Schur P-positivity and involution Stanley symmetric functions

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Abstract

The *(fixed-point-free) involution Stanley symmetric functions* $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$ are the stable limits of the analogues of Schubert polynomials for the orbits of the orthogonal and symplectic groups in the flag variety. These symmetric functions are also generating functions for involution words, and are indexed by the (fixed-point-free) involutions in the symmetric group. It holds by construction that both $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$ are sums of Stanley symmetric functions and therefore Schur positive. Our main result is to prove the much less trivial fact that these power series are Schur P-positive, that is, nonnegative linear combinations of Schur P-functions. More specifically, we give an algorithm to efficiently compute the decomposition of $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$ into Schur P-summands, and prove that this decomposition is triangular with respect to the dominance order on partitions. As an application, we give pattern avoidance conditions which characterize the involution Stanley symmetric functions which are equal to Schur P-functions. We deduce as a corollary that the (fixed-point-free) involution Stanley symmetric function of the reverse permutation is a Schur P-function indexed by a shifted staircase shape. These results lead to alternate proofs of theorems of Ardila–Serrano and DeWitt on the relationship between skew Schur functions and Schur P-functions. We also prove new Pfaffian formulas for certain involution Stanley symmetric functions and related involution Schubert polynomials.

Contents

*This author was partially supported by NSF grant 1148634.

1 Introduction

In the seminal paper [46], Stanley defined for each permutation w in the symmetric group S_n a certain symmetric function F_w. These symmetric functions are the stable limits of Schubert polynomials, and so arise naturally in the study of the geometry of the type A complete flag variety. They also occur in representation theory as the characters of generalized Schur modules, and are related to the $U_q(A_n)$-crystals introduced by Morse and Schilling in [37]. More concretely, these objects are useful to consider when trying to count the reduced words of permutations. Stanley's construction was originally motivated as a tool for proving the following result:

Theorem 1.1 (Stanley [46]). The cardinalities $(r_n)_{n\geq 1} = (1, 1, 2, 16, 768, 292864, \dots)$ of the set of reduced words for the reverse permutation $w_n = n \cdots 321 \in S_n$ have the exact formula

$$r_n = \tbinom{n}{2}! \cdot 1^{1-n} \cdot 3^{2-n} \cdot 5^{3-n} \cdots (2n-3)^{-1}$$

which also gives the number of standard Young tableaux of shape $\delta_n = (n-1, \dots, 2, 1, 0)$.

Let us explain how F_w is related to the proof of this theorem. Let $s_i = (i, i+1) \in S_n$ for $i \in \{1, 2, \dots, n-1\}$ and write $\mathcal{R}(w)$ for the set of reduced words for $w \in S_n$, that is, the sequences of simple transpositions $(s_{i_1}, s_{i_2}, \dots, s_{i_\ell})$ of minimal possible length ℓ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. For an arbitrary sequence of simple transpositions $\mathbf{a} = (s_{a_1}, s_{a_2}, \dots, s_{a_\ell})$, let $f_{\mathbf{a}} \in \mathbb{Z}[[x_1, x_2, \dots]]$ denote the formal power series given by summing the monomials $x_{i_1} x_{i_2} \cdots x_{i_\ell}$ over all positive integers $i_1 \leq i_2 \leq \cdots \leq i_\ell$ satisfying $i_j < i_{j+1}$ whenever $a_j < a_{j+1}$.

Definition 1.2. The *Stanley symmetric function* of $w \in S_n$ is $F_w = \sum_{\mathbf{a} \in \mathcal{R}(w)} f_{\mathbf{a}}$.

Our notation differs from Stanley's in [46] by an inversion of indices. It is not obvious from this definition that F_w is a symmetric function (for an alternate definition making this clear, see Section 2.2), but it is evident that the size of $|\mathcal{R}(w)|$ is the coefficient of $x_1x_2\cdots x_\ell$ in F_w, where $\ell = \ell(w)$ is the length of w. To find this coefficient we should expand F_w in terms of one of the familiar bases of the algebra of symmetric functions. In general, this is difficult to do explicitly, but much can be said in special cases. Recall that a permutation in S_n is *vexillary* if it is 2143-avoiding, and *Grassmannian* if it has at most one right descent.

Theorem 1.3 (Lascoux and Schützenberger [29])**.** $F_w \in \mathbb{N}\text{-span}\{F_v : v \text{ is Grassmannian}\}$.

Theorem 1.4 (Stanley [46])**.** F_w is a Schur function if and only if w is vexillary.

Grassmannian permutations are vexillary, so these theorems imply the following corollary, first proved by Edelman and Greene [11], who also gave the first bijective proof of Theorem 1.1.

Corollary 1.5 (Edelman and Greene [11])**.** Each F_w is Schur positive.

With a little more notation, one can make an even stronger statement. Write $<$ for the dominance order on integer partitions. Combining results from [29, 46] gives the following.

Theorem 1.6 (Stanley [46], Lascoux and Schützenberger [29])**.** Let $w \in S_n$ and let a_j be the number of positive integers $i < j$ such that $w(i) > w(j)$. If λ is the transpose of the partition given by sorting $(a_1, a_2, \ldots, a_n)$, then $F_w \in s_\lambda + \mathbb{N}\text{-span}\{s_\mu : \mu < \lambda\}$.

Coming full circle, the reverse permutation $w_n \in S_n$ is certainly vexillary, and so the preceding theorem has this corollary, which implies Theorem 1.1 by the familiar hook length formula:

Corollary 1.7. If n is a positive integer then $F_{w_n} = s_{\delta_n}$ for $\delta_n = (n-1, \ldots, 2, 1, 0)$.

We mention all of this as prelude to our main results, which arise out of formally similar counting problems. There are by now a multitude of generalizations [4, 28] of the symmetric functions F_w. The one which will be of interest here comes from the following construction for involutions in Coxeter groups.

Let $\mathcal{I}_n = \{w \in S_n : w^2 = 1\}$ denote the set of involutions in S_n. It is well-known (see Section 2.3) that there exists a unique associative product $\circ : S_n \times S_n \to S_n$ such that $w \circ s_i = ws_i$ if $w(i) < w(i+1)$, $w \circ s_i = w$ if $w(i) > w(i+1)$, and $(v \circ w)^{-1} = w^{-1} \circ v^{-1}$. For each $y \in \mathcal{I}_n$ define $\hat{\mathcal{R}}(y)$ as the set of sequences of simple transpositions $(s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})$ of minimal possible length ℓ such that $y = s_{i_\ell} \circ \cdots \circ s_{i_2} \circ s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_\ell}$. Up to minor differences in notation, the elements of $\hat{\mathcal{R}}(y)$ are the same as what Richardson and Springer [41, 42] call "admissible sequences", what Hultman [19, 20, 21] calls "$\underline{S}$-expressions, what Hu and Zhang [17, 18] call "$\mathbf{I}_*$-expressions," and what we [13, 14, 15] have been calling *involution words*.

There are a few different reasons why one might consider this definition. Geometrically, the notion comes up (e.g., in [9, 41]) when one studies the action of the orthogonal group $\mathrm{O}_n(\mathbb{C})$ on the flag variety. The orbits of this action are indexed by $\mathcal{I}_n$, and the "Bruhat order" induced by reverse inclusion of orbit closures is closely related to the order on $\mathcal{I}_n$ induced by subword containment of involution words. In representation theory, involution words arise in the study of the Iwahori-Hecke algebra modules constructed by Lusztig and Vogan in [30, 31, 32]; see, for example, the applications in [17, 18, 36]. Finally, in combinatorics, these objects are interesting in view of identities like the following, which we proved in [13]. Note here that $w_n = n \cdots 321$ belongs to $\mathcal{I}_n$.

Theorem 1.8 (See [13])**.** The numbers $(\hat{r}_n)_{n\geq 1} = (1, 1, 2, 8, 80, 2688, \dots)$ giving the size of $\hat{\mathcal{R}}(w_n)$ satisfy $\hat{r}_n = \binom{P+Q}{P} r_p r_q$ where $p = \lceil \frac{n+1}{2} \rceil$, $q = \lfloor \frac{n+1}{2} \rfloor$, $P = \binom{p}{2}$, $Q = \binom{q}{2}$, and r_n is as in Theorem 1.1.

This result shows that $\hat{r}_n$ is the number of standard bitableaux of shape (δ_p, δ_q), which is also the dimension of the largest complex irreducible representation of the hyperoctahedral group of rank $P + Q$. These numbers form a subsequence of [45, A066051]. To prove Theorem 1.8, we introduced in [13] the following analogue of F_w:

Definition 1.9. The *involution Stanley symmetric function* of $y \in \mathcal{I}_n$ is $\hat{F}_y = \sum_{\mathbf{a}\in\hat{\mathcal{R}}(y)} f_{\mathbf{a}}$.

As with Definition 1.2, while it is evident that $|\hat{\mathcal{R}}(y)|$ can be extracted as a coefficient of $\hat{F}_y$, this formulation does not make clear that this power series is a symmetric function, or reveal the important fact that each $\hat{F}_y$ is a multiplicity-free sum of (ordinary) Stanley symmetric functions. (An alternate definition which indicates these properties appears in Section 2.3.) These observations show that $\hat{F}_y$ is manifestly Schur positive. Our primary aim in this work is to prove that the symmetric functions $\hat{F}_y$ have a stronger positivity property.

Within the ring of symmetric functions is the subalgebra $\mathbb{Q}[p_1, p_3, p_5, \dots]$ generated by the odd power-sum functions. This algebra arises in few different places in the literature (e.g., [4, 24, 39, 44, 47, 48]), and has a distinguished basis $\{P_\lambda\}$ indexed by strict integer partitions (that is, partitions with all distinct parts), whose elements P_λ are called *Schur P-functions*. See Section 2.5 for the precise definition. With this notation we can summarize our main results. Define a permutation y to be *I-Grassmannian* if it has the form $y = (\phi_1, m+1)(\phi_2, m+2)\cdots(\phi_r, m+r)$ for some positive integers $0 < \phi_1 < \phi_2 < \cdots < \phi_r \leq m$. In Section 4.1, we prove the following:

Theorem 1.10. $\hat{F}_y \in \mathbb{N}\text{-span}\left\{\hat{F}_v : v \text{ is I-Grassmannian}\right\}$.

Define $y \in \mathcal{I}_n$ to be *I-vexillary* if $\hat{F}_y$ is a Schur P-function. The following complementary statement paraphrases Theorems 4.24 and 4.59.

Theorem 1.11. There is a pattern avoidance condition characterizing I-vexillary involutions. All I-Grassmannian involutions as well as the reverse permutations w_n are I-vexillary.

For the (finite) list of patterns that must be avoided, see Corollary 4.60. In Section 4.4, we use this list to derive a new proof of a theorem of DeWitt [10], classifying the skew Schur functions which are Schur P-functions. The last two theorems together imply the following:

Corollary 1.12. Each $\hat{F}_y$ is Schur P-positive, that is, $\hat{F}_y \in \mathbb{N}\text{-span}\{P_\lambda : \lambda \text{ is a strict partition}\}$.

In Section 4.3, we prove the following analogue of Theorem 1.6. One can use this theorem to recover some results of Ardila and Serrano [1]; see Corollary 4.52.

Theorem 1.13. Let $y \in \mathcal{I}_n$ and let b_i be the number of positive integers j with $j \leq i < y(j)$ and $j < y(i)$. If μ is the transpose of the partition given by sorting $(b_1, b_2, \dots, b_n)$, then μ is strict and $\hat{F}_y \in P_\mu + \mathbb{N}\text{-span}\{P_\lambda : \lambda < \mu\}$ where $<$ is the dominance order on strict partitions.

Our proof of Theorem 1.10 is constructive and, combined with the previous theorem, gives an efficient algorithm for computing the expansion of any $\hat{F}_y$ into Schur P-summands. This represents a massive generalization of our main results in [13], which computed $\hat{F}_y$ in a rather limited special case, the most important example of which occurs when $y = w_n$. We can now derive a formula for $\hat{F}_{w_n}$ as an almost trivial corollary, from which Theorem 1.8 follows as a simple exercise:

Corollary 1.14. It holds that $\hat{F}_{w_n} = P_{(n-1,n-3,n-5,\dots)} = s_{\delta_p} s_{\delta_q}$ for $p = \lceil \frac{n+1}{2} \rceil$ and $q = \lfloor \frac{n+1}{2} \rfloor$.

Proof. The first equality holds by Theorems 1.11 and 1.13. The second equality is a consequence of [13, Theorem 1.4] or [47, Theorem 9.3]. $\square$

The first proofs we give for the preceding results are algebraic. In Section 4.6, we present a second, bijective proof of Corollary 1.12 based on Patrias and Pylyavskyy's notion of *shifted Hecke insertion* [38]. This alternate proof shows that the coefficients in the Schur P-expansion of $\hat{F}_y$ are the cardinalities of certain sets of shifted tableaux; see Corollary 4.89.

There is a parallel story when we consider only fixed-point-free involutions in symmetric groups. Assume n is even, let $\Theta_n = (1,2)(3,4)\cdots(n-1,n)$, and write $\mathcal{F}_n$ for the S_n-conjugacy class of Θ_n. For $z \in \mathcal{F}_n$, define $\hat{\mathcal{R}}_{\texttt{FPF}}(z)$ as the set of sequences of simple transpositions $(s_{i_1}, s_{i_2}, \dots, s_{i_\ell})$ of minimal possible length ℓ such that $z = s_{i_\ell} \cdots s_{i_2} s_{i_1} \Theta_n s_{i_1} s_{i_2} \cdots s_{i_\ell}$. These sequences are what one should consider as the "reduced words" of a fixed-point-free involution. They arise, in the same way as involution words, in geometry when one studies the action of the symplectic group on the flag variety, and in representation theory when one studies the "quasi-parabolic" Iwarori-Hecke algebra modules defined by Rains and Vazirani in [40]. The reverse permutation w_n belongs to $\mathcal{F}_n$ when n is even, and the following identity holds regarding the cardinality of $\hat{\mathcal{R}}_{\texttt{FPF}}(w_n)$:

Theorem 1.15 (See [13])**.** The numbers $(\hat{r}^{\texttt{FPF}}_n)_{n=2,4,6\dots} = (1, 2, 80, 236544, 108973522944, \dots)$ giving the size of $\hat{\mathcal{R}}_{\texttt{FPF}}(w_n)$ for n even satisfy $\hat{r}^{\texttt{FPF}}_n = \hat{r}_{n-1}$ where $\hat{r}_n$ is as in Theorem 1.8.

To prove this result, we introduced in [13] a second variant of the symmetric functions F_w:

Definition 1.16. The *FPF-involution Stanley symmetric function* of $z \in \mathcal{F}_n$ is $\hat{F}^{\texttt{FPF}}_z = \sum_{\mathbf{a} \in \hat{\mathcal{R}}_{\texttt{FPF}}(z)} f_{\mathbf{a}}$.

See Section 2.4 for an equivalent definition which shows more clearly that $\hat{F}^{\texttt{FPF}}_z$ is a symmetric function. As with $\hat{F}_y$, each $\hat{F}^{\texttt{FPF}}_z$ is Schur positive by definition, and our contribution will be to show that these symmetric functions are, much less trivially, Schur P-positive.

Both Theorems 1.10 and 1.11 have "fixed-point-free" analogues concerning $\hat{F}^{\texttt{FPF}}_z$, which we summarize as follows. (Unexpectedly, the proofs of these statements turn out to be significantly more complicated than their predecessors.) For positive integers $0 < \phi_1 < \phi_2 < \cdots < \phi_r < m$, let y be the I-Grassmannian involution $(\phi_1, m+1)(\phi_2, m+2)\cdots(\phi_r, m+r)$. Write n for whichever of $m+r$ or $m+r+1$ is even, and define $z = y(f_1, f_2)(f_3, f_4)\cdots(f_{n-2r-1}, f_{n-2r}) \in \mathcal{F}_n$ where $f_1 < f_2 < \cdots < f_{n-2r} \leq n$ are the fixed points of y. Define a permutation to be *FPF-Grassmannian* if it occurs as an involution z of this form. We prove the following theorem in Section 5.20.

Theorem 1.17. $\hat{F}^{\texttt{FPF}}_z \in \mathbb{N}\text{-span}\left\{ \hat{F}^{\texttt{FPF}}_v : v \text{ is FPF-Grassmannian} \right\}$.

As with Theorem 1.10, our proof of this result gives a finite algorithm for decomposing any $\hat{F}_z$ into its FPF-Grassmannian summands. Define an involution $z \in \mathcal{F}_n$ to be *FPF-vexillary* if $\hat{F}^{\texttt{FPF}}_z$ is a Schur P-function. The next statement summarizes Theorems 5.20 and 5.55.

Theorem 1.18. There is a pattern avoidance condition characterizing FPF-vexillary involutions. All FPF-Grassmannian involutions as well as the reverse permutations w_{2n} are FPF-vexillary.

Corollary 5.56 gives the minimal list of 16 patterns that must be avoided. Combining the last two theorems gives this corollary, which we had conjectured in [13]:

Corollary 1.19. Each $\hat{F}^{\mathtt{FPF}}_z$ is Schur P-positive.

As in the involution case, the preceding corollary has this refinement, proved in Section 5.3:

Theorem 1.20. Let $z \in \mathcal{F}_n$ and let c_i be the number of positive integers j with $j < i < y(j)$ and $j < y(i)$. If ν is the transpose of the partition given by sorting $(c_1, c_2, \dots, c_n)$, then ν is strict and $\hat{F}^{\mathtt{FPF}}_z \in P_\nu + \mathbb{N}\text{-span}\{P_\lambda : \lambda < \nu\}$ where $<$ is the dominance order on strict partitions.

Finally, we return to Theorem 1.15, which is an easy consequence of the following corollary:

Corollary 1.21. If n is even then $\hat{F}^{\mathtt{FPF}}_{w_n} = \hat{F}_{w_{n-1}} = P_{(n-2,n-4,n-6,\dots)}$.

Proof. The result follows from Corollary 1.14 and Theorems 1.18 and 1.20. $\square$

The proofs of our main results depend crucially on an interpretation of the symmetric functions $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$ as stable limits of polynomials introduced by Wyser and Yong [50] to represent the cohomology classes of certain orbits closures in the flag variety. In our previous work [15], we proved transition formulas for these cohomology representatives, which we refer to as *(fixed-point-free) involution Schubert polynomials*. After some preliminaries in Section 2, we review these transition formulas in Section 3 and use them to derive some relevant identities for $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$. We prove the theorems sketched in this introduction in Sections 4 and 5. Along the way, we also establish a few other results, such as Pfaffian formulas for certain involution Stanley symmetric functions and Schubert polynomials (see Sections 4.5 and 5.5).

Our results suggest several open problems. It would be interesting to know if our definitions of I- and FPF-Grassmannian involutions have a geometric interpretation, specifically in terms of the orbits of the orthogonal and symplectic groups in the flag variety. It is possible to prove Theorems 1.10 and 1.17 bijectively via the *involution Little maps* introduced in our previous work [15], and we present a bijective proof of Corollary 1.12 in Section 4.6. By contrast, the problem of finding a bijective proof for the fact that $\hat{F}^{\mathtt{FPF}}_z$ is Schur P-positive is completely open. In another direction, the literature on Stanley symmetric functions suggests that there should exist a meaningful representation-theoretic interpretation of $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$, but this remains to be found.

Acknowledgements

We thank Dan Bump, Michael Joyce, Vic Reiner, Alex Woo, Ben Wyser, and Alex Yong for many helpful conversations during the development of this paper.

2 Preliminaries

Let $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z}$ denote the respective sets of positive, nonnegative, and all integers. For $n \in \mathbb{P}$, let $[n] = \{1, 2, \dots, n\}$. The *support* of a permutation $w : X \to X$ is the set $\mathrm{supp}(w) = \{i \in X : w(i) \neq i\}$. Define $S_\mathbb{Z}$ as the group of permutations of $\mathbb{Z}$ with finite support, and let $S_\infty \subset S_\mathbb{Z}$ be the subgroup of permutations with support contained in $\mathbb{P}$. We view S_n as a subset of S_∞.

Throughout, we let $s_i = (i, i+1) \in S_\mathbb{Z}$ for $i \in \mathbb{Z}$. Let $\mathcal{R}(w)$ be the set of reduced words for $w \in S_\mathbb{Z}$, write $\ell(w)$ for the common length of these words, and define $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. We let $\mathrm{Des}_L(w)$ and $\mathrm{Des}_R(w)$ denote the *left* and *right descent sets* of $w \in S_\mathbb{Z}$, consisting of the simple transpositions s_i such that $\ell(s_i w) < \ell(w)$ and $\ell(w s_i) < \ell(w)$, respectively. It is useful to recall that $s_i \in \mathrm{Des}_R(w)$ for $w \in S_\mathbb{Z}$ if and only if $w(i) > w(i+1)$, and that $\ell(w) = |\mathrm{Inv}(w)|$ where $\mathrm{Inv}(w) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i < j \text{ and } w(i) > w(j)\}$.

Let $<$ denote the *(strong) Bruhat order* on $S_\mathbb{Z}$, that is, the weakest partial order on $S_\mathbb{Z}$ with $w < wt$ if t is a transposition such that $\ell(w) < \ell(wt)$. We write $u \lessdot v$ for $u, v \in S_\mathbb{Z}$ if $\{w \in S_\mathbb{Z} : u \leq w < v\} = \{u\}$. If $u, v \in S_\mathbb{Z}$ and $(s_{i_1}, s_{i_2}, \dots, s_{i_k}) \in \mathcal{R}(v)$, then $u \lessdot v$ if and only if there exists an index $j \in [k]$ such that $(s_{i_1}, \dots, \widehat{s_{i_j}}, \dots, s_{i_k}) \in \mathcal{R}(u)$, where $\widehat{\ }$ denotes omission. The poset $(S_\mathbb{Z}, \leq)$ contains S_∞ as a lower ideal and is graded with rank function ℓ. Consequently $u \lessdot v$ if and only if $u < v$ and $\ell(v) = \ell(u) + 1$. It is well-known that if $t = (a, b) \in S_\mathbb{Z}$ for some integers $a < b$, then $u \lessdot ut$ if and only if $u(a) < u(b)$ and no $i \in \mathbb{Z}$ exists with $a < i < b$ and $u(a) < u(i) < u(b)$.

2.1 Divided difference operators

We recall a few technical facts about *divided difference operators* from the references [25, 33, 34, 35]. Let $\mathcal{L} = \mathbb{Z}\left[x_1, x_2, \dots, x_1^{-1}, x_2^{-1}, \dots\right]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in a countable set of commuting indeterminates, and let $\mathcal{P} = \mathbb{Z}[x_1, x_2, \dots]$ be the subring of polynomials in $\mathcal{L}$. The group S_∞ acts on $\mathcal{L}$ by permuting variables, and one defines

$$\partial_i f = (f - s_i f)/(x_i - x_{i+1}) \qquad \text{for } i \in \mathbb{P} \text{ and } f \in \mathcal{L}.$$

The *divided difference operator* ∂_i defines a map $\mathcal{L} \to \mathcal{L}$ which restricts to a map $\mathcal{P} \to \mathcal{P}$. It is clear by definition that $\partial_i f = 0$ if and only if $s_i f = f$.

Lemma 2.1. If $f \in \mathcal{L}$ is homogeneous and $\partial_i f \neq 0$ then $\partial_i f$ is homogeneous of degree $\deg(f) - 1$.

Lemma 2.2. If $i \in \mathbb{P}$ and $f, g \in \mathcal{L}$ then $\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g$.

Corollary 2.3. If $i \in \mathbb{P}$ and $f, g \in \mathcal{L}$ are such that $\partial_i f = 0$, then $\partial_i(fg) = f\partial_i g$.

The divided difference operators satisfy $\partial_i^2 = 0$ as well as the usual braid relations for S_∞, and so if $w \in S_\infty$ then $\partial_{i_1}\partial_{i_2} \cdots \partial_{i_k}$ is the same map $\mathcal{L} \to \mathcal{L}$ for all reduced words $(s_{i_1}, s_{i_2}, \dots, s_{i_k}) \in \mathcal{R}(w)$. We denote this map by $\partial_w : \mathcal{L} \to \mathcal{L}$ for $w \in S_\infty$. For $n \in \mathbb{P}$, let $w_n = n \cdots 321 \in S_n$ be the reverse permutation and define $\Delta_n = \prod_{1 \leq i < j \leq n}(x_i - x_j)$. The following identity is [35, Proposition 2.3.2].

Lemma 2.4 (See [35])**.** If $n \in \mathbb{P}$ and $f \in \mathcal{L}$ then $\partial_{w_n} f = \Delta_n^{-1} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma f$.

For $i \in \mathbb{P}$ the *isobaric divided difference operator* $\pi_i : \mathcal{L} \to \mathcal{L}$ is defined by

$$\pi_i(f) = \partial_i(x_i f) = f + x_{i+1}\partial_i f \qquad \text{for } f \in \mathcal{L}.$$

Observe that $\pi_i f = f$ if and only if $s_i f = f$, in which case $\pi_i(fg) = f\pi_i(g)$ for $g \in \mathcal{L}$. By Lemma 2.1, if $f \in \mathcal{L}$ is homogeneous with $\pi_i f \neq 0$, then $\pi_i f$ is homogeneous of the same degree. The isobaric divided difference operators also satisfy the braid relations for S_∞, so we may define $\pi_w = \pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ for any $(s_{i_1}, s_{i_2}, \dots, s_{i_k}) \in \mathcal{R}(w)$. Moreover, $\pi_i^2 = \pi_i$. Consequently:

Lemma 2.5. Let $w \in S_\infty$. If $s_i \in \mathrm{Des}_R(w)$ then $\pi_w \pi_i = \pi_w$. If $s_i \in \mathrm{Des}_L(w)$ then $\pi_i \pi_w = \pi_w$.

Given $a, b \in \mathbb{P}$ with $a < b$, define $\partial_{b,a} = \partial_{b-1}\partial_{b-2} \cdots \partial_a$ and $\pi_{b,a} = \pi_{b-1}\pi_{b-2} \cdots \pi_a$. For numbers $a, b \in \mathbb{P}$ with $a \geq b$, we set $\partial_{b,a} = \pi_{b,a} = \mathrm{id}$. It is convenient here to note the following identity.

Lemma 2.6. If $a \leq b$ and $f \in \mathcal{L}$ are such that $\partial_i f = 0$ for $a < i < b$, then $\pi_{b,a} f = \partial_{b,a}\left(x_a^{b-a} f\right)$.

Proof. Assume $a < b$. Using Lemma 2.2 and induction, one checks that $\pi_{b,a} f = \pi_{b,a+1}\pi_a f = \partial_{b,a+1}\left(x_{a+1}^{b-a-1}\pi_a f\right) = \partial_{b,a}\left(x_a^{b-a} f\right) - \partial_{b,a+1}\left(\partial_a\left(x_a^{b-a-1}\right)x_a f\right) = \partial_{b,a}\left(x_a^{b-a} f\right) - (x_a f)\partial_{b,a}\left(x_a^{b-a-1}\right)$. Lemma 2.1 implies that $\partial_{b,a}\left(x_a^{b-a-1}\right) = 0$, so $\pi_{b,a} f = \partial_{b,a}(x_a^{b-a} f)$ as desired. $\square$

For a sequence of integers $a = (a_1, a_2, \dots)$ of either finite length or with only finitely many nonzero terms, we let $x^a = x_1^{a_1} x_2^{a_2}\cdots \in \mathcal{L}$. For $n \in \mathbb{P}$, define $\delta_n = (n-1, \dots, 2, 1, 0)$.

Lemma 2.7. If $n \in \mathbb{P}$ and $f \in \mathcal{L}$ then $\pi_{w_n} f = \partial_{w_n}(x^{\delta_n} f)$.

Proof. Assume $n > 1$ and let $c = s_1 s_2 \cdots s_{n-1} \in S_n$. One checks that $\pi_{w_n} = \pi_c \pi_{w_{n-1}}$ and $\partial_{w_n} = \partial_c \partial_{w_{n-1}}$ and, using Corollary 2.3, that $\pi_c f = \partial_c(x_1 x_2 \cdots x_{n-1} f)$ for $f \in \mathcal{L}$. Hence, by induction, $\pi_{w_n} f = \pi_c \pi_{w_{n-1}} f = \pi_c \partial_{w_{n-1}}(x^{\delta_{n-1}} f) = \partial_c \partial_{w_{n-1}}(x_1 x_2 \cdots x_{n-1} x^{\delta_{n-1}} f) = \partial_{w_n}(x^{\delta_n} f)$. $\square$

2.2 Schubert polynomials and Stanley symmetric functions

The *Schubert polynomial* corresponding to $y \in S_n$ is the polynomial $\mathfrak{S}_y = \partial_{y^{-1}w_n} x^{\delta_n} \in \mathcal{P}$, where as above we let $w_n = n\cdots 321 \in S_n$ and $x^{\delta_n} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1$. This formula for $\mathfrak{S}_y$ is independent of the choice of n such that $y \in S_n$, and we consider the Schubert polynomials to be a family indexed by S_∞. Some useful references for the basic properties of $\mathfrak{S}_w$ include [2, 5, 25, 33, 35]. Since $\partial_i^2 = 0$, it follows directly from the definition that

$$\mathfrak{S}_1 = 1 \qquad \text{and} \qquad \partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } s_i \in \mathrm{Des}_R(w) \\ 0 & \text{if } s_i \notin \mathrm{Des}_R(w) \end{cases} \qquad \text{for each } i \in \mathbb{P}. \tag{2.1}$$

Conversely, one can show that $\{\mathfrak{S}_w\}_{w\in S_\infty}$ is the unique family of homogeneous polynomials indexed by S_∞ satisfying (2.1); see [25, Theorem 2.3] or the introduction of [4]. One checks as an exercise that $\deg \mathfrak{S}_w = \ell(w)$ and $\mathfrak{S}_{s_i} = x_1 + x_2 + \cdots + x_i$ for $i \in \mathbb{P}$. The polynomials $\mathfrak{S}_w$ for $w \in S_\infty$ are linearly independent, and form a $\mathbb{Z}$-basis for $\mathcal{P}$ [35, Proposition 2.5.4].

Let $\mathbb{Z}[[x_1, x_2, \dots]]$ be the ring of formal power series of bounded degree in the commuting variables x_i for $i \in \mathbb{P}$. Let $\Lambda \subset \mathbb{Z}[[x_1, x_2, \dots]]$ be the usual subring of *symmetric functions*. A sequence of power series $f_1, f_2, \dots$ has a limit $\lim_{n\to\infty} f_n \in \mathbb{Z}[[x_1, x_2, \dots]]$ if for each fixed monomial the corresponding coefficient sequence is eventually constant. For $n \in \mathbb{N}$, let $\rho_n : \mathbb{Z}[[x_1, x_2, \dots]] \to \mathbb{Z}[x_1, x_2, \dots, x_n]$ denote the homomorphism induced by setting $x_i = 0$ for $i > n$.

Lemma 2.8. Let $f_1, f_2, \dots \in \mathcal{L}$ be a sequence of homogeneous Laurent polynomials and suppose for some $N \in \mathbb{N}$ it holds that $f_N \neq 0$ and that $\rho_n f_{n+1} = f_n$ and $x^{\delta_n} f_{n+1} \in \mathcal{P}$ for all $n \geq N$. Then $F = \lim_{n\to\infty} \pi_{w_n} f_n$ exists and belongs to Λ, and satisfies $\rho_n F = \pi_{w_n} f_n$ for all $n \geq N$.

This lemma is false without a condition like homogeneity to control $\deg f_n$.

Proof. Since $\rho_n f_{n+1} = f_n$ for $n \geq N$ and since each f_n is homogeneous, we must have $f_n \neq 0$ and $\deg f_n = \deg f_N$ for all $n \geq N$. Note by Lemma 2.4 that if $n \geq N$ then $\pi_{w_n} f_n \in \mathcal{P}$ is invariant under the action of S_n. As such, to prove the lemma it suffices to show that $\rho_n \pi_{n+1} f_{n+1} = \pi_n f_n$ for all $n \geq N$. This is straightforward from Lemmas 2.4 and 2.7 on noting that $\rho_n \Delta_{n+1} = x_1 x_2 \cdots x_n \Delta_n$ and that if $w \in S_{n+1}$ but $w \notin S_n$ for some $n \geq N$ then by hypothesis $\rho_n w(x^{\delta_{n+1}} f_{n+1}) = 0$. $\square$

Corollary 2.9. If $p \in \mathcal{P}$ is any polynomial then $\lim_{n\to\infty} \pi_{w_n} p$ exists and belongs to Λ.

For Schubert polynomials, the limit in this corollary has a noteworthy alternate form.

Definition 2.10. For $w \in S_{\mathbb{Z}}$ and $N \in \mathbb{Z}$, let $w \gg N \in S_{\mathbb{Z}}$ denote the map $i \mapsto w(i-N)+N$.

Note that $\mathrm{supp}(w \gg N) = \{i + N : i \in \mathrm{supp}(w)\}$. The following lemma is equivalent to [33, Eq. (4.25)], or to the combination of [13, Proposition 2.12, Corollary 3.38, and Theorem 3.40].

Lemma 2.11 (See [33])**.** If $w \in S_\infty$, $\mathrm{Des}_R(w) \subset \{s_1, \dots, s_n\}$, and $N \geq n$, then $\pi_{w_n}\mathfrak{S}_w = \rho_n \mathfrak{S}_{w\gg N}$.

By Lemmas 2.8 and 2.11 we deduce the following:

Theorem-Definition 2.12 (See [33, 46])**.** If $w \in S_{\mathbb{Z}}$ then $F_w = \lim_{N\to\infty} \mathfrak{S}_{w\gg N}$ is a well-defined symmetric function, which we refer to as the *Stanley symmetric function* of w.

It follows from results in [5] that this definition gives the same power series as Definition 1.2. It is clear that $F_w = F_{w\gg N}$ for any $N \in \mathbb{Z}$. By Lemma 2.11, if $w \in S_\infty$ then $F_w = \lim_{n\to\infty} \pi_{w_n}\mathfrak{S}_w$.

2.3 Involution Schubert polynomials

Let $\mathcal{I}_n$, $\mathcal{I}_\infty$, and $\mathcal{I}_{\mathbb{Z}}$ denote the sets of involutions in S_n, S_∞, and $S_{\mathbb{Z}}$. The involutions in these groups are the permutations whose cycles all have at most two elements. For $y \in \mathcal{I}_{\mathbb{Z}}$ define

$$\mathrm{Cyc}_{\mathbb{Z}}(y) = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : i \leq j = y(i)\} \qquad \text{and} \qquad \mathrm{Cyc}_{\mathbb{P}}(y) = \mathrm{Cyc}_{\mathbb{Z}}(y) \cap (\mathbb{P}\times\mathbb{P}).$$

It is often convenient to identify elements of $\mathcal{I}_n$, $\mathcal{I}_\infty$, or $\mathcal{I}_{\mathbb{Z}}$ with the partial matchings on $[n]$, $\mathbb{P}$, or $\mathbb{Z}$ in which distinct vertices are connected by an edge whenever they form a nontrivial cycle. By convention, we draw such matchings so that the vertices are points on a horizontal axis, ordered from left to right, and the edges appear as convex curves in the upper half plane. For example,

$(1,6)(2,7)(3,4) \in \mathcal{I}_7$ is represented as 

We often omit the numbers labeling the vertices in matchings corresponding to involutions in $\mathcal{I}_\infty$.

The next four propositions can all be recast as more general statements about twisted involutions in arbitrary Coxeter groups, and appear in this form in [41, 42] or [19, 20, 21, 22].

Proposition-Definition 2.13 (See [26])**.** There exists a unique associative product $\circ : S_{\mathbb{Z}} \times S_{\mathbb{Z}} \to S_{\mathbb{Z}}$ such that $u \circ v = uv$ if $\ell(uv) = \ell(u) + \ell(v)$ and $s_i \circ s_i = s_i$ for all $i \in \mathbb{Z}$.

Clearly $s \circ w = w \circ t = w$ if $s \in \mathrm{Des}_L(w)$ and $t \in \mathrm{Des}_R(w)$. If $(t_1, t_2, \dots, t_k) \in \mathcal{R}(w)$ then $w = t_1 \circ t_2 \circ \cdots \circ t_k = t_1 t_2 \dots t_k$. The exchange principle for Coxeter groups implies the following:

Proposition 2.14. If $y \in \mathcal{I}_{\mathbb{Z}}$ and $s = s_i$ then $s \circ y \circ s = \begin{cases} sys & \text{if } s \notin \mathrm{Des}_R(y) \text{ and } ys \neq sy \\ ys & \text{if } s \notin \mathrm{Des}_R(y) \text{ and } sy = ys \\ y & \text{if } s \in \mathrm{Des}_R(y). \end{cases}$

Thus, if $y \in \mathcal{I}_{\mathbb{Z}}$ then $s \circ y \circ s \in \mathcal{I}_{\mathbb{Z}}$, and by induction on length one may deduce:

Proposition 2.15. If $y \in \mathcal{I}_{\mathbb{Z}}$ then $y = w^{-1} \circ w$ for some $w \in S_{\mathbb{Z}}$.

Let $\kappa(y)$ denote the number of nontrivial cycles of $y \in \mathcal{I}_{\mathbb{Z}}$, and define $\hat{\ell} = \frac{1}{2}(\ell + \kappa)$. The following statement, whose proof is left as an easy exercise, shows that $\hat{\ell}$ is a map $\mathcal{I}_{\mathbb{Z}} \to \mathbb{N}$.

Proposition 2.16. If $y \in \mathcal{I}_\mathbb{Z}$ and $s \in \{s_i : i \in \mathbb{Z}\} - \mathrm{Des}_R(y)$ then $\hat\ell(s \circ y \circ s) = \hat\ell(y) + 1$.

For $y \in \mathcal{I}_\mathbb{Z}$, let $\mathcal{A}(y)$ denote the set of permutations $w \in S_\mathbb{Z}$ of minimal length such that $y = w^{-1} \circ w$. Note that the set $\hat{\mathcal{R}}(y)$ defined in the introduction is precisely $\hat{\mathcal{R}}(y) = \bigcup_{w\in\mathcal{A}(y)} \mathcal{R}(w)$. We refer to the elements of $\mathcal{A}(y)$ as the *atoms* of y. It follows by induction from the preceding results that $\mathcal{A}(y)$ is finite and nonempty, and that its elements all have length $\hat\ell(y)$.

Definition 2.17. The *involution Schubert polynomial* of $y \in \mathcal{I}_\infty$ is $\hat{\mathfrak{S}}_y = \sum_{w\in\mathcal{A}(y)} \mathfrak{S}_w$.

Example 2.18. We have $321 = s_2 \circ s_1 \circ s_1 \circ s_2 = s_1 \circ s_2 \circ s_2 \circ s_1$ and $\mathcal{A}(321) = \{132, 312\}$, so

$$\hat{\mathfrak{S}}_{321} = \mathfrak{S}_{132} + \mathfrak{S}_{312} = x_1^2 + x_1x_2.$$

The essential algebraic properties of the polynomials $\hat{\mathfrak{S}}_y$ are given by [13, Theorem 3.11]:

Theorem 2.19 (See [13])**.** The involution Schubert polynomials $\{\hat{\mathfrak{S}}_y\}_{y\in\mathcal{I}_\infty}$ are the unique family of homogeneous polynomials indexed by $\mathcal{I}_\infty$ such that if $i \in \mathbb{P}$ and $s = s_i$ then

$$\hat{\mathfrak{S}}_1 = 1 \qquad \text{and} \qquad \partial_i \hat{\mathfrak{S}}_y = \begin{cases} \hat{\mathfrak{S}}_{sys} & \text{if } s \in \mathrm{Des}_R(y) \text{ and } sy \neq ys \\ \hat{\mathfrak{S}}_{ys} & \text{if } s \in \mathrm{Des}_R(y) \text{ and } sy = ys \\ 0 & \text{if } s \notin \mathrm{Des}_R(y). \end{cases} \tag{2.2}$$

Observe that if $s_i \notin \mathrm{Des}_R(x)$ then $\partial_i \hat{\mathfrak{S}}_{s\circ y\circ s} = \hat{\mathfrak{S}}_y$. Since $\mathfrak{S}_w$ has degree $\ell(w)$, it follows that $\hat{\mathfrak{S}}_y$ has degree $\hat\ell(y)$. As the sets $\mathcal{A}(y)$ for $y \in \mathcal{I}_\infty$ are pairwise disjoint, the polynomials $\hat{\mathfrak{S}}_y$ for $y \in \mathcal{I}_\infty$ are linearly independent.

The involution Schubert polynomials were introduced in a rescaled form by Wyser and Yong in [50], where they were denoted $\Upsilon_{y;(\mathrm{GL}_n,\mathrm{O}_n)}$. Wyser and Yong's definition was motivated by the study of the action of the orthogonal group $\mathrm{O}_n(\mathbb{C})$ on the flag variety $\mathrm{Fl}(n) = \mathrm{GL}_n(\mathbb{C})/B$, with $B \subset \mathrm{GL}_n(\mathbb{C})$ denoting the Borel subgroup of lower triangular matrices. It follows from [50] that the involution Schubert polynomials are cohomology representatives for the closures of the $\mathrm{O}_n(\mathbb{C})$-orbits in $\mathrm{Fl}(n)$, and so are special cases of an older formula of Brion [6, Theorem 1.5]. See the discussion in [13, 15].

The symmetric functions $\hat{F}_y$ presented in the introduction are related to the polynomials $\hat{\mathfrak{S}}_y$ by the following formula, which is equivalent to Definition 1.9 since $\hat{\mathcal{R}}(y) = \bigcup_{w\in\mathcal{A}(y)} \mathcal{R}(w)$.

Definition 2.20. The *involution Stanley symmetric function* of $y \in \mathcal{I}_\mathbb{Z}$ is the power series

$$\hat{F}_y = \sum_{w\in\mathcal{A}(y)} F_w = \lim_{N\to\infty} \hat{\mathfrak{S}}_{y\gg N} \in \Lambda.$$

The second equality in this definition holds by Theorem-Definition 2.12. Note that $\hat{F}_y$ is a homogeneous symmetric function of degree $\hat\ell(y)$. If $y \in \mathcal{I}_\infty$, then $\hat{F}_y = \lim_{n\to\infty} \pi_{w_n} \hat{\mathfrak{S}}_y$.

2.4 FPF-involution Schubert polynomials

The families $\{\hat{\mathfrak{S}}_y\}_{y\in\mathcal{I}_\infty}$ and $\{\hat{F}_y\}_{y\in\mathcal{I}_\mathbb{Z}}$ each have "fixed-point-free" variants that arise when one studies the action on the flag variety of the symplectic group instead of the orthogonal group. To describe these, let $\mathcal{F}_n$ for $n \in \mathbb{P}$ denote the set of involutions $z \in \mathcal{I}_n$ with $z(i) \neq i$ for all $i \in [n]$. Define $\mathcal{F}_\infty$ and $\mathcal{F}_\mathbb{Z}$ as the S_∞- and $S_\mathbb{Z}$-conjugacy classes of the permutation $\Theta : \mathbb{Z} \to \mathbb{Z}$ given by

$$\Theta : i \mapsto i - (-1)^i.$$

Note that $\mathcal{F}_n$ is empty if n is odd, and that if $z \in \mathcal{F}_\mathbb{Z}$ and $N \in \mathbb{Z}$ then $z \gg N \in \mathcal{F}_\mathbb{Z}$ (cf. Definition 2.10) if and only if N is even. While technically $\mathcal{F}_n \not\subset \mathcal{F}_\infty$, there is a natural inclusion

$$\iota : \mathcal{F}_n \hookrightarrow \mathcal{F}_\infty \tag{2.3}$$

mapping $z \in \mathcal{F}_n$ to the permutation of $\mathbb{Z}$ whose restrictions to $[n]$ and to $\mathbb{Z}\setminus[n]$ coincide respectively with those of z and Θ. In symbols, we have $\iota(z) = z \cdot \Theta \cdot s_1 \cdot s_3 \cdot s_5 \cdots s_{n-1}$ for $z \in \mathcal{F}_n$. We refer to elements of $\mathcal{F}_n$, $\mathcal{F}_\infty$, and $\mathcal{F}_\mathbb{Z}$ as *fixed-point-free (FPF) involutions*.

Define $\mathrm{Inv}(z)$, $\mathrm{Des}_R(z)$, $\mathrm{Cyc}_\mathbb{Z}(z)$, and $\mathrm{Cyc}_\mathbb{P}(z)$ for $z \in \mathcal{F}_\mathbb{Z}$ exactly as for elements of $S_\mathbb{Z}$, so that $\mathrm{Des}_R(z) = \{s_i : (i, i+1) \in \mathrm{Inv}(z)\}$. Let $\mathrm{Inv}_{\mathtt{FPF}}(z) = \mathrm{Inv}(z) - \mathrm{Cyc}_\mathbb{Z}(z)$. The set $\mathrm{Inv}_{\mathtt{FPF}}(z)$ is finite with an even number of elements, and is empty if and only if $z = \Theta$. For $z \in \mathcal{F}_\mathbb{Z}$, we define

$$\hat\ell_{\mathtt{FPF}}(z) = \tfrac{1}{2}|\mathrm{Inv}_{\mathtt{FPF}}(z)| \qquad \text{and} \qquad \mathrm{Des}_R^{\mathtt{FPF}}(z) = \{s_i \in \mathrm{Des}_R(z) : (i, i+1) \notin \mathrm{Cyc}_\mathbb{Z}(z)\}.$$

These definitions are related by the following lemma, whose elementary proof is omitted.

Proposition 2.21. If $z \in \mathcal{F}_\mathbb{Z}$ then $\hat\ell_{\mathtt{FPF}}(szs) = \begin{cases} \hat\ell_{\mathtt{FPF}}(z) - 1 & \text{if } s \in \mathrm{Des}_R^{\mathtt{FPF}}(z) \\ \hat\ell_{\mathtt{FPF}}(z) & \text{if } s \in \mathrm{Des}_R(z) - \mathrm{Des}_R^{\mathtt{FPF}}(z) \\ \hat\ell_{\mathtt{FPF}}(z) + 1 & \text{if } s \in \{s_i : i \in \mathbb{Z}\} - \mathrm{Des}_R(z). \end{cases}$

Define $\mathcal{A}_{\mathtt{FPF}}(z)$ for $z \in \mathcal{F}_\mathbb{Z}$ as the set of permutations $w \in S_\mathbb{Z}$ of minimal length with $z = w^{-1}\Theta w$. This set is nonempty and finite, and its elements all have length $\hat\ell_{\mathtt{FPF}}(z)$. Note that the set $\hat{\mathcal{R}}_{\mathtt{FPF}}(z)$ defined in the introduction is the union $\hat{\mathcal{R}}_{\mathtt{FPF}}(z) = \bigcup_{w \in \mathcal{A}_{\mathtt{FPF}}(z)} \mathcal{R}(w)$.

Definition 2.22. The *FPF-involution Schubert polynomial* of $z \in \mathcal{F}_\infty$ is $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \sum_{w \in \mathcal{A}_{\mathtt{FPF}}(z)} \mathfrak{S}_w$.

For $z \in \mathcal{F}_n$, we set $\mathcal{A}_{\mathtt{FPF}}(z) = \mathcal{A}_{\mathtt{FPF}}(\iota(z))$ and $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \hat{\mathfrak{S}}^{\mathtt{FPF}}_{\iota(z)}$.

Example 2.23. We have $\iota(4321) = s_1 s_2 \Theta s_2 s_1 = s_3 s_2 \Theta s_2 s_3$ and $\mathcal{A}_{\mathtt{FPF}}(4321) = \{312, 1342\}$, so

$$\hat{\mathfrak{S}}^{\mathtt{FPF}}_{4321} = \mathfrak{S}_{312} + \mathfrak{S}_{1342} = x_1^2 + x_1x_2 + x_1x_3 + x_2x_3.$$

The polynomials $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ have the following characterization via divided differences.

Theorem 2.24 (See [13])**.** The FPF-involution Schubert polynomials $\{\hat{\mathfrak{S}}^{\mathtt{FPF}}_z\}_{z \in \mathcal{F}_\infty}$ are the unique family of homogeneous polynomials indexed by $\mathcal{F}_\infty$ such that if $i \in \mathbb{P}$ and $s = s_i$ then

$$\hat{\mathfrak{S}}^{\mathtt{FPF}}_\Theta = 1 \qquad \text{and} \qquad \partial_i \hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \begin{cases} \hat{\mathfrak{S}}^{\mathtt{FPF}}_{szs} & \text{if } s \in \mathrm{Des}_R(z) \text{ and } (i, i+1) \notin \mathrm{Cyc}_\mathbb{Z}(z) \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Wyser and Yong defined these polynomials in [50], where they were denoted $\Upsilon_{z;(\mathrm{GL}_n, \mathrm{Sp}_n)}$. They showed, when n is even, that the FPF-involution Schubert polynomials indexed by $\mathcal{F}_n$ are cohomology representatives for the $\mathrm{Sp}_n(\mathbb{C})$-orbit closures in $\mathrm{Fl}(n)$, and therefore are instances of an older, more general construction of Brion [6]. See the discussion in [13, 15].

As with $\hat F_y$ in the previous section, the symmetric functions $\hat F^{\mathtt{FPF}}_z$ from the introduction have the following alternate definition, which relates them directly to the polynomials $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$.

Definition 2.25. The *FPF-involution Stanley symmetric function* of $z \in \mathcal{F}_\mathbb{Z}$ is the power series

$$\hat{F}^{\mathtt{FPF}}_z = \sum_{w \in \mathcal{A}_{\mathtt{FPF}}(w)} F_w = \lim_{N\to\infty} \hat{\mathfrak{S}}^{\mathtt{FPF}}_{z \gg 2N} \in \Lambda.$$

The second equality again holds by Theorem-Definition 2.12. If $z \in \mathcal{F}_\infty$, then additionally $\hat{F}^{\mathtt{FPF}}_z = \lim_{n\to\infty} \pi_{w_n} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z$. Note that both $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ and $\hat{F}^{\mathtt{FPF}}_z$ are homogeneous of degree $\hat{\ell}_{\mathtt{FPF}}(z)$.

2.5 Schur P-functions

Our main results will relate $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$ to the *Schur P-functions* in Λ. These symmetric functions were introduced in work of Schur on the projective representations of the symmetric group [44] but have since arisen in a variety of other contexts (see, e.g., [4, 24, 39, 48]). We briefly review some of their properties from [47, §6] and [34, §III.8]. For integers $0 \leq r \leq n$, let

$$G_{r,n} = \prod_{i\in[r]} \prod_{j\in[n-i]} \left(1 + x_i^{-1} x_{i+j}\right) \in \mathcal{L}.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let $\ell(\lambda)$ denote the largest index $i \in \mathbb{P}$ with $\lambda_i \neq 0$. The partition λ is *strict* if $\lambda_i \neq \lambda_{i+1}$ for all $i < \ell(\lambda)$. Recall that $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r}$ for $r = \ell(\lambda)$.

Definition 2.26. For a strict partition λ with $r = \ell(\lambda)$ parts, let $P_\lambda = \lim_{n\to\infty} \pi_{w_n}\left(x^\lambda G_{r,n}\right) \in \Lambda$. The symmetric function P_λ is the *Schur P-function* corresponding to λ.

By Lemma 2.8, this formula for P_λ gives a well-defined, homogeneous symmetric function of degree $\sum_i \lambda_i$, and $\rho_n P_\lambda = \pi_{w_n}\left(x^\lambda G_{r,n}\right)$ for $n \geq r = \ell(\lambda)$. We emphasize this definition of P_λ for its compatibility with our definition of F_w in Section 2.2. One can show that the Schur function s_λ is given by a similar limit: namely, $s_\lambda = \lim_{n\to\infty} \pi_{w_n} x^\lambda$. Some other similarities exist between s_λ and P_λ. Whereas the Schur functions form a $\mathbb{Z}$-basis for Λ, the Schur P-functions form a $\mathbb{Z}$-basis for the subring $\mathbb{Q}[p_1, p_3, p_5, \dots] \cap \Lambda$ generated by the odd-indexed power sum symmetric functions [47, Corollary 6.2(b)]. Each Schur P-function P_λ is itself Schur positive [34, Eq. (8.17), §III.8].

Alternatively, P_λ may be defined as a generating function for certain shifted tableaux. Recall that the *diagram* of a partition λ is the set $D_\lambda = \{(i,j) \in \mathbb{P}\times\mathbb{P} : j \leq \lambda_i\}$. If λ is a strict partition, then its *shifted diagram* is the set $D'_\lambda = \{(i, i+j-1) : (i,j) \in D_\lambda\}$. We orient the elements of D_λ and D'_λ in the same way as the positions in a matrix, and refer to the ith row or jth column of these sets according to this convention. A *tableau* (respectively, *shifted tableau*) of shape λ is a map $D_\lambda \to \mathbb{P}$ (respectively, $D'_\lambda \to \mathbb{Z}\setminus\{0\}$). We write T_{ij} for the image of (i,j) under T, and refer to this number as the entry of T in position (i,j).

A shifted tableau T is *increasing* if its entries are positive and strictly increasing along each row and column. Let $\prec$ be the total order on $\mathbb{Z}\setminus 0$ with $-1 \prec 1 \prec -2 \prec 2 \prec \dots$. A shifted tableau T is *semi-standard* if the following conditions hold:

- The entries of T are weakly increasing with respect to $\prec$ along each row and column.
- No two positions in the same column of T contain the same positive number.
- No two positions in the same row of T contain the same negative number.
- Every entry of T on the main diagonal $\{(i,i) : i \in \mathbb{P}\}$ is positive.

An (unshifted) tableau is defined to be *semi-standard* in the same way, but with the added constraint that its entries are all positive. Finally, a (shifted) tableau T of shape λ is *standard* if it is semi-standard and $(i,j) \mapsto |T_{ij}|$ is a bijection $D_\lambda \to [n]$ or $D'_\lambda \to [n]$ for some $n \in \mathbb{N}$, as appropriate. Let $\mathrm{SSYT}(\lambda)$ and $\mathrm{SYT}(\lambda)$ be the sets of semi-standard and standard tableaux of shape λ, respectively. Similarly, when λ is strict, let $\mathrm{Inc}(\lambda)$, $\mathrm{SSMT}(\lambda)$, and $\mathrm{SMT}(\lambda)$ be the sets of increasing, semi-standard, and standard shifted tableaux of shape λ. Shifted tableaux as we have defined them are sometimes called *shifted marked tableaux*—hence our use of the letters "SMT."

Example 2.27. Every semi-standard shifted tableau of shape $\lambda = (2,1)$ has the form

$$\begin{array}{|c|c|}\hline a & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array} \quad\text{or}\quad \begin{array}{|c|c|}\hline a & -b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array} \quad\text{or}\quad \begin{array}{|c|c|}\hline a & a \\ \hline \multicolumn{1}{c|}{} & b \\ \cline{2-2}\end{array} \quad\text{or}\quad \begin{array}{|c|c|}\hline a & -b \\ \hline \multicolumn{1}{c|}{} & b \\ \cline{2-2}\end{array}$$

for some positive integers $a < b < c$. The set $\mathrm{SMT}(\lambda)$ contains two elements, given by the first two tableaux shown here with $(a,b,c) = (1,2,3)$.

The following power series are the *fundamental quasi-symmetric functions (of degree n)*:

Definition 2.28. For $n \in \mathbb{P}$ and $S \subset [n-1]$ define $f_{n,S} = \sum x_{i_1} x_{i_2} \dots x_{i_n} \in \mathbb{Z}[[x_1, x_2, \dots]]$, where the sum is over all weakly increasing sequences $(i_1, i_2, \dots, i_n) \in \mathbb{P}^n$ with $i_j < i_{j+1}$ for all $j \in S$.

When n is clear from context, we write f_S in place of $f_{n,S}$.

Remark 2.29. Note that if $\mathbf{a} = (s_{a_1}, s_{a_2}, \dots, s_{a_n})$ is a sequence of simple transpositions, then the power series $f_{\mathbf{a}}$ defined in the introduction is $f_{n,S}$ for $S = \{i \in [n-1] : a_i < a_{i+1}\}$.

For a (shifted) tableau T, define x^T as the monomial given by the product $\prod_{(i,j)} x_{|T_{ij}|}$ over all (i,j) in T's domain. When T is standard, its *descent set* if the set $\mathrm{Des}(T)$ of positive integers i such that either (1) i and $i+1$ both appear in T with i in a row strictly above $i+1$, (2) $-i$ and $-(i+1)$ both appear in T with $-i$ in a column strictly to the left of $-(i+1)$, or (3) i and $-(i+1)$ both appear in T. Note that if $T \in \mathrm{SYT}(\lambda)$ then only condition (1) can occur, and if $n = |\lambda|$ then $\mathrm{Des}(T)$ is the complement in $[n-1]$ of the descent set of the transpose of T, which is also a standard tableau. The next identity [27, Proposition 1.5] is well-known.

Proposition 2.30. If λ is a partition of n then $s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T = \sum_{T \in \mathrm{SYT}(\lambda)} f_{\mathrm{Des}(T)}$.

The following proposition is also not new, but is perhaps less widely known.

Proposition 2.31. If λ is a strict partition of n then

$$P_\lambda = \sum_{T \in \mathrm{SSMT}(\lambda)} x^T = \sum_{T \in \mathrm{SMT}(\lambda)} f_{\mathrm{Des}(T)} = \sum_{T \in \mathrm{SMT}(\lambda)} f_{[n-1]-\mathrm{Des}(T)}.$$

Proof. The first equality is [34, Eq. (8.16′), §III.8] or [47, Eq. (6.4)], and the second follows as an exercise. Since the fundamental quasi-symmetric functions are linearly independent and since P_λ is invariant under the linear automorphism of Λ with $s_\mu \mapsto s_{\mu^T}$ [34, Example 3(a), §III.8], the third equality follows upon noting that $s_{\mu^T} = \sum_{T \in \mathrm{SYT}(\mu)} f_{[|\mu|-1]-\mathrm{Des}(T)}$. ∎

Example 2.32. If $\lambda = (1,1)$ then $\mathrm{SYT}(\lambda)$ contains a single element whose descent set is $\{1\}$, so $s_{(1,1)} = \sum_{i<j} x_i x_j$. If $\lambda = (2,1)$ then Example 2.27 shows that $\mathrm{SMT}(\lambda)$ has two elements, whose descent sets are $\{2\}$ and $\{1\}$, so $P_{(2,1)} = 2\sum_{i<j<k} x_i x_j x_k + \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2$.

In Section 4.6, we will need one other family. Fix a strict partition λ. A *set-valued shifted tableau* T of shape λ is a map from D'_λ to the set of finite, nonempty subsets of $\mathbb{Z} \setminus \{0\}$. A set-valued shifted tableau T of shape λ is *increasing* if each shifted tableau U of shape λ with $U_{ij} \in T_{ij}$ for all (i,j) is increasing, and *standard (of rank n)* if each shifted tableau U of shape λ with $U_{ij} \in T_{ij}$ for all (i,j) is semi-standard and the map $x \mapsto |x|$ is a bijection $\bigsqcup_{(i,j)\in D'_\lambda} T_{ij} \to [n]$, where $\bigsqcup$ denotes disjoint union. Let $\mathrm{SetMT}_n(\lambda)$ be the set of standard set-valued shifted tableaux of rank n. Any shifted tableau may be viewed as a set-valued shifted tableau whose entries are all singleton sets, and with respect to this identification it holds that $\mathrm{SMT}(\lambda) = \mathrm{SetMT}_n(\lambda)$ for $n = |\lambda|$.

3 Transition formulas

In this section, we review the transition formulas for $\hat{\mathfrak{S}}_y$ and $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ proved in [15], in order to derive some related identities for the symmetric functions $\hat{F}_y$ and $\hat{F}^{\mathtt{FPF}}_z$. To begin, we briefly discuss the classical results of this kind pertaining to $\mathfrak{S}_w$ and F_w.

3.1 Permutations

Given $y \in S_\mathbb{Z}$ and $r \in \mathbb{Z}$, define $\Phi^\pm(y,r)$ as the set of $w \in S_\mathbb{Z}$ such that $\ell(w) = \ell(y) + 1$ and $w = y(r,s)$ for some $s \in \mathbb{Z}$ with $\pm(r-s) < 0$. The following transition formula for Schubert polynomials appears, e.g., as [25, Corollary 3.3], and is equivalent to *Monk's rule* (see [35, §2.7]).

Theorem 3.1. If $y \in S_\infty$ and $r \in \mathbb{P}$ then

$$x_r \mathfrak{S}_y = \sum_{w\in\Phi^+(y,r)} \mathfrak{S}_w - \sum_{w\in\Phi^-(y,r)} \mathfrak{S}_w$$

where we set $\mathfrak{S}_w = 0$ for $w \in S_\mathbb{Z} - S_\infty$.

Taking limits transforms Theorem 3.1 into the following identity:

Theorem 3.2. If $y \in S_\mathbb{Z}$ and $r \in \mathbb{Z}$ then $\sum_{w\in\Phi^-(y,r)} F_w = \sum_{w\in\Phi^+(y,r)} F_w$.

Proof. One checks that $\Phi^\pm(y \gg N, r+N) = \{w \gg N : w \in \Phi^\pm(y,r)\}$ for all $y \in S_\mathbb{Z}$ and $r, N \in \mathbb{Z}$. It follows by Theorem 3.1 that $\sum_{w\in\Phi^+(y,r)} F_w - \sum_{w\in\Phi^-(y,r)} F_w = \lim_{N\to\infty} x_{r+N}\mathfrak{S}_{y\gg N} = 0$. $\square$

3.2 Involutions

To state a transition formula for the involution Schubert polynomials $\hat{\mathfrak{S}}_y$, we need to review a few technical properties of the partial order given by restricting the Bruhat order $<$ on $S_\mathbb{Z}$ to $\mathcal{I}_\mathbb{Z}$. Our notation follows Section 2.3. More general results of Hultman imply the following useful facts:

Theorem 3.3 (Hultman [19, 20, 21])**.** The following properties hold:

(a) $(\mathcal{I}_\mathbb{Z}, <)$ is a graded poset with rank function $\hat{\ell}$.

(b) Fix $y, z \in \mathcal{I}_\mathbb{Z}$ and $w \in \mathcal{A}(z)$. Then $y \le z$ if and only if there exists $v \in \mathcal{A}(y)$ with $v \le w$.

We write $y \lessdot_\mathcal{I} z$ if $z \in \mathcal{I}_\mathbb{Z}$ covers $y \in \mathcal{I}_\mathbb{Z}$ in the partial order given by restricting $<$ to $\mathcal{I}_\mathbb{Z}$, that is, if $\{w \in \mathcal{I}_\mathbb{Z} : y \le w < z\} = \{y\}$. Note that while $y \lessdot_\mathcal{I} z \Rightarrow y < z$ and $y \lessdot z \Rightarrow y \lessdot_\mathcal{I} z$, it does not hold that $y \lessdot_\mathcal{I} z \Rightarrow y \lessdot z$ for $y, z \in \mathcal{I}_\mathbb{Z}$. The following is equivalent to [15, Theorem 1.3]:

Theorem-Definition 3.4 (See [15]). Let $y \in \mathcal{I}_\mathbb{Z}$ and choose integers $i < j$. There exists at most one involution $z \in \mathcal{I}_\mathbb{Z}$ such that the set $\{w \in \mathcal{A}(y) : w \lessdot w(i,j) \in \mathcal{A}(z)\}$ is nonempty. We define $\tau_{ij}(y)$ to be this involution z if it exists, and otherwise set $\tau_{ij}(y) = y$.

Note that $\hat\ell(\tau_{ij}(y)) - \hat\ell(y) \in \{0,1\}$. A second theorem is needed to explain how $\tau_{ij}(y)$ can be efficiently computed. Let $E \subset \mathbb{Z}$ be a finite set of size n, and write ϕ_E and ψ_E for the unique order-preserving bijections $\phi_E : [n] \to E$ and $\psi_E : E \to [n]$. Given $w \in S_\mathbb{Z}$, we define

$$[w]_E = \psi_{w(E)} \circ w \circ \phi_E \in S_n \subset S_\mathbb{Z}. \tag{3.1}$$

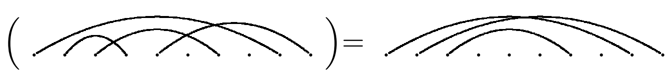
We call $[w]_E$ the *standardization* of w with respect to E. This notation is intended to distinguish $[w]_E$ from the restriction of w to E, which we instead denote as $w|_E : E \to \mathbb{Z}$. The next theorem is a consequence of our results in [15, Section 3].

Theorem 3.5 (See [15]). Fix $y \in \mathcal{I}_\mathbb{Z}$ and $i < j$ in $\mathbb{Z}$. Let $A = \{i, j, y(i), y(j)\}$ and $B = \mathbb{Z} \setminus A$, and define $a = \psi_A(i)$ and $b = \psi_A(j)$. Then $\tau_{ij}(y)$ is the unique element of $S_\mathbb{Z}$ with

$$[\tau_{ij}(y)]_A = \tau_{ab}([y]_A) \qquad \text{and} \qquad \tau_{ij}(y)|_B = y|_B.$$

This result shows that to compute $\tau_{ij}(y)$ from y in general, we only need to know a formula for $\tau_{ij}(y)$ in this case when $y \in \mathcal{I}_n$ and $[n] = \{i, j, y(i), y(j)\}$. This information is completely specified in Table 1. As a corollary, it follows that $\tau_{ij}(y) \gg N = \tau_{i+N,j+N}(y \gg N)$ for all $N \in \mathbb{Z}$.

Example 3.6. If $y = (1,9)(2,4)(3,7)(5,10) \in \mathcal{I}_{10}$ then $\tau_{2,10}(y) = (1,9)(2,10)(3,7)$, that is:

$\tau_{2,11}$ 

Apart from some differences in notation, the map $\tau_{ij} : \mathcal{I}_\mathbb{Z} \to \mathcal{I}_\mathbb{Z}$ is essentially the same as the map ct_{ij} which Incitti defines in [23]; see the discussion in [15, Section 3.1]. Incitti's work implies the following theorem, which we also stated as [15, Theorem 3.16]. This result shows that the covering relations in $(\mathcal{I}_\mathbb{Z}, <)$ are completely described by the operators τ_{ij}.

Theorem 3.7 (Incitti [23]). Let $y, z \in \mathcal{I}_\mathbb{Z}$. The following are then equivalent:

(a) $y \lessdot_\mathcal{I} z$.

(b) $z = \tau_{ij}(y)$ for some $i < j$ in $\mathbb{Z}$ and $\hat\ell(z) = \hat\ell(y) + 1$.

(c) $z = \tau_{ij}(y)$ for some $i < j$ in $\mathbb{Z}$ with $y(i) \le i$ and $y \lessdot z(i,j)$.

(d) $z = \tau_{ij}(y)$ for some $i < j$ in $\mathbb{Z}$ with $j \le y(j)$ and $y \lessdot z(i,j)$.

Now, given $y \in \mathcal{I}_\mathbb{Z}$ and $r \in \mathbb{Z}$, we define

$$\hat\Phi^+(y,r) = \Big\{ z \in \mathcal{I}_\mathbb{Z} : \hat\ell(z) = \hat\ell(y) + 1 \text{ and } z = \tau_{rj}(y) \text{ for an integer } j > r \Big\}$$
$$\hat\Phi^-(y,r) = \Big\{ z \in \mathcal{I}_\mathbb{Z} : \hat\ell(z) = \hat\ell(y) + 1 \text{ and } z = \tau_{ir}(y) \text{ for an integer } i < r \Big\}.$$

These sets are both nonempty [15, Proposition 3.26], and if $z \in \hat\Phi^\pm(y,r)$ then $y \lessdot_\mathcal{I} z$. Moreover, Theorem 3.7 implies that if $(p,q) \in \mathrm{Cyc}_\mathbb{Z}(y)$ then the following holds:

$A = \{i, j, y(i), y(j)\}$	$[y]_A$	(i, j)	$[\tau_{ij}(y)]_A$	σ such that $\tau_{ij}(y) = y\sigma$
$\{a < b\}$		(a, b)		(a, b)
$\{a < b < c\}$		$(b, c), (a, c)$		(a, c, b)
		$(a, b), (a, c)$		(a, b, c)
$\{a < b < c < d\}$		(b, c)		$(a, d)(b, c)$
		$(a, c), (b, d), (a, d)$		(a, c, d, b)
		$(a, b), (c, d), (a, d)$		$(a, b)(c, d)$

Table 1: Nontrivial values of $\tau_{ij}(y)$. Fix $y \in \mathcal{I}_\mathbb{Z}$ and $i < j$ in $\mathbb{Z}$, and define $A = \{i, j, y(i), y(j)\}$. The first column labels the elements of A. The third column rewrites (i, j) in this labeling. The second and fourth columns identify the matchings which represent $[y]_A$ and $[\tau_{ij}(y)]_A$. For values of y and $i < j$ that do not correspond to any rows in this table, we have defined $\tau_{ij}(y) = y$.

1. If $q < j$ and $z = \tau_{qj}(y)$, then $z \in \hat{\Phi}^+(y, q)$ if and only if $y \lessdot y(q, j)$.
2. If $i < p$ and $z = \tau_{ip}(y)$, then $z \in \hat{\Phi}^-(y, p)$ if and only if $y \lessdot y(i, p)$.

For $(p, q) \in \mathbb{P} \times \mathbb{P}$, let $x_{(p,q)} = x_p = x_q$ if $p = q$ and otherwise set $x_{(p,q)} = x_p + x_q$. The following transition formula for involution Schubert polynomials is [15, Theorem 3.28].

Theorem 3.8 (See [15])**.** If $y \in \mathcal{I}_\infty$ and $(p, q) \in \mathrm{Cyc}_\mathbb{P}(y)$ then

$$x_{(p,q)}\hat{\mathfrak{S}}_y = \sum_{z \in \hat{\Phi}^+(y,q)} \hat{\mathfrak{S}}_z - \sum_{z \in \hat{\Phi}^-(y,p)} \hat{\mathfrak{S}}_z$$

where we set $\hat{\mathfrak{S}}_z = 0$ for all $z \in \mathcal{I}_\mathbb{Z} - \mathcal{I}_\infty$.

Example 3.9. If $y = (2, 3)(4, 7) \in \mathcal{I}_7$ then

$$\begin{aligned} \hat{\Phi}^+(y, 3) &= \{\tau_{3,4}(y),\ \tau_{3,5}(y),\ \tau_{3,7}(y)\} = \{(24)(37),\ (2, 5)(4, 7),\ (2, 7)\} \\ \hat{\Phi}^-(y, 2) &= \{\tau_{1,2}(y)\} = \{(1, 3)(4, 7)\} \end{aligned}$$

so $(x_2 + x_3)\hat{\mathfrak{S}}_{(2,3)(4,7)} = \hat{\mathfrak{S}}_{(24)(37)} + \hat{\mathfrak{S}}_{(2,5)(4,7)} + \hat{\mathfrak{S}}_{(2,7)} - \hat{\mathfrak{S}}_{(13)(47)}$.

Our main new results will depend crucially on the following identity.

Theorem 3.10. If $y \in \mathcal{I}_\mathbb{Z}$ and $(p,q) \in \mathrm{Cyc}_\mathbb{Z}(y)$ then $\sum_{z\in\hat\Phi^-(y,p)} \hat F_z = \sum_{z\in\hat\Phi^+(y,q)} \hat F_z$.

Proof. It holds that $\hat\Phi^\pm(y \gg N, r+N) = \{w \gg N : w \in \hat\Phi^\pm(y,r)\}$ for all $y \in \mathcal{I}_\mathbb{Z}$ and $r, N \in \mathbb{Z}$. By Theorem 3.8, it follows that $\sum_{z\in\hat\Phi^+(y,q)} \hat F_z - \sum_{z\in\hat\Phi^-(y,p)} \hat F_z = \lim_{N\to\infty} x_{(p+N,q+N)}\hat{\mathfrak{S}}_{y\gg N} = 0$. □

3.3 Fixed-point-free involutions

We turn to a transition formula for the polynomials $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$. Our notation now follows Section 2.4. Recall, especially, the definitions of $\mathcal{F}_\mathbb{Z}$, $\mathcal{A}_{\mathtt{FPF}}(z)$, $\hat\ell_{\mathtt{FPF}}(z)$, and the inclusion $\iota : \mathcal{F}_n \to \mathcal{F}_\infty$. We define the *Bruhat order* $<$ on $\mathcal{F}_\mathbb{Z}$ as the weakest partial order with $z < tzt$ when $z \in \mathcal{F}_\mathbb{Z}$ and $t \in S_\mathbb{Z}$ is a transposition such that $\hat\ell_{\mathtt{FPF}}(z) < \hat\ell_{\mathtt{FPF}}(tzt)$. Rains and Vazirani's results in [40] imply the following theorem, which we also stated as [15, Theorem 4.6].

Theorem 3.11 (See [40])**.** Let $n \in 2\mathbb{P}$. The following properties hold:

(a) $(\mathcal{F}_\mathbb{Z}, <)$ is a graded poset with rank function $\hat\ell_{\mathtt{FPF}}$.

(b) If $y, z \in \mathcal{F}_n \subset \mathcal{I}_\infty$ then $y \le z$ holds in $(S_\mathbb{Z}, <)$ if and only if $\iota(y) \le \iota(z)$ holds in $(\mathcal{F}_\mathbb{Z}, <)$.

(c) Fix $y, z \in \mathcal{F}_\mathbb{Z}$ and $w \in \mathcal{A}_{\mathtt{FPF}}(z)$. Then $y \le z$ if and only if there exists $v \in \mathcal{A}_{\mathtt{FPF}}(y)$ with $v \le w$.

Note that both $\iota(\mathcal{F}_n)$ and $\mathcal{F}_\infty$ are lower ideals in $(\mathcal{F}_\mathbb{Z}, <)$. We write $y \lessdot_\mathcal{F} z$ for $y, z \in \mathcal{F}_\mathbb{Z}$ if $\{w \in \mathcal{F}_\mathbb{Z} : y \le w < z\} = \{y\}$. If $y, z \in \mathcal{F}_n$ for some $n \in 2\mathbb{P}$, then we write $y \lessdot_\mathcal{F} z$ if $\iota(y) \lessdot_\mathcal{F} \iota(z)$.

Example 3.12. The set $\mathcal{F}_4 = \{(1,2)(3,4) < (1,3)(2,4) < (1,4)(2,3)\}$ is totally ordered by $<$.

Let $z \in \mathcal{F}_\mathbb{Z}$. Cycles $(a,b), (i,j) \in \mathrm{Cyc}_\mathbb{Z}(z)$ with $a < i$ are *crossing* if $a < i < b < j$ and *nesting* if $a < i < j < b$. It is a useful exercise to check for $z \in \mathcal{F}_\mathbb{Z}$ that $\hat\ell_{\mathtt{FPF}}(z) = 2n + c$ where n and c are the respective numbers of unordered pairs of nesting and crossing cycles of z. The following description of the covering relations $(\mathcal{F}_\mathbb{Z}, <)$ is equivalent to [3, Corollary 2.3].

Proposition 3.13 (See [3])**.** Let $y \in \mathcal{F}_\mathbb{Z}$. Fix integers $i < j$ and define $A = \{i, j, y(i), y(j)\}$ and $z = (i,j)y(i,j)$. Then $\hat\ell_{\mathtt{FPF}}(z) = \hat\ell_{\mathtt{FPF}}(y) + 1$ if and only if the following conditions hold:

(a) $y(i) < y(j)$ and no $e \in \mathbb{Z}$ exists with $i < e < j$ and $y(i) < y(e) < y(j)$.

(b) Either $[y]_A = (1,2)(3,4) \lessdot_\mathcal{F} [z]_A = (1,3)(2,4)$ or $[y]_A = (1,3)(2,4) \lessdot_\mathcal{F} [z]_A = (1,4)(2,3)$.

Remark 3.14. If condition (a) holds then $(i,j) \notin \mathrm{Cyc}_\mathbb{Z}(y)$ so necessarily $|A| = 4$, and condition (b) asserts that $[y]_A \lessdot_\mathcal{F} [z]_A$, which occurs if and only if $[y]_A$ and $[z]_A$ coincide with

 $\lessdot_\mathcal{F}$  or  $\lessdot_\mathcal{F}$  .

In the first case $[(i,j)]_A \in \{(1,4),(2,3)\}$, and in the second $[(i,j)]_A \in \{(1,2),(3,4)\}$.

We introduce a second variation of the sets $\Phi^\pm(w,r)$. Given $y \in \mathcal{F}_\mathbb{Z}$ and $r \in \mathbb{Z}$, let

$$\begin{aligned} \hat\Psi^+(y,r) &= \left\{ z \in \mathcal{F}_\mathbb{Z} : \hat\ell_{\mathtt{FPF}}(z) = \hat\ell_{\mathtt{FPF}}(y) + 1 \text{ and } z = (r,j)y(r,j) \text{ for an integer } j > r \right\} \\ \hat\Psi^-(y,r) &= \left\{ z \in \mathcal{F}_\mathbb{Z} : \hat\ell_{\mathtt{FPF}}(z) = \hat\ell_{\mathtt{FPF}}(y) + 1 \text{ and } z = (i,r)y(i,r) \text{ for an integer } i < r \right\}. \end{aligned} \tag{3.2}$$

These sets are both nonempty, and if z belongs to either of them then $y \lessdot_\mathcal{F} z$ by construction. The following transition formula for FPF-involution Schubert polynomials is [15, Theorem 4.17].

Theorem 3.15 (See [15])**.** If $y \in \mathcal{F}_\infty$ and $(p,q) \in \mathrm{Cyc}_{\mathbb{P}}(y)$ then

$$(x_p + x_q)\hat{\mathfrak{S}}^{\mathtt{FPF}}_y = \sum_{z \in \hat{\Psi}^+(y,q)} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z - \sum_{z \in \hat{\Psi}^-(y,p)} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z$$

where we set $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = 0$ for all $z \in \mathcal{F}_{\mathbb{Z}} - \mathcal{F}_\infty$.

Example 3.16. Set $\hat{\Psi}^\pm(y,r) = \hat{\Psi}^\pm(\iota(y),r)$ for $y \in \mathcal{F}_n$. If $y = (1,2)(3,7)(5,4)(6,8) \in \mathcal{F}_8$ then

$$\hat{\Psi}^+(y,7) = \{(7,8)y(7,8)\} = \{(1,2)(3,8)(4,5)(6,7)\}$$
$$\hat{\Psi}^-(y,3) = \{(2,3)y(2,3)\} = \{(1,3)(2,7)(5,4)(6,8)\}$$

so $(x_3 + x_7)\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,7)(5,4)(6,8)} = \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,8)(4,5)(6,7)} - \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,3)(2,7)(5,4)(6,8)}$.

We also have a fixed-point-free analogue of Theorem 3.10.

Theorem 3.17. If $y \in \mathcal{F}_{\mathbb{Z}}$ and $(p,q) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ then $\sum_{z \in \hat{\Psi}^-(y,p)} \hat{F}^{\mathtt{FPF}}_z = \sum_{z \in \hat{\Psi}^+(y,q)} \hat{F}^{\mathtt{FPF}}_z$.

Proof. We have $\hat{\Psi}^\pm(y \gg 2N, r+2N) = \{w \gg 2N : w \in \hat{\Psi}^\pm(y,r)\}$ for $y \in \mathcal{I}_{\mathbb{Z}}$ and $r, N \in \mathbb{Z}$, so it follows that $\sum_{z \in \hat{\Psi}^+(y,q)} \hat{F}^{\mathtt{FPF}}_z - \sum_{z \in \hat{\Psi}^-(y,p)} \hat{F}^{\mathtt{FPF}}_z = \lim_{N \to \infty}(x_{p+2N} + x_{q+2N})\hat{\mathfrak{S}}^{\mathtt{FPF}}_{y \gg 2N} = 0$. $\square$

4 Schur P-positivity for involution Stanley symmetric functions

Edelman and Greene proved in [11] that the Stanley symmetric functions F_w are Schur positive, and it is immediate from Definition 2.20 that the involution Stanley symmetric functions $\hat{F}_y$ share this positivity property. The main goal of this section is to prove the stronger result that each $\hat{F}_y$ is *Schur P-positive*, i.e., a linear combination of Schur P-functions with nonnegative integer coefficients.

4.1 I-Grassmannian involutions

The following definitions are standard; see, for example, [25, 35].

Definition 4.1. The *diagram* of $w \in S_\infty$ is the set $D(w) = \{(i, w(j)) : (i,j) \in \mathrm{Inv}(w)\}$.

We identify $D(w)$ with the set of positions in a matrix, so that we may speak of the ith row or jth column to refer to the subsets $\{(i,n) \in D(w) : n \in \mathbb{P}\}$ or $\{(n,j) \in D(w) : n \in \mathbb{P}\}$.

Definition 4.2. The *code* of $w \in S_\infty$ is the sequence $c(w) = (c_1, c_2, c_3, \dots)$ in which c_i is the number of positions in the ith row of $D(w)$.

Of course, $D(w)$ is a finite set and $c(w)$ has only finitely many nonzero terms. We make no distinction between $(a_1, a_2, \dots, a_k)$ and the infinite sequence $(a_1, a_2, \dots, a_k, 0, 0, \dots)$.

Definition 4.3. The *essential set* of $D \subset \mathbb{P} \times \mathbb{P}$ is the set $\mathrm{Ess}(D)$ of southeast corners in D, that is, the positions $(i,j) \in D$ such that $(i+1,j) \notin D$ and $(i,j+1) \notin D$.

Example 4.4. If $w = 4231$ then $\mathrm{Ess}(D(w)) = \{(3,1),(1,3)\}$ and $c(w) = (3,1,1)$.

Definition 4.5. A permutation $w \in S_{\mathbb{Z}}$ is *Grassmannian* if $|\mathrm{Des}_R(w)| \le 1$.

The proof of the next statement is an instructive exercise; see, e.g., [35, Chapter 2].

Proposition-Definition 4.6. For $w \in S_\infty$ and $n \in \mathbb{P}$, the following are equivalent:

(a) $\mathrm{Des}_R(w) = \{s_n\}$.

(b) $w(1) < w(2) < \cdots < w(n) > w(n+1) < w(n+2) < w(n+3) < \cdots$.

(b) $c(w) = (c_1, c_2, \dots, c_n, 0, 0, \dots)$ where $c_1 \leq c_2 \leq \cdots \leq c_n \neq 0$.

(c) $\mathrm{Ess}(D(w))$ is nonempty and contained in $\{(n, j) : j \in \mathbb{P}\}$.

A permutation $w \in S_\infty$ with these equivalent properties is called *n-Grassmannian*. The identity $1 \in S_\infty$ is by convention the unique *0-Grassmannian* permutation.

Let $\lambda(w) = (w(n) - n + 1, \dots, w(2) - 1, w(1)) = (c_n, \dots, c_2, c_1)$ for an n-Grassmannian permutation $w \in S_\infty$ with code $c(w) = (c_1, c_2, \dots)$. Also define $\lambda(1) = \emptyset = (0, 0, \dots)$. The map $w \mapsto \lambda(w)$ is a bijection from n-Grassmannian permutations in S_∞ to partitions with at most n parts. Recall the definition of the map $\rho_n : \mathbb{Z}[[x_1, x_2, \dots]] \to \mathbb{Z}[x_1, x_2, \dots, x_n]$ from Section 2.2. The main object of this section is to prove an involution analogue of the following theorem.

Theorem 4.7 (See [35])**.** If $w \in S_\infty$ is n-Grassmannian, then $\mathfrak{S}_w = \rho_n s_{\lambda(w)}$ and $F_w = s_{\lambda(w)}$.

To start, we recall the following variations of Definitions 4.1 and 4.2 from [13, Section 3.2]:

Definition 4.8. The *(involution) diagram* of $y \in \mathcal{I}_\infty$ is the set $\hat{D}(y) = \{(i, j) \in D(y) : j \leq i\}$. Equivalently, $(i, j) \in \mathbb{P} \times \mathbb{P}$ belongs to $\hat{D}(y)$ if and only if $j \leq i < y(j)$ and $j < y(i)$.

Definition 4.9. The *(involution) code* of $y \in \mathcal{I}_\infty$ is the sequence $\hat{c}(y) = (c_1, c_2, \dots)$ in which c_i is the number of positions in the ith row of $\hat{D}(y)$.

Example 4.10. If $y = (1, 4)$ then $\hat{D}(y) = \{(1, 1), (2, 1), (3, 1)\}$ and $\hat{c}(y) = (1, 1, 1)$.

These are the appropriate analogues of diagrams and codes in the context of involution Schubert polynomials. For example, the degree of $\hat{\mathfrak{S}}_y$ is $|\hat{D}(y)| = \hat{\ell}(y)$ [13, Proposition 3.6]. The diagram $\hat{D}(y)$ also affords an explicit product formula for certain involution Schubert polynomials. An involution $y \in \mathcal{I}_\infty$ is *dominant* if $\hat{D}(y)$ is the transpose of the shifted diagram of a strict partition, as defined in Section 2.5. The following results are [13, Proposition 3.25] and [13, Theorem 3.26]:

Proposition 4.11 (See [13])**.** An involution in $\mathcal{I}_\infty$ is dominant if and only if it is 132-avoiding.

Recall that $x_{(i,j)}$ is either $x_i = x_j$ (if $i = j$) or $x_i + x_j$ (if $i \neq j$).

Theorem 4.12 (See [13])**.** If $y \in \mathcal{I}_\infty$ is dominant then $\hat{\mathfrak{S}}_y = \prod_{(i,j) \in \hat{D}(y)} x_{(i,j)}$.

The *lexicographic order* on S_∞ is the total order induced by identifying $w \in S_\infty$ with its one-line representation $w(1)w(2)w(3)\cdots$. The following is a consequence of [14, Theorem 6.10].

Lemma 4.13 (See [14])**.** Suppose $y \in \mathcal{I}_\infty$ and $\mathrm{Cyc}_\mathbb{P}(y) = \{(a_i, b_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \cdots$. The lexicographically minimal element of $\mathcal{A}(y)$ is the inverse of the permutation whose one-line representation is given by the sequence $b_1 a_1 b_2 a_2 b_3 a_3 \cdots$ with a_i omitted whenever $a_i = y(a_i) = b_i$.

Denote the lexicographically minimal element of $\mathcal{A}(y)$ as $\alpha_{\min}(y)$.

Example 4.14. If $y = (1, 4)$ then $b_1a_1b_2a_2b_3a_3 = 412233$ and $\alpha_{\min}(y) = 4123^{-1} = 2341$.

We say that a pair $(i, j) \in \mathbb{Z}$ is a *visible inversion* of $y \in \mathcal{I}_\mathbb{Z}$ if $i < j$ and $y(j) \leq \min\{i, y(i)\}$.

Lemma 4.15. The set of visible inversions of $y \in \mathcal{I}_\infty$ is equal to $\mathrm{Inv}(\alpha_{\min}(y))$.

Proof. Fix $y \in \mathcal{I}_\infty$ and let $\mathrm{Cyc}_\mathbb{P}(y) = \{(a_i, b_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \cdots$. Note that all visible inversions of y are contained in $\mathbb{P} \times \mathbb{P}$. Let $m < n$ be positive integers and let $j, k \in \mathbb{P}$ be such that $m \in \{a_k, b_k\}$ and $n \in \{a_j, b_j\}$. By Lemma 4.13, we have $(m, n) \in \mathrm{Inv}(\alpha_{\min}(y))$ if and only if $j \leq k$, which holds if and only if $a_j \leq a_k$. Note that $a_j = \min\{n, y(n)\}$ and $a_k = \min\{m, y(m)\}$.

If (m, n) is a visible inversion of y then $y(m) > y(n)$ and $n = b_j > m \geq a_j = y(n)$, so $a_j \leq \min\{m, y(m)\} = a_k$ as desired. Conversely, if $a_j \leq a_k$, then $n \neq a_j$ since $a_k \leq m$, so we must have $y(n) = a_j \leq a_k = \min\{m, y(m)\}$ which means that (m, n) is a visible inversion of y. □

The preceding lemma implies the following result, which is also [13, Lemma 3.8].

Lemma 4.16 (See [13])**.** If $y \in \mathcal{I}_\infty$ then $\hat{c}(y) = c(\alpha_{\min}(y))$.

We say that $i \in \mathbb{Z}$ is a *visible descent* of $y \in \mathcal{I}_\mathbb{Z}$ if $(i, i+1)$ is a visible inversion, and define $\mathrm{Des}_V(y) = \{s_i : i \in \mathbb{Z} \text{ is a visible descent of } y\}$. We note two facts about this set.

Lemma 4.17. If $y \in \mathcal{I}_\infty$ then $\mathrm{Des}_V(y) = \mathrm{Des}_R(\alpha_{\min}(y))$.

Proof. This follows from Lemma 4.15 since $s_i \in \mathrm{Des}_R(w)$ if and only if $(i, i+1) \in \mathrm{Inv}(w)$. □

Lemma 4.18. If $y \in \mathcal{I}_\infty$ then the ith row of $\mathrm{Ess}(\hat{D}(y))$ is nonempty if and only if $s_i \in \mathrm{Des}_V(y)$.

Proof. If $s_i \in \mathrm{Des}_V(y)$ then $(i, y(i+1)) \in \hat{D}(y)$ but all positions of the form $(i+1, j) \in \hat{D}(y)$ have $j < y(i+1)$, so the ith row of $\mathrm{Ess}(\hat{D}(y))$ is nonempty. Conversely, if the ith row of $\mathrm{Ess}(\hat{D}(y))$ is nonempty, then there exists $(i, j) \in \hat{D}(y)$ with $(i+1, j) \notin \hat{D}(y)$. This occurs if and only if $j = y(k)$ for some $k > i$ with $y(i) > y(k)$ and $i \geq y(k) \geq y(i+1)$, in which case evidently $s_i \in \mathrm{Des}_V(y)$. □

We may now give analogues of Definition 4.5 and Proposition-Definition 4.6.

Definition 4.19. An involution $y \in \mathcal{I}_\mathbb{Z}$ is *I-Grassmannian* if $|\mathrm{Des}_V(y)| \leq 1$.

For $y \in \mathcal{I}_\infty$, this definition is equivalent to the one in the introduction by the following.

Proposition-Definition 4.20. For $y \in \mathcal{I}_\infty$ and $n \in \mathbb{P}$, the following are equivalent:

(a) $\mathrm{Des}_V(y) = \{s_n\}$.

(b) $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r)$ for integers $r \in \mathbb{P}$ and $1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n$.

(c) $\hat{c}(y) = (c_1, c_2, \ldots, c_n, 0, 0, \ldots)$ where $c_1 \leq c_2 \leq \cdots \leq c_n \neq 0$.

(d) $\mathrm{Ess}(\hat{D}(y))$ is nonempty and contained in $\{(n, j) : j \in \mathbb{P}\}$.

(e) The lexicographically minimal atom $\alpha_{\min}(y) \in \mathcal{A}(y)$ is n-Grassmannian.

We refer to involutions $y \in \mathcal{I}_\infty$ with these equivalent properties as *n-I-Grassmannian*, and consider $1 \in \mathcal{I}_\infty$ to be the unique *0-I-Grassmannian* involution.

Proof. The equivalences (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) follow from Proposition-Definition 4.6 and Lemmas 4.16, 4.17, and 4.18. Proving that (a) $\Leftrightarrow$ (b) is a straightforward exercise from the definitions. $\square$

Remark 4.21. The number g_n of I-Grassmannian elements of $\mathcal{I}_n$ satisfies $g_n = g_{n-1} + g_{n-2} + n - 2$ for $n \geq 3$. The sequence $(g_n)_{n\geq 1} = (1, 2, 4, 8, 15, 27, 47, 80, \dots)$ appears as [45, A000126].

Any involution in $S_\mathbb{Z}$ which is Grassmannian in the ordinary sense is also I-Grassmannian. Moreover, $y \in \mathcal{I}_\mathbb{Z}$ is I-Grassmannian if and only if $y \gg N$ is I-Grassmannian for all $N \in \mathbb{Z}$.

Corollary 4.22. If $y \in \mathcal{I}_\mathbb{Z}$ is I-Grassmannian and $E \subset \mathbb{Z}$ is a finite set with $y(E) = E$, then the standardized involution $[y]_E$ is also I-Grassmannian.

Proof. The result is evident from Proposition-Definition 4.20(b) and the observation just noted. $\square$

Let $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r) \in \mathcal{I}_\infty$ where $r \in \mathbb{P}$ and $1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n$. Define the *shape* of the n-I-Grassmannian involution y to be the strict partition

$$\mu(y) = (n + 1 - \phi_1, n + 1 - \phi_2, \dots, n + 1 - \phi_r).$$

One can check that $\mu(y)$ is the transpose of the partition given by reversing $\hat{c}(y)$. The map $y \mapsto \mu(y)$ is a bijection from n-I-Grassmannian involutions to strict partitions whose parts all have size at most n. Recall the definition of the operators $\pi_{b,a}$ from Section 2.1.

Lemma 4.23. In the notation just given, we have $\hat{\mathfrak{S}}_y = \pi_{\phi_1,1}\pi_{\phi_2,2}\cdots\pi_{\phi_r,r}\left(x^{\mu(y)}G_{r,n}\right)$.

Proof. Let $\Sigma(\phi) = \sum_{i=1}^r (\phi_i - i)$. If $\Sigma(\phi) = 0$, then $y = (1, n+1)(2, n+2)\cdots(r, n+r)$ and the lemma asserts that $\hat{\mathfrak{S}}_y = x_1^n x_2^{n-1}\cdots x_r^{n-r+1}G_{r,n}$, which holds by Theorem 4.12 since $\hat{D}(y) = \{(i+j, i) : 1 \leq i \leq r \text{ and } 0 \leq j \leq n - i\}$. Suppose $\Sigma(\phi) > 0$ and let $i \in [r]$ be the smallest index such that $i < \phi_i$. It suffices to show that $\hat{\mathfrak{S}}_y = \pi_{\phi_i,i}\pi_{\phi_{i+1},i+1}\cdots\pi_{\phi_r,r}\left(x^{\mu(y)}G_{r,n}\right)$. Let

$$v = (1, n+1)(2, n+2)\cdots(i, n+i)(\phi_{i+1}, n+i+1)(\phi_{i+2}, n+i+2)\cdots(\phi_r, n+r) \in \mathcal{I}_\infty.$$

Equation (2.2) implies $\hat{\mathfrak{S}}_y = \partial_{\phi_i,i}\hat{\mathfrak{S}}_v$, and by induction $\hat{\mathfrak{S}}_v = \pi_{\phi_{i+1},i+1}\pi_{\phi_{i+2},i+2}\cdots\pi_{\phi_r,r}\left(x^{\mu(v)}G_{r,n}\right)$. Since $x^{\mu(v)} = x_i^{\phi_i - i}x^{\mu(y)}$ and since multiplication by x_i commutes with $\pi_{\phi_j,j}$ when $i < j$, we have

$$\hat{\mathfrak{S}}_y = \partial_{\phi_i,i}\hat{\mathfrak{S}}_v = \partial_{\phi_i,i}\left(x_i^{\phi_i - i}\pi_{\phi_{i+1},i+1}\pi_{\phi_{i+2},i+2}\cdots\pi_{\phi_r,r}\left(x^{\mu(y)}G_{r,n}\right)\right). \tag{4.1}$$

Since $\partial_j\left(x_i^{i-\phi_i}\hat{\mathfrak{S}}_v\right) = x_i^{i-\phi_i}\partial_j\hat{\mathfrak{S}}_v = 0$ for $i + 1 \leq j < \phi_i$ as $s_j \notin \mathrm{Des}_R(v)$, the desired identity $\hat{\mathfrak{S}}_y = \pi_{\phi_i,i}\pi_{\phi_{i+1},i+1}\cdots\pi_{\phi_r,r}\left(x^{\mu(y)}G_{r,n}\right)$ follows from (4.1) by Lemma 2.6. $\square$

If $y \in \mathcal{I}_\mathbb{Z} - \{1\}$ is I-Grassmannian, then $y \gg N$ is n-I-Grassmannian for some $n \in \mathbb{P}$ and $N \in \mathbb{N}$, and we define $\mu(y) = \mu(y \gg N)$. We also set $\mu(1) = \emptyset = (0, 0, \dots)$.

Theorem 4.24. If $y \in \mathcal{I}_\mathbb{Z}$ is I-Grassmannian, then $\hat{F}_y = P_{\mu(y)}$.

Proof. Since $\hat{F}_y = \hat{F}_{y\gg N}$ for all $N \in \mathbb{Z}$, we may assume that $y \in \mathcal{I}_\infty$ is n-I-Grassmannian. If $\mu(y)$ has r parts, then Lemmas 2.5 and 4.23 imply that $\pi_{w_n}\hat{\mathfrak{S}}_y = \pi_{w_n}\left(x^{\mu(y)}G_{r,n}\right)$ for all $n \geq r$, and the theorem follows by taking the limit as $n \to \infty$. $\square$

Remark 4.25. Unlike in Theorem 4.7, it may happen that $\hat{\mathfrak{S}}_y \neq \rho_n P_{\mu(y)}$ when $y \in \mathcal{I}_\infty$ is n-I-Grassmannian.

4.2 Schur P-positivity

In this section we describe an algorithm to expand $\hat{F}_y$ into a nonnegative linear combination of Schur P-functions. Our approach is inspired by Lascoux and Schützenberger's original proof of Theorem 1.3 from [29], which we sketch as follows. Throughout, we consider $\mathbb{Z} \times \mathbb{Z}$ to be ordered lexicographically. Recall the definition of $\Phi^{\pm}(w, r)$ from Section 3.1. For $w \in S_{\mathbb{Z}}$, define

$$\mathfrak{T}_1(w) = \begin{cases} \varnothing & \text{if } w \text{ is Grassmannian} \\ \Phi^-(w(r,s), r) & \text{otherwise} \end{cases}$$

where in the second case, (r, s) is the (lexicographically) maximal element of $\mathrm{Inv}(w)$. One can check that if (r, s) is the maximal inversion of $w \in S_{\mathbb{Z}}$, then $w(r, s) \lessdot w$ and $\Phi^+(w(r, s), r) = \{w\}$ and $r \in \mathbb{Z}$ is the largest integer such that $w(r) > w(r + 1)$.

Definition 4.26. The *Lascoux-Schützenberger tree* $\mathfrak{T}(w)$ of $w \in S_{\mathbb{Z}}$ is the tree with root w, in which the children of any vertex $v \in S_{\mathbb{Z}}$ are the elements of $\mathfrak{T}_1(v)$.

A given permutation may correspond to more than one vertex in $\mathfrak{T}(w)$. One can show that $\mathfrak{T}(w)$ is always finite [29]. Since $F_w = \sum_{v \in \mathfrak{T}_1(w)} F_v$ for any non-Grassmannian permutation by Theorem 3.2, it follows that $F_w = \sum_v F_v$ where the sum is over the finite set of leaf vertices v in $\mathfrak{T}(w)$. The leaves of $\mathfrak{T}(w)$ are Grassmannian permutations by construction, so Theorem 1.3 follows.

Example 4.27. The Lascoux-Schützenberger tree $\mathfrak{T}(w)$ of $w = 1254376 \in S_7$ is shown below. The maximal inversion of each vertex is underlined.

$$\begin{tikzcd}[arrows=dash]
 & 12543\underline{76} \arrow[dl] \arrow[d] \arrow[dr] & \\
1254\underline{63}7 \arrow[d] & 125\underline{6}3\underline{4}7 & 12\underline{6}43\underline{5}7 \arrow[d] \\
135\underline{42}67 \arrow[d] & & 13\underline{6}24\underline{5}7 \\
23\underline{5}1\underline{4}67 & &
\end{tikzcd}$$

It follows from Theorem 4.7 that $F_{1254376} = s_{(3,2,2)} + s_{(3,3,1,1)} + s_{(4,2,1)}$.

Fix $z \in \mathcal{I}_{\mathbb{Z}}$. Recall that an inversion $(i, j) \in \mathrm{Inv}(z)$ is visible if $z(j) \leq \min\{i, z(i)\}$, and that $i \in \mathbb{Z}$ is a visible descent of z if $(i, i + 1)$ is a visible inversion. It follows by Lemma 4.17 that if z has no visible descents then $\alpha_{\min}(z) = 1$ so $z = 1$.

Lemma 4.28. Let $z \in \mathcal{I}_{\mathbb{Z}} - \{1\}$ and suppose $j \in \mathbb{Z}$ is the smallest integer such that $z(j) < j$. Then $j - 1$ is the minimal visible descent of z.

Proof. By hypothesis $z(j) \leq j - 1 \leq z(j - 1)$ so $j - 1$ is a visible descent of z, and if $i < j - 1$ then $i + 1 \leq z(i + 1)$ so i is not a visible descent as $z(i + 1) \not\leq i$. $\square$

Lemma 4.29. Suppose $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ is the maximal visible inversion of $z \in \mathcal{I}_{\mathbb{Z}} - \{1\}$. Let m be the largest element of $\mathrm{supp}(z)$. Then q is the maximal visible descent of z while r is the maximal integer with $z(r) \leq \min\{q, z(q)\}$, and we have $z(q + 1) < z(q + 2) < \cdots < z(m) \leq q$. In addition, either (a) $z(q) < q < r \leq m$, (b) $z(q) = q < r = m$, or (c) $q < z(q) = r = m$.

Proof. Since $(q+1, r)$ is not a visible inversion of z, we have $z(q+1) \leq \min\{q, z(q)\}$ so q is a visible descent. If d is another visible descent of z then $(d, d+1)$ is a visible inversion, so $d \leq i$. It is clear by definition that r is maximal such that $z(r) \leq \min\{q, z(q)\}$. We must have $z(q+1) < z(q+2) < \cdots < z(m) \leq q$ since otherwise z would have a visible inversion greater than (q, r). It follows that $r = m$ if $z(q) = q$, and that $z(q) = r = m$ if $q < z(q)$. $\square$

To each nontrivial element of $\mathcal{I}_{\mathbb{Z}}$, we associate a Bruhat covering relation in the following way.

Proposition-Definition 4.30. Suppose (q, r) is the maximal visible inversion of $z \in \mathcal{I}_{\mathbb{Z}} - \{1\}$. There exists a unique involution $\eta(z) \in \mathcal{I}_{\mathbb{Z}}$ such that $\eta(z) < z$ and $z = \tau_{qr}(\eta(z))$. The involution $\eta(z)$ is as specified in Table 2, and it holds that $\eta(z)(q) \leq q$ and $\eta(z)(q) < z(q) \leq \eta(z)(r)$.

Proof. By the definition of τ_{ij}, if an involution $y \in \mathcal{I}_{\mathbb{Z}}$ exists such that $y < z$ and $z = \tau_{qr}(y)$, then y is unique and belongs to the set $\mathcal{Y} \subset S_{\mathbb{Z}}$ of permutations with the same restriction as z to the complement of $A = \{q, z(q), r, z(r)\}$ in $\mathbb{Z}$. Since (q, r) is the maximal visible inversion of z, we have either $z(r) = q < z(q) = r$ or $z(r) < q = z(q) < r$ or $z(r) < z(q) < q < r$. Consulting Table 1, we deduce that $\eta(z)$ exists and is given by the element $y \in \mathcal{Y}$ with $[y]_A = 12$ in the first case, $[y]_A = 213$ in the second case, and $[y]_A = 3412$ in the third case, as specified in Table 2. $\square$

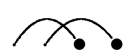
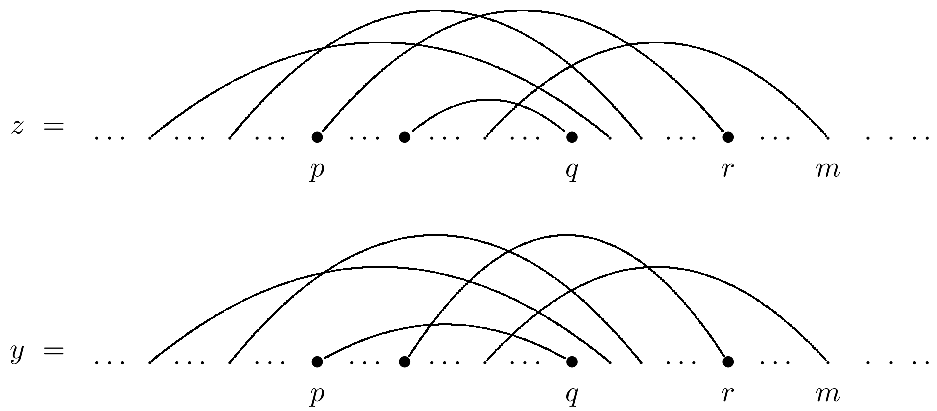
$A = \{q, r, z(q), z(r)\}$	$[z]_A$	(q, r)	$[\eta(z)]_A$	σ such that $\eta(z) = z\sigma$
$\{a < b\}$		(a, b)		(a, b)
$\{a < b < c\}$		(b, c)		(a, b, c)
$\{a < b < c < d\}$		(c, d)		$(a, b)(c, d)$

Table 2: Values of $\eta(z)$. Fix $z \in \mathcal{I}_{\mathbb{Z}}$ with maximal visible inversion (q, r). Let $A = \{q, r, z(q), z(r)\}$. The first column labels the elements of A. The third column rewrites (q, r) in this labeling. The last two columns determine $\eta(z)$ as characterized in Proposition-Definition 4.30. In the second and fourth columns, we use $\bullet$ symbols to mark the vertices corresponding to q and r.

As $\eta(z)$ is only defined if z has a visible inversion, we view η as a map $\mathcal{I}_{\mathbb{Z}} - \{1\} \to \mathcal{I}_{\mathbb{Z}}$.

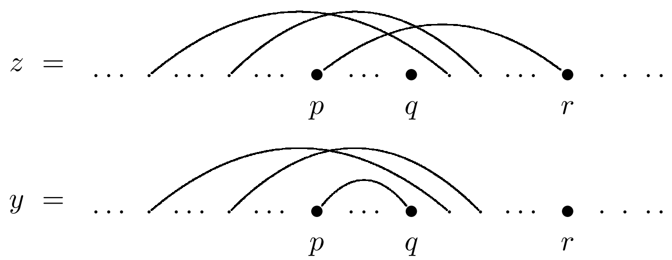
Remark 4.31. Suppose $z \in \mathcal{I}_{\mathbb{Z}} - \{1\}$ has maximal visible inversion (q, r). Let $p = z(r)$, $y = \eta(z)$, and $m = \max \operatorname{supp}(z)$. Lemma 4.29 completely determines the values of $y(i)$ and $z(i)$ for all $i \geq q$, and there are three qualitatively distinct cases for what can happen.

(a) If $z(q) < q < r \leq m$ then $y = (q, r)z(q, r)$ and z correspond to the pictures



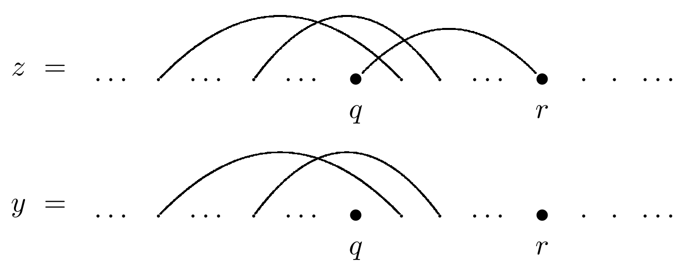
In our diagrams of this kind, each ellipsis "..." stands for zero or more unspecified vertices. Lemma 4.29 implies that $z(q+1) < z(q+2) < \cdots < z(r) < z(q)$, and that if $r < m$ then $z(q) < z(r+1) < z(r+2) < \cdots < z(m) < q$.

(b) If $z(q) = q < r = m$ then $y = (q, r)z(q, r)$ and z may be represented as



In this case, $z(q+1) < z(q+2) < \cdots < z(r) < q$, so $z(i) < q$ if $p < i < q$.

(c) If $q < z(q) = r = m$ so that $p = q$, then $y = z(q, r)$ and z may be represented as



In this case $z(q+1) < z(q+2) < \cdots < z(r-1) < q$.

Lemma 4.32. If (q, r) is the maximal visible inversion of $z \in \mathcal{I}_\infty - \{1\}$ and $w = \alpha_{\min}(z)$ is the minimal atom of z, then $w(q, r) = \alpha_{\min}(\eta(z))$ is the minimal atom of $\eta(z)$.

Proof. Let $\mathrm{Cyc}_{\mathbb{P}}(z) = \{(a_i, b_i) : i \in \mathbb{P}\}$ and $\mathrm{Cyc}_{\mathbb{P}}(\eta(z)) = \{(c_i, d_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \dots$ and $c_1 < c_2 < \dots$. By Lemma 4.13, it suffices to show that interchanging q and r and removing all repeated letters after their first appearance in $b_1a_1b_2a_2\cdots$ gives the same word as removing the repeated letters in $d_1c_1d_2c_2\cdots$. This is straightforward from Remark 4.31. For example, if $p = z(r) < q = z(q) < r$, then for some $n \in \mathbb{P}$ we have $b_na_nb_{n+1}a_{n+1} = rpqq$, $d_nc_nd_{n+1}c_{n+1} = qprr$, and $(a_i, b_i) = (c_i, d_i)$ for all $i \neq n$, in which case the desired property is clear. $\square$

Recall the definition of the sets $\hat{\Phi}^+(y, r)$ and $\hat{\Phi}^-(y, r)$ from Section 3.2.

Lemma 4.33. If $z \in \mathcal{I}_\mathbb{Z} - \{1\}$ has maximal visible inversion (q, r) then $\hat{\Phi}^+(\eta(z), q) = \{z\}$.

Proof. This follows from Theorem 3.7, Remark 4.31, and the definitions of $\eta(z)$ and $\hat{\Phi}^+(y, q)$. $\square$

We may now define an involution analogue of the set $\mathfrak{T}_1(w)$. For $z \in \mathcal{I}_\mathbb{Z}$, let

$$\hat{\mathfrak{T}}_1(z) = \begin{cases} \varnothing & \text{if } z \text{ is I-Grassmannian} \\ \hat{\Phi}^-(y, p) & \text{otherwise} \end{cases}$$

where in the second case, we set $y = \eta(z)$ and $p = y(q)$ with q the maximal visible descent of z. Note that if z is not I-Grassmannian then $\hat{\mathfrak{T}}_1(z) \neq \varnothing$.

Definition 4.34. The *involution Lascoux-Schützenberger tree* $\hat{\mathfrak{T}}(z)$ of $z \in \mathcal{I}_\mathbb{Z}$ is the tree with root z, in which the children of any vertex $v \in \mathcal{I}_\mathbb{Z}$ are the elements of $\hat{\mathfrak{T}}_1(v)$.

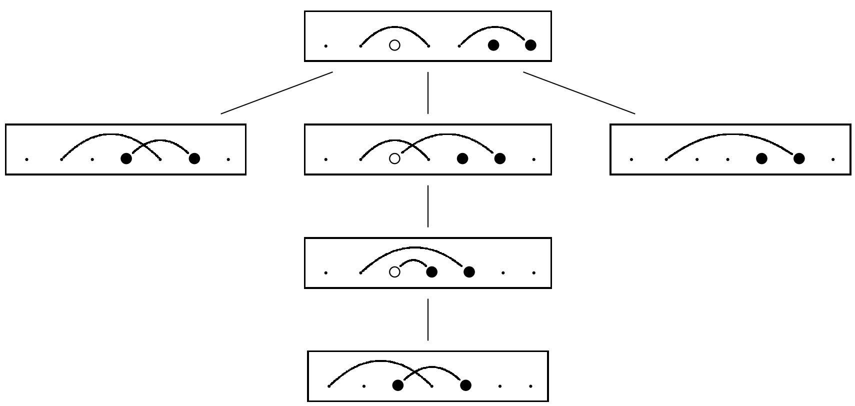


Figure 1: The involution Lascoux-Schützenberger tree $\hat{\mathfrak{T}}(z)$ for $z = (2,4)(5,7) \in \mathcal{I}_7$. The maximal visible inversion of each vertex is marked with $\bullet$, and the minimal visible descent in each non-leaf is marked with $\circ$. From Theorem 4.24 and Corollary 4.35, one computes that $\hat{F}_z = 2P_{(3,1)} + P_{(4)}$.

As with $\mathfrak{T}(w)$, an involution is allowed to correspond to more than one vertex in $\hat{\mathfrak{T}}(z)$. All vertices v in $\hat{\mathfrak{T}}(z)$ satisfy $\hat{\ell}(v) = \hat{\ell}(z)$ by construction, so 1 is not a vertex unless $z = 1$. An example tree $\hat{\mathfrak{T}}(z)$ is shown in Figure 1. Recall that $x_{(p,q)}$ is $x_p + x_q$ if $p \neq q$ and $x_p = x_q$ otherwise.

Corollary 4.35. Suppose $z \in \mathcal{I}_\mathbb{Z}$ is an involution which is not I-Grassmannian, whose maximal visible descent is $q \in \mathbb{Z}$. The following identities then hold:

(a) $\hat{\mathfrak{S}}_z = x_{(p,q)}\hat{\mathfrak{S}}_y + \sum_{v \in \hat{\mathfrak{T}}_1(z)} \hat{\mathfrak{S}}_v$ where $y = \eta(z)$ and $p = y(q)$.

(b) $\hat{F}_z = \sum_{v\in\hat{\mathfrak{T}}_1(z)} \hat{F}_v$.

Proof. The result is immediate from Theorems 3.8 and 3.10 and Lemma 4.33. ☐

To show that $\hat{\mathfrak{T}}(z)$ is a finite tree, we depend on a sequence of technical lemmas. Note that $\eta(z) = 1$ if and only if z is a transposition, in which case z is I-Grassmannian.

Lemma 4.36. Suppose $z \in \mathcal{I}_\mathbb{Z}$ is not I-Grassmannian, so that $\eta(z) \neq 1$.

(a) The maximal visible descent of $\eta(z)$ is less than or equal to that of z.

(b) The minimal visible descent of $\eta(z)$ is equal to that of z.

Proof. We may assume that $z \in \mathcal{I}_\infty$. In view of Proposition-Definition 4.20 and Lemmas 4.15 and 4.32, it suffices to show that if (i, j) is the maximal inversion of a permutation $w \in S_\infty$ which is not Grassmannian, then the maximal (respectively, minimal) descent of $w(i, j)$ is at most (respectively, equal to) that of w. This is a straightforward exercise which is left to the reader. ☐

Lemma 4.37. If $y \in \mathcal{I}_\mathbb{Z}$ and $n < p \leq q = y(p) < r$ then $\tau_{np}(y)(q) \leq y(q)$ and $\tau_{np}(y)(r) = y(r)$.

Proof. The result follows from the definition of τ_{np}; see Table 1. ☐

Lemma 4.38. Suppose $z \in \mathcal{I}_\mathbb{Z}$ is not I-Grassmannian. Let i and j be the minimal and maximal visible descents of z, and suppose $v \in \hat{\mathfrak{T}}_1(z)$. If d is a visible descent of v, then $i \leq d \leq j$.

Proof. Let $p = \eta(z)(j) \leq j$, let d be a visible descent of v, and let $n < p$ be such that $v = \tau_{np}(\eta(z))$. By Lemma 4.37, we have $v(k) = \eta(z)(k)$ for all $k > j$. As the maximal visible descent of $\eta(z)$ is at most j by Lemma 4.36(a), we deduce that $d \leq j$.

Define $a, b \in \mathbb{Z}$ as the smallest integers such that $\eta(z)(a) < a$ and $v(b) < b$. It follows from Lemmas 4.28 and 4.36(b) that $i = a - 1$ and that $b - 1$ is the minimal visible descent of v, so to prove that $i \leq d$ it suffices to show that $a \leq b$. This is clear from the definition of τ_{np} except when n and p are both fixed points of $\eta(z)$, in which case it could occur that $b = p$. In this situation, however, we would have $p = \eta(z)(j) = j$, so $a \leq b$ would hold anyway since $a = i + 1 \leq j$. ☐

For any $z \in \mathcal{I}_\mathbb{Z}$, let $\hat{\mathfrak{T}}_0(z) = \{z\}$ and define $\hat{\mathfrak{T}}_n(z) = \bigcup_{v\in\hat{\mathfrak{T}}_{n-1}(z)} \hat{\mathfrak{T}}_1(v)$ for $n \geq 1$.

Lemma 4.39. Suppose $z \in \mathcal{I}_\mathbb{Z}$ and $v \in \hat{\mathfrak{T}}_1(z)$. Let (q, r) be the maximal visible inversion of z, and let (q_1, r_1) be any visible inversion of v. Then $q_1 < q$ or $r_1 < r$. Hence, if $n \geq r - q$ then the maximal visible descent of every element of $\hat{\mathfrak{T}}_n(z)$ is strictly less than q.

Proof. Lemmas 4.29 and 4.38 imply that $q_1 \leq q$, so suppose $q_1 = q$. Since $v(q) \leq \eta(z)(q) < \eta(z)(r) = v(r)$ by Proposition-Definition 4.30 and Lemma 4.37, (q, r) is not a visible inversion of v. If $s > r$, then $z(q) < z(s)$ by Lemma 4.29, while $v(q) \leq \eta(z)(q) \leq z(q)$ and $v(s) = \eta(z)(s) = z(s)$ by Lemma 4.37 and the definition of $\eta(z)$, so (q, s) is also not a visible inversion of v. Thus $r_1 < r$. ☐

Theorem 4.40. The involution Lascoux-Schützenberger tree $\hat{\mathfrak{T}}(z)$ is finite for all $z \in \mathcal{I}_\mathbb{Z}$, and it holds that $\hat{F}_z = \sum_v \hat{F}_v$ where the sum is over the finite set of leaf vertices v in $\hat{\mathfrak{T}}(z)$.

Proof. It follows by induction from Lemmas 4.38 and 4.39 that for some sufficiently large n either $\hat{\mathfrak{T}}_n(z) = \varnothing$ or all elements of $\hat{\mathfrak{T}}_n(z)$ are I-Grassmannian, and in the latter case $\hat{\mathfrak{T}}_{n+1}(z) = \varnothing$. The tree $\hat{\mathfrak{T}}(z)$ is therefore finite, so the identity $\hat{F}_z = \sum_v \hat{F}_v$ follows from Corollary 4.35. ☐

The theorem implies this corollary, which we stated in the introduction as Theorem 1.10.

Corollary 4.41. If $z \in \mathcal{I}_\mathbb{Z}$ then $\hat{F}_z \in \mathbb{N}\text{-span}\left\{\hat{F}_y : y \in \mathcal{I}_\mathbb{Z} \text{ is I-Grassmannian}\right\}$.

4.3 Triangularity

We can say something a bit more specific about the expansion of $\hat{F}_y$ into Schur P-functions. Recall the definitions of $c(w)$ for $w \in S_\infty$ and $\hat{c}(y)$ for $y \in \mathcal{I}_\infty$ from Section 4.1. The *shape* of $w \in S_\infty$ is the partition $\lambda(w)$ given by sorting $c(w)$. For involutions, we have this alternative:

Definition 4.42. Let $\mu(y)$ for $y \in \mathcal{I}_\infty$ be the transpose of the partition given by sorting $\hat{c}(y)$.

These constructions are consistent with our earlier definitions of $\lambda(w)$ and $\mu(y)$ when w is Grassmannian and y is I-Grassmannian. Stanley [46] gave bounds on the Schur expansion of F_w in terms of the shape of w. Let $<$ denote the usual dominance order on partitions, and write μ^T for the transpose of a partition μ. Recall that $\lambda \leq \mu$ if and only if $\mu^T \leq \lambda^T$ [34, Eq. (1.11), §I.1].

Theorem 4.43 (Stanley [46])**.** If $w \in S_\infty$ and $\lambda'(w) = \lambda(w^{-1})^T$, then $\lambda(w) \leq \lambda'(w)$ and it holds that $F_w \in s_{\lambda(w)} + s_{\lambda'(w)} + \mathbb{N}\text{-span}\left\{s_\nu : \lambda(w) < \nu < \lambda'(w)\right\}$.

Stanley only established the form of this expansion; the positivity of its coefficients follows from results of Edelman and Greene [11]. We will prove an analogous result for the decomposition of $\hat{F}_y$ into Schur P-functions.

Define $<_\mathcal{A}$ on S_∞ as the transitive relation generated by setting $v <_\mathcal{A} w$ when the one-line representation of v^{-1} can be transformed to that of w^{-1} by replacing a consecutive subsequence of the form cab with $a < b < c$ by bca, or equivalently when $v < s_{i+1}v = s_i w > w$ for some $i \in \mathbb{P}$. For example, $3412 = (3412)^{-1} <_\mathcal{A} (3241)^{-1} = 4213$. Recall the definition of $\alpha_{\min}(y)$ from Lemma 4.13. In prior work, we showed [14, Theorem 6.10] that $<_\mathcal{A}$ is a partial order and that $\mathcal{A}(y) = \{w \in S_\infty : \alpha_{\min}(y) \leq_\mathcal{A} w\}$ for all $y \in \mathcal{I}_\infty$.

Lemma 4.44. Let $y \in \mathcal{I}_\infty$. If $v, w \in \mathcal{A}(y)$ and $v <_\mathcal{A} w$, then $\lambda(v) < \lambda(w)$.

Proof. Fix $v, w \in \mathcal{A}(y)$ with $v <_\mathcal{A} w$. It suffices to consider the case when w covers v, so assume $v < s_{i+1}v = s_i w > w$ for some $i \in \mathbb{P}$. Let $a = w^{-1}(i+2)$, $b = w^{-1}(i)$, and $c = w^{-1}(i+1)$, so that $a < b < c$. Note that if $u \in S_\infty$ and $u < us_j$ for some $j \in \mathbb{P}$, then the diagram $D(us_j)$ is given by transposing rows j and $j+1$ of the union $D(u) \cup \{(j+1, u(j))\}$. It follows that $D(v^{-1})$ is given by permuting rows i, $i+1$, and $i+2$ of $D(w^{-1}) \cup \{(i+1, b)\} - \{(i, a)\}$. There are evidently at least two more positions in column a than b of $D(w^{-1})$, so as $D(u^{-1}) = D(u)^T$ for any $u \in S_\infty$, we deduce that $\lambda(v) = \lambda(w) - e_j + e_k$ for some indices $j < k$, and hence that $\lambda(v) < \lambda(w)$. ☐

Theorem 4.45. If $y \in \mathcal{I}_\infty$ and $\mu = \mu(y)$ then $\mu^T \leq \mu$ and $\hat{F}_y \in s_{\mu^T} + s_\mu + \mathbb{N}\text{-span}\left\{s_\lambda : \mu^T < \lambda < \mu\right\}$.

Proof. Let $y \in \mathcal{I}_\infty$. Since $\hat{F}_y = \sum_{w \in \mathcal{A}(y)} F_w$, it follows from Theorem 4.43 and Lemma 4.44 that $\hat{F}_y \in s_\nu + \mathbb{N}\text{-span}\{s_\lambda : \nu < \lambda\}$ for $\nu = \lambda(\alpha_{\min}(y))$. Write $\omega : \Lambda \to \Lambda$ for the linear map with $\omega(s_\lambda) = s_{\lambda^T}$ for each partition λ. If λ is a strict partition then $\omega(P_\lambda) = P_\lambda$ [34, Example 3(a), §III.8], so $\omega(\hat{F}_y) = \hat{F}_y$ by Corollary 1.12. It follows that $\nu \leq \nu^T$ and $\hat{F}_y \in s_\nu + s_{\nu^T} + \mathbb{N}\text{-span}\left\{s_\lambda : \nu < \lambda < \nu^T\right\}$. It remains to show that $\mu(y)^T = \nu$, but this is clear from the definition of $\mu(y)$ and Lemma 4.16. ☐

The following is equivalent to Theorem 1.13 in the introduction.

Corollary 4.46. If $y \in \mathcal{I}_\infty$ then $\mu(y)$ is strict and $\hat{F}_y \in P_{\mu(y)} + \mathbb{N}\text{-span}\left\{P_\lambda : \lambda < \mu(y)\right\}$.

Proof. Since $P_\lambda \in s_\lambda + \mathbb{N}\text{-span}\{s_\nu : \nu < \lambda\}$ for any strict partition λ [34, Eq. (8.17)(ii), §III.8] and since $\hat{F}_y$ is Schur P-positive, the result holds by Theorem 4.45. ☐

Remark 4.47. This is the easiest way we know of showing that $\mu(y)$ is a strict partition. There should exist a more direct, combinatorial proof of this fact, using just the definition of $\mu(y)$.

We mention some applications to skew Schur functions. As is standard, we write $\mu \subset \lambda$ and say that λ *contains* μ if λ and μ are partitions with $\mu_i \leq \lambda_i$ for all $i \in \mathbb{P}$. When $\mu \subset \lambda$, we let $\lambda \setminus \mu$ and $s_{\lambda\setminus\mu}$ denote the corresponding skew shape and skew Schur function. We say that λ *strictly contains* μ if $0 = \mu_i = \lambda_i$ or $0 \leq \mu_i < \lambda_i$ for each $i \in \mathbb{P}$. For a partition μ which is strictly contained in $\delta_{n+1} = (n, n-1, \dots, 2, 1)$, we define

$$y_{\mu,n} = (a_1, b_1)(a_2, b_2)\cdots(a_n, b_n) \in \mathcal{F}_{2n} \tag{4.2}$$

where $b_i = n + i - \mu_i^T$ for $i \in [n]$ and $a_1 < a_2 < \cdots < a_n$ are the numbers in $[2n] \setminus \{b_1, b_2, \dots, b_n\}$ labeled in increasing order. Note that $b_1 < b_2 < \cdots < b_n = 2n$, and that $\mu_i^T < n + 1 - i$ and $2i - 1 < b_i$ for each $i \in [n]$. Thus $a_i < b_i$ for each i, so $y_{\mu,n} \in \mathcal{F}_{2n}$ is well-defined.

Example 4.48. If $\mu = (5, 3, 2, 2) \subset \delta_7$, then $\mu^T = (4, 4, 2, 1, 1)$ and $y_{\mu,6} = (a_1, b_1)(a_2, b_2)\cdots(a_6, b_6)$ for $(b_1, b_2, \dots, b_6) = (3, 4, 7, 9, 10, 12)$ and $(a_1, a_2, \dots, a_6) = (1, 2, 5, 6, 8, 11)$.

Two subsets of $\mathbb{P} \times \mathbb{P}$ are *equivalent* if one can be transformed to the other by permuting its rows and columns. Equivalent skew shapes index equal skew Schur functions [5, Proposition 2.4].

Proposition 4.49. Let $n \in \mathbb{N}$, suppose μ is a partition strictly contained in δ_{n+1}, and set $y = y_{\mu,n}$. Then y is 321-avoiding, the sets $\hat{D}(y)$ and $\delta_{n+1} \setminus \mu$ are equivalent, and $\hat{F}_y = s_{\delta_{n+1}\setminus\mu}$.

Proof. It is evident that $y = y_{\mu,n}$ is 321-avoiding since $a_i < b_i$ and $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for all i. For the same reason, we have $(i, j) \in \hat{D}(y)$ only if $\{i, j\} \subset A$ where $A = \{a_1, a_2, \dots, a_n\}$, and the positions in column a_i of $\hat{D}(y)$ are the pairs (a_j, a_i) with $i \leq j$ and $a_j < b_i$. Since exactly $n + 1 - i - \mu_i^T$ elements $a \in A$ satisfy $a_i \leq a < b_i$, we deduce that the map $(a_j, a_i) \mapsto (n + 1 - j, i)$ is a bijection $\hat{D}(y) \to \delta_{n+1} \setminus \mu$. It follows that $\hat{D}(y)$ and $\delta_{n+1} \setminus \mu$ are equivalent subsets of $\mathbb{P} \times \mathbb{P}$, so $\hat{F}_y = s_{\delta_{n+1}\setminus\mu}$ by [13, Proposition 3.31] and the discussion in [5, Section 2]. $\square$

Lemma 4.50. Let $m \in \mathbb{P}$ and suppose $\mu \subset \delta_m$ is a partition with $\mu \neq \delta_m$. There exists $n \in \mathbb{P}$ and a partition ν strictly contained in δ_n such that $\delta_m \setminus \mu$ and $\delta_n \setminus \nu$ are equivalent shapes.

Proof. If $\mu_i = m - i$ for some $i \in [m-1]$ then $\delta_m \setminus \mu$ is equivalent to $\delta_{m-1} \setminus \nu$ for $\nu = (\mu_1 - 1, \dots, \mu_{i-1} - 1, \mu_{i+1}, \dots, \mu_{m-1})$. The lemma follows by repeatedly applying this observation. $\square$

Proposition 4.51. For each $n \in \mathbb{P}$ and partition $\mu \subset \delta_n$, there exists $y \in \mathcal{I}_{\mathbb{Z}}$ with $s_{\delta_n\setminus\mu} = \hat{F}_y$.

Proof. Since $s_\varnothing = \hat{F}_1 = 1$, it suffices by Lemma 4.50 to prove that $s_{\delta_n\setminus\mu} = \hat{F}_y$ for some $y \in \mathcal{I}_{\mathbb{Z}}$ when μ is strictly contained in δ_{n+1}. This holds for $y = y_{\mu,n}$ by Proposition 4.49. $\square$

For a finite set $D \subset \mathbb{P}\times\mathbb{P}$, let $\gamma(D)$ be the transpose of the partition given by sorting the numbers of positions in each row of D. For example, if $\mu = (3, 3, 1)$ then $\gamma(\delta_6 \setminus \mu) = (2, 2, 2, 1, 1)^T = (5, 3)$. If $\mu \subset \delta_n$ then $\gamma(\delta_n \setminus \mu)$ is the same as what DeWitt calls the n-*complement* of μ [10, Definition IV.11], and is always a strict partition, since its parts count the positions of $\delta_n \setminus \mu$ on each southwest-to-northeast diagonal. The following is a weaker version of [10, Theorem V.5], and also closely related to the main result of Ardila and Serrano's paper [1].

Corollary 4.52 (DeWitt [10])**.** If $\mu \subset \delta_n$ and $\gamma = \gamma(\delta_n \setminus \mu)$, then $s_{\delta_n\setminus\mu} \in P_\gamma + \mathbb{N}\text{-span}\{P_\nu : \nu < \gamma\}$.

Proof. Both $\gamma(D)$ and s_D (when D is a skew shape) are invariant under equivalences between subsets of $\mathbb{P} \times \mathbb{P}$, so this follows from Corollary 4.46, Proposition 4.49, and Lemma 4.50. $\square$

4.4 I-vexillary involutions

By [13, Theorem 3.36], the involutions $z \in \mathcal{I}_\mathbb{Z}$ for which $\hat{F}_z$ is a Schur function are precisely those which are Grassmannian in the ordinary sense of having at most one right descent. This condition is quite restrictive, as $z \in \mathcal{I}_\mathbb{Z}$ is Grassmannian if and only if z is I-Grassmannian with shape $\mu(z) = \delta_k = (k-1,\dots,2,1,0)$ for some $k \in \mathbb{P}$ [13, Proposition 3.34]. In this section we consider the more general problem of classifying the involutions $z \in \mathcal{I}_\mathbb{Z}$ for which $\hat{F}_z = P_\mu$ for some strict partition μ. As in the introduction, we refer to involutions with this property as *I-vexillary.*

Remark 4.53. All I-Grassmannian involutions are I-vexillary by Theorem 4.24. The sequence $(v_n)_{n\geq 1} = (1, 2, 4, 10, 24, 63, 159, 423, 1099, 2962, 7868, \dots)$, with v_n counting the I-vexillary elements of $\mathcal{I}_n$, does not appear to be related to any existing entry in [45].

Recall that if $E \subset \mathbb{Z}$ is a finite set of size n then ψ_E is the unique order-preserving bijection $E \to [n]$. In the next three lemmas, we maintain the following notation: let $z \in \mathcal{I}_\mathbb{Z} - \{1\}$ be a nontrivial involution with maximal visible inversion (q, r), set $y = \eta(z)$, and write $p = y(q)$ so that $\hat{\mathfrak{T}}_1(z) = \hat{\Phi}^-(y, p)$ if z is not I-Grassmannian. Recall that $p \leq q$ by Proposition-Definition 4.30.

Lemma 4.54. Let $E \subset \mathbb{Z}$ be a finite set with $\{q, r\} \subset E$ and $z(E) = E$. Then $(\psi_E(q), \psi_E(r))$ is the maximal visible inversion of $[z]_E$ and it holds that $[\eta(z)]_E = \eta([z]_E)$.

Proof. The first assertion holds since the set of visible inversions of z contained in E and the set of all visible inversions of $[z]_E$ are in bijection via the map $\psi_E \times \psi_E$, which preserves lexicographic order. Since $\{q, r, z(q), z(r)\} \subset E$, we have $[\eta(z)]_E = \eta([z]_E)$ by the definition of η. $\square$

Write $L(z)$ for the set of integers $i < p$ with $\tau_{ip}(y) \in \hat{\Phi}^-(y, p)$ and, given a set $E \subset \mathbb{Z}$, define

$$\mathfrak{C}(z, E) = \{\tau_{ip}(y) : i \in E \cap L(z)\}.$$

Also let $\mathfrak{C}(z) = \mathfrak{C}(z, \mathbb{Z})$. Note that $\mathfrak{C}(z) = \hat{\mathfrak{T}}_1(z)$ if z is not I-Grassmannian. Thus, the only way that $\mathfrak{C}(z)$ could be empty would be if z were I-Grassmannian, but even in this case $\mathfrak{C}(z) \neq \varnothing$:

Lemma 4.55. If $z \in \mathcal{I}_\mathbb{Z} - \{1\}$ is I-Grassmannian, then $|\mathfrak{C}(z)| = 1$.

Proof. Suppose $z \in \mathcal{I}_\mathbb{Z} - \{1\}$ is I-Grassmannian, so that $z = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_k, n+k)$ for some integers $k \in \mathbb{P}$ and $\phi_1 < \phi_2 < \cdots < \phi_k \leq n$ by Proposition-Definition 4.20. Then $(q, r) = (n, n+k)$ and $z(r) = \phi_k$, and if $i \in \mathbb{Z}$ is maximal such that $i = z(i) < \phi_k$, then $\mathfrak{C}(z) = \{v\}$ where $v = (\phi_1, n+1)\cdots(\phi_{k-1}, n+k-1)(i, n)$. $\square$

Lemma 4.56. Let $E \subset \mathbb{Z}$ be a finite set such that $\{q, r\} \subset E$ and $z(E) = E$.

(a) The operation $v \mapsto [v]_E$ restricts to an injective map $\mathfrak{C}(z, E) \to \mathfrak{C}([z]_E)$.

(b) If E contains $L(z)$, then the injective map in (a) is a bijection.

Proof. It follows from Lemma 4.54 and the definitions of τ_{ip} and η that if $v \in \mathfrak{C}(z, E)$ then $[v]_E \in \mathfrak{C}([z]_E)$. The standardization map $v \mapsto [v]_E$ is injective on the set of permutations preserving E, which contains $\mathfrak{C}(z, E)$, so part (a) holds.

We prove the contrapositive of part (b). Suppose $a < b = \psi_E(p)$ are integers such that $\tau_{ab}([y]_E) \in \mathfrak{C}([z]_E)$ but $\tau_{ab}([y]_E)$ does not belong to the image of $\mathfrak{C}(z, E)$ under the map $v \mapsto [v]_E$. Let $i \in E$ be such that $\psi_E(i) = a$. We have $\tau_{ab}([y]_E) = [\tau_{ip}(y)]_E$, and by Theorem 3.7 it holds that $[y]_E \lessdot [y]_E(a,b)$ and therefore $[y]_E(a) < [y]_E(b)$ and $y(i) < y(p)$. Since $y \lessdot y(i,p)$ would imply that $\tau_{ip}(y) \in \mathfrak{C}(z,E)$ by Theorem 3.7, there must exist an integer j with $i < j < p$ and $y(i) < y(j) < y(p)$. Let j be the maximal integer with these properties; then $y \lessdot y(j,p)$ and so $j \in L(z)$ by Theorem 3.7. However, it cannot hold that $j \in E$ since this would contradict the fact that $[y]_E \lessdot [y]_E(a,b)$, so $L(z) \not\subset E$. $\square$

We say that $z \in \mathcal{I}_{\mathbb{Z}}$ *contains a bad pattern* if there exists a finite set $E \subset \mathbb{Z}$ which is z-invariant and which contains at most four z-orbits, such that $[z]_E$ is not I-vexillary. In this situation we refer to the set E as a *bad pattern* for z. We state two technical lemmas about this definition.

Lemma 4.57. If $z \in \mathcal{I}_{\mathbb{Z}}$ is such that $|\hat{\mathfrak{T}}_1(z)| \geq 2$, then z contains a bad pattern.

Proof. Let (q,r) be the maximal visible inversion of z, let $y = \eta(z)$, and let $p = y(q) \leq q$ so that $\hat{\mathfrak{T}}_1(z) = \hat{\Phi}^-(y,p)$. By hypothesis, there exist integers $i < j < p$ such that $\tau_{ip}(y)$ and $\tau_{jp}(y)$ are distinct elements of $\mathfrak{C}(z) = \hat{\mathfrak{T}}_1(z)$. The set $E = \{i, z(i), j, z(j), p, q, r, z(r)\}$ is z-invariant and it holds by Lemma 4.56(a) that $2 \leq |\mathfrak{C}(z,E)| \leq |\mathfrak{C}([z]_E)|$. Lemma 4.55 implies that $[z]_E$ is not I-Grassmannian, so $\hat{\mathfrak{T}}_1([z]_E) = \mathfrak{C}([z]_E)$ and therefore E is a bad pattern for z. $\square$

Lemma 4.58. Suppose $z \in \mathcal{I}_{\mathbb{Z}}$ is such that $\hat{\mathfrak{T}}_1(z) = \{v\}$ is a singleton set. Then z contains no bad patterns if and only if v contains no bad patterns.

Proof. It is a reasonable computer calculation to check the following claim by brute force: if $z \in \mathcal{I}_{12} - \{1\}$ and $\mathfrak{C}(z) = \{v\}$ is a singleton set, then z contains no bad patterns if and only if v contains no bad patterns. (There are 73,843 such involutions z to check.) We will deduce the lemma as a consequence of this empirical fact.

Assume $z \in \mathcal{I}_{\mathbb{Z}}$ is such that $\hat{\mathfrak{T}}_1(z) = \{v\}$ is a singleton set. By construction, v and z have the same action on all integers outside a set $A \subset \mathbb{Z}$ of size at most 6. If z (respectively, v) contains a bad pattern which is disjoint from A then v (respectively, z), clearly does as well. If z contains a bad pattern B which is not disjoint from A, then since $|B| \leq 8$ and since both A and B are z-invariant, the set $E = A \cup B$ can have size at most 12. In this case, using Lemma 4.56(b), one checks that $\mathfrak{C}([z]_E) = \{[v]_E\}$ and that $[z]_E$ contains a bad pattern, so we deduce by the claim in the first paragraph that both $[v]_E$ and v contain bad patterns. If instead v contains a bad pattern disjoint from A, then it follows by a similar argument that z contains a bad pattern. $\square$

We arrive at the main result of this section.

Theorem 4.59. An involution $z \in \mathcal{I}_{\mathbb{Z}}$ is I-vexillary if and only if $[z]_E$ is I-vexillary for all sets $E \subset \mathbb{Z}$ with $z(E) = E$ and $|E| = 8$.

Proof. Define $h(z)$ for $z \in \mathcal{I}_{\mathbb{Z}}$ to be the height of the tree $\hat{\mathfrak{T}}(z)$, i.e., the largest integer $n \in \mathbb{N}$ such that $\hat{\mathfrak{T}}_n(z) \neq \varnothing$. Note that $h(z) = 0$ if and only if $z \in \mathcal{I}_{\mathbb{Z}}$ is I-Grassmannian. Let $\mathcal{X}_n$ (respectively, $\mathcal{Y}_n$) be the set of involutions $z \in \mathcal{I}_{\mathbb{Z}}$ with $h(z) = n$ which are I-vexillary (respectively, which contain no bad patterns). The theorem is equivalent to the assertion that $\mathcal{X}_n = \mathcal{Y}_n$ for all $n \in \mathbb{N}$.

By Corollary 4.22 and Theorem 4.24, $\mathcal{X}_0$ and $\mathcal{Y}_0$ are both equal to the set of I-Grassmannian involutions in $\mathcal{I}_{\mathbb{Z}}$. Let $n > 0$ and $z \in \mathcal{I}_{\mathbb{Z}}$. Theorem 4.40 implies that $z \in \mathcal{X}_n$ if and only if $\hat{\mathfrak{T}}_1(z)$ is a singleton set contained in $\mathcal{X}_{n-1}$, while Lemmas 4.57 and 4.58 imply that $z \in \mathcal{Y}_n$ if and only if $\hat{\mathfrak{T}}_1(z)$ is a singleton set contained in $\mathcal{Y}_{n-1}$. Since $\mathcal{X}_{n-1} = \mathcal{Y}_{n-1}$ by induction, $\mathcal{X}_n = \mathcal{Y}_n$ as desired. $\square$

Corollary 4.60. An involution $z \in \mathcal{I}_\mathbb{Z}$ is I-vexillary if and only if for all finite sets $E \subset \mathbb{Z}$ with $z(E) = E$ the standardization $[z]_E$ is not any of the following eleven permutations:

$$\begin{array}{llll}
(1,2)(3,5), & (1,4)(3,6), & (1,5)(2,4)(3,7), & (1,6)(2,5)(3,8)(4,7), \\
(1,3)(4,5), & (1,4)(2,3)(5,6), & (1,5)(3,7)(4,6), & (1,6)(2,4)(3,8)(5,7), \\
 & (1,2)(3,6)(4,5), & & (1,3)(2,5)(4,7)(6,8). \\
 & (1,2)(3,4)(5,6), & &
\end{array}$$

Proof. Using Theorems 4.24 and 4.40, we have checked by a computer calculation that $z \in \mathcal{I}_8$ is not I-vexillary if and only if there exists a z-invariant subset $E \subset \mathbb{Z}$ such that $[z]_E$ is one of the given involutions. The corollary therefore follows by Theorem 4.59. □

Corollary 4.61. Suppose $z \in \mathcal{I}_\mathbb{Z}$ is 321-avoiding. Then z is I-vexillary if and only if for all finite z-invariant sets $E \subset \mathbb{Z}$, it holds that $[z]_E$ is neither $(1,2)(3,4)(5,6)$ nor $(1,3)(2,5)(4,7)(6,8)$.

Proof. The other nine permutations in Corollary 4.60 are not 321-avoiding, so the result follows. □

As an application, we give an alternate proof of a theorem of DeWitt [10]. A partition is a *rectangle* if its nonzero parts are all equal. The next statement is equivalent to [10, Theorem V.3].

Theorem 4.62 (DeWitt [10])**.** Fix a partition $\mu \subset \delta_m$. The skew Schur function $s_{\delta_m \setminus \mu}$ is a Schur P-function if and only if $\delta_m \setminus \mu$ is equivalent to $\delta_n \setminus \rho$ for a rectangle $\rho \subset \delta_n$ for some $n \in \mathbb{P}$.

Proof. Let μ be a partition strictly contained in δ_{n+1} for some $n \in \mathbb{N}$, and define $y = y_{\mu,n}$ as in (4.2). By Proposition 4.49 and Lemma 4.50, it suffices to show that y is I-vexillary if and only if μ is a rectangle. If μ is a rectangle with k parts of size j, then the numbers b_i in (4.2) have the form $\{b_1, b_2, \dots, b_n\} = (n - k + [j]) \cup (n + j + [n - j])$, and it is an easy exercise to check that the 321-avoiding involution y satisfies the conditions in Corollary 4.61 so is I-vexillary.

Suppose that μ is not a rectangle. Let a_i and b_i be as in (4.2) so that $a_1 = 1$ and $b_i = 2n$. It is helpful to note that if $\mathcal{G}$ is the graph on $[2n]$ with an edge from i to $i+1$ for each $i \in [2n-1]$, then μ is not a rectangle if and only if the induced subgraph of $\mathcal{G}$ on $\{a_1, a_2, \dots, a_n\}$ has at least three connected components. Let $i \in [n]$ be maximal such that $a_i = i$ and let $j \in [n]$ be minimal such that $b_j = n + j$. If $i = 1$, then $[y]_E = (1,2)(3,4)(5,6)$ for $E = \{a_1, b_1, a_2, b_2, a_n, b_n\}$. If $j = n$ then $[y]_E = (1,2)(3,4)(5,6)$ for $E = \{a_1, b_1, a_{n-1}, b_{n-1}, a_n, b_n\}$. If $i > 1$ and $j < n$, then one checks that $[y]_E$ is $(1,2)(3,4)(5,6)$ or $(1,3)(2,5)(4,7)(6,8)$ when E is one of $\{a_1, b_1, a_{i+1}, b_{i+1}, a_n, b_n\}$, $\{a_1, b_1, a_{j-1}, b_{j-1}, a_n, b_n\}$, or $\{a_1, b_1, a_{i+1}, b_{i+1}, a_{j-1}, b_{j-1}, a_n, b_n\}$. In either case, we conclude by Corollary 4.61 that y is not I-vexillary, as required. □

4.5 Pfaffian formulas

Let $y \in \mathcal{I}_\infty$ be I-Grassmannian. In this section we prove a formula for $\hat{\mathfrak{S}}_y$ inspired by a determinantal expression for the Schur P-function $\hat{F}_y = P_{\mu(y)}$. Recall that $\mathcal{F}_n$ is the set of fixed-point-free involutions in S_n. The *Pfaffian* of a skew-symmetric $n \times n$ matrix A is the expression

$$\operatorname{pf} A = \sum_{z \in \mathcal{F}_n} (-1)^{\hat{\ell}(z)} \prod_{z(i) < i \in [n]} A_{i,z(i)}. \tag{4.3}$$

It is a classical fact that $\det A = (\operatorname{pf} A)^2$. Since $\det A = 0$ when A is skew-symmetric but n is odd, the definition (4.3) is consistent with the fact that $\mathcal{F}_n$ is empty for n odd.

Example 4.63. If $A = (a_{ij})$ is a 2×2 skew-symmetric matrix then $\operatorname{pf} A = a_{12} = -a_{21}$. If $A = (a_{ij})$ is a 4×4 skew-symmetric matrix then $\operatorname{pf} A = a_{21}a_{43} - a_{31}a_{42} + a_{41}a_{32}$.

All matrices of interest in this section are skew-symmetric, and we write $[a_{ij}]_{1\le i<j\le n}$ to denote the unique $n\times n$ skew-symmetric matrix with a_{ij} in entry (i, j) for $i < j$ (and, necessarily, with $-a_{ij}$ in entry (j, i), and 0 in each diagonal entry). Observe that in this notation $[1]_{1\le i<j\le n}$ is neither the identity matrix nor the matrix whose entries are all 1's.

Lemma 4.64. Suppose $n \in \mathbb{P}$ is even. Then $\operatorname{pf}[1]_{1\le i<j\le n} = \sum_{z\in\mathcal{F}_n} (-1)^{\hat\ell(z)+\frac{n}{2}} = 1$.

Proof. Let $\mathcal{X}_n = \{z \in \mathcal{F}_n : z(n-1) = n\}$ and $\mathcal{Y}_n = \mathcal{F}_n - \mathcal{X}_n$. Conjugation and multiplication by s_{n-1} define bijections $\mathcal{Y}_n \to \mathcal{Y}_n$ and $\mathcal{F}_{n-2} \to \mathcal{X}_n$ reversing the sign of $(-1)^{\hat\ell(z)}$. Hence $\operatorname{pf}[1]_{1\le i<j\le n} = \sum_{z\in\mathcal{F}_n} (-1)^{\hat\ell(z)+\frac{n}{2}} = \sum_{z\in\mathcal{X}_n} (-1)^{\hat\ell(z)+\frac{n}{2}} = \operatorname{pf}[1]_{1\le i<j\le n-2}$, and the result follows by induction. □

Let $\phi = (\phi_1, \phi_2, \dots)$ be an integer sequence which has finitely many nonzero terms. If ϕ is of finite length r, then we identify ϕ with the infinite sequence with $\phi_i = 0$ for all $i > r$. Define

$$\ell(\phi) = \max\{i \in \mathbb{P} : \phi_i \neq 0\} \qquad \text{and} \qquad \ell^+(\phi) = \begin{cases} \ell(\phi) + 1 & \text{if } \ell(\phi) \text{ is odd} \\ \ell(\phi) & \text{otherwise.} \end{cases}$$

As a notational convenience we write $P_{\lambda_1\lambda_2\cdots\lambda_r}$ in place of $P_\lambda = P_{(\lambda_1,\lambda_2,\dots,\lambda_r)}$ for a strict partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$. The following identity appears as [34, Eq. (8.11), §III.8].

Theorem 4.65 (Macdonald [34])**.** If λ is a strict partition then $P_\lambda = \operatorname{pf}[P_{\lambda_i\lambda_j}]_{1\le i<j\le \ell^+(\lambda)}$.

This theorem is an analogue of the Jacobi-Trudi identity for Schur functions, which may be written succinctly as $s_\lambda = \det[s_{\lambda_i-i+j}]$. The formula in Theorem 4.65 is what Schur gave as the original definition of P_λ in [44], after specifying P_λ for strict partitions λ with $\ell(\lambda) \le 2$.

Example 4.66. For $\lambda = (3, 2, 1)$, Theorem 4.65 gives $P_\lambda = P_{(3,2)}P_{(1)} - P_{(3,1)}P_{(2)} + P_{(2,1)}P_{(3)}$.

When $y \in \mathcal{I}_\infty$ is I-Grassmannian, Theorem 4.65 expresses $\hat F_y$ as a Pfaffian in terms of involution Stanley symmetric functions of I-Grassmannian involutions with at most two nontrivial cycles. We introduce some notation to make this idea more explicit. Fix

$$n, r \in \mathbb{P} \qquad \text{and} \qquad \phi \in \mathbb{P}^r \text{ with } 0 < \phi_1 < \phi_2 < \cdots < \phi_r \le n. \tag{4.4}$$

Note that we set $\phi_i = 0$ for $i > r$. Let $y = (\phi_1, n + 1)(\phi_2, n + 2)\cdots(\phi_r, n + r) \in \mathcal{I}_\infty$ and define

$$\hat{\mathfrak{S}}[\phi_1, \phi_2, \dots, \phi_r; n] = \mathfrak{S}_y \qquad \text{and} \qquad \hat F[\phi_1, \phi_2, \dots, \phi_r; n] = \hat F_y.$$

When r is odd, we also set $\hat{\mathfrak{S}}[\phi_1, \phi_2, \dots, \phi_r, 0; n] = \mathfrak{S}_y$ and $\hat F[\phi_1, \phi_2, \dots, \phi_r, 0; n] = \hat F_y$. Since $\hat F_y = P_{(n+1-\phi_1,\dots,n+1-\phi_r)}$ by Theorem 4.24, Theorem 4.65 implies the following identity.

Corollary 4.67. In the setup of (4.4), $\hat F[\phi_1, \phi_2, \dots, \phi_r; n] = \operatorname{pf}\left[\hat F[\phi_i, \phi_j; n]\right]_{1\le i<j\le\ell^+(\phi)}$.

Our main result in this section is to show that the preceding formula is true even before stabilizing, that is, with $\hat F[\cdots; n]$ replaced by $\hat{\mathfrak{S}}[\cdots; n]$. In the following lemmas, let $\mathfrak{M}[\phi; n] = \mathfrak{M}[\phi_1, \phi_2, \dots, \phi_r; n]$ denote the skew-symmetric matrix $\left[\hat{\mathfrak{S}}[\phi_i, \phi_j; n]\right]_{1\le i<j\le\ell^+(\phi)}$.

Lemma 4.68. Maintain the notation of (4.4), and suppose $p \in [n-1]$. Then

$$\partial_p (\mathrm{pf}\, \mathfrak{M}[\phi; n]) = \begin{cases} \mathrm{pf}\, \mathfrak{M}[\phi + e_i; n] & \text{if } p = \phi_i \notin \{\phi_2 - 1, \dots, \phi_r - 1\} \text{ for some } i \in [r] \\ 0 & \text{otherwise} \end{cases}$$

where $e_i = (0, \dots, 0, 1, 0, 0, \dots)$ is the standard basis vector whose ith coordinate is 1.

Proof. Write $\mathfrak{M} = \mathfrak{M}[\phi; n]$. For indices $1 \leq i < j \leq \ell^+(\phi)$, it follows from (2.2) that $\partial_p \mathfrak{M}_{ij} = \partial_p \hat{\mathfrak{S}}[\phi_i, \phi_j; n]$ is $\hat{\mathfrak{S}}[\phi_i + 1, \phi_j]$ if $p = \phi_i \neq \phi_j - 1$, $\hat{\mathfrak{S}}[\phi_i, \phi_j + 1]$ if $p = \phi_j$, and 0 otherwise. Thus, if $p \notin \{\phi_1, \phi_2, \dots, \phi_r\}$, then all entries of $\mathfrak{M}$ are symmetric in x_p and x_{p+1}, so $\partial_p (\mathrm{pf}\, \mathfrak{M}) = 0$. Assume $p = \phi_k$ for some $k \in [r]$. Then $\partial_p \mathfrak{M}_{ij} = 0$ unless $i = k$ or $j = k$, so it follows from Corollary 2.3 and (4.3) that $\partial_p (\mathrm{pf}\, \mathfrak{M}) = \mathrm{pf}\, \mathfrak{N}$ where $\mathfrak{N}$ is the matrix formed by replacing the entries in the kth row and the kth column of $\mathfrak{M}$ by their images under ∂_p. If $k < r$ and $\phi_k = \phi_{k+1} - 1$, then columns k and $k+1$ of $\mathfrak{N}$ are identical, so $\mathrm{pf}\, \mathfrak{M} = \mathrm{pf}\, \mathfrak{N} = 0$ since $(\mathrm{pf}\, \mathfrak{N})^2 = \det \mathfrak{N} = 0$. If $k = r$ or if $k < r$ and $\phi_k \neq \phi_{k+1} - 1$, then $\mathfrak{N} = \mathfrak{M}[\phi + e_k; n]$. In either case the desired identity holds. ☐

Recall that $\mathcal{P} = \mathbb{Z}[x_1, x_2, \dots]$. A polynomial $f \in \mathcal{P}$ is *divisible* by $D \in \mathcal{P}$ if $f/D \in \mathcal{P}$.

Lemma 4.69. Let $n \in \mathbb{P}$ and $D = x_1(x_1 + x_2)(x_1 + x_3) \cdots (x_1 + x_n)$. Then $\mathrm{pf}\, \mathfrak{M}[1; n] = D$, and if $b \in \mathbb{P}$ is such that $1 < b \leq n$, then $\mathrm{pf}\, \mathfrak{M}[1, b; n]$ is divisible by D.

Note that $\mathfrak{M}[1; n]$ and $\mathfrak{M}[1, b; n]$ are both 2×2 skew-symmetric matrices; cf. Example 4.63.

Proof. It follows from Theorem 4.12 that $\mathrm{pf}\, \mathfrak{M}[1; n] = \mathfrak{S}_{(1,n+1)} = D$ and, when $n \geq 2$, that $\mathrm{pf}\, \mathfrak{M}[1, 2; n] = \mathfrak{S}_{(1,n+1)(2,n+2)} = x_2(x_2 + x_3) \cdots (x_2 + x_n) D$. Assume $2 < b \leq n$ so that $\mathrm{pf}\, \mathfrak{M}[1, b; n] = \partial_{b-1}(\mathrm{pf}\, \mathfrak{M}[1, b-1; n])$ by Lemma 4.68. By induction $\mathrm{pf}\, \mathfrak{M}[1, b-1; n] = qD$ for some $q \in \mathcal{P}$, so since since D is symmetric in x_{b-1} and x_b, we have $\mathrm{pf}\, \mathfrak{M}[1, b; n] = \partial_{b-1}(qD) = (\partial_{b-1} q) D$ as desired. ☐

If $i : \mathbb{P} \to \mathbb{N}$ is a map with $i^{-1}(\mathbb{P}) \subset [n]$ for some finite n, then we define $x^i = x_1^{i(1)} x_2^{i(2)} \cdots x_n^{i(n)}$. Given a nonzero polynomial $f = \sum_{i : \mathbb{P} \to \mathbb{N}} c_i x^i \in \mathcal{P}$, let $j : \mathbb{P} \to \mathbb{N}$ be the lexicographically minimal index such that $c_j \neq 0$ and define $\mathrm{lt}(f) = c_j x^j$. We refer to $\mathrm{lt}(f)$ as the *least term* of f. Set $\mathrm{lt}(0) = 0$, so that $\mathrm{lt}(fg) = \mathrm{lt}(f)\,\mathrm{lt}(g)$ for any $f, g \in \mathcal{P}$. The following is [13, Proposition 3.14].

Lemma 4.70 (See [13])**.** If $y \in \mathcal{I}_\infty$ then $\mathrm{lt}(\hat{\mathfrak{S}}_y) = x^{\hat{c}(y)} = \prod_{(i,j) \in \hat{D}(y)} x_i$.

Lemma 4.71. Let $i, j, n \in \mathbb{P}$. The following identities then hold:

(a) If $i \leq n$ then $\mathrm{lt}(\hat{\mathfrak{S}}[i; n]) = x_i x_{i+1} \cdots x_n$.

(b) If $i < j \leq n$ then $\mathrm{lt}(\hat{\mathfrak{S}}[i, j; n]) = (x_i x_{i+1} \cdots x_n)(x_j x_{j+1} \cdots x_n)$.

Proof. If $i \leq n$ then $\hat{\mathfrak{S}}[i; n] = \hat{\mathfrak{S}}_y$ for $y = (i, n+1)$, and if $i < j \leq n$ then $\hat{\mathfrak{S}}[i, j; n] = \hat{\mathfrak{S}}_z$ for $z = (i, n+1)(j, n+2)$. One checks that $\hat{D}(y) = \{(i,i), (i+1, i), \dots, (n, i)\}$ and $\hat{D}(z) = \{(i,i), (i+1, i), \dots, (n, i)\} \cup \{(j,j), (j+1, j), \dots, (n, j)\}$, so the result follows by Lemma 4.70. ☐

The following proves the base case of this section's main result.

Lemma 4.72. If $n \in \mathbb{P}$ and $r \in [n]$ then $\hat{\mathfrak{S}}[1, 2, \dots, r; n] = \mathrm{pf}\, \mathfrak{M}[1, 2, \dots, r; n]$.

Proof. Let $y = (1, n+1)(2, n+2)\cdots(r, n+r) \in \mathcal{I}_\infty$ and $D_i = x_i(x_i + x_{i+1})(x_i + x_{i+2})\cdots(x_i + x_n)$ for $i \in [n]$, so that $D_n = x_n$. As noted in the proof of Theorem 4.23, Theorem 4.12 implies that $\hat{\mathfrak{S}}[1, 2, \dots, r; n] = \hat{\mathfrak{S}}_y = D_1 D_2 \cdots D_r$. Write $\mathfrak{M} = \mathfrak{M}[1, 2, \dots, r; n]$. Lemma 4.68 implies that $\partial_i(\operatorname{pf}\mathfrak{M}) = 0$ for each $i \in [r-1]$, so $\operatorname{pf}\mathfrak{M}$ is symmetric in the variables $x_1, x_2, \dots, x_r$. By Lemma 4.69, every entry in the first column of $\mathfrak{M}$ is divisible by D_1, so $\operatorname{pf}\mathfrak{M}$ is also divisible by D_1. Since $s_i(D_i)$ is divisible by D_{i+1} for $i \in [n-1]$ and since $D_1, D_2, \dots, D_r$ are pairwise coprime, we deduce that $\operatorname{pf}\mathfrak{M}$ is divisible by $\hat{\mathfrak{S}}[1, 2, \dots, r; n]$. Since both of these polynomials are homogeneous and one divides the other, to prove they are equal it suffices to show that they have the same least term.

Let $m \in \mathbb{P}$ be whichever of r or $r+1$ is even and choose $z \in \mathcal{F}_m$. If $j \in [m]$ and $i = z(j) < j$, then $\mathfrak{M}_{ij}$ is either $\hat{\mathfrak{S}}[i, j; n]$ (if $j < m$) or $\hat{\mathfrak{S}}[i; n]$ (if $j = m$). We compute by Lemma 4.71 that $\operatorname{lt}\left(\prod_{z(i)<i\in[m]} \mathfrak{M}_{z(i),i}\right) = \prod_{z(i)<i\in[m]} \operatorname{lt}\left(\mathfrak{M}_{z(i),i}\right) = (x_1x_2\cdots x_n)(x_2x_3\cdots x_n)\cdots(x_r x_{r+1}\cdots x_n)$ which is precisely $\operatorname{lt}(\hat{\mathfrak{S}}[1, 2, \dots, r; n]) = \operatorname{lt}(D_1)\operatorname{lt}(D_2)\cdots\operatorname{lt}(D_r)$. Since $\sum_{z\in\mathcal{F}_m}(-1)^{\hat{\ell}(z)+\frac{m}{2}} = 1$ by Lemma 4.64, we deduce that $\operatorname{lt}(\operatorname{pf}\mathfrak{M}) = \operatorname{lt}(\hat{\mathfrak{S}}[1, 2, \dots, r; n])$ as needed. $\square$

Here, finally, is the other Pfaffian formula in the title of this section.

Theorem 4.73. In the setup of (4.4), $\hat{\mathfrak{S}}[\phi_1, \phi_2, \dots, \phi_r; n] = \operatorname{pf}\left[\hat{\mathfrak{S}}[\phi_i, \phi_j; n]\right]_{1\leq i<j\leq \ell^+(\phi)}$.

Proof. Writing $\hat{\mathfrak{S}}[\phi; n]$ in place of $\hat{\mathfrak{S}}[\phi_1, \phi_2, \dots, \phi_r; n]$, we must show that $\hat{\mathfrak{S}}[\phi; n] = \operatorname{pf}\mathfrak{M}[\phi; n]$. As in the proof of Lemma 4.23, we proceed by induction on $\Sigma(\phi) = \sum_{i=1}^r(\phi_i - i)$. If $\Sigma(\phi) = 0$ then $\phi = (1, 2, \dots, r)$ so $\hat{\mathfrak{S}}[\phi; n] = \operatorname{pf}\mathfrak{M}[\phi; n]$ by Lemma 4.72. Suppose instead that $\Sigma(\phi) > 0$. Let $i \in [r]$ be the smallest index such that $i < \phi_i$ and set $p = \phi_i - 1$. Theorem 2.19 then implies that $\hat{\mathfrak{S}}[\phi; n] = \partial_p\hat{\mathfrak{S}}[\phi - e_i; n]$, while Lemma 4.68 implies that $\operatorname{pf}\mathfrak{M}[\phi; n] = \partial_p(\operatorname{pf}\mathfrak{M}[\phi - e_i; n])$. We may assume that $\hat{\mathfrak{S}}[\phi - e_i; n] = \operatorname{pf}\mathfrak{M}[\phi - e_i; n]$ by induction, so $\hat{\mathfrak{S}}[\phi; n] = \operatorname{pf}\mathfrak{M}[\phi; n]$ as needed. $\square$

Example 4.74. For $\phi = (1, 2, 3)$ and $n = 3$ the theorem reduces to the identity

$$\hat{\mathfrak{S}}_{(1,4)(2,5)(3,6)} = \operatorname{pf}\begin{pmatrix} 0 & \hat{\mathfrak{S}}_{(1,4)(2,5)} & \hat{\mathfrak{S}}_{(1,4)(3,5)} & \hat{\mathfrak{S}}_{(1,4)} \\ -\hat{\mathfrak{S}}_{(1,4)(2,5)} & 0 & \hat{\mathfrak{S}}_{(2,4)(3,5)} & \hat{\mathfrak{S}}_{(2,4)} \\ -\hat{\mathfrak{S}}_{(1,4)(3,5)} & -\hat{\mathfrak{S}}_{(2,4)(3,5)} & 0 & \hat{\mathfrak{S}}_{(3,4)} \\ -\hat{\mathfrak{S}}_{(1,4)} & -\hat{\mathfrak{S}}_{(2,4)} & -\hat{\mathfrak{S}}_{(3,4)} & 0 \end{pmatrix}.$$

By Theorem 4.12, both of these expressions evaluate to $x_1x_2x_3(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$.

It is an open question whether there exists a simple, general formula for the involution Schubert polynomial $\hat{\mathfrak{S}}[i, j; n]$. If this were known, then the preceding result together with Corollary 4.35(a) would give an effective algorithm for computing $\hat{\mathfrak{S}}_y$ when y is any element of $\mathcal{I}_\infty$.

4.6 Insertion algorithms

In this section we describe an insertion algorithm for involution words, which we call *shifted Coxeter-Knuth insertion*, in order to prove bijectively that $\hat{F}_y$ is Schur P-positive. Conveniently, it turns out that most of the work required to construct this algorithm has already been done elsewhere in the literature. Specifically, we can realize shifted Coxeter-Knuth insertion by restricting the domain of *shifted Hecke insertion* as defined by Patrias and Pylyavskyy [38]. Shifted Hecke insertion is itself a shifted analogue of *Hecke insertion* as introduced in [7].

We sketch the definition of shifted Hecke insertion from [38]. The simplest implementation requires three methods: a bumping rule, an insertion rule, and a final algorithm. In what follows, we write $:=$ to denote the assignment of an expression on the right to a variable on the left.

Algorithm 4.75 (Bumping rule)**.** This algorithm takes a number, a binary digit, and a possibly empty increasing sequence as inputs, and outputs three values of the same types.

Inputs: $p \in \mathbb{P}$, $\texttt{dir} \in \{0, 1\}$, and $M = (m_1 < m_2 < \cdots < m_n) \in \mathbb{P}^n$.

Pseudo-code:

B1: If $p > m_n$ then set $M' := (m_1, m_2, \dots, m_n, p)$ and $q := 0$.
B2: Else if $p = m_n$ then set $M' := M$ and $q := 0$.
B3: Else if $p < m_n$:
  B4: Let $i \in [n]$ be such that $m_{i-1} < p \leq m_i$ where $m_0 := -\infty$.
  B5: If $i = 1$ then set $\texttt{dir} := 1$.
  B6: If $p = m_i$ then set $M' := M$ and $q := m_{i+1}$.
  B7: Else if $p < m_i$ then set $M' := (m_1, \dots, m_{i-1}, p, m_{i+1}, \dots, m_n)$ and $q := m_i$.
B8: Return $(q, \texttt{dir}, M')$.

We denote the output of this algorithm as $\texttt{Bump}(p, \texttt{dir}, M)$.

Informally, when inserting p into M, we bump the first entry larger than p and replace it with p if the resulting sequence is increasing. Here, q is the entry bumped while $\texttt{dir}$ is used in the insertion rule.

Algorithm 4.76 (Insertion rule)**.** This algorithm takes a number and an increasing shifted tableau as inputs, and outputs an index, a binary digit, and an increasing shifted tableau.

Inputs: $p \in \mathbb{P}$ and P an increasing shifted tableau.

Pseudocode:

I1: Set $j := 0$ and $\texttt{dir} := 0$.
I2: While $p > 0$:
  I3: Set $j := j + 1$, and define R and C as the jth row and column of P.
  I4: If $\texttt{dir} = 0$:
    I5: Let $(p, \texttt{dir}, R') := \texttt{Bump}(p, \texttt{dir}, R)$.
    I6: Let P' be the shifted tableau given by replacing the jth row of P by R'.
  I7: Else if $\texttt{dir} = 1$:
    I8: Let $(p, \texttt{dir}, C') := \texttt{Bump}(p, \texttt{dir}, C)$.
    I9: Let P' be the shifted tableau given by replacing the jth column of P by C'.
  I10: If P' is increasing then set $P := P'$.
I11: Return $(j, \texttt{dir}, P)$.

We denote the output of this algorithm as $\texttt{Insert}(p, P)$.

Here, we apply the bumping rule row by row until there is no output ($q = 0$) or $\texttt{dir} = 1$, at which point we begin bumping column by column until there is no output. Using the bumping rule and insertion rule, we now define shifted Hecke insertion. Recall the definitions of $\mathrm{Inc}(\lambda)$ and $\mathrm{SetMT}_n(\lambda)$ from Section 2.5.

Algorithm 4.77 (Shifted Hecke insertion)**.** This algorithm takes a word as input and outputs an increasing shifted tableau and a standard set-valued shifted tableau, both of the same shape.

Inputs: $\mathbf{a} = (a_1, a_1, \dots, a_n) \in \mathbb{P}^n$.

Pseudocode:

S1: Set $P := \varnothing$, $Q := \varnothing$, $\lambda := \varnothing$, and $\texttt{dir} := 0$.

S2: For $i := 1, 2, \dots, n$:

  S3: Set $(j, \texttt{dir}, P) := \texttt{Insert}(a_i, P)$ and $\lambda := \mathrm{shape}(P)$, and let R and C denote the jth row and column of P.

  S4: If $\texttt{dir} = 0$, add i to the last position in column $|R|$ of Q, so that $\mathrm{shape}(Q) = \lambda$.

  S5: If $\texttt{dir} = 1$, add $-i$ to the last position in row $|C|$ of Q, so that $\mathrm{shape}(Q) = \lambda$.

S6: Return $(P, Q) \in \mathrm{Inc}(\lambda) \times \mathrm{SetMT}_n(\lambda)$.

We denote the output of this algorithm as $SH(\mathbf{a}) = (P_{SH}(\mathbf{a}), Q_{SH}(\mathbf{a}))$, and refer to $P_{SH}(\mathbf{a})$ as the *insertion tableau* and $Q_{SH}(\mathbf{a})$ as the *recording tableau*.

In most insertion algorithms, each new entry in the recording tableau goes in the same position as the entry just inserted into the insertion tableau. However, with set-valued recording tableaux, this position need not be a corner. Steps $S4$ and $S5$ resolve this by translating the position of the recording tableau's new entry to the bottom of its column (row insertion) or end of its row (column insertion). See Figure 2 for an example. The following key property is [38, Theorem 5.18].

Theorem 4.78 (Patrias and Pylyavskyy [38])**.** For all $n \in \mathbb{N}$, shifted Hecke insertion is a bijection

$$SH : \mathbb{P}^n \to \bigcup_{\lambda \text{ strict}} \mathrm{Inc}(\lambda) \times \mathrm{SetMT}_n(\lambda).$$

Define a *word* to be a finite sequence of positive integers. The *descent set* of a word $\mathbf{a} = (a_1, a_2, \dots, a_n)$ is $\mathrm{Des}(\mathbf{a}) = \{i \in [n-1] : a_i > a_{i+1}\}$. Let $T \in \mathrm{SetMT}_n(\lambda)$ be a standard set-valued shifted tableau, and say that x appears in position (i, j) of T if $x \in T_{ij}$. Following [12, Section 3.2], we define the *descent set* of T to be the set $\mathrm{Des}(T)$ consisting of all positive integers i which satisfy one of the following mutually exclusive conditions:

- i and $i + 1$ both appear in T and i is in a row strictly above $i + 1$.
- i and $-(i + 1)$ both appear in T.
- $-i$ and $-(i + 1)$ both appear in T and $-(i + 1)$ is in a row strictly above $-i$.
- $-i$ and $-(i + 1)$ appear in the same row of T but not in the same position.

One can check that if every entry of T is a singleton set, then this definition reduces to the descent set of a standard shifted tableau defined in Section 2.5. We recall three properties of shifted Hecke insertion from [12]. The first is noted in the discussion prior to [12, Theorem 3.7]:

Figure 2: We compute $SH(\mathbf{a}) = (P_{SH}(\mathbf{a}), Q_{SH}(\mathbf{a}))$ for $\mathbf{a} = (5, 4, 1, 3, 4, 5, 2, 1, 2)$. For convenience, we write i' in place of $-i$ in all shifted tableaux, and define $\mathbf{a}[i] = (a_1, a_2, \dots, a_i)$.

$$P_{SH}(\mathbf{a}[1]) = \begin{array}{|c|}\hline 5 \\ \hline \end{array} \qquad Q_{SH}(\mathbf{a}[1]) = \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$P_{SH}(\mathbf{a}[2]) = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \qquad Q_{SH}(\mathbf{a}[2]) = \begin{array}{|c|c|}\hline 1 & 2' \\ \hline \end{array}$$

$$P_{SH}(\mathbf{a}[3]) = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline \end{array} \qquad Q_{SH}(\mathbf{a}[3]) = \begin{array}{|c|c|c|}\hline 1 & 2' & 3' \\ \hline \end{array}$$

$$P_{SH}(\mathbf{a}[4]) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \qquad Q_{SH}(\mathbf{a}[4]) = \begin{array}{|c|c|c|}\hline 1 & 2' & 3' \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

$$P_{SH}(\mathbf{a}[5]) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3} \end{array} \qquad Q_{SH}(\mathbf{a}[5]) = \begin{array}{|c|c|c|}\hline 1 & 2' & 3' \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3} \end{array}$$

$$P_{SH}(\mathbf{a}[6]) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array} \qquad Q_{SH}(\mathbf{a}[6]) = \begin{array}{|c|c|c|c|}\hline 1 & 2' & 3' & 6 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

$$P_{SH}(\mathbf{a}[7]) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array} \qquad Q_{SH}(\mathbf{a}[7]) = \begin{array}{|c|c|c|c|}\hline 1 & 2' & 3' & 67' \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{2-3} \end{array}$$

$$P_{SH}(\mathbf{a}[8]) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{2}{c}{} \\ \cline{2-3} \end{array} \qquad Q_{SH}(\mathbf{a}[8]) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2' & 3' & 67' & 8' \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{2}{c}{} \\ \cline{2-3} \end{array}$$

$$P_{SH}(\mathbf{a}[9]) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{2}{c}{} \\ \cline{2-3} \end{array} \qquad Q_{SH}(\mathbf{a}[9]) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2' & 3' & 67' & 8' \\ \hline \multicolumn{1}{c|}{} & 4 & 59' & \multicolumn{2}{c}{} \\ \cline{2-3} \end{array}$$

Proposition 4.79 (see [12])**.** For any word $\mathbf{a}$, we have $\mathrm{Des}(\mathbf{a}) = \mathrm{Des}(Q_{SH}(\mathbf{a}))$.

The following equivalence relation was introduced in [8].

Definition 4.80. The *weak K-Knuth moves* are the relations on words given by

1. $(a, b, \dots) \mathrel{\hat{\equiv}} (b, a, \dots)$
2. $(\dots, a, c, b, \dots) \mathrel{\hat{\equiv}} (\dots, c, a, b, \dots)$
3. $(\dots, b, a, c, \dots) \mathrel{\hat{\equiv}} (\dots, b, c, a, \dots)$
4. $(\dots, a, b, a, \dots) \mathrel{\hat{\equiv}} (\dots, b, a, b, \dots)$
5. $(\dots, a, a, \dots) \mathrel{\hat{\equiv}} (\dots, a, \dots)$

for any integers $a < b < c$, where in these expressions corresponding ellipses denote matching subsequences. Two words $\mathbf{a}$ and $\mathbf{b}$ are *weak K-Knuth equivalent*, denoted $\mathbf{a} \mathrel{\hat{\equiv}} \mathbf{b}$, if there exists a sequence of weak K-Knuth moves transforming $\mathbf{a}$ to $\mathbf{b}$.

The following statement is [12, Corollary 2.18]; its converse does not hold.

Proposition 4.81 (see [12])**.** Let $\mathbf{a}, \mathbf{b}$ be words such that $P_{SH}(\mathbf{a}) = P_{SH}(\mathbf{b})$. Then $\mathbf{a} \mathrel{\hat{\equiv}} \mathbf{b}$.

For an integer-valued tableau T, define $\rho(T)$ as the sequence $(a_1, a_2, \dots, a_k)$ given by reading the rows of T from bottom to top, reading the entries in each row from left to right. For example, the reading word of $P_{SH}(\mathbf{a})$ in Figure 2 is $(3, 5, 1, 2, 3, 4, 5)$. The following is implicit in [12].

Proposition 4.82 (See [12])**.** If λ is a strict partition and $P \in \mathrm{Inc}(\lambda)$, then $P_{SH}(\rho(P)) = P$.

Our first new result in this section in the following observation about weak K-Knuth equivalence.

Lemma 4.83. Let $\mathbf{a} = (a_1, a_2, \dots, a_k)$ and $\mathbf{b} = (b_1, b_2, \dots, b_l)$ be words with $\mathbf{a} \mathrel{\hat{\equiv}} \mathbf{b}$. If $v, w \in S_\infty$ are given by $v = s_{a_1} \circ s_{a_2} \circ \dots \circ s_{a_k}$ and $w = s_{b_1} \circ s_{b_2} \circ \dots \circ s_{b_l}$, then $v^{-1} \circ v = w^{-1} \circ w$.

Proof. We may assume that $\mathbf{a}$ and $\mathbf{b}$ differ by a single weak K-Knuth move. If this move is (2)-(5) in Definition 4.80 then $v = w$. In the remaining case, one can check directly that $v^{-1} \circ v = w^{-1} \circ w$. $\square$

Recall the definition of $\hat{\mathcal{R}}(y)$ from the introduction. Restricting shifted Hecke insertion to this set gives both a shifted variant of Edelman-Greene insertion [11] and a "reduced word" variant of Sagan-Worley insertion [43, 49]. To refer to this map, we introduce the following terminology.

Definition 4.84. For $y \in \mathcal{I}_\infty$ with $n = \hat{\ell}(y)$, *involution Coxeter-Knuth insertion* is the map

$$\hat{\mathcal{R}}(y) \longrightarrow \mathbb{P}^n \xrightarrow{SH} \bigcup_{\lambda \text{ strict}} \mathrm{Inc}(\lambda) \times \mathrm{SetMT}_n(\lambda) \tag{4.5}$$

where the first arrow is the inclusion $(s_{a_1}, s_{a_2}, \dots, s_{a_n}) \mapsto (a_1, a_2, \dots, a_n)$.

With slight abuse of notation, we denote the map (4.5) also by SH. For $\mathbf{a} \in \hat{\mathcal{R}}(y)$, define $\hat{P}(\mathbf{a})$ and $\hat{Q}(\mathbf{a})$ as the increasing/set-valued shifted tableaux such that $(\hat{P}(\mathbf{a}), \hat{Q}(\mathbf{a})) = SH(\mathbf{a})$. In the following results, just to make our notation consistent, redefine $\rho(T)$ for an integer-valued tableau T to be the sequence $(s_{a_1}, s_{a_2}, \dots, s_{a_n})$ where $(a_1, a_2, \dots, a_n)$ is the usual reading word of T.

Lemma 4.85. Let $y \in \mathcal{I}_\infty$ and $\mathbf{a} \in \hat{\mathcal{R}}(y)$. Then $\rho(\hat{P}(\mathbf{a})) \in \hat{\mathcal{R}}(y)$ and $\hat{Q}(\mathbf{a}) \in \bigcup_{\lambda \text{ strict}} \mathrm{SMT}(\lambda)$.

Proof. Note that $\hat{Q}(\mathbf{a})$ is a standard set-valued shifted tableau by Theorem 4.78, and so belongs to $\mathrm{SMT}(\lambda)$ for some λ if and only if all of its entries are singleton sets. Write $\mathbf{a} = (s_{a_1}, s_{a_2}, \dots, s_{a_n})$ and define $\mathbf{b} = (s_{b_1}, s_{b_2}, \dots, s_{b_m}) = \rho(\hat{P}(\mathbf{a}))$. By definition $n \geq m$, and it holds that all entries of $\hat{Q}(\mathbf{a})$ are singletons if and only if $n = m$. Let $v = s_{a_1} \circ s_{a_2} \circ \dots \circ s_{a_n}$ and $w = s_{b_1} \circ s_{b_2} \circ \dots \circ s_{b_m}$. Propositions 4.81 and 4.82 imply that $(a_1, a_2, \dots, a_n) \mathrel{\hat{\equiv}} (b_1, b_2, \dots, b_m)$, so $y = v^{-1} \circ v = w^{-1} \circ w$ by Lemma 4.83. Since $v \in \mathcal{A}(y)$, this implies that $n = m$, so $\mathbf{b} \in \hat{\mathcal{R}}(y)$ as desired. $\square$

Putting together all of the preceding facts, we arrive at the main result of this section.

Theorem 4.86. Let $y \in \mathcal{I}_\infty$. Then involution Coxeter-Knuth insertion is a bijection

$$\hat{\mathcal{R}}(y) \to \bigcup_{\lambda \text{ strict}} \left\{ (P, Q) \in \mathrm{Inc}(\lambda) \times \mathrm{SMT}(\lambda) : \rho(P) \in \hat{\mathcal{R}}(y) \right\}.$$

Note that the union on the right is only over the finite set of strict partitions λ of $\hat{\ell}(y)$.

Proof. The given map is a well-defined injection by Theorem 4.78 and Lemma 4.85. To see that it is surjective, let λ be a strict partition and suppose $(P, Q) \in \mathrm{Inc}(\lambda) \times \mathrm{SMT}(\lambda)$ is such that $\rho(P) \in \hat{\mathcal{R}}(y)$, so that $|\lambda| = \hat{\ell}(y)$. Since SH is a bijection, there exists a unique word $\mathbf{a}$ with $P_{SH}(\mathbf{a}) = P$ and $Q_{SH}(\mathbf{a}) = Q$. No entry of Q contains multiple values, so the length of $\mathbf{a}$ must also be $|\lambda| = \hat{\ell}(y)$. By Propositions 4.81 and 4.82 and Lemma 4.83, replacing the entries of $\mathbf{a}$ by simple transpositions therefore gives an element of $\hat{\mathcal{R}}(y)$ whose image under SH is (P, Q). $\square$

Remark 4.87. These results show that involution Coxeter-Knuth insertion may be defined by a slightly simpler procedure than SH. Since $\hat{Q}(\mathbf{a}) \in \bigcup_\lambda \mathrm{SMT}(\lambda)$ for $\mathbf{a} \in \hat{\mathcal{R}}(y)$, when computing involution Coxeter-Knuth insertion the following holds: (1) step B2 is superfluous in the Bumping rule, (2) in step I10 of the Insertion rule, P' is always increasing, and (3) in steps S4/5 of Shifted Hecke insertion, the last position in column/row j is also the last position in its respective row/column.

Example 4.88. For the involution word $\mathbf{a} = (s_3, s_5, s_4, s_1, s_2, s_3) \in \mathcal{R}(246135) \subset \hat{\mathcal{R}}(456123)$, the sequence of tableaux obtained by involution Coxeter-Knuth insertion is as follows:

$$\begin{ytableau} 3 \end{ytableau} \to \begin{ytableau} 3 & 5 \end{ytableau} \to \begin{ytableau} 3 & 4 \\ \none & 5 \end{ytableau} \to \begin{ytableau} 1 & 3 & 4 \\ \none & 5 \end{ytableau} \to \begin{ytableau} 1 & 2 & 4 \\ \none & 3 & 5 \end{ytableau} \to \begin{ytableau} 1 & 2 & 3 \\ \none & 3 & 4 \\ \none & \none & 5 \end{ytableau} = P_{SH}(\mathbf{a})$$

$$\begin{ytableau} 1 \end{ytableau} \to \begin{ytableau} 1 & 2 \end{ytableau} \to \begin{ytableau} 1 & 2 \\ \none & 3 \end{ytableau} \to \begin{ytableau} 1 & 2 & 4' \\ \none & 3 \end{ytableau} \to \begin{ytableau} 1 & 2 & 4' \\ \none & 3 & 5' \end{ytableau} \to \begin{ytableau} 1 & 2 & 4' \\ \none & 3 & 5' \\ \none & \none & 6 \end{ytableau} = Q_{SH}(\mathbf{a}).$$

As a corollary, we get a new proof that $\hat{F}_y$ is Schur P-positive.

Corollary 4.89. Let $y \in \mathcal{I}_\infty$. Then $\hat{F}_y = \sum_\lambda \beta_{y,\lambda} P_\lambda$ where the sum is over all strict partitions λ and $\beta_{y,\lambda}$ is the number of increasing shifted tableaux P of shape λ with $\rho(P) \in \hat{\mathcal{R}}(y)$.

Proof. Since $\hat{F}_y = \sum_{\mathbf{a} \in \hat{\mathcal{R}}(y)} f_{\mathbf{a}} = \sum_{\mathbf{a} \in \hat{\mathcal{R}}(y)} f_{[n-1]-\mathrm{Des}(\mathbf{a})}$ for $n = \hat{\ell}(y)$, where $\mathrm{Des}(\mathbf{a})$ is defined in the usual way, the result is immediate from Propositions 2.31 and 4.79 and Theorem 4.86. $\square$

Corollary 4.90. Involution Coxeter-Knuth insertion is a bijection $\hat{\mathcal{R}}(w_n) \to \mathrm{SMT}(\hat{\delta}_n)$ for each $n \in \mathbb{N}$, where $\hat{\delta}_n$ denotes the strict partition $(n-1, n-3, n-5, \dots)$.

Proof. This follows from Theorem 4.86 and Corollary 4.89 since $\hat{F}_{w_n} = P_{(n-1,n-3,n-5,\dots)}$. $\square$

Define *shifted Coxeter-Knuth equivalence* to be the equivalence relation generated by (1)-(4) in Definition 4.80. Note that for involution words, it necessarily holds that $b \in \{a+1, a-1\}$ in relation (4). We conjecture this analogue of both [11, Theorem 6.24] and [43, Theorem 7.2].

Conjecture 4.91. Two involution words $\mathbf{a}$, $\mathbf{b}$ are shifted Coxeter-Knuth equivalent if and only if they have the same insertion tableau under the map SH, that is, $P(\mathbf{a}) = P(\mathbf{b})$.

If this conjecture were true, then one would be able to apply the approach outlined in [16] to relate shifted Coxeter-Knuth insertion to involution Little bumps, as defined in [15].

5 Schur P-positivity in the fixed-point-free case

Most of the results in Section 4 have analogues for the symmetric functions $\hat{F}^{\mathtt{FPF}}_z$, which we present in this section. Curiously, some statements here seem to be more difficult to prove than their predecessors, despite the poset $(\mathcal{F}_{\mathbb{Z}}, <)$ being in many ways less complicated than $(\mathcal{I}_{\mathbb{Z}}, <)$.

5.1 FPF-Grassmannian involutions

Our first task is to give a fixed-point-free analogue of Theorem 4.24. For this, we must identify a sufficiently general class of "Grassmannian" elements in $\mathcal{F}_{\mathbb{Z}}$ for which $\hat{F}^{\mathtt{FPF}}_z$ is a Schur P-function. We begin by reviewing the following notions of diagrams and codes from [13, Section 3.2].

Definition 5.1. The *(FPF-involution) diagram* of $z \in \mathcal{F}_\infty$ is the set $\hat{D}_{\mathtt{FPF}}(z)$ whose elements are the pairs $(i,j) \in \mathbb{P} \times \mathbb{P}$ with $j < i < z(j)$ and $j < z(i)$.

Note that $\hat{D}_{\mathtt{FPF}}(z) = \{(i, z(j)) : (i,j) \in \mathrm{Inv}_{\mathtt{FPF}}(z),\ z(j) < i\}$, with $\mathrm{Inv}_{\mathtt{FPF}}(z)$ as in Section 2.4.

Definition 5.2. The *(FPF-involution) code* of $z \in \mathcal{F}_\infty$ is the sequence $\hat{c}_{\mathtt{FPF}}(z) = (c_1, c_2, \dots)$ in which c_i is the number of positions in the ith row of $\hat{D}_{\mathtt{FPF}}(z)$.

Define $\hat{D}_{\mathtt{FPF}}(z) = \hat{D}_{\mathtt{FPF}}(\iota(z))$ and $\hat{c}_{\mathtt{FPF}}(z) = \hat{c}_{\mathtt{FPF}}(\iota(z))$ for $z \in \mathcal{F}_n$ when $n \in 2\mathbb{P}$; then $\hat{D}_{\mathtt{FPF}}(z)$ is the subset of positions in $D(z)$ strictly below the diagonal. The code $\hat{c}_{\mathtt{FPF}}(z)$ may be recovered from $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ as the exponent of the lexicographically smallest nonzero monomial [13, Proposition 3.14].

Example 5.3. If $z = (1,4)(2,5)(3,6)$ then $\hat{D}_{\mathtt{FPF}}(z) = \{(2,1),(3,1),(3,2)\}$ and $\hat{c}_{\mathtt{FPF}}(z) = (0,1,2)$.

A fixed-point-free involution z in $\mathcal{F}_n$ or $\mathcal{F}_\infty$ is *FPF-dominant* if $\{(i-1,j) : (i,j) \in \hat{D}_{\mathtt{FPF}}(z)\}$ is the transpose of the shifted diagram of a strict partition. (Note that $\hat{D}_{\mathtt{FPF}}(z)$ contains no positions in its first row.) In contrast to the notion of dominance for elements of $\mathcal{I}_\infty$, we do not know of any pattern avoidance condition characterizing the FPF-dominant elements of $\mathcal{F}_\infty$.

Example 5.4. Both $(1,7)(2,4)(3,5)(6,8)$ and $(1,7)(2,5)(3,4)(6,8)$ are FPF-dominant but not dominant. The only elements of $\mathcal{F}_{2k}$ for $k \in \mathbb{P}$ which are 132-avoiding are those of the form $(1,k+1)(2,k+2)\cdots(k,2k)$. These dominant involutions are also FPF-dominant.

We do have an analogue of Theorem 4.12, however.

Theorem 5.5 (See [13])**.** If $z \in \mathcal{F}_\infty$ is FPF-dominant then $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \prod_{(i,j) \in \hat{D}_{\mathtt{FPF}}(z)} (x_i + x_j)$.

Proof sketch. This a slightly stronger statement than the result we proved as [13, Theorem 3.26], which gave the same formula but only for the dominant (i.e., 132-avoiding) elements of $\mathcal{F}_n$. However, the more general formula follows by the same argument with minor changes. ☐

The next lemma is a consequence of [14, Theorem 6.22].

Lemma 5.6 (See [14])**.** Suppose $z \in \mathcal{F}_\infty$ and $\mathrm{Cyc}_{\mathbb{P}}(z) = \{(a_i, b_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \cdots$. The lexicographically minimal element of $\mathcal{A}_{\mathtt{FPF}}(z)$ is the inverse of the permutation whose one-line representation is $a_1 b_1 a_2 b_2 a_3 b_3 \cdots$.

The same statement, but with "$a_1 b_1 a_2 b_2 a_3 b_3 \cdots$" replaced by "$a_1 b_1 a_2 b_2 \cdots a_k b_k$," describes the minimal element of $\mathcal{A}_{\mathtt{FPF}}(z) = \mathcal{A}_{\mathtt{FPF}}(\iota(z))$ for $z \in \mathcal{F}_{2k}$ when $k \in \mathbb{P}$. In either case, we denote this lexicographically minimal atom as $\beta_{\min}(z) \in \mathcal{A}_{\mathtt{FPF}}(z)$.

Example 5.7. If $z = (1,4)(2,3) \in \mathcal{F}_4$ then $a_1b_1a_2b_2 = 1423$ and $\beta_{\min}(z) = 1423^{-1} = 1342$.

Typically $\hat{D}_{\mathtt{FPF}}(z) \neq D(\beta_{\min}(z))$, but the following holds by [13, Lemma 3.8].

Lemma 5.8 (See [13])**.** If $z \in \mathcal{F}_\infty$ then $\hat{c}_{\mathtt{FPF}}(z) = c(\beta_{\min}(z))$.

Say that a pair $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is an *FPF-visible inversion* of $z \in \mathcal{F}_\mathbb{Z}$ if $i < j$ and $z(j) < \min\{i, z(i)\}$. We omit the proof of the following result, which is similar to that of Lemma 4.15.

Lemma 5.9. The set of FPF-visible inversions of $z \in \mathcal{F}_\infty$ is $\mathrm{Inv}(\beta_{\min}(z))$.

We say that $i \in \mathbb{Z}$ is an *FPF-visible descent* of $z \in \mathcal{F}_\mathbb{Z}$ if $(i, i+1)$ is an FPF-visible inversion, and define $\mathrm{Des}_V^{\mathtt{FPF}}(z) = \{s_i : i \in \mathbb{Z} \text{ is an FPF-visible descent of } z\ \} \subset \mathrm{Des}_R^{\mathtt{FPF}}(z)$. The following results are similar to Lemmas 4.17 and 4.18 and have nearly the same proofs, which we omit.

Lemma 5.10. If $z \in \mathcal{F}_\infty$ then $\mathrm{Des}_V^{\mathtt{FPF}}(z) = \mathrm{Des}_R(\beta_{\min}(z))$.

Lemma 5.11. If $z \in \mathcal{F}_\infty$ then the ith row of $\mathrm{Ess}(\hat{D}_{\mathtt{FPF}}(z))$ is nonempty if and only if $s_i \in \mathrm{Des}_V^{\mathtt{FPF}}(z)$.

These preliminaries lead to the following analogue of Proposition-Definition 4.20.

Proposition 5.12. For $z \in \mathcal{F}_\infty$ and $n \in \mathbb{P}$, the following are equivalent:

(a) $\mathrm{Des}_V^{\mathtt{FPF}}(z) = \{s_n\}$.

(b) $\hat{c}_{\mathtt{FPF}}(z)$ has the form $(0, c_2, \dots, c_n, 0, 0, \dots)$ where $c_2 \leq \dots \leq c_n \neq 0$.

(c) $\mathrm{Ess}(\hat{D}_{\mathtt{FPF}}(z))$ is nonempty and contained in $\{(n,j) : j \in \mathbb{P}\}$.

(d) The lexicographically minimal atom $\beta_{\min}(z) \in \mathcal{A}_{\mathtt{FPF}}(z)$ is n-Grassmannian.

Proof. The result follows from Proposition-Definition 4.6 and Lemmas 5.8, 5.10, and 5.11. $\square$

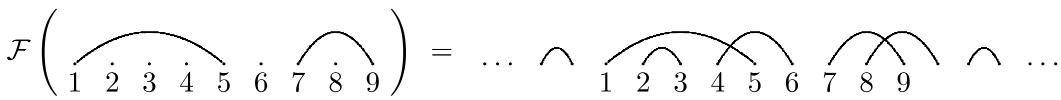
From an algebraic perspective, the conditions in the preceding result give a natural concept of a "Grassmannian" fixed-point-free involution. It turns out, however, that we need a slightly more general definition to extend the constructions in Section 4.2. To describe this, we introduce maps

$$\mathcal{F} : \mathcal{I}_\mathbb{Z} \to \mathcal{F}_\mathbb{Z} \qquad \text{and} \qquad \mathcal{I} : \mathcal{F}_\mathbb{Z} \to \mathcal{I}_\mathbb{Z}$$

via the following pair of definitions.

Definition 5.13. For $y \in \mathcal{I}_\mathbb{Z}$, let m be any even integer with $m < i$ for all $i \in \mathrm{supp}(y)$, write ϕ for the order-preserving bijection $\mathbb{Z} \to \mathbb{Z} \setminus \mathrm{supp}(y)$ with $\phi(0) = m$, and define $\mathcal{F}(y)$ as the unique element of $\mathcal{F}_\mathbb{Z}$ with $\mathcal{F}(y)(i) = y(i)$ for $i \in \mathrm{supp}(y)$ and $\mathcal{F}(y) \circ \phi = \phi \circ \Theta$.

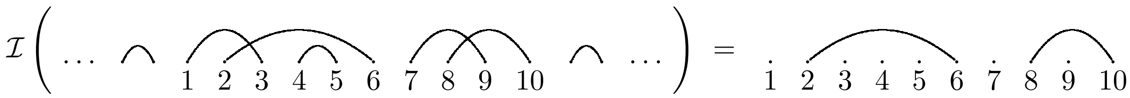
The map $\mathcal{F} : \mathcal{F}_\mathbb{Z} \to \mathcal{I}_\mathbb{Z}$ extends the inclusion (2.3) in the sense that $\mathcal{F}(y) = \iota(y)$ for $y \in \mathcal{F}_n$. Observe that $\mathcal{F}(z)$ is formed from z by turning every pair of adjacent fixed points into a cycle; there are two ways of doing this, and we choose the way which makes $(2i-1, 2i)$ into a cycle for all sufficiently large $i \in \mathbb{Z}$. For example, we have



The map $\mathcal{F}$ has a right inverse given by the following.

Definition 5.14. For $z \in \mathcal{F}_\mathbb{Z}$, define $\mathcal{I}(z) \in \mathcal{I}_\mathbb{Z}$ as the involution whose nontrivial cycles are precisely the pairs $(p, q) \in \mathrm{Cyc}_\mathbb{Z}(z)$ for which there exists $(a, b) \in \mathrm{Cyc}_\mathbb{Z}(z)$ with $p < b < q$.

Equivalently, $\mathcal{I}(z)$ is the involution which restricts to the same map as z on its support, and whose fixed points are the integers $i \in \mathbb{Z}$ such that $\max\{i, z(i)\} < z(j)$ for all $j \in \mathbb{Z}$ with $\min\{i, z(i)\} < j < \max\{i, z(i)\}$. For example, we have



We see in these examples that $\mathcal{I}$ and $\mathcal{F}$ restrict to maps $\mathcal{F}_\infty \to \mathcal{I}_\infty$ and $\mathcal{I}_\infty \to \mathcal{F}_\infty$, respectively.

Proposition 5.15. Let $z \in \mathcal{F}_\mathbb{Z}$. Then $\mathcal{I}(z) = 1$ if and only if $z = \Theta$.

Proof. If $z \neq \Theta$ and i is the largest integer such that $i < z(i) \neq i+1$, then necessarily $z(i+1) < z(i)$, so $(i, z(i))$ is a nontrivial cycle of $\mathcal{I}(z)$, which is therefore not the identity. $\square$

Proposition 5.16. The composition $\mathcal{F} \circ \mathcal{I}$ is the identity map $\mathcal{F}_\mathbb{Z} \to \mathcal{F}_\mathbb{Z}$.

Proof. Fix $z \in \mathcal{I}_\infty$ and let $\mathcal{C}$ be the set of cycles $(p, q) \in \mathrm{Cyc}_\mathbb{Z}(z)$ such that p and q are fixed points in $\mathcal{I}(z)$. By definition, if (p, q) and (p', q') are distinct elements of $\mathcal{C}$ then $p < q < p' < q'$ or $p' < q' < p < q$. The claim that $\mathcal{F} \circ \mathcal{I}(z) = z$ is a straightforward consequence of this fact. $\square$

The correct fixed-point-free variant of Definition 4.19 is now as follows.

Definition 5.17. An involution $z \in \mathcal{F}_\mathbb{Z}$ is *FPF-Grassmannian* if $\mathcal{I}(z) \in \mathcal{I}_\mathbb{Z}$ is I-Grassmannian.

This definition is equivalent to the one in the introduction, once we define an element of $\mathcal{F}_n$ to be FPF-Grassmannian whenever its image under $\iota : \mathcal{F}_n \to \mathcal{F}_\infty$ is.

Remark 5.18. The sequence $(g^{\mathtt{FPF}}_{2n})_{n\geq 1} = (1, 3, 12, 41, 124, 350, 952, 2540, \dots)$ with $g^{\mathtt{FPF}}_n$ the number of FPF-Grassmannian elements of $\iota(\mathcal{F}_n) \subset \mathcal{F}_\mathbb{Z}$ seems unrelated to any existing sequence in [45].

The FPF-Grassmannian elements of $\mathcal{F}_\mathbb{Z}$ are the fixed-point-free involutions to which we can associate a meaningful shape. Suppose $z \in \mathcal{F}_\infty - \{\Theta\}$ is such that $\mathcal{I}(z)$ is n-I-Grassmannian, so that by Proposition-Definition 4.20

$$\mathcal{I}(z) = (\phi_1, n+1)(\phi_2, n+2) \cdots (\phi_r, n+r) \in \mathcal{I}_\infty$$

for integers $n, r \in \mathbb{P}$ and $1 \leq \phi_1 < \cdots < \phi_r \leq n$. We define the *shape* of z to be the strict partition

$$\nu(z) = (n - \phi_1, n - \phi_2, \dots, n - \phi_r).$$

Note that $\nu(z) + 1^r$ is the shape $\mu(y)$ of $y = \mathcal{I}(z)$ from Section 4.1, where $1^r = (1, 1, \dots, 1) \in \mathbb{Z}^r$. For the next lemma, recall the definition of the operators $\pi_{b,a}$ from Section 2.1.

Lemma 5.19. In the notation just given, we have $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \pi_{\phi_1,1}\pi_{\phi_2,2} \cdots \pi_{\phi_r,r}\left(x^{\nu(z)} G_{r,n}\right)$.

Proof. Note that if $f_1 < f_2 < \cdots < f_k$ are the fixed points of $\mathcal{I}(z)$ in $\{1, 2, \dots, n\}$, then k is necessarily even and $(f_1, f_2), (f_3, f_4), \dots, (f_{k-1}, f_k) \in \mathrm{Cyc}_\mathbb{Z}(z)$. It follows that if $\phi_i = i$ for all $i \in [r]$ then z is FPF-dominant with diagram $\hat{D}_{\mathtt{FPF}}(z) = \{(i + j, i) : i \in [r] \text{ and } j \in [n - i]\}$, in which case the lemma reduces to the identity $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = x_1^{n-1} x_2^{n-2} \cdots x_r^{n-r} G_{r,n}$ which is evident from Theorem 5.5. If $i < \phi_i$ for some $i \in [r]$, then one derives the result by induction as in the proof of Lemma 4.23, but using (2.4) in place of (2.2). $\square$

If $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ is FPF-Grassmannian, then $\mathcal{I}(z \gg N)$ is n-I-Grassmannian for some $n \in \mathbb{P}$ and $N \in 2\mathbb{N}$, and we define $\nu(z) = \nu(z \gg N)$. We also set $\nu(\Theta) = \emptyset = (0, 0, \dots)$.

Theorem 5.20. If $z \in \mathcal{F}_\mathbb{Z}$ is FPF-Grassmannian, then $\hat{F}^{\mathtt{FPF}}_z = P_{\nu(z)}$.

Proof. Since $\hat{F}^{\mathtt{FPF}}_z = \hat{F}^{\mathtt{FPF}}_{z \gg N}$ for all $N \in 2\mathbb{Z}$, we may assume that $z \in \mathcal{F}_\infty$ and that $\mathcal{I}(z)$ is n-I-Grassmannian. If $\nu(z)$ has r parts, then Lemmas 2.5 and 5.19 imply that $\pi_{w_n} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \pi_{w_n}\left(x^{\nu(z)} G_{r,n}\right)$ for all $n \geq r$, and the theorem follows by taking the limit as $n \to \infty$. $\square$

We conclude this section with a result clarifying the relationship between FPF-Grassmannian involutions and the elements of $\mathcal{F}_\mathbb{Z}$ with at most one FPF-visible descent.

Lemma 5.21. Fix $z \in \mathcal{F}_\infty$. Let $E = \{i \in \mathbb{P} : |z(i) - i| \neq 1\}$ and define $y \in \mathcal{I}_\infty$ as the involution with $y(i) = z(i)$ if $i \in E$ and $y(i) = i$ otherwise. Then $z = \mathcal{F}(y)$ and $\mathrm{Des}^{\mathtt{FPF}}_V(z) = \mathrm{Des}_V(y)$.

Proof. It is evident that $z = \mathcal{F}(y)$. Suppose $s_i \in \mathrm{Des}_V(y)$. Since $y(i+1) \neq i$ for all $i \in \mathbb{P}$ by definition, we must have $y(i+1) < \min\{i, y(i)\}$, so $i + 1 \in E$, and therefore either $i \in E$ or $z(i) = i - 1$. It follows in either case that $z(i+1) < \min\{i, z(i)\}$ so $s_i \in \mathrm{Des}^{\mathtt{FPF}}_V(z)$. Conversely, suppose $s_i \in \mathrm{Des}^{\mathtt{FPF}}_V(z)$ so that $i + 1 \in E$. If $i \in E$ then $s_i \in \mathrm{Des}_V(y)$ holds immediately, and if $i \notin E$ then $z(i+1) < z(i) = i-1$, in which case $y(i+1) = z(i+1) < i = y(i)$ so $s_i \in \mathrm{Des}_V(y)$. $\square$

Proposition 5.22. An involution $z \in \mathcal{F}_\mathbb{Z}$ has $|\mathrm{Des}^{\mathtt{FPF}}_V(z)| \leq 1$ if and only if z is FPF-Grassmannian and $\nu(z)$ is a strict partition whose consecutive parts each differ by odd numbers.

Proof. We may assume that $z \in \mathcal{F}_\infty - \{\Theta\}$. It is straightforward to check that if z is FPF-Grassmannian and the consecutive parts of $\nu(z)$ differ by odd numbers then $|\mathrm{Des}^{\mathtt{FPF}}_V(z)| \leq 1$. For the converse statement, define $y \in \mathcal{I}_\infty$ as in Lemma 5.21 so that $z = \mathcal{F}(y)$. By Proposition 4.20 and Lemma 5.21, we have $\mathrm{Des}^{\mathtt{FPF}}_V(z) = \{s_n\}$ if and only if $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r)$ for integers $r \in \mathbb{P}$ and $0 = \phi_0 < \phi_1 < \phi_2 < \cdots < \phi_r \leq n$. If y has this form then $\phi_i - \phi_{i-1}$ is odd by construction and either $\mathcal{I}(z) = y$ or $\mathcal{I}(z) = (\phi_2, n+2)(\phi_3, n+3)\cdots(\phi_r, n+r)$, so z is FPF-Grassmannian and the consecutive parts of $\nu(z)$ differ by odd numbers. $\square$

Remark 5.23. Using the previous result, one can show that the number k_n of elements of $\mathcal{F}_n$ with at most one FPF-visible descent satisfies the recurrence $k_{2n} = 2k_{2n-2} + 2n - 3$ for $n \geq 2$. The corresponding sequence $(k_{2n})_{n \geq 1} = (1, 3, 9, 23, 53, 115, 241, 495, \dots)$ is [45, A183155].

5.2 Schur P-positivity

Here, we describe a fixed-point-free version of the tree $\hat{\mathfrak{T}}(z)$ from Section 4.2. As usual, we order $\mathbb{Z} \times \mathbb{Z}$ lexicographically. Recall that $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is an FPF-visible inversion of $z \in \mathcal{F}_\mathbb{Z}$ if $i < j$ and $z(j) < \min\{i, z(i)\}$, and that $i \in \mathbb{Z}$ is an FPF-visible descent of z if $(i, i+1)$ is an FPF-visible inversion. By Lemma 5.10, every $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ has at least one FPF-visible descent.

Lemma 5.24. Let $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ and suppose $j \in \mathbb{Z}$ is the smallest integer such that $z(j) < j - 1$. Then $j - 1$ is the minimal FPF-visible descent of z.

Proof. By hypothesis, either $z(j) < j - 2 = z(j-1)$ or $z(j) < j - 1 < z(j-1)$, so $j - 1$ is an FPF-visible descent of z. If $k-1$ is another FPF-visible descent of z, then $z(k) < k-1$ so $j \leq k$. $\square$

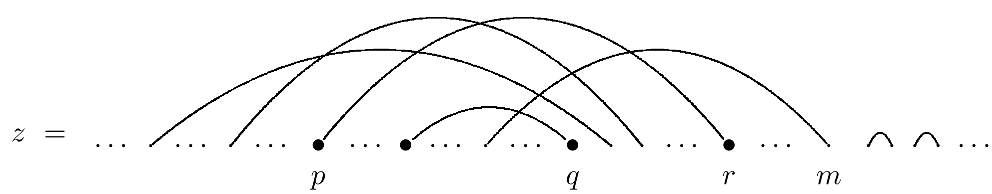
Lemma 5.25. Suppose $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ is the maximal FPF-visible inversion of $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$. Let m be the largest even integer such that $z(m) \neq m - 1$. Then q is the maximal FPF-visible descent of z while r is the maximal integer with $z(r) < \min\{q, z(q)\}$, and $z(q+1) < z(q+2) < \cdots < z(m) \leq q$. In addition, we have either (a) $z(q) < q < r \leq m$ or (b) $q < z(q) = r + 1 = m$.

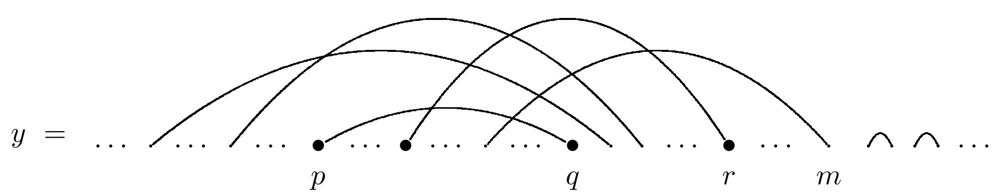
Proof. Everything but the last sentence in this result follows as in the proof of Lemma 4.29, *mutatis mutandis*. It remains to show that if $q < z(q)$ then $z(q) = r + 1 = m$. Assume $q < z(q)$. It cannot hold that $r < z(q) - 1$, since then either $(q, r+1)$ or $(r+1, z(q))$ would be an FPF-visible inversion of z, contradicting the maximality of (q, r). It also cannot hold that $z(q) < r$, as then $(z(q), r)$ would be an FPF-visible inversion of z. Hence $r = z(q) - 1$. If $j > z(q)$, then since $z(i) < q$ for all $q < i < z(q)$ and since $(z(q), j)$ cannot be an FPF-visible inversion of z, we must have $z(j) > z(q)$. From this observation and the fact that z has no FPF-visible descents greater than q, we deduce that $z(j) = \Theta(j)$ for all $j > z(q)$, which implies that $z(q) = m$ as required. □

Definition 5.26. Let $\eta_{\mathsf{FPF}} : \mathcal{F}_\mathbb{Z} - \{\Theta\} \to \mathcal{F}_\mathbb{Z}$ be the map $\eta_{\mathsf{FPF}} : z \mapsto (q, r)z(q, r)$ where (q, r) is the maximal FPF-visible inversion of z.

Remark 5.27. Suppose $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ has maximal FPF-visible inversion (q, r). Let $p = z(r)$ and $y = \eta_{\mathsf{FPF}}(z) = (q, r)z(q, r)$ and write m for the largest even integer such that $z(m) \neq m - 1$. The two cases of Lemma 5.25 correspond to the following pictures:

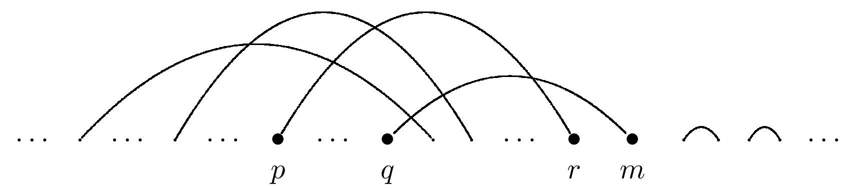
(a) If $z(q) < q < r \leq m$ then y and z may be represented as

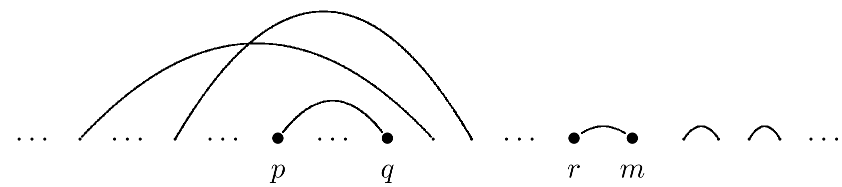




This case is essentially identical to Remark 4.31(a), except that on the right hand side there is an infinite series of cycles of the form $(i, i+1)$ rather than fixed points. We have $z(q+1) < z(q+2) < \cdots < z(r) < z(q)$, and if $r < m$ then $z(q) < z(r+1) < z(r+2) < \cdots < z(m) < q$.

(b) If $q < z(q) = r+1 = m$ then y and z may be represented as

$z =$ 

$y =$ 

Here, we have $z(q+1) < z(q+2) < \cdots < z(r) = p < q$, so $z(i) < q$ whenever $p < i < q$.

Recall the definition of $\beta_{\min}(z)$ from Lemma 5.6.

Proposition 5.28. If (q,r) is the maximal FPF-visible inversion of $z \in \mathcal{F}_\infty - \{\Theta\}$ and $w = \beta_{\min}(z)$ is the minimal element of $\mathcal{A}_{\mathsf{FPF}}(z)$, then $w(q,r) = \beta_{\min}(\eta_{\mathsf{FPF}}(z))$ is the minimal atom of $\eta_{\mathsf{FPF}}(z)$.

Proof. Let $\mathrm{Cyc}_{\mathbb{P}}(z) = \{(a_i, b_i) : i \in \mathbb{P}\}$ and $\mathrm{Cyc}_{\mathbb{P}}(\eta_{\mathsf{FPF}}(z)) = \{(c_i, d_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \dots$ and $c_1 < c_2 < \dots$. By Lemma 5.6, it suffices to show that interchanging q and r in the word $a_1b_1a_2b_2\cdots$ gives $c_1d_1c_2d_2\cdots$, which is straightforward from Remark 5.27. $\square$

Recall the definition of the sets $\hat{\Psi}^+(y,r)$ and $\hat{\Psi}^-(y,r)$ from (3.2).

Lemma 5.29. If $z \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ has maximal FPF-visible inversion (q,r) then $\hat{\Psi}^+(\eta_{\mathsf{FPF}}(z), q) = \{z\}$.

Proof. This holds by Proposition 3.13, Remark 5.27, and the definitions of $\eta_{\mathsf{FPF}}(z)$ and $\hat{\Psi}^+(y,q)$. $\square$

The fixed-point-free analogue of $\hat{\mathfrak{T}}_1(z)$ is given as follows. For $z \in \mathcal{F}_{\mathbb{Z}}$, let

$$\hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z) = \begin{cases} \varnothing & \text{if } z \text{ is FPF-Grassmannian} \\ \hat{\Psi}^-(y,p) & \text{otherwise} \end{cases}$$

where in the second case, we define $y = \eta_{\mathsf{FPF}}(z)$ and $p = y(q)$ where q denotes the maximal FPF-visible descent of z. This construction leads to the following variant of Definition 4.34.

Definition 5.30. The *FPF-involution Lascoux-Schützenberger tree* $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ of $z \in \mathcal{F}_{\mathbb{Z}}$ is the tree with root z, in which the children of any vertex $v \in \mathcal{F}_{\mathbb{Z}}$ are the elements of $\hat{\mathfrak{T}}_1^{\mathsf{FPF}}(v)$.

For $z \in \mathcal{F}_n$ we define $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z) = \hat{\mathfrak{T}}^{\mathsf{FPF}}(\iota(z))$. A given involution is allowed to correspond to more than one vertex in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$. All vertices v in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ satisfy $\hat{\ell}_{\mathsf{FPF}}(v) = \hat{\ell}_{\mathsf{FPF}}(z)$ construction, so if $z \neq \Theta$ then Θ is not a vertex in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$. An example tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ is shown in Figure 3.

Corollary 5.31. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ is a fixed-point-free involution which is not FPF-Grassmannian, whose maximal FPF-visible descent is $q \in \mathbb{Z}$. The following identities then hold:

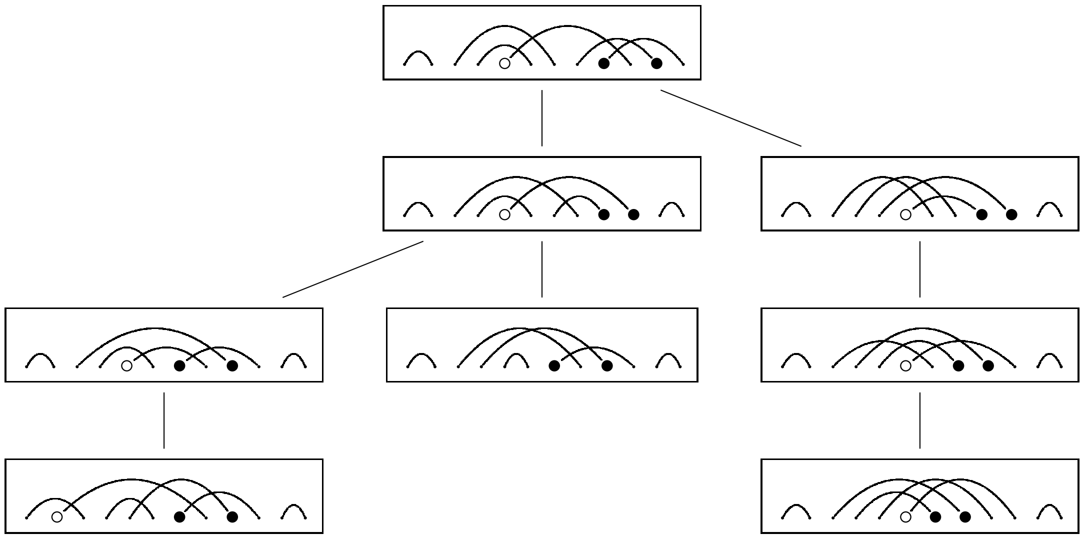


Figure 3: The tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ for $z = (1,2)(3,7)(4,6)(5,10)(8,11)(9,12) \in \mathcal{F}_{12} \hookrightarrow \mathcal{F}_{\mathbb{Z}}$. We draw all vertices as elements of $\mathcal{F}_{12} \subset \mathcal{I}_{12}$ for convenience. The maximal FPF-visible inversion of each vertex is marked with $\bullet$, and the minimal FPF-visible descent is marked with $\circ$ (when this is not also maximal). By Theorem 5.20 and Corollary 5.31, we have $\hat{F}^{\mathsf{FPF}}_z = P_{(5,2)} + P_{(4,3)} + P_{(4,2,1)}$.

(a) $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = (x_p + x_q)\hat{\mathfrak{S}}^{\mathsf{FPF}}_y + \sum_{v \in \hat{\mathfrak{T}}^{\mathsf{FPF}}_1(z)} \hat{\mathfrak{S}}^{\mathsf{FPF}}_v$ where $y = \eta_{\mathsf{FPF}}(z)$ and $p = y(q)$.

(b) $\hat{F}^{\mathsf{FPF}}_z = \sum_{v \in \hat{\mathfrak{T}}^{\mathsf{FPF}}_1(z)} \hat{F}^{\mathsf{FPF}}_v$.

Proof. The result follows from Theorems 3.15 and 3.17 and Lemma 5.29. □

In Section 4.2, we proved that $\hat{\mathfrak{T}}(z)$ was finite by showing that the intervals between the minimal and maximal visible descents of each vertex form a descending chain as one moves down the tree. This "descending interval" property not hold for FPF-visible descents in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$: a child in this tree may have strictly smaller FPF-visible descents than its parent. However, a similar property does hold if we instead consider the visible descents of the image of $z \in \mathcal{F}_{\mathbb{Z}}$ under the map $\mathcal{I} : \mathcal{F}_{\mathbb{Z}} \to \mathcal{I}_{\mathbb{Z}}$ from Definition 5.14. In the following lemmas, we note a few properties of such descents.

Lemma 5.32. Let $z \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ and suppose $(i,j) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ is the cycle with j minimal such that $i < b < j$ for some $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$. Then $j-1$ is the minimal visible descent of $\mathcal{I}(z)$.

Proof. The claim follows by Lemma 4.28 since j is the smallest integer such that $\mathcal{I}(z)(j) < j$. □

Lemma 5.33. If $z \in \mathcal{F}_{\mathbb{Z}}$ then $i \in \mathbb{Z}$ is a visible descent of $\mathcal{I}(z)$ if and only if one of these holds:

(a) $z(i+1) < z(i) < i$.

(b) $z(i) < z(i+1) < i$ and $\{t \in \mathbb{Z} : z(i) < t < i\} \subset \{z(t) : i < t\}$.

(c) $z(i+1) < i < z(i)$ and $\{t \in \mathbb{Z} : z(i+1) < t < i+1\} \not\subset \{z(t) : i+1 < t\}$.

Proof. It is straightforward to check that $i \in \mathbb{Z}$ is a visible descent of $\mathcal{I}(z)$ if and only if either (a) $z(i+1) < z(i) < i$; (b) $z(i) < z(i+1) < i$ and i is a fixed point of $\mathcal{I}(z)$; or (c) $z(i+1) < i < z(i)$ and $i+1$ is not a fixed point of $\mathcal{I}(z)$. The given conditions are equivalent to these statements. $\square$

Corollary 5.34. Let $y, z \in \mathcal{F}_{\mathbb{Z}}$ and let $i, j \in \mathbb{Z}$ be integers with $i < j$. Suppose $y(t) = z(t)$ for all integers $t > i$. Then j is a visible descent of $\mathcal{I}(y)$ if and only if j is a visible descent of $\mathcal{I}(z)$.

Proof. By Lemma 5.33, whether or not j is a visible descent of $\mathcal{I}(z)$ depends only on the action of z on integers greater than or equal to j. $\square$

Corollary 5.35. Let $z \in \mathcal{F}_{\mathbb{Z}}$ and suppose i is a visible descent of $\mathcal{I}(z)$. Then either i or $i-1$ is an FPF-visible descent of z. Therefore, if j is the maximal FPF-visible descent of z, then $i \leq j+1$.

Proof. It follows from Lemma 5.33 that i is an FPF-visible descent of z unless $z(i) < z(i+1) < i$ and $\{t \in \mathbb{Z} : z(i) < t < i\} \subset \{z(t) : i < t\}$, in which case $i-1$ is an FPF-visible descent of z. $\square$

The following statement is the first of two key technical lemmas in this section.

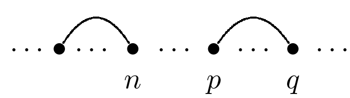
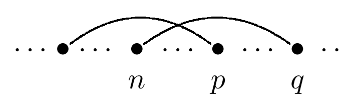
Lemma 5.36. Let $y \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ and $(p,q) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ and suppose $v = (n,p)y(n,p) \in \hat{\Psi}^-(y,p)$.

(a) If $i \in \mathbb{Z} \setminus \{n, y(n), p, q\}$ is such that $\mathcal{I}(y)(i) = i$, then $\mathcal{I}(v)(i) = i$.

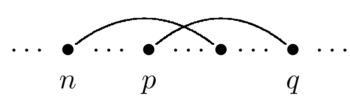
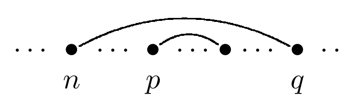
(b) If j and k are the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(v)$ and $j \leq q-1$, then $j \leq k$.

Remark 5.37. Part (b) is false if $j \geq q$: consider $y = (6,7)\Theta(6,7)$ and $(n,p,q) = (2,3,4)$. There is no analogous inequality governing the minimal FPF-visible descents of y and v.

Proof. Since $y \lessdot_{\mathcal{F}} v = (n,p)y(n,p) \in \hat{\Psi}^-(y,p)$, it follows from Proposition 3.13 that either $y(n) < n < p < q$, in which case $n < p < v(p) < q = v(n)$ and y and v correspond to the diagrams

$y =$  and $v =$  (5.1)

or $n < p < y(n) < q$, in which case $n < p < v(p) < q = v(n)$ and we instead have

$y =$  and $v =$  (5.2)

Let $A = \{n, y(n), p, q\} = \{n, p, v(p), q\}$ and note that $y(i) = v(i)$ for all $i \in \mathbb{Z} \setminus A$. Suppose $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ is such that $b \notin A$ and $b < y(i)$ for all $a < i < b$, so that a and b are both fixed points of $\mathcal{I}(y)$. Then (a,b) is also a cycle of v, and to prove part (a) it suffices to check that $b < v(i)$ for all $i \in A$ with $a < i < b$. This holds if $i \in \{n, y(n)\}$ since then $y(i) < v(i)$, and we cannot have $a < q < b$ since $y(q) < q$. Suppose $a < p < b$; it remains to show that $b < v(p)$. Since $b < y(i)$ for all $a < i < b$ by hypothesis, it follows that if y and v are as in (5.1) then $n < a < p < b < q$, and that if y are v are as in (5.2) then $a < p < b < y(n)$. The first of these cases cannot occur in view of Proposition 3.13(a), since $y \lessdot_{\mathcal{F}} v$. In the second case $y(n) = v(p)$ so $b < v(p)$ as needed.

To prove part (b), note that $\Theta \notin \{y, v\}$ so neither $\mathcal{I}(y)$ nor $\mathcal{I}(v)$ is the identity. Let j and k be the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(v)$ and assume $j \leq q-1$. Write S_y for the set of integers $i \in \mathbb{Z} \setminus A$ such that $\mathcal{I}(y)(i) < i$, and let $T_y = S_y \setminus A$ and $U_y = S_y \cap A$. Define S_v, T_v, and

U_v similarly. Lemma 4.28 implies that $j \leq k$ if and only if $\min S_y \leq \min S_v$. Since $j \leq q-1$ we have $\min S_y \leq q$. It follows from part (a) that $T_v \subset T_y$, so $\min T_y \leq \min T_v$.

There are two cases to consider. First suppose $y(n) < n < p < q$ and $v(p) < n < p < q = v(n)$. It is then evident from (5.1) that $\{q\} \subset U_v \subset \{p, q\}$. Since $\min S_y \leq q$ by hypothesis, to prove that $\min S_y \leq \min S_v$ it suffices to show that if $p \in U_v$ then $\min S_y < p$. Since $y \lessdot_{\mathcal{F}} v$, neither y nor v can have any cycles (a, b) with $y(n) < a < p$ and $n < b < p$. It follows that if $p \in U_v$ then y and v share a cycle (a, b) with either (i) $a < b$ and $y(n) < b < n$, or (ii) $a < y(n) < n < b < p$. If (i) occurs then $n \in U_y$ while if (ii) occurs then $\min T_y < p$, so $\min S_y < p$ as desired.

Suppose instead that $n < p < y(n) < q$ and $n < p < v(p) < q = v(n)$. In view of (5.2), we then have $\{q\} \subset U_v \subset \{y(n), q\}$. As $\min S_y \leq q$, to prove that $\min S_y \leq \min S_v$ it now suffices to show that if $y(n) \in U_v$ then $y(n) \in U_y$. This implication is clear from (5.2), since if $y(n) = v(p) \in U_v$ then y and v must share a cycle (a, b) with $a < b$ and $p < b < y(n)$. $\square$

Lemma 5.38. Let $y \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ and $(p, q) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ and suppose $z = (q, r)y(q, r) \in \hat{\Psi}^+(y, q)$. The involution $\mathcal{I}(y)$ has a visible descent less than $q - 1$ if and only if $\mathcal{I}(z)$ does, and in this case the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(z)$ are equal.

Proof. Let $\mathcal{C}_w$ for $w \in \mathcal{F}_{\mathbb{Z}}$ be the set of cycles $(a, b) \in \mathrm{Cyc}_{\mathbb{Z}}(w)$ with $b < q$. By Lemma 5.32, the set $\mathcal{C}_w$ determines whether or not $\mathcal{I}(w)$ has a visible descent less than $q - 1$ and, when this occurs, the value of $\mathcal{I}(w)$'s smallest visible descent. Since $q < r$ we have $\mathcal{C}_y = \mathcal{C}_z$, so the result follows. $\square$

Our second key technical lemma is the following.

Lemma 5.39. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ is not FPF-Grassmannian, so that $\eta_{\mathtt{FPF}}(z) \neq \Theta$. Let (q, r) be the maximal FPF-visible inversion of z and define $y = \eta_{\mathtt{FPF}}(z) = (q, r)z(q, r)$.

(a) The maximal visible descent of $\mathcal{I}(z)$ is q or $q + 1$.

(b) The maximal visible descent of $\mathcal{I}(y)$ is at most q.

(c) The minimal visible descent of $\mathcal{I}(y)$ is equal to that of $\mathcal{I}(z)$, and is at most $q - 1$.

Proof. Adopt the notation of Remark 5.27. To prove the first two parts, let j and k be the maximal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(z)$, respectively. In case (a) of Remark 5.27, it follows by inspection that $j \leq q = k$, with equality unless $r = q + 1$ and there exists at least one cycle $(a, b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ such that $p < b < q$. In case (b) of Remark 5.27, one of the following occurs:

- If $p = q - 1 = r - 2$, then $j < q - 1 < k = q + 1$.
- If $p = q - 1 < r - 2$, then $j = q$ and $k \in \{q, q + 1\}$.
- If $p < q - 1$, then $j = k = q$.

We conclude that $j \leq q$ and $k \in \{q, q + 1\}$ as required.

Let j and k now be the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(z)$, respectively. Part (c) is immediate from Lemmas 5.29 and 5.38 if $j < q - 1$ or $k < q - 1$, so assume that j and k are both at least $q - 1$. Suppose $z(q) < q < r \leq m$ so that we are in case (a) of Remark 5.27, when q is the maximal visible descent of $\mathcal{I}(z)$. Since z is not FPF-Grassmannian, we must have $k = q - 1$, so by Lemma 5.32 there exists $(a, b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ with $z(q) < b < q$. Since $y(q) = p < z(q)$, it follows that $j \leq q - 1$; as the reverse inequality holds by hypothesis, we get $j = k = q - 1$ as desired.

Suppose instead that we are in case (b) of Remark 5.27. Since $q < z(q)$, it cannot hold that $q-1$ is visible descent of $\mathcal{I}(z)$, so we must have $k \geq q$. As z is not FPF-Grassmannian, it follows from part (a) that $k = q$ and that $q+1$ is the maximal visible descent of $\mathcal{I}(z)$. This is impossible, however, since we can only have $k = q$ if there exists $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ with $z(q+1) < b < q+1$, while $q+1$ can only be a visible descent of $\mathcal{I}(z)$ if no such cycle exists. $\square$

Lemma 5.40. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ is not FPF-Grassmannian and $v \in \hat{\mathfrak{T}}^{\mathtt{FPF}}_1(z)$. Let i and j be the minimal and maximal visible descents of $\mathcal{I}(z)$. If d is a visible descent of $\mathcal{I}(v)$, then $i \leq d \leq j$.

Proof. Let (q,r) be the maximal FPF-visible descent of z, set $y = (q,r)z(q,r) = \eta_{\mathtt{FPF}}(z)$ and $p = y(q) = z(r)$, and let $n < p < q$ be the unique integer such that $v = (n,p)y(n,p)$. Since $y \lessdot_{\mathcal{F}} v$, it must hold that $y(n) < q$, so $v(t) = y(t)$ for all $t > q$. The maximal visible descent of $\mathcal{I}(y)$ is at most $q \leq j$ by Lemma 5.39, so the same is true of the maximal visible descent of $\mathcal{I}(v)$ by Corollary 5.34. On the other hand, the minimal visible descent of $\mathcal{I}(y)$ is $i \leq q-1$ by Lemma 5.39, so by Lemma 5.36 the minimal visible descent of $\mathcal{I}(v)$ is at least i. $\square$

For any $z \in \mathcal{F}_{\mathbb{Z}}$, let $\hat{\mathfrak{T}}^{\mathtt{FPF}}_0(z) = \{z\}$ and define $\hat{\mathfrak{T}}^{\mathtt{FPF}}_n(z) = \bigcup_{v\in\hat{\mathfrak{T}}^{\mathtt{FPF}}_{n-1}(z)} \hat{\mathfrak{T}}^{\mathtt{FPF}}_1(v)$ for $n \geq 1$.

Lemma 5.41. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ and $v \in \hat{\mathfrak{T}}^{\mathtt{FPF}}_1(z)$. Let (q,r) be the maximal FPF-visible inversion of z, and let (q_1,r_1) be any FPF-visible inversion of v. Then $q_1 < q$ or $r_1 < r$. Hence, if $n \geq r-q$ then the maximal FPF-visible descent of every element of $\hat{\mathfrak{T}}_n(z)$ is strictly less than q.

Proof. It is considerably easier to track the FPF-visible inversions of z and v than the visible inversions of $\mathcal{I}(z)$ and $\mathcal{I}(v)$, and this result follows essentially by inspecting Remark 5.27. In more detail, let $y = \eta_{\mathtt{FPF}}(z) = (q,r)z(q,r)$ and $p = z(r) = y(q)$. Since $y \lessdot_{\mathcal{F}} v = (n,p)y(n,p)$ for some $n < p$, we must have $v(i) = y(i)$ for all $i > q$, and so it is apparent from Remark 5.27 that $q_1 \leq q$. If $q_1 = q$, then necessarily $v(q) < p < v(i)$ for all $i \geq r$, and so it follows that $r_1 < r$. $\square$

Theorem 5.42. The FPF-involution Lascoux-Schützenberger tree $\hat{\mathfrak{T}}^{\mathtt{FPF}}(z)$ is finite for all $z \in \mathcal{F}_{\mathbb{Z}}$, and it holds that $\hat{F}^{\mathtt{FPF}}_z = \sum_v \hat{F}^{\mathtt{FPF}}_v$ where the sum is over the finite set of leaf vertices v in $\hat{\mathfrak{T}}^{\mathtt{FPF}}(z)$.

Proof. Our argument is the same as in the proof of Theorem 4.40, though with different lemmas. By induction, using Corollary 5.35 and Lemmas 5.40 and 5.41, we deduce that for some sufficiently large n either $\hat{\mathfrak{T}}^{\mathtt{FPF}}_n(z) = \varnothing$ or all elements of $\hat{\mathfrak{T}}^{\mathtt{FPF}}_n(z)$ are FPF-Grassmannian, whence $\hat{\mathfrak{T}}^{\mathtt{FPF}}_{n+1}(z) = \varnothing$. The tree $\hat{\mathfrak{T}}^{\mathtt{FPF}}(z)$ is therefore finite, so the identity $\hat{F}^{\mathtt{FPF}}_z = \sum_v \hat{F}^{\mathtt{FPF}}_v$ holds by Corollary 5.31. $\square$

As a corollary, we recover Theorem 1.17 from the introduction.

Corollary 5.43. If $z \in \mathcal{F}_{\mathbb{Z}}$ then $\hat{F}^{\mathtt{FPF}}_z \in \mathbb{N}\text{-span}\left\{\hat{F}^{\mathtt{FPF}}_y : y \in \mathcal{F}_{\mathbb{Z}} \text{ is FPF-Grassmannian}\right\}$.

5.3 Triangularity

As with $\hat{F}_y$, we can show that the expansion of $\hat{F}^{\mathtt{FPF}}_z$ into Schur P-functions not only has positive coefficients, but is unitriangular with respect to the dominance order $<$ on (strict) partitions. Recall the definitions of $\lambda(w)$ for $w \in S_\infty$ and $\hat{c}_{\mathtt{FPF}}(z)$ for $z \in \mathcal{F}_\infty$ from Sections 4.3 and 5.1.

Definition 5.44. Let $\nu(z)$ for $z \in \mathcal{F}_\infty$ be the transpose of the partition given by sorting $\hat{c}_{\mathtt{FPF}}(z)$.

One can show that this construction is consistent with our earlier definition of $\nu(z)$ when $z \in \mathcal{F}_\infty$ is FPF-Grassmannian. Define $<_{\mathcal{A}_{\mathsf{FPF}}}$ on S_∞ as the transitive relation generated by setting $v <_{\mathcal{A}_{\mathsf{FPF}}} w$ when the one-line representation of v^{-1} can be transformed to that of w^{-1} by replacing a consecutive subsequence starting at an odd index of the form $adbc$ with $a < b < c < d$ by $bcad$, or equivalently when $s_i v > v > s_{i+1}v > s_{i+2}s_{i+1}v = s_i s_{i+1} w < s_{i+1} w < w < s_i w$ for an odd number $i \in \mathbb{P}$. For example, $235164 = (412635)^{-1} <_{\mathcal{A}_{\mathsf{FPF}}} (413526)^{-1} = 253146$, but $(12534)^{-1} \not<_{\mathcal{A}_{\mathsf{FPF}}} (13425)^{-1}$. Recall the definition of $\beta_{\min}(z)$ from Lemma 5.6. In our earlier work, we showed [14, Theorem 6.22] that $<_{\mathcal{A}_{\mathsf{FPF}}}$ is a partial order and that $\mathcal{A}_{\mathsf{FPF}}(z) = \{w \in S_\infty : \beta_{\min}(z) \leq_{\mathcal{A}_{\mathsf{FPF}}} w\}$ for all $z \in \mathcal{F}_\infty$.

Lemma 5.45. Let $z \in \mathcal{F}_\infty$. If $v, w \in \mathcal{A}_{\mathsf{FPF}}(z)$ and $v <_{\mathcal{A}_{\mathsf{FPF}}} w$, then $\lambda(v) < \lambda(w)$.

Proof. Suppose $v, w \in \mathcal{A}_{\mathsf{FPF}}(z)$ are such that $s_i v > v > s_{i+1}v > s_{i+2}s_{i+1}v = s_i s_{i+1} w < s_{i+1} w < w < s_i w$ for an odd number $i \in \mathbb{P}$, so that $v <_{\mathcal{A}_{\mathsf{FPF}}} w$. Define $a = w^{-1}(i+2)$, $b = w^{-1}(i)$, $c = w^{-1}(i+1)$, and $d = w^{-1}(i+3)$ so that $a < b < c < d$. By considerations similar to those in the proof of Lemma 4.44, it follows that the diagram $D(v^{-1})$ is given by permuting rows i, $i+1$, $i+2$, and $i+3$ of $D(w^{-1}) \cup \{(i+3, b), (i+3, c)\} - \{(i, a), (i+1, a)\}$, and hence that $\lambda(v)$ is given by sorting $\lambda(w) - 2e_j + e_k + e_l$ for some indices $j < k < l$ with $\lambda(w)_j - 2 \geq \lambda(w)_k \geq \lambda(w)_l$. It is straightforward to check that in this case $\lambda(v) < \lambda(w)$, and this suffices to prove the lemma. ☐

Theorem 5.46. If $z \in \mathcal{F}_\infty$ and $\nu = \nu(z)$ then $\nu^T \leq \nu$ and it holds that $\hat{F}^{\mathsf{FPF}}_z \in s_{\nu^T} + s_\nu + \mathbb{N}\text{-span}\left\{s_\lambda : \nu^T < \lambda < \nu\right\}$.

Proof. Lemma 5.8 implies that $\nu(z)^T = \lambda(\beta_{\min}(z))$ for all $z \in \mathcal{F}_\infty$. Given this fact, the result follows by the same argument as Theorem 4.45. ☐

As a corollary, we obtain Theorem 1.20 from the introduction.

Corollary 5.47. If $z \in \mathcal{F}_\infty$ then $\nu(z)$ is strict and $\hat{F}^{\mathsf{FPF}}_z \in P_{\nu(z)} + \mathbb{N}\text{-span}\{P_\lambda : \lambda < \nu(y)\}$.

Proof. This follows from Theorem 5.46 by the same proof as Corollary 4.46. ☐

As with $\mu(y)$ for $y \in \mathcal{I}_\infty$, we do not know of any simple, elementary way of showing that $\nu(z)$ for $z \in \mathcal{F}_\infty$ is a strict partition, though this necessarily holds by the preceding result.

5.4 FPF-vexillary involutions

Define an element z of $\mathcal{F}_n$ or $\mathcal{F}_\mathbb{Z}$ to be *FPF-vexillary* if $\hat{F}^{\mathsf{FPF}}_z = P_\mu$ for a strict partition μ. In this section, we derive a pattern avoidance condition classifying such involutions. Most proofs here are similar to those in Section 4.4; only Lemmas 5.49 and 5.51 require genuinely new arguments.

Remark 5.48. All FPF-Grassmannian involutions, as well as all elements of $\mathcal{F}_n$ for $n \in \{2, 4, 6\}$, are FPF-vexillary. The sequence $(v^{\mathsf{FPF}}_{2n})_{n \geq 1} = (1, 3, 15, 92, 617, 4354, \dots)$, with v^{FPF}_n counting the FPF-vexillary elements of $\mathcal{F}_n$, again seems unrelated to any existing entry in [45].

As in earlier sections, if $E \subset \mathbb{Z}$ is a finite set of size n then we write ψ_E for the unique order-preserving bijection $E \to [n]$. Recall that if $z \in S_\mathbb{Z}$ then $[z]_E \in S_n$. We also define

$$[[z]]_E = \iota([z]_E) \in \mathcal{F}_\infty \qquad \text{for } z \in \mathcal{F}_\mathbb{Z} \text{ and finite sets } E \subset \mathbb{Z} \text{ with } z(E) = E.$$

Note that if $z \in \mathcal{F}_\mathbb{Z}$ and $E \subset \mathbb{Z}$ is finite with $z(E) = E$, then $|E|$ must be even so $[z]_E \in \mathcal{F}_{|E|}$.

Lemma 5.49. If $z \in \mathcal{F}_\mathbb{Z}$ is FPF-Grassmannian and $E \subset \mathbb{Z}$ is a finite set with $z(E) = E$, then the fixed-point-free involution $[[z]]_E$ is also FPF-Grassmannian.

Proof. Suppose $z \in \mathcal{F}_\mathbb{Z}$ is FPF-Grassmannian and $E \subset \mathbb{Z}$ is finite and z-invariant. We may assume that $z \in \mathcal{F}_\infty$ and $E \subset \mathbb{P}$. Fix a set $F = \{1, 2, \dots, 2n\}$ where $n \in \mathbb{P}$ is large enough that $E \subset F$ and $[[z]]_F = z$. Note that for any z-invariant set $D \subset E$ we have $[[z]]_D = [[z']]_{D'}$ for $z' = [[z]]_E$ and $D' = \psi_E(D)$. Inductively applying this property, we see that it suffices to show that $[[z]]_E$ is FPF-Grassmannian when $E = F \setminus \{a, b\}$ with $\{a, b\} \subset F$ a nontrivial cycle of z. In this special case, it is a relatively straightforward exercise to check that $\mathcal{I}([[z]]_E)$ is either $[\mathcal{I}(z)]_E$ (which is I-Grassmannian by Corollary 4.22) or the I-Grassmannian involution formed by replacing the left-most cycle of $[\mathcal{I}(z)]_E$ by two fixed points. Thus $[[z]]_E$ is FPF-Grassmannian as needed. □

We fix the following notation in Lemmas 5.50, 5.52, and 5.53. Let $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ and write $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ for the maximal FPF-visible inversion of z. Set $y = \eta_{\mathsf{FPF}}(z) = (q, r)z(q, r) \in \mathcal{F}_\mathbb{Z}$ and define $p = y(q) < q$ so that $\hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z) = \hat{\Psi}^-(y, p)$ if z is not FPF-Grassmannian.

Lemma 5.50. Let $E \subset \mathbb{Z}$ be a finite set with $\{q, r\} \subset E$ and $z(E) = E$. Then $(\psi_E(q), \psi_E(r))$ is the maximal FPF-visible inversion of $[[z]]_E$ and it holds that $[[\eta_{\mathsf{FPF}}(z)]]_E = \eta_{\mathsf{FPF}}([[z]]_E)$.

Proof. The results follows by the same logic as Lemma 4.54; we skip the details. □

Mimicking the notation in Section 4.4, we write $L^{\mathsf{FPF}}(z)$ for the set of integers $i < p$ with $(i, p)y(i, p) \in \hat{\Psi}^-(y, p)$, and for any subset $E \subset \mathbb{Z}$ we define

$$\mathfrak{C}^{\mathsf{FPF}}(z, E) = \left\{ (i, p)y(i, p) : i \in E \cap L^{\mathsf{FPF}}(z) \right\}.$$

Also let $\mathfrak{C}^{\mathsf{FPF}}(z) = \mathfrak{C}^{\mathsf{FPF}}(z, \mathbb{Z})$, so that $\mathfrak{C}^{\mathsf{FPF}}(z) = \hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z)$ if z is not FPF-Grassmannian.

Lemma 5.51. If $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ is FPF-Grassmannian, then $|\mathfrak{C}^{\mathsf{FPF}}(z)| = 1$.

Proof. Assume $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ is FPF-Grassmannian. By Proposition 5.16 we have $z = \mathcal{F}(g)$ for an I-Grassmannian involution $g \in \mathcal{I}_\mathbb{Z}$. Using this fact and the observations in Remark 5.27, one checks that $\mathfrak{C}^{\mathsf{FPF}}(z) = \{(i, p)y(i, p)\}$ where i is the greatest integer less than p such that $y(i) < q$. □

Lemma 5.52. Let $E \subset \mathbb{Z}$ be a finite set such that $\{q, r\} \subset E$ and $z(E) = E$.

(a) The operation $v \mapsto [[v]]_E$ restricts to an injective map $\mathfrak{C}^{\mathsf{FPF}}(z, E) \to \mathfrak{C}^{\mathsf{FPF}}([[z]]_E)$.

(b) If E contains $L^{\mathsf{FPF}}(z)$, then the injective map in (a) is a bijection.

Proof. The proof is similar to the argument given to show Lemma 4.56, except that one swaps Lemma 5.50 for Lemma 4.54 and Proposition 3.13 for Theorem 3.7. We omit the details. □

We say that $z \in \mathcal{F}_\mathbb{Z}$ *contains a bad FPF-pattern* if there exists a finite set $E \subset \mathbb{Z}$ with $z(E) = E$ and $|E| \leq 12$, such that $[[z]]_E$ is not FPF-vexillary. We refer E as a *bad FPF-pattern* for z.

Lemma 5.53. If $z \in \mathcal{F}_\mathbb{Z}$ is such that $|\hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z)| \geq 2$, then z contains a bad FPF-pattern.

Proof. This follows by the same argument as Lemma 4.57, but using Lemmas 5.51 and 5.52 instead of Lemmas 4.55 and 4.56. We omit the details. □

Lemma 5.54. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ is such that $\hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z) = \{v\}$ is a singleton set. Then z contains no bad FPF-patterns if and only if v contains no bad FPF-patterns.

Proof. We have checked the following claim by brute force: if $z \in \mathcal{F}_{16} - \{\Theta\}$ and $\mathfrak{C}^{\mathtt{FPF}}(z) = \{v\}$ is a singleton set, then z contains no bad FPF-patterns if and only if v contains no bad FPF-patterns. There are 940,482 such involutions z to check, which is substantially more than in the proof of Lemma 4.58, but still a tractable number for the computations involved. From our claim, the lemma follows by the same argument as in the proof of Lemma 4.58, *mutatis mutandis*. □

Theorem 5.55. An involution $z \in \mathcal{F}_{\mathbb{Z}}$ is FPF-vexillary if and only if $[[z]]_E$ is FPF-vexillary for all sets $E \subset \mathbb{Z}$ with $z(E) = E$ and $|E| = 12$.

Proof. Apply the same argument as in the proof of Theorem 4.59, changing what needs to be changed (e.g., substituting $\hat{\mathfrak{T}}^{\mathtt{FPF}}(z)$ for $\hat{\mathfrak{T}}(z)$), and using Theorems 5.20 and 5.42 and Lemmas 5.49, 5.53, and 5.54 in place of Theorems 4.24 and 4.40, Corollary 4.22, and Lemmas 4.57 and 4.58. □

Corollary 5.56. An involution $z \in \mathcal{F}_{\mathbb{Z}}$ is FPF-vexillary if and only if for all finite sets $E \subset \mathbb{Z}$ with $z(E) = E$ the standardization $[z]_E$ is not any of the following sixteen permutations:

$$\begin{array}{lll}
(1,3)(2,4)(5,8)(6,7), & (1,5)(2,3)(4,7)(6,8), & (1,6)(2,4)(3,8)(5,7),\\
(1,3)(2,5)(4,7)(6,8), & (1,5)(2,3)(4,8)(6,7), & (1,6)(2,5)(3,8)(4,7),\\
(1,3)(2,5)(4,8)(6,7), & (1,5)(2,4)(3,7)(6,8), & (1,3)(2,4)(5,7)(6,9)(8,10),\\
(1,3)(2,6)(4,8)(5,7), & (1,5)(2,4)(3,8)(6,7), & (1,3)(2,5)(4,6)(7,9)(8,10),\\
(1,4)(2,3)(5,7)(6,8), & (1,6)(2,3)(4,8)(5,7), & (1,3)(2,4)(5,7)(6,8)(9,11)(10,12).\\
(1,4)(2,3)(5,8)(6,7), & &
\end{array}$$

Proof. It follows by a computer calculation using the formulas in Theorems 5.20 and 5.42 that $z \in \iota(\mathcal{F}_{12}) \subset \mathcal{F}_\infty$ is not FPF-vexillary if and only if there exists a z-invariant subset $E \subset \mathbb{Z}$ such that $[z]_E$ is one of the given involutions. The corollary follows from this fact by Theorem 5.55. □

5.5 Pfaffian formulas

Recall the definition of the Pfaffian $\operatorname{pf} A$ of a skew-symmetric matrix A from (4.3). As in the involution case, both $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ and $\hat{F}^{\mathtt{FPF}}_z$ can be expressed by certain Pfaffian formulas when z is FPF-Grassmannian. We fix the following notation for the duration of this section: first, let

$$n, r \in \mathbb{P} \qquad \text{and} \qquad \phi \in \mathbb{P}^r \text{ with } 0 < \phi_1 < \phi_2 < \cdots < \phi_r < n. \tag{5.3}$$

Note that this setup differs from (4.4) in Section 4.5 in that we do not allow $\phi_r = n$. Set $\phi_i = 0$ for $i > r$. Define $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r) \in \mathcal{I}_\infty$ and $z = \mathcal{F}(y)$, and let

$$\hat{\mathfrak{S}}^{\mathtt{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n] = \hat{\mathfrak{S}}^{\mathtt{FPF}}_z \qquad \text{and} \qquad \hat{F}^{\mathtt{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n] = \hat{F}^{\mathtt{FPF}}_z.$$

When r is odd, we also set $\hat{\mathfrak{S}}^{\mathtt{FPF}}[\phi_1, \phi_2, \dots, \phi_r, 0; n] = \hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ and $\hat{F}^{\mathtt{FPF}}[\phi_1, \phi_2, \dots, \phi_r, 0; n] = \hat{F}^{\mathtt{FPF}}_z$.

Proposition 5.57. In the notation just given, $z \in \mathcal{F}_\infty$ is FPF-Grassmannian with shape $\nu(z) = (n-\phi_1, n-\phi_2, \dots, n-\phi_r)$. Moreover, each FPF-Grassmannian element of $\mathcal{F}_\infty - \{\Theta\}$ occurs as such an involution z for a unique choice of $n, r \in \mathbb{P}$ and $\phi \in \mathbb{P}^r$ as in (5.3).

Proof. Let $X = [n] \setminus \{\phi_1, \phi_2, \ldots, \phi_r\}$ and note by construction that $n \in X$. If $|X|$ is even then $\mathcal{I}(z) = y$. If $|X|$ is odd and at least 3, then $\mathcal{I}(z) = y \cdot (n, n+r+1)$. If $|X| = 1$, finally, then $\phi = (1, 2, \ldots, n-1)$ and $\mathcal{I}(z) = (2, n+2)(3, n+3) \cdots (n, 2n)$. In each of these cases, $\mathcal{I}(z)$ is evidently I-Grassmannian, and we have $\nu(z) = \mu(\mathcal{I}(z)) - 1^r = (n - \phi_1, n - \phi_2, \ldots, n - \phi_r)$ as desired. The second assertion also follows from these observations, since by Proposition 5.16 a FPF-Grassmannian element of $\mathcal{F}_\infty$ is uniquely determined by its image under $\mathcal{I} : \mathcal{F}_\infty \to \mathcal{I}_\infty$, which must be an n-I-Grassmannian involution which has an even number of fixed points in $[n]$ and which is not equal to $(i+1, n+1)(i+2, n+2) \cdots (n, 2n-i)$ for any $i \in [n]$. $\square$

As in earlier sections, let $\ell^+(\phi)$ be whichever of r or $r+1$ is even, and let $[a_{ij}]_{1 \le i < j \le n}$ denote the skew-symmetric matrix with a_{ij} in position (i,j) and $-a_{ij}$ in position (j,i) for $i < j$.

Corollary 5.58. In the setup of (5.3), $\hat{F}^{\mathsf{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n] = \mathrm{pf}\left[\hat{F}^{\mathsf{FPF}}[\phi_i, \phi_j; n]\right]_{1 \le i < j \le \ell^+(\phi)}$.

Proof. The result follows from Theorems 4.65 and 5.20, given the preceding proposition. $\square$

Our goal is to prove that the identity in this corollary holds with $\hat{F}^{\mathsf{FPF}}[\cdots; n]$ replaced by $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\cdots; n]$. Our strategy is similar to the one in Section 4.5, and in the following lemmas, we let $\mathfrak{M}^{\mathsf{FPF}}[\phi; n] = \mathfrak{M}^{\mathsf{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n]$ denote the skew-symmetric matrix $\left[\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]\right]_{1 \le i < j \le \ell^+(\phi)}$.

Lemma 5.59. Maintain the notation of (5.3), and suppose $p \in [n-1]$. Then

$$\partial_p \left(\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi; n]\right) = \begin{cases} \mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi + e_i; n] & \text{if } p = \phi_i \notin \{\phi_2 - 1, \ldots, \phi_r - 1\} \text{ for some } i \in [r] \\ 0 & \text{otherwise} \end{cases}$$

where $e_i = (0, \ldots, 0, 1, 0, 0, \ldots)$ is the standard basis vector whose ith coordinate is 1.

Proof. The result follows by the same proof as Lemma 4.68, but using (2.4) instead of (2.2). $\square$

Lemma 5.60. Let $n \ge 2$ and $D = (x_1 + x_2)(x_1 + x_3) \cdots (x_1 + x_n)$. Then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1; n] = D$, and if $b \in \mathbb{P}$ is such that $1 < b < n$, then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b; n]$ is divisible by D.

Proof. Theorem 5.5 implies that $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1; n] = D$ and, when $n > 2$, that $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, 2; n] = (x_2 + x_3) \cdots (x_2 + x_n) D$. If $2 < b < n$ then it follows from Lemma 5.59 by induction that $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b; n]$ is divisible by D, using an argument similar to the one in the proof of Lemma 4.69. $\square$

Define $\mathrm{lt}(f)$ for $f \in \mathbb{Z}[x_1, x_2, \ldots]$ as in Lemma 4.71. The following is [13, Proposition 3.14].

Lemma 5.61 (See [13])**.** If $z \in \mathcal{F}_\infty$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}_z) = x^{\hat{c}_{\mathsf{FPF}}(z)} = \prod_{(i,j) \in \hat{D}_{\mathsf{FPF}}(z)} x_i$.

Let $\mathscr{M}$ denote the set of monomials $x^i = x_1^{i(1)} x_2^{i(2)} \cdots$ for maps $i : \mathbb{P} \to \mathbb{N}$ with $i^{-1}(\mathbb{P})$ finite. Define $\prec$ as the "lexicographic" order on $\mathscr{M}$, that is, the order with $x^i \prec x^j$ when there exists $n \in \mathbb{P}$ such that $i(t) = j(t)$ for $1 \le t < n$ and $i(n) < j(n)$. Note that $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}_z) \in \mathscr{M}$. Also, observe that if $a, b, c, d \in \mathscr{M}$ and $a \preceq c$ and $b \preceq d$, then $ab \preceq cd$ with equality if and only if $a = c$ and $b = d$.

Lemma 5.62. Let $i, j, n \in \mathbb{P}$. The following identities then hold:

(a) If $i < n$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[i; n]) \succeq x_{i+1} x_{i+2} \cdots x_n$, with equality if and only if i is odd.

(b) If $i < j < n$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathtt{FPF}}[i,j;n]) \succeq (x_{i+1}x_{i+2}\cdots x_n)(x_{j+1}x_{j+2}\cdots x_n)$, with equality if and only if i is odd and j is even.

Proof. The result follows by routine calculations using Lemma 5.61. For example, suppose $i < j < n$ and let $y = (i, n+1)(j, n+2)$ and $z = \mathcal{F}(y)$, so that $\hat{\mathfrak{S}}^{\mathtt{FPF}}[i,j;n] = \hat{\mathfrak{S}}^{\mathtt{FPF}}_z$. If i is even and $j = i+1$, then $\hat{D}_{\mathtt{FPF}}(z) = \{(i,i-1),(i+1,i-1)\}\cup\{(i+1,i),(i+3,i),\dots,(n,i)\}\cup\{(i+3,i+1),\dots,(n,i+1)\}$ so $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathtt{FPF}}[i,j;n]) = (x_ix_{i+1}x_{i+3}\cdots x_n)(x_jx_{j+2}\cdots x_n)$. The other cases follow by similar analysis. $\square$

Lemma 5.63. If $n \in \mathbb{P}$ and $r \in [n-1]$ then $\hat{\mathfrak{S}}^{\mathtt{FPF}}[1,2,\dots,r;n] = \mathrm{pf}\,\mathfrak{M}^{\mathtt{FPF}}[1,2,\dots,r;n]$.

Proof. Our argument is similar to the proof of Lemma 4.72. Let $D_i = (x_i+x_{i+1})(x_i+x_{i+2})\cdots(x_i+x_n)$ for $i \in [n-1]$ and $\mathfrak{M}^{\mathtt{FPF}} = \mathfrak{M}^{\mathtt{FPF}}[1,2,\dots,r;n]$. Theorem 5.5 implies that $\hat{\mathfrak{S}}^{\mathtt{FPF}}[1,2,\dots,r;n] = D_1D_2\cdots D_r$, and it follows by Lemmas 5.59 and 5.60, using the same reasoning as in the proof of Lemma 4.72, that $\mathrm{pf}\,\mathfrak{M}^{\mathtt{FPF}}$ is divisible by $\hat{\mathfrak{S}}^{\mathtt{FPF}}[1,2,\dots,r;n]$. To prove the lemma, it suffices to show that $\mathrm{pf}\,\mathfrak{M}^{\mathtt{FPF}}$ and $\hat{\mathfrak{S}}^{\mathtt{FPF}}[1,2,\dots,r;n]$ have the same least term.

Let $m \in \mathbb{P}$ be whichever of r or $r+1$ is even and choose $z \in \mathcal{F}_m$. It follows by Lemma 4.71 that $\mathrm{lt}\left(\prod_{z(i)<i\in[m]} \mathfrak{M}^{\mathtt{FPF}}_{z(i),i}\right) \succeq (x_2x_3\cdots x_n)(x_3x_4\cdots x_n)\cdots(x_{r+1}x_{r+2}\cdots x_n) = \mathrm{lt}(\hat{\mathfrak{S}}[1,2,\dots,r;n])$, with equality if and only if i is odd and j is even whenever $i < j = z(i)$. The only element $z \in \mathcal{F}_m$ with the latter property is the "trivial" involution $z = (1,2)(3,4)\cdots(m-1,m)$, so we deduce from the definition (4.3) of pf that $\mathrm{lt}(\mathrm{pf}\,\mathfrak{M}^{\mathtt{FPF}}) = \mathrm{lt}(\hat{\mathfrak{S}}^{\mathtt{FPF}}[1,2,\dots,r;n])$ as needed. $\square$

We arrive at the "fixed-point-free" analogue of Theorem 4.73.

Theorem 5.64. In the setup of (5.3), $\hat{\mathfrak{S}}^{\mathtt{FPF}}[\phi_1,\phi_2,\dots,\phi_r;n] = \mathrm{pf}\left[\hat{\mathfrak{S}}^{\mathtt{FPF}}[\phi_i,\phi_j;n]\right]_{1\le i<j\le \ell^+(\phi)}$.

Proof. The result follows by induction from Lemmas 5.59 and 5.63 via the same argument as in the proof of Theorem 4.73, after minor changes in notation. $\square$

Example 5.65. For $\phi = (1,2,3)$ and $n = 4$ the theorem reduces to the identity

$$\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,5)(2,6)(3,7)(4,8)} = \mathrm{pf}\begin{pmatrix} 0 & \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,5)(2,6)(3,4)} & \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,5)(2,4)(3,6)} & \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,5)(2,3)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,5)(2,6)(3,4)} & 0 & \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,4)(2,5)(3,6)} & \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,3)(2,5)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,5)(3,6)(2,4)} & -\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,4)(2,5)(3,6)} & 0 & \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,5)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,5)(2,3)(4,6)} & -\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,3)(2,5)(4,6)} & -\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,5)(4,6)} & 0 \end{pmatrix}$$

where for $z \in \mathcal{F}_n$ we define $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \hat{\mathfrak{S}}^{\mathtt{FPF}}_{\iota(z)}$. By Theorem 5.5, both of these expressions evaluate to $(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)$.

As in the involution case, it is an open problem to find a simple, closed formula for the FPF-involution Schubert polynomial $\hat{\mathfrak{S}}^{\mathtt{FPF}}[i,j;n]$.

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