EQUIVARIANT EULER CHARACTERISTICS OF SUBSPACE POSETS

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Abstract. We compute the (primary) equivariant Euler characteristics of the building for the general linear group over a finite field.

1. Introduction

Let G be a finite group, Π a finite G-poset, and $r \geq 1$ a natural number. The *rth equivariant reduced Euler characteristic* of the G-poset Π as defined by Atiyah and Segal [2] is the normalized sum

$$\widetilde{\chi}_r(\Pi, G) = \frac{1}{|G|} \sum_{X \in \operatorname{Hom}(\mathbf{Z}^r, G)} \widetilde{\chi}(C_\Pi(X(\mathbf{Z}^r)) \tag{1.1}$$

of the reduced Euler characteristics of the $X(\mathbf{Z}^r)$-fixed Π-subposets, $C_\Pi(X(\mathbf{Z}^r))$, as X runs through the set of all homomorphisms of $\mathbf{Z}^r$ to G.

In this note we specialize to posets of linear subspaces of finite vector spaces. Let q be a prime power, $n \geq 1$ a natural number, $V_n(\mathbf{F}_q)$ the n-dimensional vector space over $\mathbf{F}_q$, $\mathrm{L}_n(\mathbf{F}_q)$ the $\mathrm{GL}_n(\mathbf{F}_q)$-poset of subspaces of $V_n(\mathbf{F}_q)$, and $\mathrm{L}_n^*(\mathbf{F}_q) = \mathrm{L}_n(\mathbf{F}_q) - \{0, V_n(\mathbf{F}_q)\}$ the proper part of $\mathrm{L}_n(\mathbf{F}_q)$ consisting of nontrivial and proper subspaces. In this context the general Definition (1.1) takes the following form:

Definition 1.2. *The rth equivariant reduced Euler characteristic of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$ is the normalized sum*

$$\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = \frac{1}{|\mathrm{GL}_n(\mathbf{F}_q)|} \sum_{X \in \operatorname{Hom}(\mathbf{Z}^r, \mathrm{GL}_n(\mathbf{F}_q))} \widetilde{\chi}(C_{\mathrm{L}_n^*(\mathbf{F}_q)}(X(\mathbf{Z}^r)))$$

of the Euler characteristics of the subposets $C_{\mathrm{L}_n^*(\mathbf{F}_q)}(X(\mathbf{Z}^r))$ of $X(\mathbf{Z}^r)$-invariant subspaces as X ranges over all homomorphisms $\mathbf{Z}^r \to \mathrm{GL}_n(\mathbf{F}_q)$ of the free abelian group $\mathbf{Z}^r$ on r generators into the general linear group.

The rth generating function at q is the associated power series

$$F_r(x) = 1 + \sum_{n \geq 1} \widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) x^n \in \mathbf{Z}[q][[x]] \tag{1.3}$$

with coefficients in integral polynomials in q.

Theorem 1.4. *$F_1(x) = 1 - x$ and $F_{r+1}(x)F_r(x) = F_r(qx)$ for $r \geq 1$.*

The first generating functions $F_r(x)$ for $1 \leq r \leq 5$ are

$$1-x, \quad \frac{1-qx}{1-x}, \quad \frac{(1-q^2x)(1-x)}{(1-qx)^2}, \quad \frac{(1-qx)^3(1-q^3x)}{(1-q^2x)^3(1-x)}, \quad \frac{(1-q^2x)^6(1-q^4x)(1-x)}{(1-q^3x)^4(1-qx)^4}$$

and from, for instance,

$$\begin{aligned} F_4(x) &= \frac{(1-qx)^3(1-q^3x)}{(1-q^2x)^3(1-x)} \\ &= 1 - (q-1)^3(x + (3q^2+1)x^2 + (6q^4 - q^3 + 3q^2 + 1)x^3 + (10q^6 - 3q^5 + 6q^4 - q^3 + 3q^2 + 1)x^4 + \cdots \end{aligned}$$

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/Users/jespermoller/projects/euler/linposet/version3/linposet03.tex.

we read off that the 4th equivariant reduced Euler characteristics for $n = 1, 2, 3, 4$ are

$$\widetilde{\chi}_4(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q)) = \begin{cases} -(q-1)^3 & n=1 \\ -(q-1)^3(3q^2+1) & n=2 \\ -(q-1)^3(6q^4-q^3+3q^2+1) & n=3 \\ -(q-1)^3(10q^6-3q^5+6q^4-q^3+3q^2+1) & n=4 \end{cases}$$

for all prime powers q.

With a shift of viewpoint we now let $F_r(x)(q) \in \mathbf{Z}[[x]]$ be the evaluation of the generating function $F_r(x) \in \mathbf{Z}[q][[x]]$ at the prime power q. The generating function in this form has an explicit presentation.

Theorem 1.5. *The rth generating function $F_r(x)(q) \in \mathbf{Z}[[x]]$ at q is*

$$F_r(x)(q) = \exp\big(-\sum_{n\geq 1}(q^n-1)^{r-1}\frac{x^n}{n}\big)$$

for all $r \geq 1$.

We also discuss p-primary equivariant reduced Euler characteristics $\widetilde{\chi}_r^p(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q))$ for any prime p (Definition 5.1). The p-primary generating function at q

$$F_r^p(x) = 1 + \sum_{n\geq 1}\widetilde{\chi}_r^p(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q))x^n \in \mathbf{Z}[[x]]$$

has a presentation similar to the one of Theorem 1.5.

Theorem 1.6. *The rth p-primary generating function $F_r^p(x)(q) \in \mathbf{Z}[[x]]$ at q is*

$$F_r^p(x)(q) = \exp\big(-\sum_{n\geq 1}(q^n-1)_p^{r-1}\frac{x^n}{n}\big)$$

for any $r \geq 1$.

In case q is a power of p, the p-part $(q^n-1)_p = 1$ for all $n \geq 1$ and $F_r^p(x) = \exp\big(-\sum_{n\geq 1}\frac{x^n}{n}\big) = \exp(\log(1-x)) = 1-x$. The more interesting case is thus when q is prime to p.

The subspace poset $\mathrm{L}_n^*(\mathbf{F}_q)$ is $\mathrm{GL}_n(\mathbf{F}_q)$-equivariantly homotopy equivalent to the subgroup poset $\mathcal{S}^{q+*}_{\mathrm{GL}_n(\mathbf{F}_q)}$ consisting of subgroups H of $\mathrm{GL}_n(\mathbf{F}_q)$ with $\mathrm{GCD}(q, |H|) > 1$ [13, Theorem 3.1]. In Theorems 1.4–1.6 we may therefore replace $\mathrm{L}_n^*(\mathbf{F}_q)$ by $\mathcal{S}^{q+*}_{\mathrm{GL}_n(\mathbf{F}_q)}$.

This note consists of six sections including an appendix with numerical tables. The next two sections prepare for the proofs of Theorems 1.4 and 1.5 in Section 4. The final Section 5 concerns p-primary equivariant reduced Euler characteristics of subspace posets. Theorem 1.6 follows from Lemmas 5.14 and 5.21.

The following notation will be used in this note:

q	is a prime power
p	is a prime number
$\nu_p(n)$	is the p-adic valuation of n
n_p	is the p-part of the natural number n ($n_p = p^{\nu_p(n)}$)
$\mathbf{F}_q$	is the finite field with q elements
$\mathbf{Z}_p$	is the ring of p-adic integers

See [16, 12] for equivariant Euler characteristics of boolean and partition posets.

2. Equivariant Euler characteristics of subspace posets

From now on, we will often write simply $\widetilde{\chi}_r(n,q)$ for $\widetilde{\chi}_r(\mathrm{L}_n^*(\mathbf{F}_q), \mathrm{GL}_n(\mathbf{F}_q))$.

Proposition 2.1. *Suppose that $r = 1$ or $n = 1$.*

(1) When $r = 1$, $\widetilde{\chi}_1(n,q) = -\delta_{1,n}$ is -1 for $n = 1$ and 0 for all $n > 1$.
(2) When $n = 1$, $\widetilde{\chi}_r(1,q) = -(q-1)^{r-1}$ for all $r \geq 1$.

Proof. The $\mathrm{GL}_n(\mathbf{F}_q)$-orbit of a flag is described by the dimensions of the subspaces in the flag. Thus the quotient $\Delta(L_n^*(\mathbf{F}_q))/\mathrm{GL}_n(\mathbf{F}_q)$ is the simplicial complex of all subsets of $\{1, \ldots, n-1\}$, an $(n-2)$-simplex. The first equivariant reduced Euler characteristic of the building is [11, Proposition 2.13] the usual reduced Euler characteristic of the quotient, $\widetilde{\chi}(\Delta^{n-2})$, which is -1 when $n = 1$ and 0 when $n > 1$.

When $n = 1$, $\mathrm{L}_1^*(\mathbf{F}_q) = \emptyset$ is empty. Since $\widetilde{\chi}(\emptyset) = -1$, the rth equivariant Euler characteristic is

$$\widetilde{\chi}_r(1,q) = -|\operatorname{Hom}(\mathbf{Z}^r, \mathrm{GL}_1(\mathbf{F}_q))|/|\mathrm{GL}_1(\mathbf{F}_q)| = -(q-1)^{r-1}$$

for all $r \geq 1$. $\square$

We now know that the first generating function is $F_1(x) = 1 + \sum_{n\geq 1} \widetilde{\chi}(n,q)x^n = 1-x$, independent of q. We aim now for a recursion leading to the other generating functions $F_r(x)$ for $r>1$. From the next lemma ensues a significant reduction of the problem.

Lemma 2.2. *Let A be an abelian subgroup of $\mathrm{GL}_n(\mathbf{F}_q)$ where $n>1$. If $(|A|,q)\neq 1$, then $C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$ is contractible and $\widetilde{\chi}(C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A))=0$.*

Proof. The assumption is that A contains an element of order s, the characteristic of $\mathbf{F}_q$. Let $F = C_{V_n(\mathbf{F}_q)}(O_s(A))$ be the subspace of vectors in $V_n(\mathbf{F}_q)$ that are fixed by the Sylow s-subgroup $O_s(A)$ of A. Then F is a nontrivial [6, Proposition VI.8.1] and proper subspace of $\mathbf{F}_q^n$ which is normalized by A since $(vg)h=(vh)g=vg$ for all $g\in A$, $h\in O_s(A)$. Then $U\geq U\cap F\leq F$ is a contraction of $C_{\mathrm{L}_n^*(\mathbf{F}_q)}(A)$. $\square$

We also need to know that the rth equivariant reduced Euler characteristic is multiplicative. For any lattice L, we write $L^* = L - \{\widehat{0},\widehat{1}\}$ for the proper part of L of all non-extreme elements.

Lemma 2.3. *The function $\widetilde{\chi}_r$ is multiplicative in the sense that*

$$\widetilde{\chi}_r\big((\prod_{i\in I} L_i)^*, \prod_{i\in I} G_i\big) = \prod_{i\in I}\widetilde{\chi}_r(L_i^*, G_i)$$

for any finite set of G_i-lattices L_i, $i\in I$, and any $r\geq 1$.

Proof. This follows immediately from the similar multiplicativity rule, $\widetilde{\chi}((\prod_{i\in I} L_i)^*) = \prod_{i\in I}\widetilde{\chi}(L_i^*)$, valid for usual Euler characteristics. Using this property, and assuming for simplicity that the index set $I=\{1,2\}$ has just two elements, we get

$$\begin{aligned}
|G_1\times G_2|\widetilde{\chi}_r((L_1\times L_2)^*, G_1\times G_2) &= \sum_{X\in\mathrm{Hom}(\mathbf{Z}^r, G_1\times G_2)} \widetilde{\chi}(C_{(L_1\times L_2)^*}(X(\mathbf{Z}^r)) \\
&= \sum_{X_1\in\mathrm{Hom}(\mathbf{Z}^r,G_1)}\sum_{X_2\in\mathrm{Hom}(\mathbf{Z}^r,G_2)} \widetilde{\chi}(C_{(L_1\times L_2)^*}((X_1\times X_2)(\mathbf{Z}^r)) \\
&= \sum_{X_1\in\mathrm{Hom}(\mathbf{Z}^r,G_1)}\sum_{X_2\in\mathrm{Hom}(\mathbf{Z}^r,G_2)} \widetilde{\chi}((C_{L_1}(X_1(\mathbf{Z}^r))\times C_{L_2}(X_2(\mathbf{Z}^r)))^*) \\
&= \sum_{X_1\in\mathrm{Hom}(\mathbf{Z}^r,G_1)}\sum_{X_2\in\mathrm{Hom}(\mathbf{Z}^r,G_2)} \widetilde{\chi}(C_{L_1^*}(X_1(\mathbf{Z}^r))\times\widetilde{\chi}(C_{L_2^*}(X_2(\mathbf{Z}^r)) \\
&= \sum_{X_1\in\mathrm{Hom}(\mathbf{Z}^r,G_1)}\widetilde{\chi}(C_{L_1^*}(X_1(\mathbf{Z}^r))\sum_{X_2\in\mathrm{Hom}(\mathbf{Z}^r,G_2)}\widetilde{\chi}(C_{L_1^*}(X_1(\mathbf{Z}^r)) = |G_1|\widetilde{\chi}_r(L_1,G_1)|G_2|\widetilde{\chi}_r(L_2,G_2)
\end{aligned}$$

for any $r\geq 1$. $\square$

We now formulate a basic recursive relation between equivariant reduced Euler characteristics of subspace posets.

Corollary 2.4. *When $r\geq 2$, the rth equivariant Euler characteristic of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$ is*

$$\widetilde{\chi}_r(n,q) = \sum_{\substack{[g]\in[\mathrm{GL}_n(\mathbf{F}_q)]\\ \mathrm{GCD}(q,|g|)=1}} \widetilde{\chi}_{r-1}(C_{L_n^*(\mathbf{F}_q)}(g), C_{\mathrm{GL}_n(\mathbf{F}_q)}(g))$$

Proof. For general reasons

$$\begin{aligned}
\widetilde{\chi}_r(n,q) &= \frac{1}{|\mathrm{GL}_n(\mathbf{F}_q)|}\sum_{g\in\mathrm{GL}_n(\mathbf{F}_q)}\sum_{X\in\mathrm{Hom}(\mathbf{Z}^{r-1},C_{\mathrm{GL}_n(\mathbf{F}_q)}(g))}\widetilde{\chi}_{r-1}(C_{L_n^*(\mathbf{F}_q)}(g), C_{\mathrm{GL}_n(\mathbf{F}_q)}(g)) \\
&= \frac{1}{|\mathrm{GL}_n(\mathbf{F}_q)|}\sum_{[g]\in[\mathrm{GL}_n(\mathbf{F}_q)]}|\mathrm{GL}_n(\mathbf{F}_q):C_{\mathrm{GL}_n(\mathbf{F}_q)}(g)|\sum_{X\in\mathrm{Hom}(\mathbf{Z}^{r-1},C_{\mathrm{GL}_n(\mathbf{F}_q)}(g))}\widetilde{\chi}_{r-1}(C_{L_n^*(\mathbf{F}_q)}(g), C_{\mathrm{GL}_n(\mathbf{F}_q)}(g)) \\
&= \sum_{[g]\in[\mathrm{GL}_n(\mathbf{F}_q)]}\widetilde{\chi}_{r-1}(C_{\mathrm{L}_n^*(\mathbf{F}_q)}(g), C_{\mathrm{GL}_n(\mathbf{F}_q)}(g))
\end{aligned}$$

where the sum ranges over conjugacy classes of elements of $\mathrm{GL}_n(\mathbf{F}_q)$. According to Lemma 2.2, only the semi-simple classes contribute to this sum. $\square$

3. Generating functions for polynomial sequences

Let $A = (A(1), A(2), \ldots) = (A(d))_{d\geq 1}$ and $a = (a(1), a(2), \ldots) = (a(n))_{n\geq 1}$ be sequences of integral polynomials in the variable q.

Definition 3.1. *The A-transform of the sequence a is the sequence $T_A(a)$ with nth element*

$$T_A(a)(n) = \sum_{\substack{\lambda\in M_n \\ \lambda=(m_1,d_1)^{e(m_1,d_1)}\cdots(m_s,d_s)^{e(m_s,d_s)} \\ e(m_1,d_1)m_1d_1+\cdots e(m_s,d_s)m_sd_s=n}} \prod_{\substack{d \\ \exists m\colon (m,d)^{e(m,d)}\in\lambda}} \binom{A(d)}{[e(m,d) \mid (m,d)^{e(m,d)}\in\lambda]} \prod_{(m,d)^{e(m,d)}\in\lambda} a(m)(q^d)^{e(m,d)}$$

where

- *the sum ranges over the set M_n of all multisets λ of pairs (m_i, d_i) with multiplicities $e(m_i, d_i)$ such that the multiset $(m_1d_1)^{e(m_1,d_1)}\cdots(m_sd_s)^{e(m_s,d_s)}$ is a partition of n*
- *the first product ranges over the set of all d occurring as second coordinate of an element of the multiset λ*
- *$[e(m,d) \mid (m,d)^{e(m,d)}\in\lambda]$ is the multiset of multiplicities of elements of λ with d as second coordinate*
- *the multinomial coefficient*
$$\binom{n}{k_1,\ldots,k_s} = \frac{n(n-1)\cdots(n+1-\sum k_i)}{k_1!k_2!\cdots k_s!}$$
- *$a(m)(q^d)$ is the polynomial $a(m)$ evaluated at the monomial q^d*

The number of multisets of pairs $(m_i, d_i)^{e(m_i,d_i)}$ such that the multiset $(m_id_i)^{e(m_i,d_i)}$ partitions n is

$$|M_n| = \sum_{n_1^{e(n_1)}\cdots n_s^{e(n_s)}\in P(n)} \prod p(e(n_i))$$

where $p(e)$ is the *number* of partitions of e and $P(n)$ the *set* of partitions of n. The Euler factorization of the generating function of this sequence (OEIS A006171) is

$$1 + \sum_{n\geq 1} |M_n| x^n = \prod_{n\geq 1} \frac{1}{(1-x^n)^{\tau(n)}}$$

where $\tau(n)$ is the number of divisors of n. The first terms are $1, 3, 5, 11, 17, 34, 52, 94, 145, 244, \ldots$.

For instance, the polynomial $T(a)_{10}(q)$ is a sum of 244 terms, one of which,

$$\binom{A(1)(q)}{2,2}\binom{A(2)(q)}{1} a(1)(q)^2 a(2)(q)^2 a(1)(q^2)^2$$

is contributed by the multiset $\lambda = (1,1)^2(2,1)^2(1,2)^2$.

The relation between a sequence a and its A-transform $T_A(a)$ is can be expressed more concisely using generating functions. Let $1 + (x) \subseteq \mathbf{Z}[q][[x]]$ be the multiplicative abelian group of polynomial power series with constant term 1. For a polynomial power series $F(x) = 1 + \sum_{n\geq 1} a(n)x^n \in 1 + (x)$ with constant term 1, write $T_A(F(x))$ for the generating function of the A-transform of its polynomial coefficient sequence $a = (a(n))_{n\geq 1}$. The defining relations for the A-transform of a polynomial sequence $(a(n))_{n\geq 1}$ or a polynomial power series $F(x) \in 1 + (x) \subseteq \mathbf{Z}[q][[x]]$ are

$$(3.2)\qquad 1 + \sum_{n\geq 1} T_A(a(n)(q))x^n = \prod_{d\geq 1}(1 + \sum_{n\geq 1} a(n)(q^d)x^{nd})^{A(d)(q)}, \qquad T_A(F(x))(q) = \prod_{d\geq 1}(F(x^d)(q^d))^{A(d)(q)}$$

Note that $T_A\colon 1 + (x) \to 1 + (x)$ is multiplicative and translation invariant in the sense that

$$(3.3)\qquad T_A(F(x)G(x)) = T_A(F(x))T_A(G(x)), \qquad T_A(F(qx)) = T_A(F(x))(qx)$$

for any two polynomial power series $F(x), G(x) \in 1 + (x)$.

From now on, $A(d)$ will not be any sequence of polynomials but we will fix

$$(3.4)\qquad A(d)(q) = \frac{1}{n}\sum_{d|n} \mu(n/d)q^d$$

to denote the number of monic irreducible polynomials of degree d in $\mathbf{F}_q[t]$ [10, Theorem 3.25]. By unique factorization in the polynomial ring $\mathbf{F}_q[t]$, any monic polynomial f factors essentially uniquely $f = \prod_i f_i^{e_i}$ as a product of the monic irreducible polynomials f_i. Since there are $A(d)(q)$ monic irreducibles of degree d, $f \to x^{\deg f}$ is multiplicative, $x^{\deg f} = \prod_i x^{e_i \deg f_i}$, we get the classical relation [14, Chp 2]

$$(3.5)\quad \frac{1}{1-qx} = \sum_{n\geq 0} q^n x^n = \sum_{f \text{ monic}} x^{\deg f} = \sum \prod_i x^{e_i \deg f_i} = \prod_{d\geq 1}(1 + x^d + x^{2d} + \cdots)^{A(d)(q)} = \prod_{d\geq 1} \frac{1}{(1-x^d)^{A(d)(q)}}$$

by using Euler factorization. We conclude that

$$T_A(1-q^ix) \overset{(3.2)}{=} \prod_{d\geq 1}(1-(q^ix)^d)^{A(d)} \overset{(3.5)}{=} 1-q^{i+1}x, \qquad T_A((1-q^ix)^j) \overset{(3.3)}{=} (1-q^{i+1}x)^j \tag{3.6}$$

for any two integers i and j. In particular, the iterated A-transforms of $1-x$ are $T_A^r(1-x) = \overbrace{T_A\cdots T_A}^{r}(1-x) = 1-q^rx$ for all $r\geq 0$.

We shall need not only the A-transform but also the $\bar{A}$-transform. The slightly modified sequence $\bar{A}$ given by

$$\bar{A}(d)(q) = \frac{1}{n}\sum_{d|n}\mu(n/d)(q^d-1) \tag{3.7}$$

differs from A only at the first term: $A(1)(q)=q$ while $\bar{A}(1)(q)=q-1$. This is reflected in the relations

$$1+\sum_{n\geq 1}T_A(a(n)(q))x^n = (1+\sum_{n\geq 1}T_{\bar{A}}(a(n)(q))x^n)(1+\sum_{n\geq 1}a(n)(q)x^n), \qquad T_A(F(x)) = T_{\bar{A}}(F(x))F(x) \tag{3.8}$$

between generating function. We now determine the iterated $\bar{A}$-transforms of $1-x$.

Lemma 3.9. *Let $F_1(x)=1-x$ and $F_{r+1}(x)=T_{\bar{A}}(F_r(x))$ for $r\geq 1$. Then*

$$F_{r+1}(x) = \frac{F_r(qx)}{F_r(x)}$$

for all $r\geq 1$.

Proof. The A-transform and the $\bar{A}$-transform of the polynomial power series $F_1(x)=1-x$ are

$$T_A(F_1(x)) \overset{(3.6)}{=} 1-qx = F_1(qx), \qquad F_2(x) = T_{\bar{A}}(F_1(x)) \overset{(3.8)}{=} \frac{T_A(F_1(x))}{F_1(x)} = \frac{1-qx}{1-x} = \frac{F_1(qx)}{F_1(x)}$$

From (3.6) and (3.8) we now get that

$$F_{r+1}(x) = \frac{F_r(qx)}{F_r(x)}$$

by induction. □

4. Semi-simple elements of $\mathrm{GL}_n(\mathbf{F}_q)$

The companion matrix of a monic polynomial $f = t^n + c_{n-1}t^{n-1}+\cdots+c_1t+c_0 \in \mathbf{F}_q[t]$ is the square matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{n-1}\end{pmatrix}$$

An element g of $\mathrm{GL}_n(\mathbf{F}_q)$ is semi-simple if the order of g is prime to q. The primary rational canonical form of a semi-simple $g\in \mathrm{GL}_n(\mathbf{F}_q)$ has the form

$$[g] = \mathrm{diag}(\overbrace{C(f_1),\cdots,C(f_1)}^{m_1},\cdots,\overbrace{C(f_i),\cdots,C(f_i)}^{m_i},\cdots)$$

where the $f_i\in\mathbf{F}_q[t]$, are distinct irreducible monic polynomials. The degree 1 polynomial t with constant term equal to 0 is, however, not allowed as the companion matrix (0) is not invertible. Thus there are $q-1$ allowed monic irreducible polynomials of degree 1. Let d_i be the degree of f_i. Because [7, Lemma 2.1]

$$C_{L_n^*(q)}(g) = L_{m_i}^*(\mathbf{F}_{q^{d_i}}), \qquad C_{\mathrm{GL}_n(\mathbf{F}_q)}(g) = \prod \mathrm{GL}_{m_i}(\mathbf{F}_{q^{d_i}})$$

the contribution to the sum of Corollary 2.4 is $\prod_i \widetilde{\chi}_{r-1}(m_i,q^{d_i})$.

We associate to $[g]$ the multiset $\lambda([g]) = (m_1,d_1),\dots,(m_i,d_i),\dots\in M_n$. For every multiset $\lambda\in M_n$ there are

$$\prod_{\substack{d \\ \exists m\colon (m,d)^{e(m,d)}\in\lambda}} \binom{\bar{A}(d)(q)}{[e(m,d)\mid (m,d)^{e(m,d)}\in\lambda]}$$

semi-simple elements of $\mathrm{GL}_n(\mathbf{F}_q)$ with the same contribution to the sum for $\widetilde{\chi}_r(n,q)$. In other words, $F_{r+1}(x) = T_{\bar{A}}(F_r(x))$ for all $r\geq 1$.

Proof of Theorem 1.4. Let F_r be the generating function of the polynomial sequence $(\widetilde{\chi}_r)_{n\geq 1}$. Then $F_1(x) = 1 - x$ according to Proposition 2.1.(2). We observed above that $F_{r+1} = T_{\bar{A}}(F_r)$ for all $r \geq 1$ and we saw in Lemma 3.9 that $F_{r+1}(x) = F_r(qx)/F_r(x)$ for all $r \geq 1$. □

See [17, Proposition 4.1] for an alternative proof of the case $r = 2$ where $F_2(x) = \frac{1-qx}{1-x} = 1 + (1-q)\sum_{n\geq 1} x^n$.

Corollary 4.1. *$(q-1)^{r-1}$ divides $F_r(x) - 1$ for all $r \geq 1$.*

Proof. Suppose that $F_r(x) - 1$ lies in the principal ideal generated by $(q-1)^i$. Then

$$F_{r+1}(x) - 1 = \frac{F_r(qx)}{F_r(x)} - 1 = \frac{F_r(qx) - F_r(x)}{F_r(x)} = \frac{\sum_{n\geq 1} a(n)(q)(q^n - 1)x^n}{F_r(x)} = \frac{\sum_{n\geq 1} a(n)(q)(q-1)(q^{n-1} + \cdots + 1)x^n}{F_r(x)}$$

lies in the principal ideal generated by $(q-1)^{i+1}$. □

Corollary 4.2. *$q^{n-1} \mid \widetilde{\chi}_r(n, q)$ for all odd $r \geq 1$ and for all $n \geq 1$.*

Proof. Let $r \geq 1$ be odd. Write the generating function $F_r(x)$ as $1 + \sum_{n\geq 1} a(n)x^n$. Then

$$F_{r+2}(x) = \frac{F_r(q^2x)F_r(x)}{F_r(qx)^2} = \frac{(1 + \sum a(n)q^{2n}x^n)(1 + \sum a(n)x^n)}{(1 + \sum a(n)x^n)^2}$$

By inspection one sees that if q^{n-1} divides $a(n)$ for all $n \geq 1$ then q^{n-1} also divides the coefficient of x^n in the above expression. □

For example, in the third generating function

$$F_3(x) - 1 = -(q-1)^2 \sum_{n\geq 1} nq^{n-1}x^n = -(q-1)^2 \frac{\partial}{\partial q} \sum_{n\geq 0} q^n x^n = -(q-1)^2 \frac{\partial}{\partial q} \frac{1}{1-qx}$$

the coefficient of x^n, $\widetilde{\chi}_3(n, q)$, is divisible by $(q-1)^2 q^{n-1}$. Section 6 contains some tables of quotient polynomials $\widetilde{\chi}_r(n, q)/(q-1)^{r-1} \in \mathbf{Z}[q]$ or $\widetilde{\chi}_r(n, q)/(q^{n-1}(q-1)^{r-1})$ for odd r.

4.1. **Other presentations of equivariant reduced Euler characteristics.** In this subsection we view q as a fixed prime power rather than as the variable of a polynomial ring. To emphasize the shift of viewpoint we write $F_r(x)(q)$, the rth generating function at q, for the power series in $\mathbf{Z}[[x]]$ obtained by evaluating $F_r(x) \in \mathbf{Z}[q][[x]]$ at the prime power q.

Proof of Theorem 1.5. Write $\log F_r(x)(q) = \sum_{n\geq 1} f_r^n \frac{x^n}{n}$. Theorem 1.4 implies that $f_1^n = -1$ and $f_{r+1}^n = (q^n - 1)f_r^n$ for all $n \geq 1$ and $r \geq 1$. Thus $f_r^n = -(q^n - 1)^{r-1}$ for all $r \geq 1$. □

We now unfold the recursion for equivariant reduced Euler characteristics implicit in Theorem 1.4.

Corollary 4.3. *The rth equivariant reduced Euler characteristics, $\widetilde{\chi}_r(n, q)$, at the prime power q are given by the recursion*

$$\widetilde{\chi}_r(n, q) = \begin{cases} -(q-1)^{r-1} & n = 1 \\ -\dfrac{1}{n} \displaystyle\sum_{1\leq j\leq n} (q^j - 1)^{r-1} \widetilde{\chi}_r(n-j, q) & n > 1 \end{cases}$$

with the convention that $\widetilde{\chi}_r(0, q) = 1$.

Proof. Apply Lemma 5.30 to the formula of Theorem 1.5. □

The infinite product presentation (Lemma 5.30) of the rth generating function at q is

$$F_r(x)(q) = \prod_{n\geq 1} (1 - x^n)^{b_r(q)(n)}, \qquad b_r(q)(n) = \frac{1}{n} \sum_{d\mid n} \mu(n/d)(q^d - 1)^{r-1}$$

For $r = 2$, $F_2(x)(q) = (1-x)^{-1}(1-qx) = \prod_{n\geq 1}(1-x^n)^{\bar{A}(d)(n)}$ and hence $b_2(q)(n) = \bar{A}(n)(q)$ is the number of monic irreducible degree n polynomials in $\mathbf{F}_q[t]$. I am not aware of any similar interpretation of $b_r(q)(n)$ when $r > 2$.

5. The p-primary equivariant reduced Euler characteristic

In this section we discuss the p-primary equivariant reduced Euler characteristics of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$.

Definition 5.1. [16, (1-5)] *The rth p-primary equivariant reduced Euler characteristic of the $\mathrm{GL}_n(\mathbf{F}_q)$-poset $\mathrm{L}_n^*(\mathbf{F}_q)$ is the normalized sum*

$$\widetilde{\chi}_r^p(n,q) = \frac{1}{|\operatorname{GL}_n(\mathbf{F}_q)|} \sum_{X \in \operatorname{Hom}(\mathbf{Z}\times\mathbf{Z}_p^{r-1},\operatorname{GL}_n(\mathbf{F}_q))} \widetilde{\chi}(C_{\mathrm{L}_n^*(q)}(X(\mathbf{Z}\times\mathbf{Z}_p^{r-1})))$$

of reduced Euler characteristics.

The rth p-primary generating function at q is the integral power series

$$F_r^p(x)(q) = 1 + \sum_{n\geq 1} \widetilde{\chi}_r^p(n,q)x^n \in \mathbf{Z}[[x]] \tag{5.2}$$

associated to the sequence $(\widetilde{\chi}_r^p(n,q))_{n\geq 1}$ of p-primary equivariant reduced Euler characteristics.

The rth p-primary equivariant *unreduced* Euler characteristic $\chi_r^p(n,q)$ agrees with the Euler characteristic of the homotopy orbit space $\mathrm{BL}_n^*(\mathbf{F}_q)_{h\,\mathrm{GL}_n(\mathbf{F}_q)}$ computed in Morava $K(r)$-theory at p [8] [12, Remark 7.2] [16, 2-3, 5-1].

Proposition 5.3. *Suppose that $r=1$ or $n=1$.*

1. *When $r=1$, $\widetilde{\chi}_1^p(n,q)=\widetilde{\chi}_1(n,q)=-\delta_{1,n}$ is -1 for $n=1$ and 0 for $n>1$.*
2. *When $n=1$, $\widetilde{\chi}_r^p(1,q)=-(q-1)_p^{r-1}$ for all p, q, and $r\geq 1$.*

Proof. When $r=1$, the p-primary equivariant reduced Euler characteristic and the equivariant reduced Euler characteristic agree and we refer to Proposition 2.1. When $n=1$,

$$\widetilde{\chi}_r^p(1,q) = -|\operatorname{Hom}(\mathbf{Z}\times\mathbf{Z}_p^{r-1},\operatorname{GL}_1(\mathbf{F}_q))|/|\operatorname{GL}_1(\mathbf{F}_q)| = -(q-1)(q-1)_p^{r-1}/(q-1) = -(q-1)_p^{r-1}$$

since $\mathrm{L}_1^*(\mathbf{F}_q)=\emptyset$ and $\widetilde{\chi}(\emptyset)=-1$. ☐

Proposition 5.3 and Lemma 2.2 show that

$$\begin{aligned} &F_1^p(x)(q) = 1-x \\ &F_r^p(x)(q) = 1-x, \qquad p \mid q, \quad r\geq 1 \\ &F_r^p(x)(q) = 1-(q-1)_p^{r-1}x+\cdots, \qquad r\geq 1 \end{aligned}$$

In particular, the first p-primary generating function at q, $F_1^p(x)(q)=F_1(x)(q)=1-x$, is independent of p and q.

When q is prime to p, write $\operatorname{ord}_{p^n}(q)$ for the order of q modulo p^n [9, Chp 4, §1].

Lemma 5.4. *When $p\nmid q$ and n is big enough, $\operatorname{ord}_{p^{n+1}}(q)=p\operatorname{ord}_{p^n}(q)$.*

Proof. Reduction modulo p^n induces group homomorphisms $(\mathbf{Z}/p^{n+1}\mathbf{Z})^\times \to (\mathbf{Z}/p^n\mathbf{Z})^\times$ that are onto if p is odd and $n\geq 1$ or $p=2$ and $n\geq 3$; see (the proofs of) [9, Theorem 2, Theorem 2' Chp 4, §1]. The kernels of these homomorphisms have order p. Viewing $\operatorname{ord}_{p^n}(q)$ as the order of the subgroup $\langle q\rangle_{p^n}\leq(\mathbf{Z}/p^n\mathbf{Z})^\times$ generated by q, it follows that $\operatorname{ord}_{p^{n+1}}(q)$ must equal either $\operatorname{ord}_{p^n}(q)$ or $p\operatorname{ord}_{p^n}(q)$. Suppose that $\operatorname{ord}_{p^n}(q)=r$ and $q^r=1+a_0p^n$ where $p\nmid a_0$. (There will eventually be an integer n with this property: Suppose that $\operatorname{ord}_p(q)=r$. Write $q^r=1+a_0p^n$ where $n\geq 0$ and $p\nmid a_0$. Then also $\operatorname{ord}_{p^n}(q)=r$.) If we assume that $\operatorname{ord}_{p^{n+1}}(q)=r$, then we can write $q^r=1+bp^{n+1}$ for some integer b. This contradicts $p\nmid a_0$. Therefore, $\operatorname{ord}_{p^{n+1}}(q)=pr$. Since also

$$\begin{aligned} q^{pr} &= (1+a_0p^n)^p = 1+\binom{p}{1}a_0p^n+\binom{p}{2}a_0^2p^{2n}+\cdots+a_0^pp^{pn} \\ &= 1+(a_0+\binom{p}{2}a_0^2p^{n-1}+\cdots+a_0^pp^{pn-n-1})p^{n+1} = 1+a_1p^{n+1} \end{aligned}$$

where $p\nmid a_1$, we can continue inductively. ☐

Let $A^p(d)(q)$ be the number of monic irreducible degree d and p-power order polynomials in the polynomial algebra $\mathbf{F}_q[t]$ over $\mathbf{F}_q$ [10, Definition 3.2]. If $q=p^e$ is a power of p then $t^{p^e}-1=(t-1)^{p^e}$ and $A^p(d)(p^e)=1,0,0,0,\ldots$. If $p\nmid q$, the number $A^p(d)(q)$ equals 0 except if $d=\operatorname{ord}_{p^n}(q)$ for some $n\geq 0$. According to [10, Theorem 3.5] we have

$$A^p(1)(q)=(q-1)_p, \qquad A^p(d)(q)=\frac{1}{d}\sum_{\substack{n\geq 0 \\ d=\operatorname{ord}_{p^n}(q)}}\varphi(p^n), \quad d>1,$$

where $\varphi(p^n) = p^n - p^{n-1}$ is Euler's φ-function. The sum is finite since $\mathrm{ord}_{p^n}(q) = d$ can occur for only finitely many n by Lemma 5.4. The $(q-1)_p$ polynomials of p-power order and degree 1 are those of the form $t - a$, with companion matrix (a), where $a \in \mathbf{F}_q^\times$ is of p-power order [10, Lemma 8.26].

Let $\mathbf{Z}_p^\times$ denote the unit topological group of the ring $\mathbf{Z}_p$ of p-adic integers. Consider the subgroup $\langle q \rangle$ of $\mathbf{Z}_p^\times$ generated by q (where q is a prime power prime to p).

Lemma 5.5. *The sequence $(A^p(d)(q))_{d\geq 1}$ depends only on the closure $\overline{\langle q \rangle}$ in $\mathbf{Z}_p^\times$ of $\langle q \rangle$.*

Proof. The sequence $(A^p(d)(q))_{d\geq 1}$ depends only on the images of $\langle q \rangle$ under the continuous [15, Chp 1, §3] homomorphisms $\mathbf{Z}_p^\times \to (\mathbf{Z}/p^n\mathbf{Z})^\times$, $n \geq 1$. But $\langle q \rangle$ and $\overline{\langle q \rangle}$ have the same image in the discrete topological space $(\mathbf{Z}/p^n\mathbf{Z})^\times$. □

We say that q_1 and q_2, prime powers prime to p, are p-equivalent if $\overline{\langle q_1 \rangle} = \overline{\langle q_2 \rangle}$ in $\mathbf{Z}_p^\times$. For a prime power q prime to p, if we let

$$\nu^p(q) = \begin{cases} (q \bmod 8, \nu_2(q^2-1)) & p = 2 \\ (\mathrm{ord}_p(q), \nu_p(q^{\mathrm{ord}_p(q)} - 1)) & p > 2 \end{cases}$$

then q_1 and q_2 are p-equivalent if and only if $\nu^p(q_1) = \nu^p(q_2)$ [5, §3]. (We write $\mathrm{ord}_p(q)$ for the order of q mod p [9, Definition p 43].)

Theorem 5.6. *The p-primary generating function $F_{r+1}^p(x)(q)$ is the $(A^p(d)(q))_{d\geq 1}$-transform*

$$F_{r+1}^p(x)(q) = T_{A^p(d)(q)}(F_r^p(x)(q))$$

of the p-primary generating function $F_r^p(x)(q)$ for all $r \geq 1$.

Proof. This is clear when $p \mid q$ where $F_r^p(x)(q) = 1 - x$ for all $r \geq 1$ and $(A^p(d)(q))_{d\geq 1} = 1, 0, 0, \ldots$. Assume now that $p \nmid q$. The analogue of Corollary 2.4 asserts that

$$\widetilde{\chi}_r^p(n,q) = \sum_{[g]\in[\mathrm{GL}_n(\mathbf{F}_q)_p]} \widetilde{\chi}_{r-1}^p(C_{\mathrm{L}_n^*(q)}(g), C_{\mathrm{GL}_n(\mathbf{F}_q)}(g))$$

where the sum ranges over the set $[\mathrm{GL}_n(\mathbf{F}_q)_p]$ of conjugacy classes of elements of p-power order. Since such elements have primary rational canonical forms build from irreducible monic polynomials of p-power order which are enumerated by the sequence $(A^p(d)(q))_{d\geq 1}$, the situation is formally exactly as in the proof of Theorem 1.4. □

According to Theorem 5.6 and Equation (3.2), the generating functions (5.2) are given by the recursion

$$F_1^p(x)(q) = 1 - x, \qquad F_{r+1}^p(x)(q) = \prod_{d\geq 1}(F_r^p(x^d)(q^d))^{A^p(d)(q)}, \quad r \geq 1 \tag{5.7}$$

Corollary 5.8. *$F_r^p(x)(q)$, the rth p-primary generating function at q, depends only on r and $\nu^p(q)$.*

We shall now determine $F_r^p(x)(q)$ in all cases. The following notation and well-known lemma will be convenient.

Definition 5.9 (L-notation)**.** *Let c be a boolean and a_1, a_2 two rational sequences. Then $L(c : a_1\,;\, a_2)$ is the rational sequence with value*

$$L(c : a_1\,;\, a_2)(n) = \begin{cases} a_1(n) & c(n) \text{ is true} \\ a_2(n) & c(n) \text{ is false} \end{cases}$$

at $n \geq 1$.

With this notation we have for instance

$$\log\frac{(1-x)^t}{1-x^t} = -\sum_{n\geq 1} L(t \mid n : 0\,;\, t)\frac{x^n}{n} \tag{5.10}$$

for all $t \geq 1$.

Lemma 5.11 (Lifting the Exponent)**.** *Let p be any prime and $n \geq 1$ any natural number.*

1. *If $a \equiv b \not\equiv 0 \bmod p$ and $\gcd(p,n) = 1$ then $\nu_p(a^n - b^n) = \nu_p(a-b)$*
2. *If p is odd and $a \equiv b \not\equiv 0 \bmod p$ then $\nu_p(a^n - b^n) = \nu_p(a-b) + \nu_p(n)$*
3. *If a and b are odd and n even then $\nu_2(a^n - b^n) = \nu_2(a-b) + \nu_2(a+b) + \nu_2(n) - 1$.*
4. *If a and b are odd and $a \equiv b \bmod 4$ then $\nu_2(a^n - b^n) = \nu_2(a-b) + \nu_2(n)$*

We shall also make use of the following two series.

Definition 5.12. *When p is a prime number and $r \geq 1$ a natural number, let*

$$Q_p(x) = \frac{(1-x)^p}{1-x^p}, \qquad G^p_r(x) = \exp\big(-\sum_{n\geq 1}(pn)_p^{r-1}\frac{x^n}{n}\big)$$

in the integral power series ring $\mathbf{Z}[[x]]$.

Lemma 5.13. *The power series $Q_p(x)$ and $G^p_r(x)$ satisfy*

(1) $Q_p(x) \equiv 1 \bmod p$

(2) $G^p_1(x) = 1 - x$ *and*

$$G^p_r(x) = (1-x)^{p^{r-1}} \prod_{n\geq 1}(1-x^{p^n})^{p^{n(r-2)}\cdot(p^{r-1}-1)} = \prod_{n\geq 0} Q_p(x^{p^n})^{p^{(n+1)(r-2)}}$$

for $r > 1$.

Proof. (1) Since $(1-x^p)Q_p(x) = (1-x)^p \equiv 1 - x^p \bmod p$ [9, Chp 4, §1, Lemma 2], the first assertion follows. (2) Assume $r > 1$. One may use Lemma 5.30 to show that

$$G^p_r(x) = (1-x)^{p^{r-1}} \prod_{n\geq 1}(1-x^{p^n})^{p^{n(r-2)}\cdot(p^{r-1}-1)}$$

Thus $G^p_r(x)$ satisfies the functional equation

$$G^p_r(x) = (Q_p(x)G^p_r(x^p))^{p^{r-2}}$$

Use this relation repeatedly

$$\begin{aligned} G^p_r(x) = (Q_p(x)G^p_r(x^p))^{p^{r-2}} &= Q_p(x)^{p^{r-2}}((Q_p(x^p)G^p_r(x^{p^2}))^{p^{2(r-2)}} \\ &= Q_p(x)^{p^{r-2}}Q_p(x^p)^{p^{2(r-2)}}((Q_p(x^{p^2})G^p_r(x^{p^3}))^{p^{3(r-2)}} = \cdots = \prod_{n\geq 0} Q_p(x^{p^n})^{p^{(n+1)(r-2)}} \end{aligned}$$

to finish the proof. □

It turns out that the prime $p = 2$ must be handled separately.

5.1. The case $p = 2$.

In this subsection the prime p will be equal to 2.

Lemma 5.14 ($p = 2$)**.** *The 2-primary generating functions at q are*

$$F^2_r(x)(q) = \exp\big(-\sum_{n\geq 1}(q^n-1)_2\frac{x^n}{n}\big)$$

for all $r \geq 1$.

Proof. The 2-equivalence classes of odd prime powers are represented by the 2-adic numbers $\pm 3^{2^e}$ [4, Lemma 1.11.(b)] with

$$\nu^2(\pm 3^{2^e}) = \begin{cases} (\pm 3, 3) & e = 0 \\ (\pm 1, 3+e) & e > 0 \end{cases}$$

It suffices to prove the theorem for these prime powers (Corollary 5.8). The nonzero values of the sequences $A^2(d)(q)$ are

$$\begin{aligned} &A^2(d)(3) = 2, 3, 2, 2, \ldots && \text{at} \quad d = 1, 2, 4, \ldots, 2^n, \ldots \\ &A^2(d)(3^{2^e}) = 2\cdot 2^{1+e}, 2^{1+e}, 2^{1+e}, \ldots && \text{at} \quad d = 1, 2, 4, \ldots, 2^n, \ldots \qquad e > 0 \\ &A^2(d)(-3) = 4, 2, 2, 2, \ldots && \text{at} \quad d = 1, 2, 4, \ldots, 2^n, \ldots \\ &A^2(d)(-3^{2^e}) = 2, 2^{2+e}-1, 2^{1+e}, 2^{1+e}, \ldots && \text{at} \quad d = 1, 2, 4, \ldots, 2^n, \ldots \qquad e > 0 \end{aligned}$$

Recursion (5.7) starts with $F_1(x)(\pm 3^{2^e}) = 1 - x$ and

$$F^2_{r+1}(x)(3^{2^e}) = \begin{cases} F^2_r(x)(3)^2 F^2_r(x^2)(3^2)^3 \prod_{n\geq 2} F^2_r(x^{2^n})(3^{2^n})^2 & e = 0 \\ F^2_r(x)(3^{2^e})^{2^{2+e}} F^2_r(x^2)(3^{2^{1+e}})^{2^{1+e}} \prod_{n\geq 2} F^2_r(x^{2^n})(3^{2^{n+e}})^{2^{1+e}} & e > 0 \end{cases}$$

$$F^2_{r+1}(x)(-3^{2^e}) = \begin{cases} F^2_r(x)(-3)^4 F^2_r(x^2)(3^2)^2 \prod_{n\geq 2} F^2_r(x^{2^n})(3^{2^n})^2 & e = 0 \\ F^2_r(x)(-3^{2^e})^2 F^2_r(x^2)(3^{2^{1+e}})^{2^{1+e}-1} \prod_{n\geq 1} F^2_r(x^{2^n})(3^{2^{n+e}})^{2^{1+e}} & e > 0 \end{cases}$$

for $r > 1$.

We now go through four cases and determine closed expressions for these 2-primary generating functions.
$q = 9^{2^e}$, $e \geq 0$: We first focus on the 2-primary generating functions $F_r^2(x)(9^{2^e}) = F_r^2(x)(3^{2^{1+e}})$ at 9^{2^e}. We have $F_1^2(x)(9) = 1 - x$ and

$$F_{r+1}^2(x)(9^{2^e}) = F_r^2(x)(9^{2^e})^{4\cdot 2^e} \prod_{n\geq 0} F_r^2(x^{2^n})(9^{2^{n+e}})^{4\cdot 2^e} = F_r^2(x)(9^{2^e})^{4\cdot 2^e} \prod_{n\geq 0} F_r^2(x^{2^n})(3^{2^{n+1+e}})^{4\cdot 2^e}$$

for all $r \geq 1$. Induction over r shows

$$F_r^2(x)(9^{2^e}) = F_r^2(x)(9)^{2^{(r-1)e}} \qquad (e \geq 0, \quad r \geq 1) \tag{5.15}$$

which implies that

$$\frac{F_{r+1}^2(x)(9)}{F_r^2(x)(9)^4} = \prod_{n\geq 0} F_r^2(x^{2^n})(9)^{4\cdot 2^{(r-1)n}}$$

Using a variant of Lemma 5.29 one gets

$$f_{r+1}^n = 4f_r^n + 4\sum_{k=0}^{\nu_2(n)} 2^{rk} f_{n/2^k}^n \qquad \text{where} \qquad \log F_r^2(x)(9) = \sum_{n\geq 1} f_r^n \frac{x^n}{n}$$

The solution to this recursion is

$$\log F_r^2(x)(9) = -\sum_{n\geq 1} (8n)_2^{r-1} \frac{x^n}{n} \qquad \text{or} \qquad F_r^2(x)(9) = \exp(-\sum_{n\geq 1} (8n)_2^{r-1} \frac{x^n}{n})$$

so that in general we get the explicit expression

$$\log F_r^2(x)(9^{2^e}) = -\sum_{n\geq 1} (2^e\cdot 8n)_2^{r-1} \frac{x^n}{n}, \qquad F_r^2(x)(9^{2^e}) = \exp(-\sum_{n\geq 1} (2^e\cdot 8n)_2^{r-1} \frac{x^n}{n}), \qquad e \geq 0 \tag{5.16}$$

by (5.15). We may thus write

$$F_r^2(x)(3^{2^e}) = \exp(-\sum_{n\geq 1} (2^e\cdot 4n)_2^{r-1} \frac{x^n}{n}) \qquad (e > 0)$$

where $(2^e\cdot 4n)_2 = ((3^{2^e})^n - 1)_2$ by Lemma 5.11.
$q = -3$: We observe that

$$\Big(\frac{F_{r+1}^2(x)(-3)}{F_r^2(x)(-3)^4}\Big)^2 = \frac{F_{r+1}^2(x^2)(9)}{F_r^2(x^2)(9)^4}$$

solve the resulting recursion

$$f_{r+1}^n - 4f_r^n = \begin{cases} -(4n)_2^r + 4(4n)_2^{r-1} & 2 \mid n \\ 0 & 2 \nmid n \end{cases} \qquad \text{where} \qquad \log F_r^2(x)(-3) = \sum_{n\geq 1} f_r^n \frac{x^n}{n}$$

and arrive at the explicit expression

$$F_r^2(x)(-3) = \exp(-\sum_{n\geq 1} (4n)_2^{r-1} \frac{x^n}{n})$$

where $(4n)_2 = ((-3)^n - 1)_2$ by Lemma 5.11.
$q = -3^{2^e}$, $e \geq 1$: The relation

$$\Big(\frac{F_{r+1}^2(x)(-3^{2^e})}{F_r^2(x)(-3^{2^e})^2}\Big)^2 = \big(F_r^2(x^2)(3^{2^{1+e}})^{2\cdot 2^e - 1} \prod_{n\geq 1} F_r^2(x^{2^n})(3^{2^{n+e}})^{2\cdot 2^e}\big)^2 = \frac{F_{r+1}^2(x^2)(9^{2^e})}{F_r^2(x^2)(9^{2^e})^2}$$

leads with input from (5.16) to the recursion

$$f_{r+1}^n - 2f_r^n = \begin{cases} -(2^e\cdot 4n)_2^r + 2(2^e\cdot 4n)_2^{r-1} & 2 \mid n \\ 0 & 2 \nmid n \end{cases} \qquad \text{where} \qquad \log F_r^2(x)(-3^{2^e}) = \sum_{n\geq 1} f_r^n \frac{x^n}{n}$$

with solution

$$\log F_r^2(x)(-3^{2^e}) = -\sum_{n\geq 1} L(2 \mid n : (2^e\cdot 4n)_2\,;\, 2)^{r-1} \frac{x^n}{n}, \qquad F_r^2(x)(-3^{2^e}) = \exp(-\sum_{n\geq 1} L(2 \mid n : (2^e\cdot 4n)_2\,;\, 2)^{r-1} \frac{x^n}{n})$$

where $L(2 \mid n : (2^e\cdot 4n)_2\,;\, 2) = ((-3^{2^e})^n - 1)_2$ by Lemma 5.11.

$\underline{q=3}$: From the observation

$$\Big(\frac{F^2_{r+1}(x)(3)}{F^2_r(x)(3)^2}\Big)^2 = \frac{F^2_{r+1}(x^2)(9)}{F^2_r(x^2)(9)^2}$$

we get the recursion

$$f^n_{r+1} - 2f^n_r = \begin{cases} -(4n)^r_2 + 2(4n)^{r-1}_2 & 2 \mid n \\ 0 & n \nmid n \end{cases} \qquad \text{where} \qquad \log F^2_r(x)(3) = \sum_{n\geq 1} f^n_r \frac{x^n}{n}$$

whose solution leads to the explicit expression

$$F^2_r(x)(3) = \exp(-\sum_{n\geq 1} L(2 \mid n : (4n)_2\, ;\, 2)^{r-1}\frac{x^n}{n})$$

where $L(2 \mid n : (4n)_2\, ;\, 2) = (3^n-1)_2$ by Lemma 5.11. □

Corollary 5.17. *The 2-primary generating functions $F^2_r(x)(-3)$ at -3 determine all other 2-primary generating functions as*

$$(1+x)^{2^{r-2}} F^2_r(x)(3) = ((1-x)F^2_r(x^2)(-3))^{2^{r-2}}$$

$$F^2_r(x)(3^{2^e}) = F^2_r(x)(-3)^{2^{e(r-1)}} \qquad (e>0)$$

$$(1+x)^{2^{r-2}} F^2_r(x)(-3^{2^e}) = ((1-x)F^2_r(x^2)(-3)^{2^{e(r-1)}})^{2^{r-2}} \qquad (e>0)$$

for all $r \geq 1$.

Proof. Combine the expressions for the generating functions from Lemma 5.14 with

$$\log\Big(\frac{1-x}{1+x}\Big)^{2^{r-2}} = \log\Big(\frac{(1-x)^2}{1-x^2}\Big)^{2^{r-2}} = -\sum_{n\geq 1} L(2 \mid n : 0\, ;\, 2)^{r-1}\frac{x^n}{n}$$

from (5.10). □

See Figure 1 for some concrete numerical values of 2-primary equivariant reduced Euler characteristics. We now address the divisibility relations suggested by the table of Figure 1.

Proposition 5.18. *For all $n \geq 1$, $r \geq 1$, $e > 0$ $4^{r-1} \mid \widetilde{\chi}^2_r(n,-3)$, $2^{(r-1)(2+e)} \mid \widetilde{\chi}^2_r(n, 3^{2^e})$, $2^{r-1} \mid \widetilde{\chi}^2_r(n,3)$, and $2^{r-1} \mid \widetilde{\chi}^2_r(n,-3^{2^e})$.*

Proof. According to Definition 5.12 and Lemma 5.14 we may write

$$F^2_r(x)(-3) = G^2_r(x)^{2^{r-1}} = \prod_{n\geq 0} Q_2(x^{2^n})^{2^{(r-2)(n+1)+(r-1)}}$$

for $r>1$. Since $2r-3 \leq (r-2)(n+1)+(r-1)$, we have $Q_2(x)^{2^{(r-2)(n+1)+(r-1)}} \equiv 1 \bmod 2^{2r-2} = 4^{r-1}$ [9, Chp 4, §1, Corollary 1]. Then also $F^2_r(x)(-3) \equiv 1 \bmod 4^{r-1}$. Furthermore, for $e>0$,

$$F^2_r(x)(3^{2^e}) = G^2_r(x)^{2^{(r-1)(1+e)}} = \prod_{n\geq 0} Q_2(x^{2^n})^{2^{(r-2)(n+1)+(r-1)(1+e)}}$$

where $(r-1)(2+e)-1 = (r-2)+(r-1)(1+e) \leq (r-2)(n+1)+(r-1)(1+e)$. It follows that $F^2_r(x)(3^{2^e}) \equiv 1 \bmod 2^{(r-1)(2+e)}$. By Lemma 5.14

$$F^2_r(x)(-3^{2^e}) = Q_2(-x)^{2^{r-2}\cdot(2^{(r-1)(1+e)}-1)} \cdot G_r(x)^{2^{(r-1)(1+e)}} \qquad (e>0)$$

Since both factors here are congruent to 1 modulo 2^{r-1} this also holds for their product, $F^2_r(x)(-3^{2^e})$. Similarly, $F^2_r(x)(3) \equiv 1 \bmod 2^{r-1}$. □

The infinite product representations of the proof of Proposition 5.18 have the form

$$F^2_r(x)(-3) = \prod_{n\geq 0}\Big(\frac{1-x^{2^n}}{1+x^{2^n}}\Big)^{2^{(r-2)(n+1)+(r-1)}} \qquad F^2_r(x)(3^{2^e}) = \prod_{n\geq 0}\Big(\frac{1-x^{2^n}}{1+x^{2^n}}\Big)^{2^{(r-2)(n+1)+(r-1)(1+e)}} \qquad (e>0)$$

since $Q_2(x) = \frac{(1-x)^2}{1-x^2} = \frac{1-x}{1+x}$.

We now focus on the second 2-primary generating functions $F^2_2(x)(\pm 3^{2^e})$. For each prime number p, let

$$P_p(x) = \prod_{n\geq 0}(1-x^{p^n})^{-1} = \prod_{n\geq 0}\sum_{j\geq 0} x^{jp^n} \tag{5.19}$$

denote the generating function of the sequence counting the number of partitions of natural numbers into powers of p [1, Theorem 1.1]. We have $P(0)=1$ and

$$(1-x)P_p(x)=P_p(x^p),\qquad \log P_p(x)=\frac{1}{p-1}\sum_{n\geq 1}(pn_p-1)\frac{x^n}{n},\quad (1-x)\exp\big(\sum_{n\geq 1}(pn)_p\frac{x^n}{n}\big)=P_p(x)^{p-1}$$

because the functional equation leads to a recursion for the coefficients of the generating function for $\log P_p(x)$.

Corollary 5.20 ($p=2$, $r=2$)**.** *The second 2-primary generating functions at $\pm 3^{2^e}$ are*

$$F_2^2(x)(-3^{2^e})=\begin{cases}\dfrac{(1-x)^2}{P_2(x)^2} & e=0\\ \dfrac{1-x}{1+x}\Big(\dfrac{1+x}{P_2(x)}\Big)^{2^{1+e}} & e>0\end{cases}\qquad F_2^2(x)(3^{2^e})=\begin{cases}\dfrac{1-x^2}{P_2(x)^2} & e=0\\ \Big(\dfrac{1-x}{P_2(x)}\Big)^{2^{1+e}} & e>0\end{cases}$$

Proof. When $r=2$, the recursion (5.7) states that

$$F_2^2(x)(-3)=(1-x)^4\prod_{n\geq 1}(1-x^{2^n})^2=(1-x)^2\prod_{n\geq 0}(1-x^{2^n})^2=\frac{(1-x)^2}{P_2(x)^2}$$

as we saw in the proof of Lemma 5.14. The final identity here follows from (5.19). □

5.2. **The case $p>2$.** Throughout this subsection, p is an *odd* prime.

Lemma 5.21. *The rth p-primary generating function at the prime power q is*

$$F_r^p(x)(q)=\exp\big(-\sum_{n\geq 1}(q^n-1)_p\frac{x^n}{n}\big)$$

for all $r\geq 1$.

Let g is a prime primitive root mod p^2 [9, Definition p 41]. Such a prime g always exists by the Dirichlet Density Theorem [9, Chp 16, §1, Theorem 1]. The congruence class of g generates $\mathbf{Z}/p^n\mathbf{Z}$ for all $n\geq 1$ [9, Chp 4, §1, Theorem 2]. Also, $s,t\geq 1$ are natural numbers with product $st=p-1$. Note that both sides of the identity in Lemma 5.21 depend only on $\overline{\langle q\rangle}$ (Lemma 5.5). By Corollary 5.8 and [4, Lemma 1.11.(a)] it suffices to consider p-primary generating functions $F_r^p(x)(q)$ at prime powers of the form $q=(g^s)^{p^e}$ where s divides $p-1$ and $e\geq 0$.

The cases $s=p-1$ and $s<p-1$ are proved in Lemma 5.22 and 5.24 respectively.

Lemma 5.22 ($s=p-1$)**.** *The rth p-primary generating function at q^{p^e}, where $q=g^{p-1}$, is*

$$F_r^p(x)(q^{p^e})=F_r^p(x)(q)^{p^{(r-1)e}}=\exp\big(-\sum_{n\geq 1}((q^{p^e})^n-1)_p^{r-1}\frac{x^n}{n}\big)=\prod_{n\geq 0}\Big(\frac{(1-x^{p^n})^p}{1-x^{p^{n+1}}}\Big)^{p^{(r-2)(n+1)+(r-1)e}}$$

and $p^{(r-1)(1+e)}\mid\widetilde{\chi}_r^p(n,q^{p^e})$ for all $n\geq 1$.

Proof. Since the order of q^{p^e} mod p^n [9, Definition p 43] is

$$\operatorname{ord}_{p^n}(q^{p^e})=\begin{cases}1 & n\leq e\\ p^{n-1-e} & n>e\end{cases}$$

we see that the nonzero values of the sequence $(A^p(d)(q^{p^e}))_{d\geq 1}$ are

$$A^p(d)(q^{p^e})=\begin{cases}(p-1)\cdot p^e+p^e & d=1\\ (p-1)\cdot p^e & d=p^m,\ m>0\end{cases}$$

Recursion (5.7) at q^{p^e}

$$F_1^p(x)(q^{p^e})=1-x,\qquad F_{r+1}^p(x)(q^{p^e})=F_r^p(x)(q^{p^e})^{p^e}\prod_{n\geq 0}F_r^p(x^{p^n})(q^{p^{n+e}})^{(p-1)\cdot p^e},\quad r\geq 1,$$

and induction over r show that

$$F_r^p(x)(q^{p^e})=F_r^p(x)(q)^{p^{(r-1)e}}$$

for all $r\geq 1$. Thus we only need to determine $F_r^p(x)(q)$. The recursive relation (5.7) at q now takes the form

$$F_1^p(x)(q)=1-x,\qquad F_{r+1}^p(x)(q)=F_r^p(x)(q)\prod_{n\geq 0}F_r^p(x^{p^n})(q)^{(p-1)\cdot p^{(r-1)n}},\quad r\geq 1,$$

Write $\log F_r^p(x)(q) = \sum_{n\geq 1} f_r^n \frac{x^n}{n}$. Then $f_n^1 = -1$ for all $n \geq 1$ and

$$f_{r+1}^n = f_r^n + (p-1)\sum_{k=0}^{\nu_p(n)} p^{kr} f_r^{n/p^k}, \qquad r \geq 1,$$

according to Lemma 5.29. The solution to this recursion is $f_r^n = -(pn)_p^{r-1}$ and we conclude that $F_r^p(x)(q) = G_r^p(x)$ and

$$F_r^p(x)(q^{p^e}) = G_r^p(x)^{p^{(r-1)e}} = \prod_{n\geq 0} Q_p(x^{p^n})^{p^{(r-2)(n+1)+(r-1)e}}$$

Since $(r-1)(1+e)-1 = (r-2)+(r-1)e \leq (r-2)(n+1)+(r-1)e$ for all $n \geq 0$ and $Q_p(x) \equiv 1 \bmod p$ (Lemma 5.13.(1)) it follows that $Q_p(x)^{p^{(r-2)(n+1)+(r-1)e}} \equiv 1 \bmod p^{(r-1)(1+e)}$ [9, Chp 4, §1, Lemma 3]. Then also $F_r^p(x)(q^{p^e}) \equiv 1 \bmod p^{(r-1)(1+e)}$.

Finally, observe that $\nu_p((q^{p^e})^n - 1) = \nu_p((g^{p-1})^{p^e n} - 1) = \nu_p(p^e n) + \nu_p(g^{p-1}-1) = \nu_p(p^e n) + 1 = \nu_p(p^{1+e}n)$ and that $((q^{p^e})^n - 1)_p = (p^{1+e}n)_p$ for all $n \geq 1$. We conclude that

$$F_r^p(x)(q^{p^e}) = F_r^p(x)(q)^{p^{(r-1)e}} = \exp\big(-\sum_{n\geq 1}(p^{1+e}n)_p^{r-1}\frac{x^n}{n}\big) = \exp\big(-\sum_{n\geq 1}((q^{p^e})^n-1)_p^{r-1}\frac{x^n}{n}\big)$$

as asserted. □

The second p-primary generating function at q^{p^e}, where $q = g^{p-1}$ as in Lemma 5.22,

$$F_2^p(x)(q^{p^e}) = F_2^p(x)(q)^{p^e} \overset{(5.7)}{=} ((1-x)\prod_{n\geq 0}(1-x^{p^n})^{p-1})^{p^e} = \Big(\frac{1-x}{P_p(x)^{p-1}}\Big)^{p^e}$$

is related to the p-adic partition function $P_p(x)$ of (5.19).

Lemma 5.23. *Let $q = g^s$ where $s \mid (p-1)$ and $s < p-1$. The nonzero values of the sequence $(A^p(d)(q^{p^e}))_{d\geq 1}$ are*

$$A^p(d)(q^{p^e}) = \begin{cases} 1 & d = 1 \\ sp^e + \frac{1}{t}(p^e-1) & d = t \\ sp^e & d = t\cdot p^m,\ m > 0 \end{cases}$$

where $st = p-1$.

Proof. This is easily proved from

$$\mathrm{ord}_{p^n}(q^{p^e}) = \begin{cases} t & 1 \leq n \leq 1+e \\ tp^{n-(1+e)} & n > 1+e \end{cases}$$

For instance,

$$A^p(t) = \frac{1}{t}\sum_{n=1}^{1+e}\varphi(p^n) = \frac{p-1}{t}\sum_{n=1}^{1+e}p^{n-1} = \frac{1}{t}(p^{1+e}-1) = \frac{1}{t}((p-1)p^e + p^e - 1) = sp^e + \frac{1}{t}(p^e-1)$$

and

$$A^p(tp^m) = \frac{1}{tp^m}\varphi(p^{m+1+e}) = \frac{p-1}{tp^m}p^{m+e} = sp^e$$

for $m > 0$. □

Lemma 5.24 ($s < p-1$)**.** *Let $u = g^s$ where $s < p-1$. The rth p-primary generating function at u^{p^e} is*

$$F_r^p(x)(u^{p^e}) = \exp\big(-\sum_{n\geq 1}((u^{p^e})^n - 1)_p\frac{x^n}{n}\big)$$

for all $r \geq 1$ and $e \geq 0$.

Proof. Let $q = g^{p-1}$ as in Lemma 5.22. The recursion formulas (5.7) at q^{p^e} and u^{p^e}

$$F_{r+1}^p(x)(q^{p^e}) = F_r^p(x)(q^{p^e})^{p^e}\prod_{n\geq 0}F_r^p(x^{p^n})(q^{p^{n+e}})^{(p-1)p^e}$$

$$F_{r+1}^p(x)(u^{p^e}) = F_r^p(x)(u^{p^e})^{p^e}F_r^p(x^t)(q^{p^e})^{\frac{1}{t}(p^e-1)}\prod_{n\geq 0}F_r^p(x^{tp^n})(q^{p^{n+e}})^{sp^e}$$

reveal that

$$\left(\frac{F^p_{r+1}(x)(u^{p^e})}{F^p_r(x)(u^{p^e})}\right)^t = \frac{F^p_{r+1}(x^t)(q^{p^e})}{F^p_r(x^t)(q^{p^e})}$$

Write $\log F^p_r(x)(u^{p^e}) = \sum_{n\geq 1} f^n_r \frac{x^n}{n}$. Then $f^n_1 = -1$ for all $n \geq 1$ and the above identity shows that

$$f^n_{r+1} - f^n_r = \begin{cases} -(p^{1+e}n)^r_p + (p^{1+e}n)^{r-1}_p & t \mid n \\ 0 & t \nmid n \end{cases}$$

The obvious solution to this recursion is $f^n_r = -L(t \mid n : (p^{1+e}n)^{r-1}_p\,;\, 1)$. Since $\mathrm{ord}_p(g^s) = t$ and $\nu_p((g^s)^{p^e tm} - 1) = \nu_p(g^{(p-1)p^e m} - 1) = \nu_p(p^e m) + \nu_p(g^{p-1} - 1) = \nu_p(p^e m) + 1 = \nu_p(p^{1+e}m) = \nu_p(p^{1+e}tm)$ by Lemma 5.11, we have $L(t \mid n : (p^{1+e}n)^{r-1}_p\,;\, 1) = ((u^{p^e})^n - 1)_p$. □

Corollary 5.25. *Suppose that $q = g^{p-1}$ and that $u = g^s$ where $st = p - 1$. Then*

$$(1 - x^t)F^p_r(x)(u^{p^e})^t = (1 - x)^t F^p_r(x^t)(q^{p^e})$$

for all $r \geq 1$ and $e \geq 0$.

Proof. Observe that

$$\log\left(F^p_r(x)(u^{p^e})^t\right) - \log\frac{(1-x)^t}{1-x^t} = \log F^p_r(x^t)(q^{p^e})$$

The left hand side is described in Lemma 5.24 and Equation (5.10) and the right hand side in Lemma 5.22. □

If we put $\bar{F}^p_r(x)(v) = F^p_r(x)(v)/(1-x)$ for all prime powers v prime to p, Corollary 5.25 states that

$$\bar{F}^p_r(x)(u^{p^e})^t = \bar{F}^p_r(x^t)(q^{p^e}) \qquad (u = g^s,\ q = g^{p-1},\ st = p-1)$$

or, more directly, that

$$\widetilde{\chi}^p_r(n, u^{p^e}) - \widetilde{\chi}^p_r(n-t, u^{p^e}) = \sum_{\substack{0\leq m \\ 0 \leq n-tm \leq t}} (-1)^{n-tm}\binom{t}{n-tm}\widetilde{\chi}^p_r(m, q^{p^e})$$

where it is understood that $\widetilde{\chi}^p_r(0, u^{p^e}) = 1 = \widetilde{\chi}^p_r(0, q^{p^e})$.

Example 5.26 ($p = 3$)**.** The 3-equivalence classes of prime powers prime to 3 are represented by 2^{3^e} and 4^{3^e} [4, Lemma 1.11.(a)] with $\nu^3(2^{3^e}) = (2, 1+e)$, $\nu^3(4^{3^e}) = (1, 1+e)$, $e \geq 0$. The 3-primary generating functions $F^3_r(x)(q^{3^e})$ for $q = 2, 4$ are for all $r \geq 1$ and $e \geq 0$ given by

$$F^3_r(x)(4^{3^e}) = \exp\big(-\sum_{n\geq 1}(3^{1+e}n)^{r-1}_3\frac{x^n}{n}\big)$$

$$(1 - x^2)F^3_r(x)(2^{3^e})^2 = (1-x)^2 F^3_r(x^2)(4^{3^e})$$

according to Lemma 5.21 and Corollary 5.25. See Figure 2 for some concrete numerical values of 3-primary equivariant reduced Euler characteristics.

Example 5.27 ($p = 5$)**.** The 5-equivalence classes of prime powers prime to 5 are represented by 2^{5^e}, 4^{5^e}, 16^{5^e} [4, Lemma 1.11.(a)] with $\nu^5(2^{5^e}) = (4, 1+e)$, $\nu^5(4^{5^e}) = (2, 1+e)$, and $\nu^5(16^{5^e}) = (1, 1+e)$, $e \geq 0$. The 5-primary generating functions $F^5_r(x)(q^{5^e})$ for $q = 2, 4, 16$ are for all $r \geq 1$ and $e \geq 0$ given by

$$F^5_r(x)(16^{5^e}) = \exp\big(-\sum_{n\geq 1}(5^{1+e}n)^{r-1}_5\frac{x^n}{n}\big)$$

$$(1 - x^2)F^5_r(x)(4^{5^e})^2 = (1-x)^2 F^5_r(x^2)(16^{5^e})$$

$$(1 - x^4)F^5_r(x)(2^{5^e})^4 = (1-x)^4 F^5_r(x^4)(16^{5^e})$$

according to Lemma 5.21 and Corollary 5.25. See Figure 3 for some concrete numerical values of 5-primary equivariant reduced Euler characteristics.

5.3. **Other presentations of p-primary equivariant reduced Euler characteristics.** We observe that the results of subsection 4.1 carry over to the p-primary case.

Corollary 5.28. *The rth p-primary equivariant reduced Euler characteristics, $\widetilde{\chi}_r^p(n,q)$, at the prime power q are given by the recursion*

$$\widetilde{\chi}_r^p(n,q) = \begin{cases} -(q-1)_p^{r-1} & n=1 \\ -\dfrac{1}{n}\displaystyle\sum_{1\le j\le n}(q^j-1)_p^{r-1}\widetilde{\chi}_r^p(n-j,q) & n>1 \end{cases}$$

with the convention that $\widetilde{\chi}_r^p(0,q)=1$.

Proof. Apply Lemma 5.30 to the formula of Theorem 1.6. □

The infinite product presentation (Lemma 5.30) of the rth generating function at q is

$$F_r^p(x)(q) = \prod_{n\ge 1}(1-x^n)^{b_r^p(q)(n)}, \qquad b_r^p(q)(n) = \frac{1}{n}\sum_{d|n}\mu(n/d)(q^d-1)_p^{r-1}$$

When $r=2$

$$F_2^p(x)(q) \overset{(5.7)}{=} \prod_{n\ge 1}(1-x^n)^{A^p(n)(q)}$$

shows that $A^p(n)(q) = b_2^p(q)(n)$ and we obtain formulas

$$A^p(n)(q) = \frac{1}{n}\sum_{d|n}\mu(n/d)(q^d-1)_p \qquad (q^n-1)_p = \sum_{d|n} dA^p(d)(q)$$

similar to the classical formulas (3.4) or (3.7) apparently due to Gauss.

5.4. **Generating functions.** This subsection consists of two easy, probably well-known, lemmas about generating functions.

Lemma 5.29. *Suppose that $F(x)$ and $G(x)$ are power series with constant term 1 and that*

$$\frac{G(x)}{F(x)} = \prod_{n\ge 0} F(x^{p^n})^{ap^{bn}}$$

for some integers a and b. If $\log F(x) = \sum_{n\ge 1} f^n\frac{x^n}{n}$ and $\log G(x) = \sum_{n\ge 1} g^n\frac{x^n}{n}$ then

$$g^n = f^n + a\sum_{k=0}^{\nu_p(n)} p^{k(b+1)} f^{n/p^k}$$

for all $n\ge 1$.

Proof. Let $Q(x) = \frac{G(x)}{F(x)}$. Then $F(x)^{-a}Q(x) = Q(x^p)^{p^b}$. The relation $-a\log F(x) + \log Q(x) = p^b\log Q(x^p)$ implies

$$q_n = a\sum_{k=0}^{\nu_p(n)} p^{k(b+1)} f^{n/p^k}$$

where $\log Q(x) = \sum_{n\ge 1} q^n\frac{x^n}{n}$. This proves the lemma as $g^n = f^n + q^n$. □

Lemma 5.30. *Let $(a_n)_{n\ge 1}$, $(b_n)_{n\ge 1}$, and $(c_n)_{n\ge 1}$ be integer sequences such that*

$$\prod_{n\ge 1}(1-x^n)^{-b_n} = \exp\big(\sum_{n\ge 1} a_n\frac{x^n}{n}\big) = 1 + \sum_{n\ge 1} c_n x^n$$

Then

$$a_n = \sum_{d|n} db_d, \qquad nb_n = \sum_{d|n}\mu(n/d)a_d, \qquad nc_n = \sum_{1\le j\le n} a_j c_{n-j}$$

where μ is the number theoretic Möbius function [9, Chp 2, §2] *and it is understood that $c_0 = 1$.*

Proof. The first identity follows from

$$\sum_{n\geq 1} na_n x^n = \sum_{n\geq 1}\sum_{k\geq 1} nb_n x^{nk}$$

obtained by applying the operator $x\frac{d}{dx}\log$ to the given identity $\exp(\sum_{n\geq 1} a_n \frac{x^n}{n}) = \prod_{n\geq 1}(1-x^n)^{-b_n}$. Möbius inversion leads to the second identity. The third identity follows from

$$\big(1+\sum_{n\geq 1} c_n x^n\big)\big(\sum_{n\geq 1} a_n x^n\big) = \sum_{n\geq 1} nc_n x^n$$

obtained by applying the operator $x\frac{d}{dx}$ to the given identity $\exp\big(\sum_{n\geq 1} a_n \frac{x^n}{n}\big) = 1+\sum_{n\geq 1} c_n x^n$. □

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6. Tables

$\widetilde{\chi}_2^2(n,q)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$q=3^{2^0}$	-2	-2	6	-2	-2	6
$q=3^{2^1}$	-8	24	-24	-40	152	-184
$q=3^{2^2}$	-16	112	-432	880	-208	-4144
$q=-3^{2^0}$	-4	4	4	-12	12	-4
$q=-3^{2^1}$	-2	-6	14	10	-34	10
$q=-3^{2^2}$	-2	-14	30	82	-194	-238

$\widetilde{\chi}_3^2(n,q)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$q=3^{2^0}$	-4	-24	116	208	-1428	-328
$q=3^{2^1}$	-64	1920	-35520	446208	-3904320	22659712
$q=3^{2^2}$	-256	32256	-2665216	162331648	-7766985984	303778507264
$q=-3^{2^0}$	-16	96	-176	-832	5424	-9184
$q=-3^{2^1}$	-4	-120	500	6928	-30228	-255336
$q=-3^{2^2}$	-4	-504	2036	125968	-514068	-20813288

Figure 1. Some second and third 2-primary equivariant reduced Euler characteristics

$\widetilde{\chi}_2^3(n,q)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$q=2^{3^0}$	-1	-1	1	0	0	-1
$q=2^{3^1}$	-1	-4	4	6	-6	-7
$q=2^{3^2}$	-1	-13	13	78	-78	-295
$q=4^{3^0}$	-3	3	-3	6	-6	3
$q=4^{3^1}$	-9	36	-90	180	-342	603
$q=4^{3^2}$	-27	351	-2943	18036	-87048	348813

$\widetilde{\chi}_3^3(n,q)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$q=2^{3^0}$	-1	-4	4	6	-6	-16
$q=2^{3^1}$	-1	-40	40	780	-780	-9988
$q=2^{3^2}$	-1	-364	364	66066	-66066	-7972936
$q=4^{3^0}$	-9	36	-108	342	-990	2376
$q=4^{3^1}$	-81	3240	-85536	1681236	-26321436	342992556
$q=4^{3^2}$	-729	265356	-64306548	11672702802	-1692852267834	204333604208352

Figure 2. Some second and third 3-primary equivariant reduced Euler characteristics

$\widetilde{\chi}_2^5(n,q)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$q=2^{5^0}$	-1	0	0	-1	1	0
$q=2^{5^1}$	-1	0	0	-6	6	0
$q=2^{5^2}$	-1	0	0	-31	31	0
$q=4^{5^0}$	-1	-2	2	1	-1	0
$q=4^{5^1}$	-1	-12	12	66	-66	-220
$q=4^{5^3}$	-1	-62	62	1891	-1891	-37820
$q=16^{5^0}$	-5	10	-10	5	-5	20
$q=16^{5^1}$	-25	300	-2300	12650	-53150	177600
$q=16^{5^2}$	-125	7750	-317750	9691375	-234531375	4690638000

$\widetilde{\chi}_3^5(n,q)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$q=2^{5^0}$	-1	0	0	-6	6	0
$q=2^{5^1}$	-1	0	0	-156	156	0
$q=2^{5^2}$	-1	0	0	-3906	3906	0
$q=4^{5^0}$	-1	-12	12	66	-66	-220
$q=4^{5^1}$	-1	-312	312	48516	-48516	-5013320
$q=4^{5^2}$	-1	-7812	7812	30509766	-30509766	-79427090820
$q=16^{5^0}$	-25	300	-2300	12650	-53250	180100
$q=16^{5^1}$	-625	195000	-40495000	6296972500	-782083987500	80815346940000
$q=16^{5^2}$	-15625	122062500	-635660812500	2482573303218750	-7756055513916093750	20191597854562540687500

FIGURE 3. Some second and third 5-primary equivariant reduced Euler characteristics

$-\chi_r(n,2)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$r=1$	1	0	0	0	0	0
2	1	1	1	1	1	1
3	1	4	12	32	80	192
4	1	13	101	645	3717	20101
5	1	40	760	11056	140848	1657216
6	1	121	5481	176921	4865465	121991097
7	1	364	38852	2742192	160444704	8459901568
8	1	1093	273421	41843005	5160361501	567145627357
9	1	3280	1918320	633421856	163640016032	37253059018752
10	1	9841	13441361	9548966001	5145988736049	2415591291535153

$-\chi_r(n,3)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$r=1$	1	0	0	0	0	0
$r=2$	2	2	2	2	2	2
$r=3$	4	24	108	432	1620	5832
$r=4$	8	224	3896	56384	738728	9084320
$r=5$	16	1920	120240	5965056	261808848	10632560256
$r=6$	32	15872	3441632	566592512	81130485920	10643326347776
$r=7$	64	129024	94627008	50623598592	23235767666496	9699351625525248
$r=8$	128	1040384	2543401856	4359012368384	6335120693648000	8325309587628007424
$r=9$	256	8355840	67464334080	366700817350656	1672593892361932032	6866060974492749299712
$r=10$	512	66977792	1775497190912	30384790121480192	432088361624308894208	5508197369484633860341760

$-\chi_r(n,4)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$r=1$	1	0	0	0	0	0
$r=2$	3	3	3	3	3	3
$r=3$	9	72	432	2304	11520	55296
$r=4$	27	1323	41067	1064043	24951915	548715627
$r=5$	81	22032	3289248	392615424	41469076224	4046629404672
$r=6$	243	350163	240939603	128147119443	59358341480787	25142376895168851
$r=7$	729	5429592	16753854672	38825589337344	77426404548337152	140143660428976386048
$r=8$	2187	83038203	1127902212027	11210270236461243	94979669231700061371	726285710380559627353275
$r=9$	6561	1259921952	74358102316608	3133261901291480064	111685827936678492989952	3577193222771146513160577024
$r=10$	19683	19027969443	4834201997243043	856136331064357729443	1274389853567871074840854111	16984678398182565955500827995299

FIGURE 4. Equivariant reduced Euler characteristics $\widetilde{\chi}_r(n,q)$ for $q=2,3,4$

$$-\widetilde{\chi}_1(n,q) = (1,0,0,0,\ldots)$$
$$-\widetilde{\chi}_2(n,q)/(q-1) = (1,1,1,1,\ldots)$$
$$(-\widetilde{\chi}_3(n,q)/((q-1)^2q^{n-1})) = (1,2,3,4,\ldots)$$

$(-\widetilde{\chi}_4(n,q)/(q-1)^3)_{1\le n\le 10} =$

$$\begin{array}{c}
1\\
3q^2+1\\
6q^4-q^3+3q^2+1\\
10q^6-3q^5+6q^4-q^3+3q^2+1\\
15q^8-6q^7+10q^6-3q^5+6q^4-q^3+3q^2+1\\
21q^{10}-10q^9+15q^8-6q^7+10q^6-3q^5+6q^4-q^3+3q^2+1\\
28q^{12}-15q^{11}+21q^{10}-10q^9+15q^8-6q^7+10q^6-3q^5+6q^4-q^3+3q^2+1\\
36q^{14}-21q^{13}+28q^{12}-15q^{11}+21q^{10}-10q^9+15q^8-6q^7+10q^6-3q^5+6q^4-q^3+3q^2+1\\
45q^{16}-28q^{15}+36q^{14}-21q^{13}+28q^{12}-15q^{11}+21q^{10}-10q^9+15q^8-6q^7+10q^6-3q^5+6q^4-q^3+3q^2+1\\
55q^{18}-36q^{17}+45q^{16}-28q^{15}+36q^{14}-21q^{13}+28q^{12}-15q^{11}+21q^{10}-10q^9+15q^8-6q^7+10q^6-3q^5+6q^4-q^3+3q^2+1
\end{array}$$

$(-\widetilde{\chi}_5(n,q)/((q-1)^4q^{n-1}))_{1\le n\le 10} =$

$$\begin{array}{c}
1\\
4q^2+4\\
10q^4-4q^3+15q^2-4q+10\\
20q^6-15q^5+40q^4-26q^3+40q^2-15q+20\\
35q^8-36q^7+86q^6-80q^5+115q^4-80q^3+86q^2-36q+35\\
56q^{10}-70q^9+160q^8-180q^7+260q^6-236q^5+260q^4-180q^3+160q^2-70q+56\\
84q^{12}-120q^{11}+269q^{10}-340q^9+500q^8-524q^7+605q^6-524q^5+500q^4-340q^3+269q^2-120q+84\\
120q^{14}-189q^{13}+420q^{12}-574q^{11}+860q^{10}-985q^9+1184q^8-1160q^7+1184q^6-985q^5+860q^4-574q^3+420q^2-189q+120\\
165q^{16}-280q^{15}+620q^{14}-896q^{13}+1365q^{12}-1660q^{11}+2060q^{10}-2180q^9+2341q^8-2180q^7+2060q^6-1660q^5+1365q^4-896q^3+620q^2-280q+165\\
220q^{18}-396q^{17}+876q^{16}-1320q^{15}+2040q^{14}-2590q^{13}+3296q^{12}-3676q^{11}+4100q^{10}-4100q^9+4100q^8-3676q^7+3296q^6-2590q^5+2040q^4-1320q^3+876q^2-396q+220
\end{array}$$

$(-\widetilde{\chi}_6(n,q)/(q-1)^5)_{1\le n\le 6} =$

$$\begin{array}{c}
1\\
5q^4+10q^2+1\\
15q^8-10q^7+45q^6-25q^5+55q^4-10q^3+10q^2+1\\
35q^{12}-45q^{11}+150q^{10}-170q^9+290q^8-235q^7+270q^6-100q^5+60q^4-10q^3+10q^2+1\\
70q^{16}-126q^{15}+395q^{14}-615q^{13}+1085q^{12}-1295q^{11}+1601q^{10}-1395q^9+1230q^8-600q^7+315q^6-101q^5+60q^4-10q^3+10q^2+1\\
126q^{20}-280q^{19}+875q^{18}-1625q^{17}+3070q^{16}-4477q^{15}+6245q^{14}-7115q^{13}+7570q^{12}-6525q^{11}+5127q^{10}-2825q^9+1465q^8-610q^7+315q^6-101q^5+60q^4-10q^3+10q^2+1
\end{array}$$

$(-\widetilde{\chi}_7(n,q)/(q-1)^6q^{n-1})_{1\le n\le 5} =$

$$\begin{array}{c}
1\\
6q^4+20q^2+6\\
21q^8-20q^7+105q^6-90q^5+211q^4-90q^3+105q^2-20q+21\\
56q^{12}-105q^{11}+420q^{10}-680q^9+1386q^8-1695q^7+2260q^6-1695q^5+1386q^4-680q^3+420q^2-105q+56\\
126q^{16}-336q^{15}+1309q^{14}-2856q^{13}+6300q^{12}-10486q^{11}+16416q^{10}-20664q^9+23507q^8-20664q^7+16416q^6-10486q^5+6300q^4-2856q^3+1309q^2-336q+126
\end{array}$$

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