $\mathbf{k}[x_1, x_2, x_3]$.
Strongly stable ideals are also stable, so all the propositions proved for stable ideals also hold here; then the computation of the bar lists is the same as done for stable ideals. Fixed a bar list (p, h, k), we first compute the integer partitions of h in k distinct

parts. Each partition $(\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$, $\alpha_1 > ... > \alpha_k$, $\sum_{i=1}^k \alpha_i = h$ represents a precise structure for the 2-bars and the 3-bars: for each $1 \le i \le k$ there are exactly α_i 2-bars over $\mathsf{B}_i^{(3)}$.

Now, fix a partition $\overline{\alpha} \in I_{(h,k)}$, $\overline{\alpha} = (\overline{\alpha_1}, ..., \overline{\alpha_k}) \in \mathbb{N}^k$, $\overline{\alpha_1} > ... > \overline{\alpha_k}$, $\sum_{i=1}^k \overline{\alpha_i} = h$. We can construct the plane partitions π of the form

$$\pi = (\pi_{i,j}) = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & ... & ... & ... & ... & ... & ... & \pi_{1,\overline{\alpha_1}} \\ 0... & \pi_{2,2} & ... & ... & ... & ... & ... & \pi_{2,2+\overline{\alpha_2}-1} & 0... \\ 0... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0... & ... & ... & \pi_{k,k} & ... & ... & \pi_{k,k+\overline{\alpha_k}-1} & 0... & ... \end{pmatrix}$$

s.t.

1. $\pi_{i,j} > 0$, $1 \le i \le k$, $i \le j \le i + \overline{\alpha_i} - 1$;
2. $\pi_{i,j} > \pi_{i,j+1}$, $1 \le i \le k$, $i \le j < i + \overline{\alpha_i} - 1$;
3. $\pi_{i,j} \ge \pi_{i+1,j}$ $1 \le i \le k-1$, $i+1 \le j \le i + \overline{\alpha_{i+1}} - 1$;
4. $n(\pi) = \sum_{i=1}^k \sum_{j=i}^{i+\overline{\alpha_i}-1} \pi_{i,j} = p$.

These plane partitions are exactly of the form of Definition 7, with $\lambda_i = i + \overline{\alpha_i} - 1 \ge i$, $1 \le i \le k$, $c = 1$ and $d = 0$.
In Remark 71, we will highlight the relation between these partitions and the ones defined in the previous section 6.
We denote by $\mathcal{S}_{(p,h,k),\overline{\alpha}}$ the set of all partitions defined above and $\mathcal{S}_{(p,h,k)} = \bigcup_{\overline{\alpha} \in I_{(h,k)}} \mathcal{S}_{(p,h,k),\overline{\alpha}}$. In other words,

$$\mathcal{S}_{(p,h,k),\overline{\alpha}} = \{\pi \in \mathcal{S}_\lambda(1,0),\ n(\pi) = p,\ \lambda_i = i + \overline{\alpha_i} - 1,\ 1 \le i \le k\}$$

$$\mathcal{S}_{(p,h,k)} = \{\pi \in \mathcal{S}_\lambda(1,0),\ n(\pi) = p,\ \lambda_i = i + \overline{\alpha_i} - 1,\ 1 \le i \le k,\ \text{for some } \overline{\alpha} \in I_{(h,k)}\}$$

Remark 71. We remark that the set of the shifted plane partitions defined here for strongly stable ideals can be easily viewed as a subset of the strict plane partitions defined in the previous section for counting stable ideals.
With the notation above, let us take a shifted plane partition $\pi := (\pi_{i,j})$, $1 \le i \le k$, $i \le j \le i + \alpha_i - 1$. There are exactly α_i elements in the i-th row and the values in row i is shifted to the right by $i-1$ positions. We define then a non-shifted plane partition $\rho := (\rho_{i,m})$ of shape $\alpha = (\alpha_1, ..., \alpha_k)$, by $\rho_{i,m} = \pi_{i,m+i-1}$ $1 \le i \le k$, $1 \le m \le \alpha_i$. We prove that $\rho \in \mathcal{P}_{(p,h,k),\alpha}$:

- $\rho_{i,m} > 0$, $1 \le i \le k$, $1 \le m \le \alpha_i$ holds true since $\pi_{i,j} > 0$, $1 \le i \le k$ $i \le j \le i + \alpha_i - 1$.
- $\rho_{i,m} > \rho_{i,m+1}$, $1 \le i \le k$, $1 \le m \le \alpha_i - 1$ is trivially true since $\pi_{i,m+i-1} > \pi_{i,m+i}$.

- $\rho_{i,m} > \rho_{i+1,m}$ $1 \leq i \leq k-1$, $1 \leq j \leq \alpha_{i+1}$ comes from $\pi_{i,m+i-1} > \pi_{i,m+i} \geq \pi_{i+1,m+i}$.
- $n(\rho) = \sum_{i=1}^{k} \sum_{m=1}^{\alpha_i} \rho_{i,j} = \sum_{i=1}^{k} \sum_{j=i}^{\alpha_i+i-1} \pi_{i,j} = p$.

On the other hand, we have to point out that there are some strict plane partitions that cannot be brought back to any shifted plane partition. For example, if we shift

$$\rho = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

we get

$$\pi = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix},$$

which is not of the type defined here and cannot be associated to any strongly stable monomial ideal.

Each plane partition $\pi \in \mathcal{S}_{(p,h,k)}$ uniquely identifies a Bar Code B:

(a) each row i represents a 3-bar $\mathsf{B}_i^{(3)}$, $1 \leq i \leq k$;

(b) for each row i, $1 \leq i \leq k$, $l_2(\mathsf{B}_i^{(3)}) = \overline{\alpha_i}$; the $\overline{\alpha_i}$ nonzero entries represent the $\overline{\alpha_i}$ 2-bars over $\mathsf{B}_i^{(3)}$, i.e $\mathsf{B}_t^{(2)}$, where $t = (\sum_{l=1}^{i-1} \overline{\alpha_l}) + j - i + 1$, $i \leq j \leq i + \overline{\alpha_i} - 1$;

(c) for each $1 \leq i \leq k$, and each $i \leq j \leq i + \overline{\alpha_i} - 1$, the number $\pi_{i,j}$ represents the number of 1-bars over $\mathsf{B}_t^{(2)}$, $t = (\sum_{l=1}^{i-1} \overline{\alpha_l}) + j - i + 1$, namely the $j-i+1$-th 2-bar lying over $\mathsf{B}_i^{(3)}$. In other words, $l_1(\mathsf{B}_t^{(2)}) = \pi_{i,j}$.

In conclusion, for each $1 \leq i \leq k$, and each $i \leq j \leq i+\overline{\alpha_i}-1$, the number $\pi_{i,j}$ means that in B there are 1-bars labelled by $(0, j-i, i-1), (1, j-i, i-1), ..., (\pi_{i,j}-1, j-i, i-1)$, but there is no 1-bar labelled by $(\pi_{i,j}, j-i, i-1)$, that is also equivalent to say that $x_1^0 x_2^{j-i} x_3^{i-1}, x_1 x_2^{j-i} x_3^{i-1}, ..., x_1^{\pi_{i,j}-1} x_2^{j-i} x_3^{i-1}$ belong to the set of terms associated to B via Bbc1 and Bbc2, but $x_1^{\pi_{i,j}} x_2^{j-i} x_3^{i-1}$ does not belong to the aforementioned set[12].

Example 72. Let us take the bar list $(p,h,k) = (6,3,2)$, $\overline{\alpha_1} = 2 > \overline{\alpha_2} = 1$, $\overline{\alpha_1} + \overline{\alpha_2} = 3 = h$. We have, for example

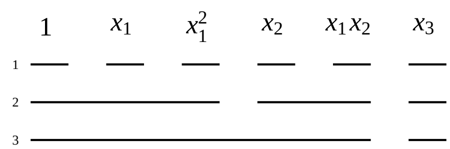
$$\pi = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

and it holds

1. $\pi_{i,j} > \pi_{i,j+1}$, $1 \leq i \leq 2$, $i \leq j < i + \overline{\alpha_i} - 1$, i.e. $\pi_{1,1} > \pi_{1,2}$;
2. $\pi_{i,j} \geq \pi_{i+1,j}$ $i = 1$, $j = 2$, i.e. $\pi_{1,2} \geq \pi_{2,2}$;
3. $n(\pi) = \sum_{i=1}^{2} \sum_{j=i}^{i+\overline{\alpha_i}-1} \pi_{i,j} = 6$.

With the notation of [31], $\lambda_1 = \lambda_2 = 2$.
The partition π uniquely identifies the Bar Code B below:

[12] Again, as for stable ideals, we will see that B is admissible and that $x_1^{\pi_{i,j}} x_2^{j-i} x_3^{i-1}$ belongs to the star set associated to B.



with $k = 2$ 3-bars $B_1^{(3)}$, $\mathsf{B}_2^{(3)}$, $l_2(B_1^{(3)}) = 2$, $l_2(B_2^{(3)}) = 1$. The bars $\mathsf{B}_1^{(2)}$ and $\mathsf{B}_2^{(2)}$ lie over $B_1^{(3)}$, whereas $\mathsf{B}_3^{(2)}$ lie over $B_2^{(3)}$. As regards 1-lengths, we have $l_1(\mathsf{B}_1^{(2)}) = \pi_{1,1} = 3$, $l_1(\mathsf{B}_2^{(2)}) = \pi_{1,2} = 2$ and $l_1(\mathsf{B}_3^{(2)}) = \pi_{2,2} = 1$. The associated set of terms, via BbC1 and BbC2 is $\mathsf{N} = \{1, x_1, x_1^2, x_2, x_1x_2, x_3\}$ and it is an order ideal.

◊

Remark 73. The Bar Code B, uniquely identified by π, has bar list $\mathsf{L}_\mathsf{B} = (p, h, k)$. The relation $\mu(3) = k$ comes from (a), $\mu(2) = h$ comes from (b), since $\alpha \in I_{(h,k)}$, so $\sum_{i=1}^k \alpha_i = h$, whereas $\mu(1) = p$ is an easy consequence of (c).

Lemma 74. *Fixed* (p, h, k) *and* $\alpha \in I_{(h,k)}$, $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$, $\alpha_1 > ... > \alpha_k$, $\sum_{i=1}^k \alpha_i = h$, *let* π *be a partition in* $\mathcal{S}_{(p,h,k),\alpha}$. *The Bar Code* B*, uniquely identified by* π*, is admissible.*

Proof. By Remark 73, $\mathsf{L}_\mathsf{B} = (p, h, k)$. Consider a 1-bar $\mathsf{B}_l^{(1)}$, $1 \leq l \leq p$ and let its e-list be $e(\mathsf{B}_l^{(1)}) = (b_{l,1}, b_{l,2}, b_{l,3})$. From the construction of B from π, we desume that $\pi_{b_{l,3}+1, b_{l,2}+b_{l_3}+1} \geq b_{l,1}+1$; moreover, we know that $(m, b_{l,2}, b_{l,3})$, $0 \leq m \leq \pi_{b_{l,3}+1, b_{l,2}+b_{l,3}+1} - 1$ are e-lists for some bars of B, so, if $b_{l,1} \geq 1$, $(b_{l,1} - 1, b_{l,2}, b_{l,3})$ is a bar list labelling a 1-bar of B.
For B being admissible, we also need two other conditions:

- if $b_{l,2} > 0$, $(b_{l,1}, b_{l,2} - 1, b_{l,3})$ labels a 1-bar of B;
- if $b_{l,3} > 0$, $(b_{l,1}, b_{l,2}, b_{l,3} - 1)$ labels a 1-bar of B.

Let us prove them:

- suppose $b_{l,2} > 0$; for $(b_{l,1}, b_{l,2} - 1, b_{l,3})$ labelling a 1-bar of B, we would need $\pi_{b_{l_3}+1, b_{l_2}+b_{l_3}} \geq b_{l_1} + 1$, but since $\pi_{b_{l_3}+1, b_{l_2}+b_{l_3}} > \pi_{b_{l_3}+1, b_{l_2}+b_{l_3}+1} \geq b_{l_1} + 1$ we are done
- suppose $b_{l,3} > 0$; for $(b_{l,1}, b_{l,2}, b_{l,3} - 1)$ labelling a 1-bar of B, we would need $\pi_{b_{l_3}, b_{l_2}+b_{l_3}} \geq b_{l_1} + 1$, but since $\pi_{b_{l_3}, b_{l_2}+b_{l_3}} > \pi_{b_{l_3}, b_{l_2}+b_{l_3}+1} \geq \pi_{b_{l_3}+1, b_{l_2}+b_{l_3}+1} \geq b_{l_1} + 1$ we are done again and B turns out to be admissible.

□

Example 75. The set of terms associated to the Bar Code constructed in example 72 is an order ideal, so the Bar Code is admissible. ◊

Theorem 76. *There is a biunivocal correspondence between* $\mathcal{S}_{(p,h,k)}$ *and the set* $\mathsf{B}_{(p,h,k)} = \{\mathsf{B} \in \mathcal{A}_3$ *s.t.* $\mathsf{L}_\mathsf{B} = (p, h, k)$, $\eta(\mathsf{B}) = \mathsf{N}(J)$, *J strongly stable*$\}$.

Proof. Let $\mathsf{B} \in \mathsf{B}_{(p,h,k)}$. We construct a plane partition

$$\pi = (\pi_{i,j}) = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \dots & \dots & \dots & \dots & \dots & \dots & \pi_{1,\alpha_1} \\ 0\dots & \pi_{2,2} & \dots & \dots & \dots & \dots & \dots & \pi_{2,2+\alpha_2-1} & 0\dots \\ 0\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0\dots & \dots & \dots & \pi_{k,k} & \dots & \dots & \pi_{k,k+\alpha_k-1} & 0\dots & \dots \end{pmatrix}$$

with k rows and $l_2(\mathsf{B}_1^{(3)})$ columns. Fixed the index i for the rows and the index j for the columns, we define $\pi_{i,j} = 0$ if $j < i$ or $i + \alpha_i - 1 < j \leq l_2(\mathsf{B}_1^{(3)})$ and $\pi_{i,j} = l_1(\mathsf{B}_t^{(2)})$ with $t = (\sum_{l=1}^{i-1} \alpha_l) + j - i + 1$ otherwise, where $\alpha_i = l_2(\mathsf{B}_i^{(3)})$, $1 \leq i \leq k$.
We observe that the partition π is uniquely determined by B and that, by Proposition 50, $\alpha \in I_{(h,k)}$; we have to prove that $\pi \in \mathcal{S}_{(p,h,k)}$.
The nonzero parts of π are positive by definition of length of a bar.

Clearly $\pi_{i,j} > \pi_{i,j+1}$, $1 \leq i \leq k$, $i \leq j < i + \alpha_i - 1$, indeed, this can be stated as $l_1(\mathsf{B}_t^{(2)}) > l_1(\mathsf{B}_{t+1}^{(2)})$ with $\mathsf{B}_t^{(2)}$ and $\mathsf{B}_{t+1}^{(2)}$ lying over the same 3-bar $\mathsf{B}_i^{(3)}$. This statement follows from Proposition 50 b) with $i = 1$.

Moreover, $\pi_{i,j} \geq \pi_{i+1,j}$ $1 \leq i \leq k-1$, $i+1 \leq j \leq i + \alpha_{i+1}$.
Indeed, if $\pi_{i,j} < \pi_{i+1,j}$ then it would happen that $x_1^{\pi_{i+1,j}-1} x_2^{j-i-1} x_3^{i} \in \mathsf{N}(J)$, but $x_1^{\pi_{i+1,j}-1} x_2^{j-i} x_3^{i-1} \notin \mathsf{N}(J)$, contradicting the strongly stable property of J. By construction, the shape of π is $\lambda = (\lambda_1, ..., \lambda_k)$ with $\lambda_i = i + \alpha_i - 1$, $1 \leq i \leq k$, so $\pi \in \mathcal{S}_\lambda(1,0)$. Moreover, $n(\pi) = p$ by definitions of bar list and 1-length.

Then, we can define a map

$$\Xi : \mathcal{B}_{(p,h,k)} \to \mathcal{S}_{(p,h,k)}$$

$$\mathsf{B} \mapsto \pi,$$

where π is constructed from B as described above. We prove that Ξ is a bijection.

It is clearly an injection by definition of lenght of a bar: two different Bar Codes have at least one bar with different length.

Now, we have to prove the surjectivity of Ξ, so let us take $\pi \in \mathcal{S}_{(p,h,k)}$. We know that it uniquely identifies a Bar Code B and by Lemma 74 that B is admissible, so we only have to prove that $\mathsf{B} \in \mathcal{B}_{(p,h,k)}$.

More precisely, we have to prove that $\mathsf{L}_\mathsf{B} = (p,h,k)$ and that $\eta(B) = \mathsf{N}(J)$, J strongly stable.

Since

1. there are k 3-bars,
2. for each row i, $1 \leq i \leq k$, $l_2(\mathsf{B}_i^{(3)}) = \alpha_i$ and $\sum \alpha_i = h$,
3. for each $1 \leq i \leq k$, and each $i \leq j \leq i+\alpha_i-1$, $l_1(\mathsf{B}_t^{(2)}) = \pi_{i,j}$, $t = (\sum_{l=1}^{i-1} \alpha_l)+j-i+1$ and $n(\pi) = p$,

then $\mathsf{L}_\mathsf{B} = (p,h,k)$.

Now, let $\mathsf{B}_l^{(1)}$ $l \in \{1,...,p\}$ be a 1-bar labelled by $e(\mathsf{B}_l^{(1)}) = (b_{l,1}, b_{l,2}, b_{l,3})$, so $\pi_{b_{l,3}+1, b_{l,2}+b_{l_3}+1} \geq b_{l,1} + 1$.

To prove that J is strongly stable, we have to prove that

- if $b_{l,3} > 0$, $(b_{l,1}+1, b_{l,2}, b_{l,3}-1)$ and $(b_{l,1}, b_{l,2}+1, b_{l,3}-1)$ are the e-lists of some 1-bars of B
- $b_{l,2} > 0$, $(b_{l,1}+1, b_{l,2}-1, b_{l,3})$ is the e-list of a 1-bar of B.

Let us prove these statements .

- suppose that $b_{l,3} > 0$ and consider $(b_{l,1}+1, b_{l,2}, b_{l,3}-1)$: we have to prove that $\pi_{b_{l_3}, b_{l_2}+b_{l_3}} \geq b_{l_1}+2$. Since $\pi_{b_{l_3}, b_{l_2}+b_{l_3}} > \pi_{b_{l_3}, b_{l_2}+b_{l_3}+1} \geq \pi_{b_{l_3}+1, b_{l_2}+b_{l_3}+1} \geq b_{l,1}+1$ we are done.
- suppose that $b_{l,3} > 0$ and consider $(b_{l,1}, b_{l,2}+1, b_{l,3}-1)$: we have to prove that $\pi_{b_{l_3}, b_{l_2}+b_{l_3}+1} \geq b_{l_1}+1$. Since $\pi_{b_{l_3}, b_{l_2}+b_{l_3}+1} \geq \pi_{b_{l_3}+1, b_{l_2}+b_{l_3}+1} \geq b_{l,1}+1$ we are done.
- suppose that $b_{l,2} > 0$ and consider $(b_{l,1}+1, b_{l,2}-1, b_{l,3})$: we have to prove that $\pi_{b_{l_3}+1, b_{l_2}+b_{l_3}} \geq b_{l_1}+2$. Since $\pi_{b_{l_3}+1, b_{l_2}+b_{l_3}} > \pi_{b_{l_3}+1, b_{l_2}+b_{l_3}+1} \geq b_{l,1}+1$ we are done.

This concludes our proof. □

Now, by Theorem 76, counting strongly stable ideals in three variables becomes an application of Theorem 12 ([32]).

Fix a constant Hilbert polynomial p. Lemma 63 allows to compute all bar lists. Fix then a bar list (p,h,k) and their shape λ. Finally, denote by $b = (1,...,1)$ and $a = (a_1,...,a_r)$ such that

$$\begin{cases} a_r = \lambda_r - r + 1, ..., \mathbf{M} - r + 1 \\ a_i = a_{i+1} + 1, ..., \mathbf{M} - i + 1, \ 1 \leq i \leq r-1 \end{cases} \tag{2}$$

$\mathbf{M} := p - \sum_{i=1}^{r} \frac{c_i(c_i+1)}{2}$, $c_1 = \lambda_1 - 1$ and $c_j = \lambda_j - j + 1$, $j = 2,...,r$, the vectors of Theorem 12. We can compute the number of strongly stable ideals by exploiting the formula in the aforementioned Theorem (see appendix A.2).

There is a simple case of shifted $(1,0)$-plane partition for which a closed formula can be easily computed.

Proposition 77. *Let $p \in \mathbb{N}\backslash\{0\}$. Then there is a biunivocal correspondence between the sets $\mathcal{S}_\lambda(1,0)$ with $\lambda = (2,2)$ and $P_{3,p-1} := \{\lambda'$ partition of $p-1$ in 3 non necessarily distinct parts $\}$.*

Proof. Let $\pi \in \mathcal{S}_\lambda(1,0)$, $\lambda = (2,2)$, then π is of the form

$$\begin{pmatrix} \pi_{1,1} & \pi_{1,2} \\ 0 & \pi_{2,2} \end{pmatrix}$$

with $\pi_{1,1} > \pi_{1,2}$, $\pi_{1,2} \geq \pi_{2,2}$, and $\pi_{1,1} + \pi_{1,2} + \pi_{2,2} = p$.
Consider the 3-uple $\pi' = (\pi_{1,1}-1, \pi_{1,2}, \pi_{2,2})$, whose sum is $\pi_{1,1} - 1 + \pi_{1,2} + \pi_{2,2} = p-1$. Since $\pi_{1,1} - 1 \geq \pi_{1,2} \geq \pi_{2,2}$ then π' is a partition of $p-1$ in three non necessarily distinct parts.
Conversely, let us consider a partition $\pi' = (\pi'_1, \pi'_2, \pi'_3) \in P_{3,p-1}$ of $p-1$ in three non

necessarily distinct parts. Then $\pi'_1 \geq \pi'_2 \geq \pi'_3$. Take $\pi'' := (\pi'_1 + 1, \pi'_2, \pi'_3)$: $\pi'_1 + 1 > \pi'_2$, $\pi'_2 \geq \pi'_3$ and $\pi'_1 + 1 + \pi'_2 + \pi'_3 = p$ so, putting it in the plane as

$$\begin{pmatrix} \pi'_1 + 1 & \pi'_2 \\ 0 & \pi'_3 \end{pmatrix}$$

we get a shifted $(1, 0)$-plane partition of shape $(2, 2)$ of p. □

The closed formula for the partitions of Proposition 77 is well known in literature.

Proposition 78 (Hardy-Wright,[25, 40])**.** *The partitions of the set* $P_{3,p-1}$ *are* $\lfloor \frac{(p-1)^2+6}{12} \rfloor$.

In general, finding closed formulas for plane partitions is rather difficult and most of them are still unknown.

8 Future work and generalizations

In this section, we present a conjecture on the relation between (strongly) stable ideals in $\mathbf{k}[x_1, ..., x_n]$, $n > 3$ and integer partitions.
We start setting an ordering on n-tuples of natural numbers, that we will need to define the required partitions.

Definition 79. *Let* $(i_1, ..., i_n), (j_1, ..., j_n) \in \mathbb{N}^n$*; we say that* $(i_1, ..., i_n) < (j_1, ..., j_n)$ *if* $i_1 \leq j_1, ..., i_n \leq j_n$ *but* $(i_1, ..., i_n) \neq (j_1, ..., j_n)$.

We can now define *strict solid partitions* (so partitions of dimension $n = 3$) and then, inductively *strict n-partitions*, for $n \geq 4$; they are the natural generalization for the partitions of Definition 6 and they will be necessary in order to state our conjecture for stable ideals.

Definition 80. *Let* $\rho = (\rho_{i,j})_{i\in\{1,...,r\}, j\in\{1,...,\beta_i\}}$ *be a* $(1, 1)$*-plane partition of shape* $\beta = (\beta_1, ..., \beta_r)$, $\beta_1 > ... > \beta_r$ *(see Definition 6). A* strict solid partition *(or* strict 3-partition*) of shape* ρ *is a* 3*-dimensional array* $\gamma = (\gamma_{i_1,i_2,i_3})$, $1 \leq i_1 \leq \beta_{i_3}$, $1 \leq i_2 \leq \rho_{i_3,i_1}$, $1 \leq i_3 \leq r$*, s.t.*

- *for each* $1 \leq l \leq r$*, the* 2*-dimensional array* $\gamma_l := (\gamma_{i_1,i_2,l})$ *is a* $(1, 1)$*-plane partition of shape* $\rho_l = (\rho_{l,1}, ..., \rho_{l,\beta_l})$.
- $\gamma_{i_1,i_2,i_3} > \gamma_{j_1,j_2,j_3}$*, for* $(i_1, i_2, i_3) < (j_1, j_2, j_3)$.

We denote by $\mathcal{P}_\rho(1, 1, 1)$ the set of strict 3-partitions of shape ρ.

Definition 81. *For* $n \geq 4$*, consider a strict* $(n - 1)$*-partition* $\rho = (\rho_{\bar{i}_1,...,\bar{i}_{n-1}})$ *with* $1 \leq \bar{i}_{n-1} \leq h$*, for some* $h > 0$.
A strict n-partition *of shape* ρ *is a n-dimensional array* $\gamma = (\gamma_{i_1,...,i_n})$ *s.t.*

- *for each* $1 \leq l \leq h$*,* $\gamma_l := (\gamma_{i_1,...,i_{n-1},l})$ *is a strict* $(n - 1)$*-partition of shape* $\rho_l = (\rho_{\bar{i}_1,...,\bar{i}_{n-2},l})$
- $\gamma_{i_1,...,i_n} > \gamma_{j_1,...,j_n}$*, for* $(i_1, ..., i_n) < (j_1, ..., j_n)$.

We denote by $\mathcal{P}_\rho(\underbrace{1,1,...,1}_{n})$ the set of strict n-partitions of shape ρ.

Example 82. Let us consider the $(1,1)$-plane partition

$$\rho = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

of shape $\beta = (3,2,1)$.
An example of strict solid partition of shape ρ is is the following γ, formed by three $(1,1)$-plane partitions $\gamma_1, \gamma_2, \gamma_3$:

$$\gamma_1 = \begin{pmatrix} \gamma_{\mathbf{1,1,1}} & \gamma_{\mathbf{1,2,1}} & \gamma_{1,3,1} & \gamma_{1,4,1} \\ \gamma_{\mathbf{2,1,1}} & \gamma_{2,2,1} & 0 & 0 \\ \gamma_{3,1,1} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{4} & \mathbf{3} & 2 & 1 \\ \mathbf{3} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} \gamma_{\mathbf{1,1,2}} & \gamma_{1,2,2} & 0 \\ \gamma_{2,1,2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{2} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} \gamma_{1,1,3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

where we mark in bold the elements of γ_i over which those of γ_{i+1} are posed, for $i = 1,2$. ◊

Example 83. Let us consider the following very simple strict solid partition ρ:

$$\rho_1 = \begin{pmatrix} \mathbf{2} & 1 \\ 1 & 0 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

An example of strict 4-partition of shape ρ is

$$\gamma_1 = \begin{pmatrix} \gamma_{\mathbf{1,1,1,1}} & \gamma_{1,2,1,1} \\ \gamma_{2,1,1,1} & 0 \end{pmatrix} \quad \begin{pmatrix} \gamma_{1,1,2,1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{4} & 2 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} \gamma_{1,1,1,2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

◊

It is possible to generalize Lemma 63 to the case of n variables, with some cumbersome computation, so that it is possible to compute the bar lists in order to count stable ideals in $\mathbf{k}[x_1,...,x_n]$.
Fixed a bar list $(p_1,...,p_n) \in \mathbb{N}^n$, $p_1,...,p_n \neq 0$ and a strict $(n-2)$-partition ρ of shape $(p_2,...,p_n)$, we define the following sets

$$\mathcal{P}_\rho(p_1,...,p_n) := \{\gamma \in \mathcal{P}_\rho(\underbrace{1,...,1}_{n-1}),\ n(\gamma) = p_1\}$$

and

$$\mathcal{P}(p_1,...,p_n) := \{\gamma \in \mathcal{P}_\rho(\underbrace{1,...,1}_{n-1}),\ \text{for some } \rho \in \mathcal{P}(p_2,...,p_n),\ \text{s.t. } n(\gamma) = p_1\},$$

where $\mathcal{P}_\rho(\underbrace{1,1,...,1}_{n-1})$ is the set of strict $(n-1)$-partitions of shape ρ.

We can then state our conjecture for stable ideals.

Conjecture 84. *There is a biunivocal correspondence between the set $\mathcal{P}_\rho(p_1,...,p_n)$ and the set $\mathcal{B}_{(p_1,...,p_n)} := \{\mathsf{B} \in \mathcal{A}_n$ s.t. $\mathsf{L}_\mathsf{B} = (p_1,...,p_n),\ \eta(\mathsf{B}) = \mathsf{N}(J),\ J$ stable$\}$.*

In an analogous (but a bit more cumbersome) way, we handle now the case of strongly stable ideals, giving the necessary generalizations of Definition 7 and stating our conjecture.

Definition 85. *Let $\pi = (\pi_{i,j})_{i\in\{1,...,r\},j\in\{1,...,\alpha_i\}}$ be a shifted $(1,0)$-plane partition of shape $\alpha = (\alpha_1,...,\alpha_r)$, $\alpha_1 \geq ... \geq \alpha_r \geq r$ (see Definition 7). A* shifted solid partition *(or* shifted 3-partition*) of shape π is a 3-dimensional array $\gamma = (\gamma_{i_1,i_2,i_3})$, $i_3 \leq i_1 \leq \alpha_{i_3}$, $i_1 \leq i_2 \leq \pi_{i_3,i_1} + i_1 - 1$, $1 \leq i_3 \leq r$, s.t.*

- *for each $1 \leq l \leq r$, the 2-dimensional array $\gamma_l := (\gamma_{i_1,i_2,l})$ is a shifted $(1,0)$-plane partition of shape $\widetilde{\pi}_l = (\pi_{l,l} + l - 1, \pi_{l,l+1} + l, ..., \pi_{l,\alpha_l} + \alpha_l - 1)$.*
- *$\gamma_{i_1,i_2,i_3} \geq \gamma_{i_1,i_2,i_3+1}$.*

We denote by $\mathcal{S}_\pi(1,1,1)$ the set of shifted 3-partitions of shape π.

Definition 86. *For $n \geq 4$, consider a shifted $(n-1)$-partition $\pi = (\pi_{\bar{i}_1,...,\bar{i}_{n-1}})$ with $1 \leq \bar{i}_{n-1} \leq h$, for some $h > 0$.*
A shifted n-partition *of shape π is a n-dimensional array $\gamma = (\gamma_{i_1,...,i_n})$ s.t.*

- *for each $1 \leq l \leq h$, $\gamma_l := (\gamma_{i_1,...,i_{n-1},l})$ is a shifted $(n-1)$-partition with shape given by the $(n-2)$-partition $\widetilde{\pi}_l = (\pi_{\bar{i}_1,...,\bar{i}_{n-2},l} + i_m - 1)$, where m is the maximal index s.t. $i_m > 1$, and such that, w.r.t. the ordering defined in Definition 79, $(l,l,...,l)$ is the minimal $(i_1,...,i_{n-1},l)$ for which $\gamma_{i_1,...,i_{n-1},l} \neq 0$;*
- *$\gamma_{i_1,...,i_n} \geq \gamma_{i_1,...,i_n+1}$.*

We denote by $\mathcal{S}_\pi(\underbrace{1,1,...,1}_{n})$ the set of shifted n-partitions of shape π.

Example 87. Let us consider the shifted $(1,0)$-plane partition

$$\pi = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

of shape $\alpha = (3,2)$.
An example of strict solid partition of shape π is the following γ, formed by two shifted $(1,0)$-plane partitions γ_1, γ_2:

$$\gamma_1 = \begin{pmatrix} \gamma_{1,1,1} & \gamma_{1,2,1} & \gamma_{1,3,1} \\ 0 & \boldsymbol{\gamma_{2,2,1}} & \boldsymbol{\gamma_{2,3,1}} \\ 0 & 0 & \gamma_{3,3,1} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \mathbf{2} & \mathbf{1} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_{2,2,2} & \gamma_{2,3,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

where we mark in bold the elements of γ_1 over which those of γ_2 are posed. ⋄

Example 88. Let us consider the following very simple shifted solid partition π:

$$\pi_1 = \begin{pmatrix} 2 & 1 \\ 0 & \mathbf{1} \end{pmatrix} \quad \pi_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

An example of strict 4-partition of shape π is[13]

$$\gamma_1 = \begin{pmatrix} \gamma_{1,1,1,1} & \gamma_{1,2,1,1} \\ 0 & \gamma_{\mathbf{2,2,1,1}} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{2,2,2,1} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 0 & \mathbf{2} \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{2,2,2,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

◊

Fixed a bar list $(p_1, ..., p_n) \in \mathbb{N}^n,\ p_1, ..., p_n \neq 0$ and a shifted $(n-2)$-partition π of shape $(p_2, ..., p_n + n - 2)$, we define the following sets

$$\mathcal{S}_\pi(p_1, ..., p_n) := \{\gamma \in \mathcal{S}_\pi(\underbrace{1, ..., 1}_{n-1}),\ n(\gamma) = p_1\}$$

and

$$\mathcal{S}(p_1, ..., p_n) := \{\gamma \in \mathcal{S}_\pi(\underbrace{1, ..., 1}_{n-1}),\ \text{for some } \pi \in \mathcal{S}(p_2, ..., p_n),\ \text{s.t. } n(\gamma) = p_1\},$$

where $\mathcal{S}_\pi(\underbrace{1, 1, ..., 1}_{n-1})$ is the set of shifted $(n-1)$-partitions of shape π.

We can then state our conjecture for strongly stable ideals.

Conjecture 89. *There is a biunivocal correspondence between the set* $\mathcal{S}_\pi(p_1, ..., p_n)$ *and the set* $\mathcal{B}_{(p_1,...,p_n)} := \{\mathsf{B} \in \mathcal{A}_n \text{ s.t. } \mathsf{L}_\mathsf{B} = (p_1, ..., p_n),\ \eta(\mathsf{B}) = \mathsf{N}(J),\ J \text{ strongly stable}\}$.

A Some explicit computation

In example 60 we have counted the (strongly) stable ideals in $\mathbf{k}[x_1, x_2]$; in the next sections, we will count the stable (section A.1) and strongly stable ideals (section A.2) in $\mathbf{k}[x_1, x_2, x_3]$ with constant affine Hilbert polynomial $p = 10$.

A.1 Stable ideals

Let us count the stable ideals in $\mathbf{k}[x_1, x_2, x_3]$ with constant affine Hilbert polynomial $p = 10$.

By Corollary 57 and Lemma 63, the possible bar lists $(p = 10, h, k)$ are:

1. $(10, 1, 1)$;

[13] According to the 3-partition shape definition $\gamma_{2,2,2,1} \geq \gamma_{2,2,1,1}$.

2. (10, 2, 1);

3. (10, 3, 1);

4. (10, 4, 1);

5. (10, 3, 2);

6. (10, 4, 2);

7. (10, 5, 2);

8. (10, 6, 3).

Indeed, for $k = 1$, the maximal value for h is $h = \left\lfloor \frac{-1+\sqrt{1+80}}{2} \right\rfloor = 4$; for $k = 2$, using Lemma 63, 2., we can deduce that h is an integer between $\frac{k(k+1)}{2} = 3$ and 5.
In order to deduce the maximal value 5, we may notice that the only partitions of 6 in $k = 2$ distinct parts are $6 = 5 + 1 = 4 + 2$ and $\mathrm{Sm}([5, 1]) = 16 > p = 10$, $\mathrm{Sm}([4, 2]) = 13 > p = 10$. For $k = 3$, using again Lemma 63, 2., we can deduce that the minimal value for h is $\frac{k(k+1)}{2} = 6$ and that the maximal value for h is again 6. Indeed, the only partition of 7 in $k = 3$ distinct parts is $7 = 4 + 2 + 1$ for which $\mathrm{Sm}([4, 2, 1]) = 14 > p = 10$.
For $k = 1$ above, we have (see Corollary 57) $Q(10, 1) + Q(10, 2) + Q(10, 3) + Q(10, 4) = 10$.
Consider now (10, 3, 2); the only possible shape[14] is $\beta = (2, 1)$, so we have

$$\begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ \rho_{2,1} & 0 \end{pmatrix}$$

We need to take $a = (8, 7)$ (see (1) of section 6) and $b = (1, 1)$ so that the determinant to compute is

$$det \begin{pmatrix} x^3\begin{bmatrix} 8 \\ 2 \end{bmatrix} & x^5\begin{bmatrix} 8 \\ 3 \end{bmatrix} \\ 1 & x\begin{bmatrix} 7 \\ 1 \end{bmatrix} \end{pmatrix}$$

and it gives $x^{22} + 2x^{21} + 3x^{20} + 5x^{19} + 7x^{18} + 9x^{17} + 12x^{16} + 13x^{15} + 14x^{14} + 14x^{13} + 14x^{12} + 12x^{11} + 11x^{10} + 8x^9 + 6x^8 + 4x^7 + 3x^6 + x^5 + x^4$, so we have 11 stable ideals with this bar list.

As for (10, 4, 2) we have $\beta = (3, 1)$, so

$$\begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,3} \\ \rho_{1,2} & 0 & 0 \end{pmatrix}$$

We fix $a = (6, 5)$ (see (1) of section 6) and, by Theorem 10, we have
$x^{20}+2x^{19}+4x^{18}+6x^{17}+9x^{16}+10x^{15}+12x^{14}+11x^{13}+10x^{12}+8x^{11}+6x^{10}+3x^9+2x^8+x^7$, so 6 plane partitions of this shape.

[14] It is the only possible partition of 3 in two distinct parts.

Then take $(10,5,2)$; we have the partition below[15]

$$M = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,3} \\ \rho_{2,1} & \rho_{2,2} & 0 \end{pmatrix}$$

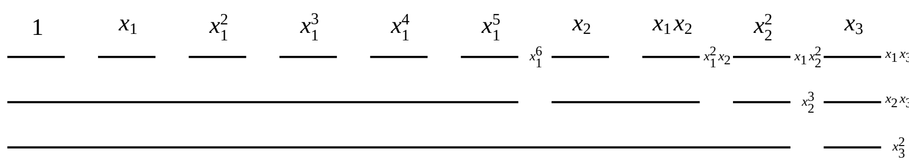
with $\beta = (3,2)$. Fixing $a = (4,3)$ (see (1) of section 6), we get $x^{14} + 2x^{13} + 2x^{12} + 2x^{11} + x^{10} + x^{9}$, so only one partition with norm 10.
We conclude with $(10,6,3)$, for which we have

$$M = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,3} \\ \rho_{2,1} & \rho_{2,2} & 0 \\ \rho_{3,1} & 0 & 0 \end{pmatrix}$$

with $\beta = (3,2,1)$; fixing $a = (3,2,1)$ (see again (1) of section 6), we get x^{10}, so again only one plane partition with this shape. Summing up, we get $10+11+6+1+1 = 29$ stable ideals in $\mathbf{k}[x_1, x_2, x_3]$, with affine Hilbert polynomial equal to 10.

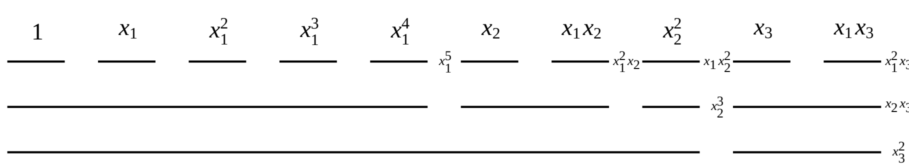
Remark 90. We notice that a tedious computation could allow us to list all 29 plane partitions and the corresponding stable ideals. To show this we limit ourselves to consider the case $(10,4,2)$, for which there are exactly 6 plane partitions:

1. The plane partition
$$\begin{pmatrix} 6 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
uniquely determines the Bar Code



which corresponds to the stable ideal $I_1 = (x_1^6, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2)$;

2. the plane partition
$$\begin{pmatrix} 5 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$
uniquely determines the Bar Code



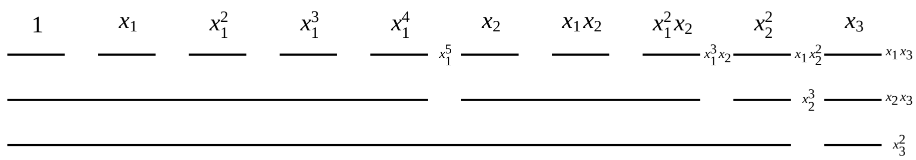
[15] Notice that also $\beta' = (4,1)$ is a potential shape; anyway there are no $(1,0)$-shifted plane partitions of 10 with shape β'.

which corresponds to the stable ideal $I_2 = (x_1^5, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_2x_3, x_3^2)$;

3. the plane partition

$$\begin{pmatrix} 5 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

uniquely determines the Bar Code

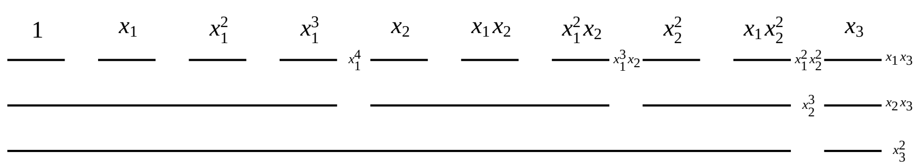


which corresponds to the stable ideal $I_3 = (x_1^5, x_1^3x_2, x_1x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2)$;

4. the plane partition

$$\begin{pmatrix} 4 & 3 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

uniquely determines the Bar Code

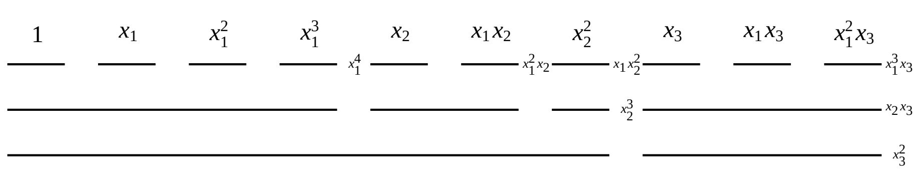


which corresponds to the stable ideal $I_4 = (x_1^4, x_1^3x_2, x_1^2x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2)$;

5. the plane partition

$$\begin{pmatrix} 4 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

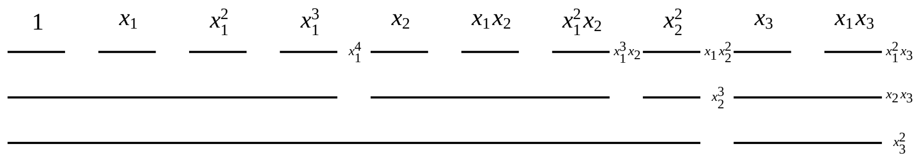
uniquely determines the Bar Code



which corresponds to the stable ideal $I_5 = (x_1^4, x_1^2x_2, x_1x_2^2, x_2^3, x_1^3x_3, x_2x_3, x_3^2)$;

6. the plane partition

$$\begin{pmatrix} 4 & 3 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$



which corresponds to the stable ideal $I_6 = (x_1^4, x_1^3x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_2x_3, x_3^2)$;

A.2 Strongly stable ideals

Let us count the strongly stable ideals in $\mathbf{k}[x_1, x_2, x_3]$ with constant affine Hilbert polynomial $p = 10$.

By Corollary 57 and Lemma 63, the possible bar lists, as for the case of stable ideals, are:

1. $(10, 1, 1)$;
2. $(10, 2, 1)$;
3. $(10, 3, 1)$;
4. $(10, 4, 1)$;
5. $(10, 3, 2)$;
6. $(10, 4, 2)$;
7. $(10, 5, 2)$;
8. $(10, 6, 3)$.

For $k = 1$ above, we proceed as for stable ideals, thanks to the equivalence of Lemma 70, getting $Q(10, 1) + Q(10, 2) + Q(10, 3) + Q(10, 4) = 10$.
Consider now $(10, 3, 2)$, for which we have the partition below

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix}$$

so $\lambda = (2, 2)$, $r = 2$, $\mathbf{M} = 8$, $a_2 = 1, ..., 7$ and $a_1 = a_2 + 1, ..., 8$ (see (2) in section 7). We report here only the computations giving nonzero result:

1. $a = (5, 1)$: $N_1 = 7$ and

$$M = \begin{pmatrix} x^3 + x^2 + x + 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that $x^{N_1} det(M) = x^7(x^3 + x^2 + x + 1)$. Therefore there is one such plane partition.

2. $a = (6, 1)$: $N_1 = 8$ and

$$M = \begin{pmatrix} x^4 + x^3 + x^2 + x + 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that $x^{N_1} det(M) = x^8(x^4 + x^3 + x^2 + x + 1)$. Therefore there is one such plane partition.

3. $a = (7, 1)$: $N_1 = 9$ and

$$M = \begin{pmatrix} x^5 + x^4 + x^3 + x^2 + x + 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that $x^{N_1} det(M) = x^9(x^5 + x^4 + x^3 + x^2 + x + 1)$. Therefore there is one such plane partition.

4. $a = (8, 1)$: $N_1 = 10$ and

$$M = \begin{pmatrix} x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that $x^{N_1} det(M) = x^{10}(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$. Therefore there is one such plane partition.

5. $a = (5, 2)$: $N_1 = 8$ and

$$M = \begin{pmatrix} x^3 + x^2 + x + 1 & 1 \\ 1 & 1 \end{pmatrix}$$

so that $x^{N_1} det(M) = x^8(x^3 + x^2 + x)$. Therefore there is one such plane partition.

6. $a = (6, 2)$: $N_1 = 9$ and

$$M = \begin{pmatrix} x^4 + x^3 + x^2 + x + 1 & 1 \\ 1 & 1 \end{pmatrix}$$

so that $x^{N_1} det(M) = x^9(x^4 + x^3 + x^2 + x)$. Therefore there is one such plane partition.

7. $a = (4, 3)$: $N_1 = 8$ and

$$M = \begin{pmatrix} x^3 + x^2 + x + 1 & x + 1 \\ 1 & 1 \end{pmatrix}$$

so that $x^{N_1} det(M) = x^8 \cdot x^2$. Therefore there is one such plane partition.

The total number we get of the partitions of type

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix}$$

is 7.
We will see below that the plane partitions of this shape can actually be counted in a simpler way.
Take then $(10, 4, 2)$
Since $4 = 3 + 1$, we only have to deal with the partitions below

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & 0 \end{pmatrix},$$

so $\lambda = (3, 2)$, $r = 2$, $\mathbf{M} = 6$, $a_2 = 1, ..., 5$ and $a_1 = a_2 + 1, ..., 6$ (see (2) in section 7). We report here only the computations giving nonzero result:

1. $a = (4, 1)$, $N_1 = 8$ and
$$M = \begin{pmatrix} x^2 + x + 1 & 0 \\ 1 & 1 \end{pmatrix},$$
so that $x^8 det(M) = x^8(x^2+x+1)$. Therefore there is only one such plane partition.

2. $a = (5, 1)$, $N_1 = 9$ and
$$M = \begin{pmatrix} (x^2 + x + 1)(x^2 + 1) & 0 \\ 1 & 1 \end{pmatrix},$$
so that $x^8 det(M) = x^9(x^2 + x + 1)(x^2 + 1)$. Therefore there is only one such plane partition.

3. $a = (5, 1)$, $N_1 = 10$ and
$$M = \begin{pmatrix} (x^4 + x^3 + x^2 + x + 1)(x^2 + 1) & 0 \\ 1 & 1 \end{pmatrix},$$
so that $x^8 det(M) = x^{10}(x^4 + x^3 + x^2 + x + 1)(x^2 + 1))$. Therefore there is only one such plane partition.

4. $a = (4, 2)$, $N_1 = 9$ and
$$M = \begin{pmatrix} (x^4 + x^3 + x^2 + x + 1)(x^2 + 1) & 0 \\ 1 & 1 \end{pmatrix},$$
so that $x^8 det(M) = x^9(x^4 + x^3 + x^2 + x + 1)(x^2 + 1))$. Therefore there is only one such plane partition.

5. $a = (5, 2)$, $N_1 = 10$ and
$$M = \begin{pmatrix} (x^2 + x + 1)(x^2 + 1) & 0 \\ 1 & 1 \end{pmatrix},$$
so that $x^8 det(M) = x^{10}(x^2 + x + 1)(x^2 + 1))$. Therefore there is only one such plane partition.

The total number of the partitions of type

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & 0 \end{pmatrix}$$

is 5.
Consider now $(10, 5, 2)$. We have the partition below

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \end{pmatrix}$$

In this case $\lambda = (3, 3)$, $r = 2$, $\mathbf{M} = 4$ and there is only one partition of this shape, coming from $a = (4, 2)$ (see (2) in section 7). Indeed, in this case $N_1 = 10$,

$$M = \begin{pmatrix} x^2 + x + 1 & 0 \\ x^2 + x + 1 & 1 \end{pmatrix}$$

and we get $x^{N_1} det(M) = x^{10}(x^2 + x + 1)$.
We conclude with $(10, 6, 3)$, for which by $6 = 3 + 2 + 1$. We obtain the matrix

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

for which $\lambda = (3, 3, 3)$, $r = 3$, $b = (1, 1, 1)$ and $\mathbf{M} = 3$. It holds then $a_3 = 1$, $a_2 = 2$, $a_1 = 3$, i.e. there is only one vector a to examine (see (2) in section 7). For $a = (3, 2, 1)$ we get $N_1 = 10$ and

$$M = \begin{pmatrix} 1 & 0 & 0 \\ x + 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

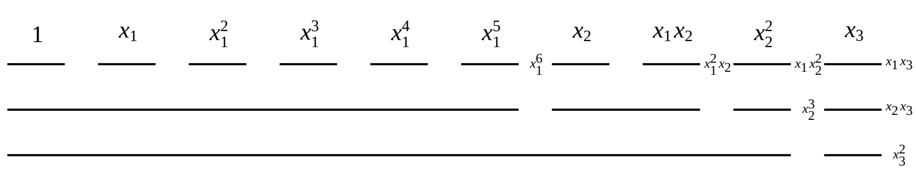
so that $x^{10} det(M) = x^{10}$. We get only one plane partition of norm 10 of this shape.
In conclusion we have exactly 24 strongly stable ideals in 3 variables with constant affine Hilbert polynomial $H_{\underline{\ }}(t) = 10$.

Remark 91. We notice that a tedious computation could allow us to list all 24 plane partitions and the corresponding strongly stable ideals. To show this we limit ourselves to consider the case $(10, 4, 2)$, for which there are exactly 5 plane partitions:

1. The plane partition

$$\begin{pmatrix} 6 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

uniquely determines the Bar Code

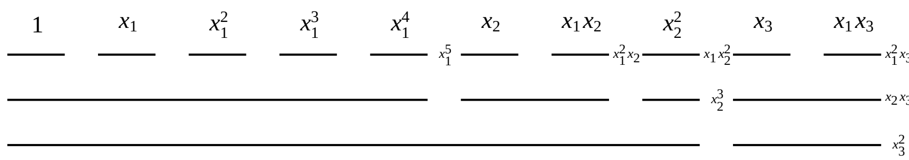


which corresponds to the stable ideal $I_1 = (x_1^6, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2)$;

2. the plane partition

$$\begin{pmatrix} 5 & 2 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

uniquely determines the Bar Code

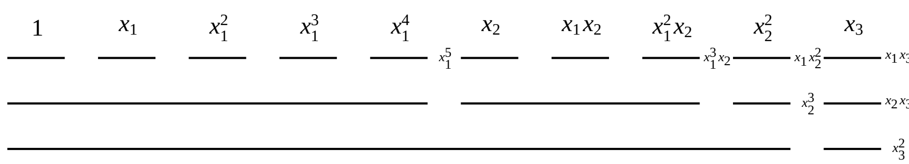


which corresponds to the stable ideal $I_2 = (x_1^5, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_2x_3, x_3^2)$;

3. the plane partition

$$\begin{pmatrix} 5 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

uniquely determines the Bar Code

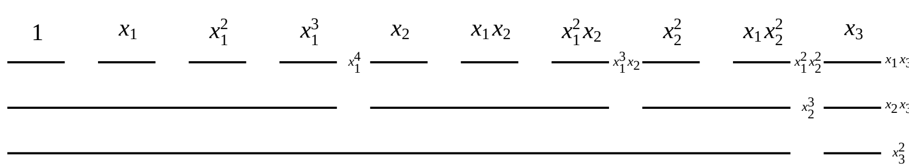


which corresponds to the stable ideal $I_3 = (x_1^5, x_1^3x_2, x_1x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2)$;

4. the plane partition

$$\begin{pmatrix} 4 & 3 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

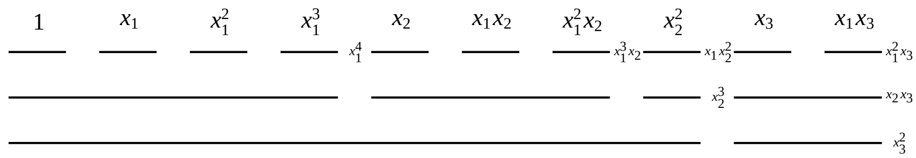
uniquely determines the Bar Code



which corresponds to the stable ideal $I_4 = (x_1^4, x_1^3x_2, x_1^2x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2)$;

5. the plane partition

$$\begin{pmatrix} 4 & 3 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$



which corresponds to the stable ideal $I_5 = (x_1^4, x_1^3x_2, x_1x_2^2, x_2^3, x_1^2x_3, x_2x_3, x_3^2)$;

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