

RESTRICTED SUMS OF FOUR SQUARES

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ABSTRACT. We refine Lagrange's four-square theorem in new ways by imposing some restrictions involving powers of two (including 1). For example, we show that each $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $|2x - y| \in \{4^k : k \in \mathbb{N}\}$ (or $x + y - z \in \{\pm 8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{t^3 : t \in \mathbb{Z}\}$), and that any positive integer can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + z$ (or $x + y + 2z$, or $x + 2y + 2z$) a power of four. We also pose some open conjectures; for example, we conjecture that any positive integer can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y - 2z$ is a power of four, and that each $n = 0, 1, 2, \dots$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $x + 24y$ are squares.

1. INTRODUCTION

The celebrated four-square theorem (cf. [N, pp. 5-7]) proved by J. L. Lagrange in 1770 states that any $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$. Recently, the author [S17b] found that this can be refined in various ways by requiring additionally that $P(x, y, z, w)$ is a square, where $P(x, y, z, w)$ is a suitable polynomial with integer coefficients. (For example, we may take $P(x, y, z, w) = x^2y^2 + y^2z^2 + z^2x^2$.) Here is a challenging conjecture posed by the author.

1-3-5 Conjecture ([S17b, Conjecture 4.3(i)]). *Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.*

In this paper we aim to refine Lagrange's four-square theorem in a new direction by imposing restrictions involving power of two (including $2^0 = 1$). [S17b, Theorem 1.1] asserts that for any $a \in \{1, 4\}$ and $m \in \{4, 5, 6\}$ we can write each $n \in \mathbb{N}$ as $ax^m + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$. Actually the proof in [S17b] works for a stronger result which requires additionally that $x \in \{2^k : k \in \mathbb{N}\} \cup \{0\}$. Similarly, by modifying the proof of [S17b, Theorem 1.2(i)] slightly we get that for

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any $a \in \{1, 2\}$ each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $a(x - y) \in \{4^k : k \in \mathbb{N}\} \cup \{0\}$.

Now we state our first theorem.

Theorem 1.1. (i) Any $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ can be written as $x^2 + y^2 + z^2 + 8^k$ with $x, y, z \in \mathbb{N}$ and $k \in \{0, 1, 2\}$.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x - y = 2^{\lfloor \text{ord}_2(n)/2 \rfloor}$, where $\text{ord}_2(n)$ is the 2-adic order of n .

(iii) Each $n \in \mathbb{N}$ not of the form $2^{6k+3} \times 7$ ($k \in \mathbb{N}$) can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$. Consequently, we can write any $n \in \mathbb{N}$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$.

(iv) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + z + w = 2^{\lfloor (\text{ord}_2(n)+1)/2 \rfloor}$.

Remark 1.1. For integers y and z of the same parity, clearly

$$y^2 + z^2 = 2 \left(\frac{y+z}{2} \right)^2 + 2 \left(\frac{y-z}{2} \right)^2.$$

So, Theorem 1.1(i) implies that any $n \in \mathbb{N}$ can be written as $x^2 + 2y^2 + 2z^2 + 8^k$ with $x, y, z \in \mathbb{N}$ and $k \in \{0, 1, 2\}$. As $\{8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{m^3 : m \in \mathbb{N}\}$, Theorem 1.1(iii) implies a conjecture stated in [S17b, Remark 1.2]. Theorem 1.1(iv) with $\text{ord}_2(n) = 1$ was first realized by Euler in a letter to Goldbach dated June 9, 1750 (cf. Question 37278 in [MathOverflow](#)). See [S, A281494] for the number of ways to represent $n \in \mathbb{Z}^+$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $|x| \leq |y| \leq |z| \leq |w|$ such that $x + y + z + w = 2^{\lfloor (\text{ord}_2(n)+1)/2 \rfloor}$. For example,

$$14 = 0^2 + 1^2 + (-2)^2 + 3^2 \quad \text{with } 0 + 1 + (-2) + 3 = 2 = 2^{\lfloor (\text{ord}_2(14)+1)/2 \rfloor}$$

and

$$107 = (-1)^2 + (-3)^2 + (-4)^2 + 9^2 \quad \text{with } (-1) + (-3) + (-4) + 9 = 1 = 2^{\lfloor (\text{ord}_2(107)+1)/2 \rfloor}.$$

It was proved in [SS] that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + z + w$ a square.

The author (cf. [S17b, Conjecture 4.1]) conjectured that for each $\varepsilon \in \{\pm 1\}$ any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $2x + \varepsilon y$ a square. Y.-C. Sun and the author [SS] confirmed this for $\varepsilon = 1$, but the case $\varepsilon = -1$ remains unsolved. Our next theorem provides an advance in this direction.

Theorem 1.2. (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $|2x - y| \in \{4^k : k \in \mathbb{N}\}$.

(ii) Let $a \in \{1, 2\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $2x - y \in \{\pm a 8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{at^3 : t \in \mathbb{Z}\}$.

Remark 1.2. Motivated by Theorem 1.2(i), we conjecture that any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $3x + y \in \{4^k : k \in \mathbb{N}\}$.

The author (cf. [S17b, Conjecture 4.3(iii)-(iv)]) conjectured that each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $x + y - z$ (or $x - y - z$) a square. In contrast, we have the following result.

Theorem 1.3. (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + z \in \{4^k : k \in \mathbb{N}\}$. Also, for any $a \in \{0, 2\}$ we can write each $n \in \mathbb{Z}^+$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $|x + y - z| \in \bigcup_{k \in \mathbb{N}} \{a^k, 4^k\}$.

(ii) Let $a \in \{1, 2\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + y - z \in \{\pm a 8^k : k \in \mathbb{N}\} \cup \{0\} \subseteq \{at^3 : t \in \mathbb{Z}\}$.

The author [S17b] conjectured that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z)$ is a square, where $P(x, y, z)$ may be any of the polynomials

$$x + y - 2z, 2x + y - z, 2x - y - z, x + 2y - 2z, 2x - y - 2z.$$

Here we give the following result.

Theorem 1.4. (i) Let $c \in \{1, 2\}$. Then each $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + 2z \in \{c4^k : k \in \mathbb{N}\}$.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{4^k : k \in \mathbb{N}\}$.

(iii) If $n \in \mathbb{Z}^+$ does not belong to $\bigcup_{k \in \mathbb{N}} \{2^{4k}, 2^{4k+3}\}$, then we can write n as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{3 \times 4^k : k \in \mathbb{N}\}$. Consequently, any integer $n > 1$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{3 \times 2^k : k \in \mathbb{N}\}$.

Remark 1.3. Y.-C. Sun and the author [SS, Theorem 1.2(ii)] showed that for each $d = 1, 2, 3$ and $m = 2, 3$, we can write any $n \in \mathbb{N}$ as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + 2y + 2z \in \{dt^m : t \in \mathbb{N}\}$.

In contrast with the 1-3-5 Conjecture, we have the following curious conjecture motivated by Theorem 1.4.

Conjecture 1.1. (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2(y - z)$ is a power of four (including $4^0 = 1$).

(ii) Let $c \in \{1, 2, 4\}$. Then each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y \leq z$ such that $2x + y - z \in \{c8^k : k \in \mathbb{N}\} \cup \{0\}$.

Remark 1.4. (i) We have verified the first part for all $n = 1, \dots, 2 \times 10^7$, and Qing-Hu Hou extended the verification for n up to 10^9 . See [S, A279612 and A279616] for related data. For example,

$$111 = 9^2 + 1^2 + 5^2 + 2^2 \quad \text{with } 9 + 2 \times 1 - 2 \times 5 = 4^0.$$

(ii) We have verified part (ii) of Conjecture 1.1 for all $n = 0, \dots, 2 \times 10^6$. See [S, A284343] for related data. For example,

$$2976 = 20^2 + 16^2 + 48^2 + 4^2 \quad \text{with } 16 < 48 \text{ and } 2 \times 20 + 16 - 48 = 8.$$

In 1917 Ramanujan [R] listed 55 possible quadruples (a, b, c, d) of positive integers with $a \leq b \leq c \leq d$ such that any $n \in \mathbb{N}$ can be written as $ax^2 + by^2 + cz^2 + dw^2$ with $x, y, z, w \in \mathbb{Z}$, and 54 of them were later confirmed by Dickson [D27] with the remaining one wrong (see also [W]).

Theorem 1.5. (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y = 1$. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + 2y^2 + 2z^2 + 2w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + z = 1$.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + 2z = 1$. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + 4y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + 2y + 2z = 1$, and any $n \in \mathbb{Z}^+$ can be written as $x^2 + 2y^2 + 2z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + 3z = 1$.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $y + z + 2w = 1$. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + 4z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $y + 2z + 2w = 1$, and any $n \in \mathbb{Z}^+$ can be written as $x^2 + 2y^2 + 2z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + z + 2w = 1$.

(iv) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $y + z + w = 1$.

(v) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + 2z^2 + 5w^2$ with $x, y, z, w \in \mathbb{Z}$ and $y + w = 1$. If any positive integer congruent to 1 modulo 7 can be written as $x^2 + 7y^2 + 14z^2$ with $x, y, z \in \mathbb{Z}$, then any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + 2z^2 + 6w^2$ with $y + w = 1$.

(vi) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 3w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + 2z = 2$.

(vii) Any integer $n > 4$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + z = t^2$ for some $t = 1, 2$.

(viii) Any integer $n > 7$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + 2z = 2t^2$ for some $t = 1, 2$.

Remark 1.5. We can also prove several other results similar to the first assertion in Theorem 1.3(v). Concerning the second assertion in Theorem 1.3(iii), we conjecture that the condition always holds, i.e., for any $n \in \mathbb{N}$ we can write $7n + 1 = x^2 + 7y^2 + 14z^2$ with $x, y, z \in \mathbb{Z}$. I. Kaplansky [K] reported that 2, 74 and 506 are the only positive integers $n \leq 10^5$ with $n \equiv 1, 2, 4 \pmod{7}$ which cannot be represented by $x^2 + 7y^2 + 14z^2$ with $x, y, z \in \mathbb{Z}$.

Theorem 1.6. (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + z + w$ a square (or a cube).

(ii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + 2y + 2z$ a square (or a cube).

Remark 1.5. Our proof of Theorem 1.6 depends heavily on a new identity similar to Euler's four-square identity. We even conjecture that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}$) with $x + y - z + w \in \{0, 1\}$.

We will prove Theorems 1.1-1.4 and 1.5-1.6 in Sections 2 and 3 respectively, and pose some related conjectures in Section 4.

2. PROOFS OF THEOREMS 1.1-1.4

It is known that the set

$$E(a, b, c) := \{n \in \mathbb{N} : n \neq ax^2 + by^2 + cz^2 \text{ for all } x, y, z \in \mathbb{Z}\} \quad (2.1)$$

is infinite for any $a, b, c \in \mathbb{Z}^+$.

Lemma 2.1 (The Gauss-Legendre Theorem). *We have*

$$E(1, 1, 1) = E_0 := \{4^k(8l + 7) : k, l \in \mathbb{N}\}. \quad (2.2)$$

Remark 2.1. This is a well-known result on sums of three squares, see, e.g., [N96, p. 23] or [MW, p. 42]).

Proof of Theorem 1.1. (i) If $n - 1 \notin E(1, 1, 1)$, then $n = 8^0 + x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{N}$.

Below we assume that $n - 1 \in E(1, 1, 1)$. Then $n - 1 = 4^k(8l + 7)$ for some $k, l \in \mathbb{N}$.

If $k > 0$, then $n - 8 \equiv n \equiv 1 \pmod{4}$ and hence $n - 8 = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{N}$.

Now we consider the case $k = 0$. In this case $n = 8l + 8$. If l is odd, then by Lemma 2.1 we can write $n - 8 = 2^3l$ as $x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{N}$. Clearly,

$$8 = 0^2 + 0^2 + 0^2 + 8, \quad 3 \times 8 = 4^2 + 0^2 + 0^2 + 8, \quad 5 \times 8 = 4^2 + 4^2 + 0^2 + 8, \quad 7 \times 8 = 4^2 + 4^2 + 4^2 + 8.$$

If l is even and at least 8, then $n - 64 = 8(l + 1) - 64 = 2^3(l - 7) \notin E(1, 1, 1)$ and hence $n = 8^2 + x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{N}$. This proves Theorem 1.1(i).

(ii) Write $n = 2^a m$ with $a \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $2 \nmid m$. Let $n_0 = n/4^{\lfloor a/2 \rfloor} = 2^{a_0} m$, where a_0 is 0 or 1 according as a is even or odd. As $4 \nmid n_0$, we have $2n_0 - 1 \not\equiv 7 \pmod{8}$. By Lemma 2.1, $2n_0 - 1 = u^2 + v^2 + (2y + 1)^2$ for some $u, v, y \in \mathbb{N}$ with $u \equiv v \pmod{2}$. Let $z = (u + v)/2$ and $w = |u - v|/2$. Then $2n_0 = 4y^2 + 4y + (z + w)^2 + (z - w)^2$ and hence $n_0 = x^2 + y^2 + z^2 + w^2$ with $x = y + 1$. It follows that

$$n = \left(2^{\lfloor a/2 \rfloor} x\right)^2 + \left(2^{\lfloor a/2 \rfloor} y\right)^2 + \left(2^{\lfloor a/2 \rfloor} z\right)^2 + \left(2^{\lfloor a/2 \rfloor} w\right)^2$$

with

$$2^{\lfloor a/2 \rfloor} x - 2^{\lfloor a/2 \rfloor} y = 2^{\lfloor a/2 \rfloor}.$$

(iii) Obviously, for any $k \in \mathbb{N}$ we have

$$2^{6k+3} \times 7 = (8^k \times 6)^2 + (8^k \times 4)^2 + (8^k \times 2)^2 + 0^2 \quad \text{with } 8^k \times 6 + 8^k \times 2 = 8^{k+1}.$$

So it suffices to prove the first assertion in Theorem 1.1(iii) by induction.

For each $n \in \{0, 1, \dots, 63\} \setminus \{56\}$, we can verify via a computer that n can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$.

Now let $n \geq 64$ be an integer not of the form $2^{6k+3} \times 7 = 64^k \times 56$ ($k \in \mathbb{N}$), and assume that any $m \in \{0, 1, \dots, n - 1\}$ not of the form $2^{6k+3} \times 7$ ($k \in \mathbb{N}$) can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$.

If $64 \mid n$, then by the induction hypothesis we can write $n/64$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x - y \in \{8^k : k \in \mathbb{N}\} \cup \{0\}$, and hence

$$n = (8x)^2 + (8y)^2 + (8z)^2 + (8w)^2 \quad \text{with } 8x - 8y = 8(x - y) \in \{8^k : k \in \mathbb{N}\} \cup \{0\}.$$

Below we suppose that $64 \nmid n$.

Case 1. $n \notin 4^k(16l + 14)$ for any $k, l \in \mathbb{N}$.

In this case, we have $2n \notin E(1, 1, 1)$ by (2.2), and hence $2n = (2y)^2 + z^2 + w^2$ for some $y, z, w \in \mathbb{N}$ with $z \equiv w \pmod{2}$. Thus

$$n = y^2 + y^2 + \left(\frac{z+w}{2}\right)^2 + \left(\frac{z-w}{2}\right)^2 \quad \text{with } y - y = 0.$$

Case 2. $n = 16l + 14$ for some $l \in \mathbb{N}$.

In this case, $2n - 1 \equiv 3 \pmod{8}$ and hence by (2.2) we can write n as $(y + 1)^2 + z^2 + w^2$ with $y, z, w \in \mathbb{N}$ and $z \equiv w \pmod{2}$. It follows that

$$n = (y + 1)^2 + y^2 + \left(\frac{z+w}{2}\right)^2 + \left(\frac{z-w}{2}\right)^2 \quad \text{with } (y + 1) - y = 0.$$

Case 3. $n = 4^k(16l + 14)$ for some $k \in \{1, 2\}$ and $l \in \mathbb{N}$.

In this case, $2n - 64 = 4^{k+1}(8l + 3k)$. In light of (2.2), $8l + 3k = x^2 + y^2 + z^2$ for some integers $x \geq y \geq z \geq 0$. If $k = 1$, then $8l + 3k = (2n - 64)/16 \geq 64/16 = 4$. If $k = 2$, then $8l + 3k \geq 3k = 6$. So, $x \geq 2$ and hence $2^{k+1}x = 2v + 8$ for some $v \in \mathbb{N}$. Therefore

$$2n - 64 = (2^{k+1}x)^2 + (2^{k+1}y)^2 + (2^{k+1}z)^2 = (2v + 8)^2 + 2(2^k y + 2^k z)^2 + 2(2^k y - 2^k z)^2$$

and hence

$$n = (v + 8)^2 + v^2 + (2^k(y + z))^2 + (2^k(y - z))^2 \quad \text{with } (v + 8) - v = 8^1.$$

The induction proof of Theorem 1.1(iii) is now complete.

(iv) We distinguish two cases.

Case 1. $4 \nmid n$.

Let

$$\delta_n = 1 - \text{ord}_2(n) = \begin{cases} 1 & \text{if } 2 \nmid n, \\ 0 & \text{if } 2 \parallel n. \end{cases}$$

Then

$$4^{\delta_n} n - 1 \equiv \begin{cases} 3 \pmod{8} & \text{if } 2 \nmid n, \\ 1 \pmod{4} & \text{if } 2 \parallel n. \end{cases}$$

By Lemma 2.1, there are $u, v, w \in \mathbb{Z}$ such that

$$4^{\delta_n} n - 1 = (2^{\delta_n} u - 1)^2 + (2^{\delta_n} v - 1)^2 + (2^{\delta_n} w - 1)^2.$$

If $2 \parallel n$, then $\delta_n = 0$ and $u + v + w \equiv n \equiv 0 \pmod{2}$. In the case $2 \nmid n$, since $(2u - 1)^2 = (2(1 - u) - 1)^2$, without loss of generality we may also assume that $u + v + w \equiv 0 \pmod{2}$. Set

$$x = \frac{u + v - w}{2}, \quad y = \frac{u - v + w}{2}, \quad z = \frac{-u + v + w}{2}.$$

Then $u = x + y$, $v = x + z$ and $w = y + z$. Therefore

$$\begin{aligned} 4^{\delta_n} n - 1 &= (2^{\delta_n}(x + y) - 1)^2 + (2^{\delta_n}(x + z) - 1)^2 + (2^{\delta_n}(y + z) - 1)^2 \\ &= (2^{\delta_n}(x + y + z) - 2)^2 + (2^{\delta_n}x)^2 + (2^{\delta_n}y)^2 + (2^{\delta_n}z)^2 - 1 \end{aligned}$$

and hence

$$n = ((x + y + z) - 2^{1-\delta_n})^2 + x^2 + y^2 + z^2 = x^2 + y^2 + z^2 + (2^{1-\delta_n} - x - y - z)^2$$

with $x + y + z + (2^{1-\delta_n} - x - y - z) = 2^{\lfloor (\text{ord}_2(n)+1)/2 \rfloor}$.

Case 2. $4 \mid n$.

Write $n = 4^k n_0$ with $k, n_0 \in \mathbb{Z}^+$ and $4 \nmid n_0$. By the above, there are $x_0, y_0, z_0, w_0 \in \mathbb{Z}$ such that

$$n_0 = x_0^2 + y_0^2 + z_0^2 + w_0^2 \quad \text{with} \quad x_0 + y_0 + z_0 + w_0 = 2^{\lfloor (\text{ord}_2(n_0)+1)/2 \rfloor}.$$

It follows that

$$n = (2^k x_0)^2 + (2^k y_0)^2 + (2^k z_0)^2 + (2^k w_0)^2$$

with

$$2^k x_0 + 2^k y_0 + 2^k z_0 + 2^k w_0 = 2^{k + \lfloor (\text{ord}_2(n_0)+1)/2 \rfloor} = 2^{\lfloor (\text{ord}_2(n)+1)/2 \rfloor}.$$

Combining the above, we have proved Theorem 1.1. \square

Lemma 2.2. *For any integer $n \geq 16$ with $16 \nmid n$, we have $\{n - 1, n - 16\} \notin E_0$, where E_0 is the set defined in (2.2).*

Proof. Clearly one of $n - 1$ or $n - 16$ is odd. If $n - 1 \equiv 7 \pmod{8}$, then $n - 16 \equiv n \equiv 8 \pmod{16}$. If $n - 16 \equiv 7 \pmod{8}$, then $n - 1 \equiv 7 + 15 \equiv 6 \pmod{8}$. Thus $\{n - 1, n - 16\} \notin E_0$ as desired. \square

Remark 2.2. It follows from Lemmas 2.1 and 2.2 that any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with x a power of two and $y, z, w \in \mathbb{N}$. A stronger result given in [S17b, Theorem 1.2(v)] states that any positive integer can be written as $4^k(1 + 4x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$.

Lemma 2.3 ([S17a, Lemma 2.1]). . Let u and v be integers with $u^2 + v^2$ a positive multiple of 5. Then $u^2 + v^2 = x^2 + y^2$ for some $x, y \in \mathbb{Z}$ with $5 \nmid xy$.

Lemma 2.4. Let $n \geq 4$ be an integer not divisible by 64.

(i) We have $\{n, n-4\} \not\subseteq E_0$.

(ii) If $16 \nmid n$, then $\{n, n-1\} \not\subseteq E_0$. If $16 \mid n$, then $\{n, n-1, n-64\} \not\subseteq E_0$.

Proof. (i) If $n = 4^k(8l+7)$ for some $k \in \mathbb{N}$ and $l \in \mathbb{N}$, then $k < 3$ as $64 \nmid n$, and hence $n-4 \notin E_0$ since

$$n-4 = \begin{cases} 8l+7-4 = 8l+3 & \text{if } k=0, \\ 4(8l+7)-4 = 4(8l+6) & \text{if } k=1, \\ 4^2(8l+7)-4 = 4(8(4l+3)+3) & \text{if } k=2. \end{cases}$$

(ii) If $n = 8l+7$ for some $l \in \mathbb{N}$, then $n-1 = 8l+6 \notin E_0$. If $n = 4(8l+7)$ for some $l \in \mathbb{N}$, then $n-1 = 32l+27 = 8(4l+3)+3 \notin E_0$. So $\{n, n-1\} \not\subseteq E_0$ if $16 \nmid n$.

Now we consider the case $16 \mid n$. As $64 \nmid n$, if $n \notin E_0$, then $n = 4^2(8l+7)$ for some $l \in \mathbb{N}$, and hence $n-64 = 4^2(8l+3) \notin E_0$.

The proof of Lemma 2.4 is now complete. \square

Proof of Theorem 1.2. (i) For $n = 1, \dots, 15$ we can easily verify the desired result.

Now fix an integer $n \geq 16$ and assume that each $m = 1, \dots, n-1$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $|2x-y| \in \{4^k : k \in \mathbb{N}\}$.

Let's first consider the case $16 \mid n$. By the the induction hypothesis, there are $x, y, z, w \in \mathbb{N}$ for which $n/16 = x^2 + y^2 + z^2 + w^2$ with $|2x-y| \in \{4^k : k \in \mathbb{N}\}$, and hence $|2(4x) - 4y| = 4|2x-y| \in \{4^k : k \in \mathbb{N}\}$.

Now we suppose that $16 \nmid n$. Then $\{5n-1, 5n-16\} \not\subseteq E_0$ by Lemma 2.2. Let $\delta = 0$ if $5n-1 \in E_0$, and $\delta = 1$ otherwise. In view of Lemma 2.1, we can write $5n-16^\delta$ as $x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$. Since a square is congruent to one of $0, 1, -1$ modulo 5. one of x^2, y^2, z^2 must be congruent to -1 modulo 5. Without loss of generality, we may assume that $x^2 + 1 \equiv y^2 + z^2 \equiv 0 \pmod{5}$. If $y^2 + z^2 \neq 0$, then by Lemma 2.3 we can write $y^2 + z^2 = y_1^2 + z_1^2$ with $y_1, z_1 \in \mathbb{Z}$ and $5 \nmid y_1 z_1$. Without loss of generality, we simply assume that $x \equiv -2 \times 4^\delta \pmod{5}$ (otherwise we use $-x$ instead of x) and that either $y = z = 0$ or $y \equiv 2z \equiv -2 \times 4^\delta \pmod{5}$.

Clearly, $r = (x + 2 \times 4^\delta)/5$, $s = (2x - 4^\delta)/5$, $u = (2y + z)/5$ and $v = (2z - y)/5$ are all integers. Observe that

$$r^2 + s^2 + u^2 + v^2 = \frac{(4^\delta)^2 + x^2}{5} + \frac{y^2 + z^2}{5} = n \quad \text{with} \quad 2r - s = 4^\delta.$$

If $x > -2$, then $x \geq 2$ since $x \equiv (-1)^{\delta-1} 2 \pmod{5}$, and hence $r, s \in \mathbb{N}$. When $x \leq -2$, clearly $s < 0$, and

$$r > 0 \iff \delta = 1 \ \& \ x = -3.$$

If $r \leq 0$ and $s \leq 0$, then

$$|2|r| - |s|| = |2(-r) - (-s)| = |2r - s| = 4^\delta.$$

Now it remains to consider the case $\delta = 1$ and $x = -3$. Note that

$$\frac{y^2 + z^2}{5} = u^2 + v^2 = n - r^2 - s^2 = n - 1^2 - (-2)^2 > 0$$

and hence $y \equiv 2z \equiv -2 \times 4^\delta \pmod{5}$. When $y \neq -3$, in the spirit of the above arguments, all the four numbers

$$\bar{r} = \frac{y + 2 \times 4^\delta}{5}, \quad \bar{s} = \frac{2y - 4^\delta}{5}, \quad \bar{u} = \frac{2x + z}{5}, \quad \bar{v} = \frac{2z - x}{5}$$

are integral, and

$$\bar{r}^2 + \bar{s}^2 + \bar{u}^2 + \bar{v}^2 = \frac{(4^\delta)^2 + y^2}{5} + \frac{x^2 + z^2}{5} = n$$

with $|2|\bar{r}| - |\bar{s}|| = |2\bar{r} - \bar{s}| = 4^\delta$. If $y = -3$, then

$$5n - 16 = x^2 + y^2 + z^2 = (-3)^2 + (-3)^2 + z^2 \equiv z^2 + 2 \not\equiv 0, 1 \pmod{4}$$

thus $5n - 1 \not\equiv 0, 3 \pmod{4}$ and hence $5n - 1 \notin E_0$ which contradicts $\delta = 1$. This concludes our induction proof of Theorem 1.2(i).

(ii) For every $n = 0, 1, \dots, 63$, we can verify the desired result directly.

Now fix an integer $n \geq 64$ and assume that the desired result holds for all smaller values of n .

If $64 \mid n$, then by the induction hypothesis we can write $n/64$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $2x - y \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\}$, and hence

$$n = (8x)^2 + (8y)^2 + (8z)^2 + (8w)^2 \quad \text{with } 2(8x) - (8y) \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\}.$$

Below we suppose that $64 \nmid n$.

By Lemma 2.4, $5n - (a\delta)^2 \notin E_0$ for some $\delta \in \{0, 1, 8\}$ satisfying

$$\delta = 8 \implies a = 1 \ \& \ 16 \mid n. \tag{2.3}$$

In view of (2.2), $5n - (a\delta)^2 = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{Z}$. Since any square is congruent to one of $\{0, \pm 1\}$ modulo 5, one of x^2, y^2, z^2 , say x^2 , is congruent to $-(a\delta)^2$ modulo 5. Without loss of generality, we simply suppose that $x \equiv -2a\delta \pmod{5}$. As $y^2 \equiv (2z)^2 \pmod{5}$, we may also suppose that $y \equiv 2z \pmod{5}$. Thus all the numbers

$$r = \frac{x + 2a\delta}{5}, \quad s = \frac{2x - a\delta}{5}, \quad u = \frac{2y + z}{5}, \quad v = \frac{y - 2z}{5}$$

are integral. Note that

$$n = \frac{(a\delta)^2 + x^2}{5} + \frac{y^2 + z^2}{5} = r^2 + s^2 + u^2 + v^2 \quad \text{with } 2r - s = a\delta \in \{a8^k : k \in \mathbb{N}\} \cup \{0\}.$$

If $x \geq a\delta/2$, then $r \geq 0$ and $s \geq 0$. If $x \leq -2a\delta$, then $r \leq 0$ and $s \leq -a\delta \leq 0$.

Now we handle the remaining case $-2a\delta < x < a\delta/2$. Clearly, $r > 0 > s$, $a\delta = 2r - s > 2$, hence $\delta = 8$, and $a = 1$ and $16 \mid n$ by (2.3). As $2r - s = 8$, we must have

$$(r, s) \in \{(1, -6), (2, -4), (3, -2)\}.$$

Note that $2 \times 2 - 4 = 0$ and $2 \times 2 - 3 = 1$. If $(r, s) = (1, -6)$, then

$$n = 1^2 + (-6)^2 + u^2 + v^2 \not\equiv 0 \pmod{4},$$

which contradicts $16 \mid n$. This ends our proof of Theorem 1.2(ii). \square

Lemma 2.5. *We have*

$$E(1, 3, 6) = \{3q + 2 : q \in \mathbb{N}\} \cup \{4^k(16l + 14) : k, l \in \mathbb{N}\} \quad (2.4)$$

and

$$E(2, 3, 6) = \{3q + 1 : q \in \mathbb{N}\} \cup E_0. \quad (2.5)$$

Remark 2.3. (2.4) and (2.5) are known results, see, e.g., L. Dickson [D39, pp. 112-113].

Proof of Theorem 1.3. (i) We can easily verify the desired result for $n = 1, \dots, 15$.

Now let $n \geq 16$ and $a \in \{0, 2\}$. Assume that each $m = 1, 2, \dots, n-1$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + z \in \{4^k : k \in \mathbb{N}\}$, and we can also write m as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $|x + y - z| \in \bigcup_{k \in \mathbb{N}} \{a^k, 4^k\}$.

We first suppose that $16 \mid n$. By the induction hypothesis, there are $x, y, z, w \in \mathbb{Z}$ and $k \in \mathbb{N}$ for which $n/16 = x^2 + y^2 + z^2 + w^2$ with $x + y + z = 4^k$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $4x + 4y + 4z = 4^{k+1}$. Also, by the induction hypothesis, there are $x, y, z, w \in \mathbb{N}$ for which $n/16 = x^2 + y^2 + z^2 + w^2$ and $|x + y - z| \in \bigcup_{k \in \mathbb{N}} \{a^k, 4^k\}$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $|4x + 4y - (4z)| = 4|x + y - z| \in \bigcup_{k \in \mathbb{N}} \{a^k, 4^k\}$.

Now we suppose that $16 \nmid n$. By Lemmas 2.2 and 2.5, for some $\delta \in \{0, 1\}$ we have $3n - 16^\delta \notin E(2, 3, 6)$. So, there are $x, y \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that

$$3n - 16^\delta = 3x^2 + 6y^2 + 2(3z - 4^\delta)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 2 \times 4^\delta z)) + 2 \times 16^\delta$$

and hence

$$n = x^2 + 2y^2 + 6z^2 - 4^{\delta+1}z + 16^\delta = x^2 + (y + z)^2 + (z - y)^2 + (4^\delta - 2z)^2$$

with $(y+z) + (z-y) + (4^\delta - 2z) = 4^\delta$.

When $z \geq 2^{2^\delta-1}$, we have $2z \geq 4^\delta$, hence

$$(y+z) + |z-y| - (2z - 4^\delta) = 4^\delta \quad \text{if } y \leq z,$$

and

$$||z-y| + (2z - 4^\delta) - (y+z)| = 4^\delta \quad \text{if } y > z.$$

When $z \leq 0$, we have $y-z \geq 0$ and $4^\delta - 2z > 0$, also

$$|y+z| + (4^\delta - 2z) - (y-z) = 4^\delta \quad \text{if } y+z \geq 0,$$

and

$$|(y-z) + |y+z| - (4^\delta - 2z)| = 4^\delta \quad \text{if } y+z < 0.$$

If $1 \leq z < 2^{2^\delta-1}$, then $\delta = 1$, $z = 1$, $4^\delta - 2z = 2$, and

$$n = x^2 + (y+1)^2 + (y-1)^2 + 2^2 \quad \text{with } (y+1) + 2 - (y-1) = 4.$$

If $\delta = z = 1$ and $y = 0$, then

$$n = x^2 + 1^2 + 1^2 + 2^2 \quad \text{with } 1 + 1 - 2 = 0 \text{ and } 1 + 2 - 1 = 2.$$

In view of the above, we have proved Theorem 1.3(i) by induction.

(ii) Via a computer we can easily verify the desired result for every $n = 0, 1, \dots, 63$.

Now fix an integer $n \geq 64$ and assume that the desired result holds for all smaller values of n .

If $64 \mid n$, then by the induction hypothesis we can write $n/64$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + y - z \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\}$, and hence

$$n = (8x)^2 + (8y)^2 + (8z)^2 + (8w)^2 \quad \text{with } 8x + 8y - 8z = 8(x + y - z) \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\}.$$

Below we suppose that $64 \nmid n$.

By Lemma 2.4, $3n - (a\delta)^2 \notin E_0$ for some $\delta \in \{0, 1, 8\}$ satisfying (2.3). In view of (2.5), for some $x, y, z \in \mathbb{Z}$ we have $3n - (a\delta)^2 = 3x^2 + 6y^2 + 2(3z - a\delta)^2$ and hence

$$n = x^2 + (y+z)^2 + (z-y)^2 + (2z - a\delta)^2.$$

When $z \geq a\delta/2$, obviously $y+z \geq 0$, $2z - a\delta \geq 0$,

$$(y+z) + (z-y) - (2z - a\delta) = a\delta \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\} \quad \text{if } y \leq z,$$

and

$$(y-z) + (2z - a\delta) - (y+z) = -a\delta \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\} \quad \text{if } y > z.$$

If $z < a\delta/2$ and $y \geq |z|$, then $\{a\delta - 2z, y + z, y - z\} \subseteq \mathbb{N}$ and

$$(a\delta - 2z) + (y + z) - (y - z) = a\delta \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\}.$$

If $z < a\delta/2$ and $z \leq -y$, then $\{a\delta - 2z, y - z, -y - z\} \subseteq \mathbb{N}$ and

$$(y - z) + (-y - z) - (a\delta - 2z) = -a\delta \in \{\pm a8^k : k \in \mathbb{N}\} \cup \{0\}.$$

Now we consider the remaining case $y < z < a\delta/2$. Since $a\delta > 2$, we have $\delta = 8$, and hence $a = 1$ and $16 \mid n$ by (2.3). In the case $y < z < 8/2 = 4$, the ordered triple $(y + z, z - y, 8 - 2z)$ is among

$$(1, 1, 6), (2, 2, 4), (3, 3, 2), (3, 1, 4), (4, 2, 2), (5, 1, 2).$$

Note that $2 + 2 - 4 = 0 = 3 + 1 - 4$. If

$$(y + z, z - y, 8 - 2z) \in \{(1, 1, 6), (3, 3, 2), (5, 1, 2)\},$$

then

$$n = x^2 + (y + z)^2 + (z - y)^2 + (8 - 2z)^2 \equiv x^2 + 2 \not\equiv 0 \pmod{4}$$

which contradicts $16 \mid n$. This concludes the proof of Theorem 1.3(ii). \square

Lemma 2.6. *Suppose that $n \in \mathbb{Z}^+$ is the sum of three squares. Then $n = a^2 + b^2 + c^2$ for some $a, b, c \in \mathbb{Z}$ and $a + b \equiv 1 \pmod{3}$.*

Proof. Write $n = x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$. If $x \equiv y \equiv z \equiv 0 \pmod{3}$, then $9 \mid n$ and hence by [S16, Lemma 2.2(ii)] we can write n as $\bar{x}^2 + \bar{y}^2 + \bar{z}^2$ with $\bar{x}, \bar{y}, \bar{z} \in \mathbb{Z}$ and $3 \nmid \bar{x}\bar{y}\bar{z}$. So, without loss of generality we may assume that $3 \nmid x$. If $x + y \equiv 0 \pmod{3}$, then $-x + y \not\equiv 0 \pmod{3}$. Thus, we may simply suppose that $x + y \not\equiv 0 \pmod{3}$ and hence $x + y$ is congruent to 1 or -1 modulo 3. If $x + y \equiv -1 \pmod{3}$, then $n = (-x)^2 + (-y)^2 + z^2$ with $(-x) + (-y) \equiv 1 \pmod{3}$. Therefore, there are $a, b, c \in \mathbb{Z}$ such that $n = a^2 + b^2 + c^2$ and $a + b \equiv 1 \pmod{3}$. \square

Proof of Theorem 1.4. (i) We prove the desired result by induction. For $n = 1, 2, \dots, 42$ we can verify the result directly via a computer.

Now let $n \geq 43$ and assume that any $m = 1, \dots, n - 1$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + 2z = c4^k$ for some $k \in \mathbb{N}$.

If $16 \mid n$, then by the induction hypothesis there are $x, y, z, w \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $n/16 = x^2 + y^2 + z^2 + w^2$ with $x + y + 2z = c4^k$, hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $4x + 4y + 2(4z) = c4^{k+1}$.

Now suppose that $16 \nmid n$.

Clearly, $3n - 2 \not\equiv 14 \pmod{16}$. If $4 \nmid 3n - 2$, then $3n - 2 \notin E(1, 3, 6)$ by (2.4). If $4 \mid 3n - 2$, then $4 \mid 3n - 32$. If $3n - 32 \equiv 14 \pmod{16}$, then $3n - 2 \equiv 12 \pmod{16}$ and hence $3n - 2 \notin E(1, 3, 6)$. Thus, for some $\delta \in \{0, 1\}$, we can write

$$3n - 2 \times 16^\delta = 3x^2 + 6y^2 + (3z - 4^\delta)^2 = 3x^2 + 6y^2 + 9z^2 - 6 \times 4^\delta z + 16^\delta$$

with $x, y, z \in \mathbb{Z}$. It follows that

$$n = x^2 + 2y^2 + 3z^2 - 2 \times 4^\delta z + 16^\delta = x^2 + (y + z)^2 + (z - y)^2 + (4^\delta - z)^2$$

with $(y + z) + (z - y) + 2(4^\delta - z) = 2 \times 4^\delta$. This proves the desired result with $c = 2$.

Below we show the desired result for $c = 1$.

Case 1. $4 \nmid n$.

In this case, $6n - 1 \not\equiv 7 \pmod{8}$ and hence $6n - 1 \notin E(2, 3, 6)$ by (2.5). So, for some $x, y, z \in \mathbb{Z}$ we have

$$6n - 1 = 6x^2 + 3(2y + 1)^2 + 2(3z + 1)^2$$

and hence

$$n = x^2 + 2y^2 + 2y + 3z^2 + 1 = x^2 + (y + z + 1)^2 + (z - y)^2 + (-z)^2$$

with $(y + z + 1) + (z - y) + 2(-z) = 1 = 4^0$.

Case 2. $4 \parallel n$.

In this case, $3n - 8 \equiv 4 \pmod{8}$ and hence $3n - 8 \notin E(1, 3, 6)$ by (2.4). Thus, for some $x, y, z \in \mathbb{Z}$ we have $3n - 8 = 3x^2 + 6y^2 + (3z - 2)^2$ and hence

$$n = x^2 + 2y^2 + 3z^2 - 4z + 4 = x^2 + (y + z)^2 + (z - y)^2 + (2 - z)^2$$

with $(y + z) + (z - y) + 2(2 - z) = 4$.

Case 3. $8 \parallel n$.

Write $n = 8q$ with q odd. Note that $q \geq 6$ and $3n > 128$ since $n \geq 43$. By Lemma 2.2, for some $\delta \in \{0, 1\}$ we have $3q - 16^\delta \notin E_0$ and hence

$$3n - 8 \times 16^\delta = 8(3q - 16^\delta) \notin E(1, 3, 6)$$

by (2.4). Thus, for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 8 \times 16^\delta = 3x^2 + 6y^2 + (3z - 2 \times 4^\delta)^2$$

and hence

$$n = x^2 + 2y^2 + 3z^2 - 4^{\delta+1}z + 4 \times 16^\delta = x^2 + (y + z)^2 + (z - y)^2 + (2 \times 4^\delta - z)^2$$

with $(y+z) + (z-y) + 2(2 \times 4^\delta - z) = 4^{\delta+1}$.

The induction proof of Theorem 1.4(i) is now completed.

(ii) For $n = 1, 2, \dots, 15$ we can easily verify the desired result.

Now let $n \geq 16$ and assume that each $m = 1, 2, \dots, n-1$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + 2y + 2z$ a power of four.

If $16 \mid n$, then by the induction hypothesis there are $x, y, z, w \in \mathbb{Z}$ and $k \in \mathbb{N}$ for which $n/16 = x^2 + y^2 + z^2 + w^2$ with $x + 2y + 2z = 4^k$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $4x + 2(4y) + 2(4z) = 4^{k+1}$.

Now we suppose that $16 \nmid n$. By Lemmas 2.1-2.2, for some $\delta \in \{0, 1\}$ we can write $9n - 16^\delta$ as $a^2 + b^2 + c^2$ with $a, b, c \in \mathbb{Z}$. Clearly, we cannot have $3 \nmid abc$. Without loss of generality, we assume that $c = 3w$ with $w \in \mathbb{Z}$. As $a^2 + b^2 \equiv -16^\delta \equiv 2 \pmod{3}$, we must have $3 \nmid ab$. We may simply suppose that $a = 3u + 2 \times 4^\delta$ and $b = 3v - 2 \times 4^\delta$ with $u, v \in \mathbb{Z}$. (Note that if $x \equiv 1 \equiv -2 \pmod{3}$ then $-x \equiv 2 \pmod{3}$.) Since

$$\begin{aligned} & 12 \times 4^\delta u - 12 \times 4^\delta v + 8(4^\delta)^2 \\ & \equiv (3u + 2 \times 4^\delta)^2 + (3v - 2 \times 4^\delta)^2 = a^2 + b^2 \equiv -(4^\delta)^2 \pmod{9}, \end{aligned}$$

we must have $u \equiv v \pmod{3}$. Set

$$y = \frac{2u + v}{3} \quad \text{and} \quad z = \frac{u + 2v}{3}.$$

Then

$$\begin{aligned} 9n - 16^\delta &= a^2 + b^2 + c^2 = (3u + 2 \times 4^\delta)^2 + (3v - 2 \times 4^\delta)^2 + 9w^2 \\ &= (3(2y - z) + 2 \times 4^\delta)^2 + (3(2z - y) - 2 \times 4^\delta)^2 + 9w^2 \\ &= 9(2y - z)^2 + 9(2z - y)^2 + 12 \times 4^\delta((2y - z) - (2z - y)) + 8(4^\delta)^2 + 9w^2 \end{aligned}$$

and hence

$$n = (2y - z)^2 + (2z - y)^2 + 4^{\delta+1}(y - z) + w^2 + (4^\delta)^2 = (2y - 2z + 4^\delta)^2 + (-y)^2 + z^2 + w^2$$

with $(2y - 2z + 4^\delta) + 2(-y) + 2z = 4^\delta \in \{4^k : k \in \mathbb{N}\}$. This proves Theorem 1.4(ii).

(iii) For any $k \in \mathbb{N}$, we obviously have

$$2^{2k+1} = (2^k)^2 + (2^k)^2 + 0^2 + 0^2 \quad \text{with} \quad 2^k + 2 \times 2^k + 2 \times 0 = 3 \times 2^k,$$

and

$$2^{2k+2} = (-2^k)^2 + (2^k)^2 + (2^k)^2 + (2^k)^2 \quad \text{with} \quad (-2^k) + 2 \times 2^k + 2 \times 2^k = 3 \times 2^k.$$

So, it suffices to prove the first assertion in Theorem 1.4(iii).

For $n \in \{1, \dots, 15\}$ with $n \neq 1, 8$, we can easily verify that n can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + 2y + 2z \in \{3 \times 4^k : k \in \mathbb{N}\}$. For example, $4 = (-1)^2 + 1^2 + 1^2 + 1^2$ with $(-1) + 2 \times 1 + 2 \times 1 = 3$.

Now let $n \geq 16$ with $n \notin S = \bigcup_{k \in \mathbb{N}} \{2^{4k}, 2^{4k+3}\}$, and assume that each $m = 1, 2, \dots, n-1$ with $m \notin S$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + 2y + 2z \in \{3 \times 4^k : k \in \mathbb{N}\}$.

If $16 \mid n$, then by the induction hypothesis there are $x, y, z, w \in \mathbb{Z}$ and $k \in \mathbb{N}$ for which $n/16 = x^2 + y^2 + z^2 + w^2$ with $x + 2y + 2z = 3 \times 4^k$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $4x + 2(4y) + 2(4z) = 3 \times 4^{k+1}$.

Now we suppose that $16 \nmid n$. By Lemmas 2.1-2.2, for some $\delta \in \{0, 1\}$ we can write $n - 16^\delta$ as the sum of three squares. Combining this with Lemma 2.6, we see that $n - 16^\delta = a^2 + b^2 + c^2$ for some $a, b, c \in \mathbb{Z}$ with $a + b \equiv 1 \pmod{3}$. Let $u = a - 2^{2\delta+1}$ and $v = b - 2^{2\delta+1}$. Then $u + v \equiv a - 2 + (b - 2) \equiv 0 \pmod{3}$, and

$$y = \frac{2u - v}{3} \quad \text{and} \quad z = \frac{2v - u}{3}$$

are both integers. Observe that

$$n - 16^\delta = (u + 2 \times 4^\delta)^2 + (v + 2 \times 4^\delta)^2 + c^2 = (2y + z + 2 \times 4^\delta)^2 + (y + 2z + 2 \times 4^\delta)^2 + c^2$$

and hence

$$n = (2y + 2z + 3 \times 4^\delta)^2 + (-y)^2 + (-z)^2 + c^2$$

with $(2y + 2z + 3 \times 4^\delta) + 2(-y) + 2(-z) = 3 \times 4^\delta$.

So far we have completed the proof of Theorem 1.4. \square

3. PROOFS OF THEOREMS 1.5 AND 1.6

Let us first recall some known results on ternary quadratic forms.

Lemma 3.1. *We have*

$$E(1, 2, 4) = \{4^k(16l + 14) : k, l \in \mathbb{N}\}, \quad (3.1)$$

$$E(1, 6, 9) = \{3q + 2 : q \in \mathbb{N}\} \cup \{9^k(9l + 3) : k, l \in \mathbb{N}\}. \quad (3.2)$$

$$E(2, 3, 12) = \{16q + 6 : q \in \mathbb{N}\} \cup \{9^k(3l + 1) : k, l \in \mathbb{N}\}, \quad (3.3)$$

$$E(1, 5, 10) = \bigcup_{k, l \in \mathbb{N}} \{25^k(5l + 2), 25^k(5l + 3)\}, \quad (3.4)$$

$$E(2, 5, 10) = \{8q + 3 : q \in \mathbb{N}\} \cup \bigcup_{k, l \in \mathbb{N}} \{25^k(5l + 1), 25^k(5l + 4)\}. \quad (3.5)$$

Remark 3.1. (3.1)-(3.5) can be found in L. E. Dickson [D39, pp. 112-113].

Lemma 3.2. *Let $n \in \mathbb{Z}^+$ and $\delta \in \{0, 1\}$. Then $6n + 1 = x^2 + 3y^2 + 6z^2$ for some $x, y, z \in \mathbb{Z}$ with $x \equiv \delta \pmod{2}$.*

Remark 3.2. This appeared in Sun [S17a, Remark 3.1].

We also need the following result in [S15].

Lemma 3.3. (i) *Any $n \in \mathbb{N}$ with $n \equiv 4 \pmod{12}$ can be written as $x^2 + 3y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ and $2 \nmid x$.*

(ii) *For $n \in \mathbb{N}$ with $n \equiv 4 \pmod{8}$, we have*

$$|\{(x, y) \in \mathbb{Z}^2 : x^2 + 3y^2 = n \text{ and } 2 \nmid xy\}| = \frac{2}{3} |\{(x, y) \in \mathbb{Z}^2 : x^2 + 3y^2 = n\}|. \quad (3.6)$$

Remark 3.3. For parts (i) and (ii) one may consult Theorem 1.7(iii) and Lemma 3.2 of Sun [S15].

Proof of Theorem 1.5. (i) By (3.1), we can write $2n - 1 = x^2 + 2z^2 + 4w^2$ with $x, z, w \in \mathbb{N}$. As x is odd, we may write $x = 2y + 1$ with $y \in \mathbb{N}$. Thus $2n - 1 = (2y + 1)^2 + 2z^2 + 4w^2$ and hence $n = x^2 + y^2 + z^2 + 2w^2$ with $x = y + 1$.

By the above, we can write any $n \in \mathbb{N}$ as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y = 1$. Clearly, z is congruent to x or y modulo 2. Without loss of generality, we may assume that $y \equiv z \pmod{2}$. Then

$$n = x^2 + 2 \left(\frac{y+z}{2} \right)^2 + 2 \left(\frac{y-z}{2} \right)^2 + 2w^2 \quad \text{with } x + \frac{y+z}{2} + \frac{y-z}{2} = 1.$$

(ii) Let $n \in \mathbb{Z}^+$. By (3.3), $6n - 1 \notin E(2, 3, 12)$. So $6n - 1 = 2u^2 + 3v^2 + 12w^2$ for some $u, v, w \in \mathbb{Z}$. As $2 \nmid v$, we may write $v = 2y + 1$ with $y \in \mathbb{Z}$. Since $3 \nmid u$, we may write u or $-u$ as $3z + 1$ with $z \in \mathbb{Z}$. Thus

$$6n - 1 = 2(3z + 1)^2 + 3(2y + 1)^2 + 12w^2$$

and hence

$$n = (y + z + 1)^2 + (z - y)^2 + (-z)^2 + 2w^2$$

with $(y + z + 1) + (z - y) + 2(-z) = 1$.

By the above, $n = x^2 + y^2 + z^2 + 2w^2$ for some $x, y, z, w \in \mathbb{Z}$ with $x + y + 2z = 1$. As $x + y$ is odd, x or y is even. Thus $n = u^2 + (2v)^2 + z^2 + 2w^2$ for some $u, v \in \mathbb{Z}$ with $u + 2v + 2z = 1$. Clearly, x or y is congruent to z modulo 2. Without any loss of generality, we may assume that $y \equiv z \pmod{2}$. Observe that

$$n = x^2 + 2 \left(\frac{z-y}{2} \right)^2 + 2 \left(\frac{y+z}{2} \right)^2 + 2w^2$$

with

$$x + \frac{z-y}{2} + 3\frac{y+z}{2} = x + y + 2z = 1.$$

(iii) As $4n - 1 \notin E(1, 2, 4)$ by (3.1), we have $4n - 1 = u^2 + 2v^2 + 4x^2$ for some $u, v, x \in \mathbb{Z}$. Since u or $-u$ is congruent to 1 modulo 4, without loss of generality we may assume that $u = 4y + 1$ with $y \in \mathbb{Z}$. As $u^2 \not\equiv -1 \pmod{4}$, we have $v = 2z + 1$ for some $z \in \mathbb{Z}$. Thus $4n - 1 = 4x^2 + (4y + 1)^2 + 2(2z + 1)^2$ and hence

$$n = x^2 + 4y^2 + 2y + 2z^2 + 2z + 1 = x^2 + (y + z + 1)^2 + (y - z)^2 + 2(-y)^2$$

with $(y + z + 1) + (y - z) + 2(-y) = 1$.

By the above, there are $x, y, z, w \in \mathbb{Z}$ with $n = x^2 + y^2 + z^2 + 2w^2$ and $y + z + 2w = 1$. As $y + z$ is odd, one of y and z is odd and another is even. If $z = 2t$ with $t \in \mathbb{Z}$, then $n = x^2 + y^2 + 4t^2 + 2w^2$ with $y + 2t + 2w = 1$. Clearly x is congruent to y or z modulo 2; if $x \equiv y \pmod{2}$ then

$$n = 2 \left(\frac{x + y}{2} \right)^2 + 2 \left(\frac{y - x}{2} \right)^2 + z^2 + 2w^2$$

with

$$\frac{x + y}{2} + \frac{y - x}{2} + z + 2w = y + z + 2w = 1.$$

This proves Theorem 1.5(iii).

(iv) We first suppose that $n \not\equiv 1 \pmod{8}$. Then $5n - 2 \not\equiv 3 \pmod{8}$ and hence $5n - 2 \notin E(2, 5, 10)$ by (3.5). So $5n - 2 = 2v^2 + 5x^2 + 10z^2$ for some $v, x, z \in \mathbb{Z}$. As $v^2 \equiv -1 \equiv 2^2 \pmod{5}$, v or $-v$ is congruent to -2 modulo 5. Without any loss of generality, we may assume that $v = 5y - 2$ with $y \in \mathbb{Z}$. Thus

$$5n = 2 + 2(5y - 2)^2 + 5x^2 + 10z^2$$

and hence

$$n = x^2 + 10y^2 - 8y + 2 + 2z^2 = x^2 + (y + z)^2 + (y - z)^2 + 2(1 - 2y)^2$$

with $(y + z) + (y - z) + (1 - 2y) = 1$.

Now we handle the case $n \equiv 1 \pmod{8}$. As $10n - 4 \notin E(1, 5, 10)$ by (3.4), there are $u, v, x \in \mathbb{Z}$ with $10n - 4 = u^2 + 5v^2 + 10x^2$. If u and v are both even, then $10x^2 \equiv 10n - 4 \equiv 2 \pmod{4}$ and hence $2 \nmid x$, thus

$$\left(\frac{u}{2} \right)^2 + \left(\frac{v}{2} \right)^2 \equiv \left(\frac{u}{2} \right)^2 + 5 \left(\frac{v}{2} \right)^2 = \frac{10(n - x^2) - 4}{4} \equiv -1 \pmod{4}$$

which is impossible. Thus $2 \nmid uv$ and we can write $v = 2z + 1$ with $z \in \mathbb{Z}$. Since $u^2 \equiv 1 \pmod{5}$, without loss of generality we may assume that $u = 10y + 1$ for some $y \in \mathbb{Z}$. Thus $10n = 4 + (10y + 1)^2 + 5(2z + 1)^2 + 10x^2$ and hence

$$n = x^2 + 10y^2 + 2y + 2z^2 + 2z + 1 = x^2 + (y + z + 1)^2 + (y - z)^2 + 2(-2y)^2$$

with $(y + z + 1) + (y - z) + (-2y) = 1$.

(v) Let $n \in \mathbb{Z}^+$. By Lemma 3.2, we can write $6n - 5 = u^2 + 3v^2 + 6x^2$ with $u, v, x \in \mathbb{Z}$ and $2 \nmid u$. As u or $-u$ is congruent to -1 modulo 6, we may simply assume that $u = 6w - 1$ for some $w \in \mathbb{Z}$. Since v is even, $v = 2z$ for some $z \in \mathbb{Z}$. Thus

$$6n - 5 = (6w - 1)^2 + 3(2z)^2 + 6x^2 = 36w^2 - 12w + 1 + 12z^2 + 6x^2$$

and hence

$$n = 6w^2 - 2w + 1 + 2z^2 + x^2 = x^2 + (1 - w)^2 + 2z^2 + 5w^2$$

with $(1 - w) + w = 1$.

Suppose that any positive integer congruent to 1 modulo 7 can be written as $x^2 + 7y^2 + 14z^2$ with $x, y, z \in \mathbb{Z}$. Then $7n - 6 = v^2 + 7x^2 + 14z^2$ for some $v, x, z \in \mathbb{Z}$. Since $v^2 \equiv 1 \pmod{7}$, without loss of generality we may assume that $v = 7w - 1$ for some $w \in \mathbb{Z}$. Thus

$$n = \frac{7x^2 + 14z^2 + (7w - 1)^2 + 6}{7} = x^2 + 2z^2 + 7w^2 - 2w + 1 = x^2 + (1 - w)^2 + 2z^2 + 6w^2$$

with $(1 - w) + w = 1$.

(vi) As $3n - 2 \notin E(1, 6, 9)$ by (3.2), there are $u, v, w \in \mathbb{Z}$ such that

$$3n - 2 = (3u + 2)^2 + 6(v + 1)^2 + 9w^2$$

and hence

$$n = 3u^2 + 4u + 4 + 2v^2 + 4v + 3w^2 = (u + v + 2)^2 + (u - v)^2 + (-u)^2 + 3w^2$$

with $(u + v + 2) + (u - v) + 2(-u) = 2$.

(vii) Note that $5 = 1^2 + (-1)^2 + 1^2 + 2 \times 1^2$ with $1 + (-1) + 1 = 1^2$.

Below we let $n \in \mathbb{N}$ with $n \geq 6$. Choose $c \in \{1, 4\}$ such that $6n - 2c^2 \equiv 0 \pmod{4}$. Note that $6n - 2c^2 \equiv c^2 \equiv 1 \pmod{3}$ and hence $6n - 2c^2 \equiv 4 \pmod{12}$. By Lemma 3.3(i), there are $r, s, t \in \mathbb{Z}$ with r odd such that $6n - 2c^2 = r^2 + 3s^2 + 3t^2$. As $s \not\equiv t \pmod{2}$, without loss of generality we simply assume that $2 \nmid s$ and $t = 2w$ with $w \in \mathbb{Z}$. Since $r^2 + 3s^2 \equiv 1 + 3 = 4 \pmod{8}$, by Lemma 3.3(ii) we can write $r^2 + 3s^2 = u^2 + 3v^2$ with $u, v \in \mathbb{Z}$ and $u \equiv v \equiv c \pmod{2}$. Clearly $v = 2y + c$ for some $y \in \mathbb{Z}$. As u or $-u$ is congruent to c modulo 3, we may write u or $-u$ as $6z + c$ with $z \in \mathbb{Z}$. Thus

$$6n - 2c^2 = (6z + c)^2 + 3(2y + c)^2 + 3(2w)^2 = 12y^2 + 12c(y + z) + 36z^2 + 12w^2 + 4c^2$$

and hence

$$n = 2y^2 + 2z^2 + 2c(y + z) + c^2 + 4z^2 + 2w^2 = (y + z + c)^2 + (z - y)^2 + (-2z)^2 + 2w^2$$

with $(y + z + c) + (z - y) + (-2z) = c \in \{t^2 : t = 1, 2\}$.

(viii) For $n = 8, 9, 10$ we can easily verify that n can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + 2z \in \{2t^2 : t = 1, 2\}$.

Now assume that $n \geq 11$. As $6n - 2^2 \not\equiv 6n - 8^2 \pmod{16}$, by (3.3) $6n - c^2 \notin E(2, 3, 12)$ for some $c \in \{2t^2 : t = 1, 2\}$. Then we can write $6n - c^2 = 2(3z + c)^2 + 3(2y + c)^2 + 12w^2$ with $y, z, w \in \mathbb{Z}$. It follows that

$$n = (y + z + c)^2 + (z - y)^2 + (-z)^2 + 2w^2$$

with $(y + z + c) + (z - y) + 2(-z) = c \in \{2t^2 : t = 1, 2\}$.

Combining the above, we have completed the proof of Theorem 1.5. \square

Lemma 3.4. *We have the new identity*

$$\begin{aligned} & (a^2 + ab + b^2)(av^2 + b(s^2 + t^2 + u^2)) \\ &= a(b(s + t - u) + (a - b)v)^2 + b(bs + au + av)^2 \\ & \quad + b(at + bu - av)^2 + b(as - bt - av)^2. \end{aligned} \quad (3.7)$$

In particular,

$$7(s^2 + t^2 + u^2 + 2v^2) = x^2 + y^2 + z^2 + 2w^2, \quad (3.8)$$

where

$$x = s + 2u + 2v, \quad y = -2t - u + 2v, \quad z = 2s - t - 2v, \quad w = s + t - u + v. \quad (3.9)$$

Proof. By expanding and simplifying the right-hand side of (3.7), we see that (3.7) does hold. Putting $a = 2$ and $b = 1$ in (3.7), we immediately obtain (3.8). \square

Lemma 3.5. *Let $m \in \{2, 3\}$. For any integer $n \geq 2 \times 7^{2m-1}$, there are integers x, y, z and $w \in \{0, 7^m\}$ such that*

$$7n = x^2 + y^2 + z^2 + 2w^2 \quad \text{with } x + 3y \equiv 0 \pmod{7}. \quad (3.10)$$

Proof. Note that $7n \geq 2 \times 7^{2m} = 2 \times 49^m$. If $7n \in E(1, 1, 1)$, then $7n = 4^k(8l + 7)$ for some $k, l \in \mathbb{N}$, and hence $7n - 2 \times 49^m \notin E(1, 1, 1)$. So, for some $w \in \{0, 7^m\}$ we can write $7n - 2w^2$ as $x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$. Note that $x^2 + y^2 + z^2 \equiv 0 \pmod{7}$.

Since $x^2 + y^2 \equiv 6z^2 \pmod{7}$, we have $x^2 \equiv (2z)^2$ and $y^2 \equiv (3z)^2$, or $x^2 \equiv (3z)^2$ and $y^2 \equiv (2z)^2$. Without any loss of generality, we may assume that $x \equiv 2z \pmod{7}$ and $y \equiv -3z \pmod{7}$, hence $x + 3y \equiv 2z - 9z \equiv 0 \pmod{7}$. This concludes our proof. \square

Proof of Theorem 1.6. For $n = 0, 1, \dots, 2 \times 7^{2m-1} - 1$ we can verify the desired results via a computer.

Below we fix an integer $n \geq 2 \times 7^{2m-1}$.

(i) By Lemma 3.5, there are $x, y, z \in \mathbb{Z}$ and $w \in \{0, 7^m\}$ satisfying (3.10). Define

$$\begin{cases} s = \frac{x+2z+2w}{7}, \\ t = \frac{2w-2y-z}{7}, \\ u = \frac{2x-y-2w}{7}, \\ v = \frac{x+y-z+w}{7}. \end{cases} \quad (3.11)$$

It is easy to see that (3.9) holds. Thus, by Lemma 3.4 we have (3.8) and hence

$$n = \frac{7n}{7} = \frac{x^2 + y^2 + z^2 + 2w^2}{7} = s^2 + t^2 + u^2 + 2v^2.$$

Note that $x + 3y \equiv w \equiv 0 \pmod{7}$ and

$$\begin{aligned} z^2 &= 7n - x^2 - y^2 - 2w^2 \equiv -x^2 - y^2 \\ &\equiv 6x^2 + 27y^2 = (2x - 3y)^2 + 2(x + 3y)^2 \\ &\equiv (2x - 3y)^2 \pmod{7}. \end{aligned}$$

Without loss of generality, we may assume that $z \equiv 2x - 3y \pmod{7}$. Now it is easy to see that $s, t, u, v \in \mathbb{Z}$ and that $s + t - u + v = w$ is an m -th power.

(ii) Choose $x \in \{0, 7^m\}$ such that $7n - x^2$ is odd. By Dickson [D39, pp. 112-113],

$$E(1, 1, 2) = \{4^k(16l + 14) : k, l \in \mathbb{N}\}.$$

So $7n - x^2 = y^2 + z^2 + 2w^2$ for some $y, z, w \in \mathbb{Z}$. As $y^2 + z^2 \equiv 5w^2 \pmod{7}$, we have $y^2 \equiv (2w)^2 \pmod{7}$ and $z^2 \equiv w^2 \pmod{7}$, or $y^2 \equiv w^2 \pmod{7}$ and $z^2 \equiv (2w)^2 \pmod{7}$. Without any loss of generality, we may assume that $y \equiv -2w \pmod{7}$ and $z \equiv -w \pmod{7}$. Now it is easy to see that the numbers s, t, u, v given by (3.9) are all integral. Note that $n = s^2 + t^2 + u^2 + 2v^2$ by Lemma 3.4. Clearly, $s + 2u + 2v = x$ is an m -th power.

The proof of Theorem 1.6 is now complete. \square

4. SOME OPEN CONJECTURES

In this section, we pose some open conjectures for further research.

Conjecture 4.1. (i) Any integer $n > 1$ can be written as $x^4 + y^3 + z^2 + 2^k$ with $x, y, z \in \mathbb{N}$ and $k \in \mathbb{Z}^+$. Also, each $n \in \mathbb{Z}^+$ can be written as $3x^4 + y^3 + z^2 + 2^k$ with $k, x, y, z \in \mathbb{N}$, and any $n \in \mathbb{Z}^+$ can be written as $4x^4 + y^3 + z^2 + 3^k$ with $k, x, y, z \in \mathbb{N}$.

(ii) Each $n \in \mathbb{Z}^+$ can be written as $x^4 + y^2 + z^2 + 2^k$ with $x, y, z \in \mathbb{N}$ and $k \in \{0, 1, 2, 3, 4, 5\}$, and any $n \in \mathbb{Z}^+$ can be written as $x^4 + 4y^2 + z^2 + 3^k$ with $x, y, z \in \mathbb{N}$ and $k \in \{0, 1, 2, 3, 4\}$.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^6 + 3y^2 + z^2 + 2^k$ with $k, x, y, z \in \mathbb{N}$, any $n \in \mathbb{Z}^+$ can be written as $2x^6 + 2y^2 + z^2 + 5^k$ with $k, x, y, z \in \mathbb{N}$.

(iv) For each $m = 6, 7, 8$, any $n \in \mathbb{Z}^+$ can be written as $4x^m + 2y^2 + z^2 + 3^k$ with $k, x, y, z \in \mathbb{N}$.

(v) Each $n \in \mathbb{Z}^+$ can be written as $4x^5 + 3y^2 + 2z^2 + 2^k$ with $k, x, y, z \in \mathbb{N}$. Also, any $n \in \mathbb{Z}^+$ can be written as $ax^5 + by^2 + z^2 + 2^k$ with $k, x, y, z \in \mathbb{N}$, whenever (a, b) is among the ordered pairs

$$(1, 3), (2, 1), (2, 2), (2, 6), (3, 2), (3, 3), (4, 2), (5, 1), (5, 3), (6, 2), \\ (6, 5), (9, 2), (10, 1), (10, 2), (12, 2), (13, 1), (13, 3), (19, 2), (20, 2).$$

(vi) Any $n \in \mathbb{Z}^+$ can be written as $ax^5 + by^2 + z^2 + 3^k$ with $k, x, y, z \in \mathbb{N}$, whenever (a, b) is among the ordered pairs $(1, 4), (4, 1), (4, 2), (5, 4)$.

(vii) For each $a = 2, 3$, any $n \in \mathbb{Z}^+$ can be written as $ax^5 + 2y^2 + z^2 + 5^k$ with $k, x, y, z \in \mathbb{N}$.

Remark 4.1. The author verified the first assertion in part (i) for n up to 2×10^7 , and Qing-Hu Hou extended the verification for n up to 10^9 ; see [S, A280356] for related data. We have verified all other assertions in parts (i)-(iv) for n up to 10^7 .

Conjecture 4.2. (i) Let $a, b, c \in \mathbb{Z}^+$ with $b \leq c$. Then, any $n \in \mathbb{Z}^+$ can be written as $ax^3 + by^2 + cz^2 + 4^k$ with $k, x, y, z \in \mathbb{N}$ if and only if (a, b, c) is among the following triples

$$(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 1, 5), (1, 1, 6), (1, 2, 3), (1, 2, 5), \\ (2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 1, 6), (4, 1, 2), (5, 1, 2), (8, 1, 2), (9, 1, 2).$$

(ii) For each $a \in \{3, 4, 6\}$, any $n \in \mathbb{Z}^+$ can be written as $x^3 + 2y^3 + z^2 + a^k$ with $k, x, y, z \in \mathbb{N}$. Also, for each $a \in \{5, 8, 9\}$, any $n \in \mathbb{Z}^+$ can be written as $x^2 + 2y^2 + z^3 + a^k$ with $k, x, y, z \in \mathbb{N}$.

(iii) For each $a = 7, 9, 11, 12, 13$, any $n \in \mathbb{Z}^+$ can be written as $ax^2 + 2y^2 + z^2 + 5^k$ with $k, x, y, z \in \mathbb{N}$. Also, for each $a = 7, 9$, any $n \in \mathbb{Z}^+$ can be written as $ax^2 + 2y^2 + z^2 + 6^k$ with $k, x, y, z \in \mathbb{N}$.

Remark 4.2. See [S, A280153] for some data related to part (ii) of Conjecture 4.2.

Conjecture 4.3. (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $P(x, y, z) \in \{2^k : k \in \mathbb{N}\}$, whenever $P(x, y, z)$ is among the polynomials

$$2x - y, 2x - 3y, x + (y - z)/3, 2x + (y - z)/3, 2x - 2y - z, \\ 4x - 2y - z, 4x - 3y - z, 4x - 4y - 3z, x + y - z, x + y - 2z, x + 2y - z, \\ x + 2y - 2z, x + 3y - z, x + 3y - 2z, x + 3y - 3z, x + 3y - 4z, x + 3y - 5z, \\ x + 4y - z, x + 4y - 2z, x + 4y - 3z, x + 4y - 4z, x + 5y - z, x + 5y - 2z, \\ x + 5y - 4z, x + 5y - 5z, x + 6y - 3z, x + 7y - 4z, x + 7y - 7z, x + 8y - z, \\ x + 9y - 2z, 2x + 3y - z, 2x + 3y - 3z, 2x + 3y - 4z, 2x + 5y - z, \\ 2x + 5y - 3z, 2x + 5y - 4z, 2x + 5y - 5z, 2x + 7y - z, 2x + 7y - 3z, \\ 2x + 7y - 7z, 2x + 9y - 3z, 2x + 11y - 5z, 3x + 4y - 3z, 7x + 8y - 7z.$$

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $Q(x, y, z, w) \in \{2^k : k \in \mathbb{N}\}$, whenever $Q(x, y, z, w)$ is among the polynomials

$$\begin{aligned} & x + y + 2z - 2w, x + y + 2z - 3w, x + y + 2z - 4w, x + 2y + 2z - 3w, \\ & x + 2y + 3z - 4w, x + 2y + 4z - 3w, x + 2y + 6z - 7w, x + 3y + 4z - 4w, \\ & x + 4y + 6z - 5w, 2x + 3y + 5z - 4w, 2x + y - z - w, 2x + y - 2z - w, \\ & 2x + y - 3z - w, 2x + 2y - 3z - 2w, 3x + y - 3z - 2w, 3x + y - 4z - 2w, \\ & 3x + 2y - 2z - w, 3x + 2y - 3z - w, 3x + 2y - 3z - 2w, 3x + y - 2z - w, \\ & 3x + 2y - 6z - w, 4x + 3y - 4z - 3w, 4x + 3y - 5z - w, 4x + 3y - 5z - 2w, \\ & 5x + 2y - 2z - w, 5x + 2y - 3z - 2w, 5x + 2y - 4z - 3w, 5x + 4y - 5z - w, \\ & 7x + y - 6z - 2w, 8x + 3y - 3z - 2w, 8x + 3y - 10z - w, 9x + y - 4z - w. \end{aligned}$$

Conjecture 4.4. (i) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $P(x, y, z, w) \in \{4^k : k \in \mathbb{N}\} \cup \{0\}$, whenever $P(x, y, z, w)$ is among the linear polynomials

$$\begin{aligned} & 2x - y, x + y - z, x - y - z, x + y - 2z, 2x + y - z, 2x - y - z, \\ & 2x - 2y - z, 2x + y - 3z, 2x + 2y - 2z, 2x + 2y - 4z, 3x - 2y - z, \\ & x + 3y - 3z, 2x + 3y - 3z, 4x + 2y - 2z, 8x + 2y - 2z, \\ & 2(x - y) + z - w, 4(x - y) + 2(z - w). \end{aligned}$$

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $|Q(x, y, z, w)| \in \{4^k : k \in \mathbb{N}\}$, whenever $Q(x, y, z, w)$ is among the polynomials

$$\begin{aligned} & x + y - z, x + y - 3z, x + 2y - 3z, x + 2y - 4z, x + 2y - 5z, \\ & x + 3y - 3z, x + 3y - 5z, x + 4y - 2z, x + 5y - 2z, x + 5y - 7z, \\ & 2x + 3y - 4z, 2x + 4y - 6z, 2x + 4y - 10z, 2x + 5y - 4z, 4x + 6y - 6z, \\ & 4x + 6y - 14z, x + 4y - 2z - w, x + 8y - 3z - w, 2x + 3y - 3z - w, \\ & x + y + 2z - 2w, x + 2y + 3z - 4w. \end{aligned}$$

(iii) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $R(x, y, z, w) \in \{\pm 8^k : k \in \mathbb{N}\} \cup \{0\}$, whenever $R(x, y, z, w)$ is among the linear polynomials

$$\begin{aligned} & x + 2y - 2z, x + 3y - 3z, 2x + 3y - 3z, 4x + 6y - 6z, 4x + 8y - 8z, \\ & 4x + 2y - 10z, 4x + 12y - 12z, 8x + 12y - 12z, 8x + 4y - 20z, 2x + y - z - w, \\ & 8x + 4y - 4z - 4w, 2x + 8y - 4z - 2w, 4x + 16y + 8z - 4w. \end{aligned}$$

Remark 4.3. This conjecture is motivated by Theorems 1.2-1.4.

Conjecture 4.5. (i) Let $a \in \mathbb{Z}^+$ and let $b, c, d \in \mathbb{N}$ with $a \geq b \geq c \geq d$ and $4 \nmid \gcd(a, b, c, d)$. Then any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $ax + by + cz + dw \in \{8^k : k \in \mathbb{N}\}$, if and only if (a, b, c, d) is among the quadruples

$$(7, 3, 2, 1), (7, 5, 2, 1), (8, 4, 2, 1), (8, 4, 3, 2), (8, 5, 4, 2), (8, 6, 2, 1), \\ (8, 6, 3, 2), (8, 6, 5, 1), (8, 6, 5, 2), (9, 8, 7, 4), (10, 4, 3, 1), (12, 4, 3, 1).$$

(ii) Let $a \in \mathbb{Z}^+$ and let $b, c, d \in \mathbb{N}$ with $a \geq b \geq c \geq d$ and $2 \nmid \gcd(a, b, c, d)$. Then any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $ax + by + cz + dw \in \{2 \times 8^k : k \in \mathbb{N}\}$, if and only if (a, b, c, d) is among the quadruples

$$(7, 3, 2, 1), (7, 5, 2, 1), (8, 4, 2, 1), (8, 4, 3, 2), (8, 5, 4, 2), (8, 6, 2, 1), \\ (8, 6, 3, 2), (8, 6, 5, 2), (9, 5, 4, 2), (14, 3, 2, 1), (14, 5, 2, 1), (14, 7, 3, 2).$$

Conjecture 4.6. (i) (24-Conjecture) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $x + 24y$ are squares.

(ii) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $49x + 48(y - z)$ are squares. Also, any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $121x + 48(y - z)$ are squares.

(iii) Every $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $-7x - 8y + 8z + 16w$ are squares.

(iv) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that both x and $(12x)^2 + (5y - 10z)^2$ are squares. Also, each $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that both x and $(35x)^2 + (12y - 24z)^2$ are squares.

(v) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \equiv y \pmod{2}$ such that both x and $x^2 + 62xy + y^2$ are squares.

Remark 4.4. This conjecture is very surprising, see [S, A281976, A281977, A281980, A282013, A282014, A282161, A282226 and A282463] for related data. We verified the 24-Conjecture for all $n = 1, \dots, 10^7$, and Qing-Hu Hou extended the verification for n up to 10^{10} . The author would like to offer 2400 US dollars as the prize for the first solution of the 24-Conjecture. We verified parts (ii) and (iii) of Conjecture 4.5 for n up to 10^7 and 10^6 respectively, and later Qing-Hu Hou extended the verification of parts (ii) and (iii) of Conjecture 4.5 for n up to 10^9 and 10^8 respectively. Note that the author [S17b] proved that any $n \in \mathbb{N}$ is the sum of a fourth power and three squares.

Conjecture 4.7. (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, w \in \mathbb{N}$ and $y, z \in \mathbb{Z}$ such that both $x + 2y$ and $z + 2w$ are squares. Also, any $n \in \mathbb{N}$ can

be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{Z}$ and $w \in \mathbb{N}$ such that both $x + 3y$ and $z + 3w$ are squares.

(ii) Each $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that both $2x + y$ and $2x + z$ are squares.

(iii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) such that both $2x - y$ and $P(z, w)$ are squares, whenever $P(z, w)$ is among the polynomials $4z^2 + 724zw + w^2$, $9z^2 + 666zw + w^2$, $21z^2 - 84zw + 64w^2$, $56z^2 - 112zw + 81w^2$.

(iv) Any $n \in \mathbb{Z}^+$ can be written as $x^4 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $9y^2 - 8yz + 8z^2$ is a square. Also, any $n \in \mathbb{N}$ can be written as $4x^4 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $79y^2 - 220yz + 205z^2$ is a square.

Remark 4.5. See [S, A283170, A283196, A283204, A283205, A282561, A282562, A282933 and A282972] for related data. The author [S17b] proved that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + 2y$ is a square, and that any $n \in \mathbb{N}$ can be written as $4x^4 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$.

Conjecture 4.8. Any $n \in \mathbb{N} \setminus \{71, 85\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $9x^2 + 16y^2 + 24z^2 + 48w^2$ is a square.

Remark 4.6. See [S, A281659] for related data and other similar conjectures.

Conjecture 4.9. (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x + y - z$ an integer cube, where $x, y, z, w \in \mathbb{N}$, $x \geq y \leq z$ and $x \equiv y \pmod{2}$.

(ii) Any nonnegative integer can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z, w)$ is an integer cube, whenever $P(x, y, z, w)$ is among the following polynomials:

$$\begin{aligned} &2x + y - z, \quad 2(2x + y - z), \quad 4(2x + y - z), \quad x + 2y - 2z, \\ &4(x + 2y - 2z), \quad x + 3y - 3z, \quad 4(x + 3y - 3z), \quad 2x + 3y - 3z, \\ &2(2x + 3y - 3z), \quad 4(2x + 3y - 3z), \quad x + 5y - 5z, \quad 4(x + 5y - 5z), \\ &2x + 4y - 10z, \quad 4x + 8y - 20z, \quad 2x + y - z - w, \quad 4(2x + y - z - w), \\ &4x + y - 2z - w, \quad 2(4x + y - 2z - w), \quad 4(4x + y - 2z - w). \end{aligned}$$

Remark 4.7. See [S, A282091] for related data. The author [S17b] proved that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with x (or $4x$) a cube.

Conjecture 4.10. (i) Let $a, b \in \mathbb{Z}^+$ with $a \leq b$ and $\gcd(a, b)$ squarefree. Then, each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $ax^3 + by^3$ a square, if and only if (a, b) is among the following ordered pairs

$$\begin{aligned} &(1, 2), (1, 8), (2, 16), (4, 23), (4, 31), \\ &(5, 9), (8, 9), (8, 225), (9, 47), (25, 88), (50, 54). \end{aligned}$$

(ii) If (a, b) is among the ordered pairs

$$(1, 8), (2, 16), (8, 1), (9, 8), (88, 25), (225, 8),$$

then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x \in \mathbb{N}$ and $y, z, w \in \mathbb{Z}$ such that $ax^3 + by^3$ is a square.

(iii) Let $a, b \in \mathbb{Z}^+$ with $\gcd(a, b)$ squarefree. Then, each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $ax^3 + b(y - z)^3$ is a square, if and only if (a, b) is among the ordered pairs

$$(1, 1), (1, 9), (2, 18), (8, 1), (9, 5), (9, 8), \\ (9, 40), (16, 2), (18, 16), (25, 16), (72, 1).$$

(iv) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y \equiv z \pmod{2}$ such that $72x^3 + (y - z)^3$ is a square. Also, we can write each $n \in \mathbb{N}$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x^3 + ((y - z)/2)^3$ is a square (or twice a square).

(v) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x^3 + (2y - z)^3$ is twice a square.

Remark 4.8. See [S, A282863 and A283617] for related data.

Conjecture 4.11. Define

$$S = \{4^\delta m^4 : \delta \in \{0, 1\} \text{ and } m \in \mathbb{N}\}.$$

Then each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $P(x, y, z) \in S$, whenever $P(x, y, z)$ is among the following polynomials

$$x^2 - y^2, x^2 + 3(y^2 - z^2), 4x^2 + y^2 - z^2, 4x^2 + 3(y^2 - z^2), \\ x^2 + 5(y^2 - z^2), \frac{x(2x + 5y)}{3}, \frac{2x(x + 3y)}{5}.$$

Remark 4.9. Clearly $\{4^k : k \in \mathbb{N}\} \cup \{0\} \subseteq S \subseteq \{m^2 : m \in \mathbb{N}\}$.

Conjecture 4.12. (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 3w^2$ ($x, y, z, w \in \mathbb{Z}$) with $P(x, y, z, w) = 1$, whenever $P(x, y, z, w)$ is among the polynomials

$$2y + z + w, 2y + z + 3w, 2x + 2y + z + w, 2x + 2y + z + 3w, 4x + 2y + z + dw \quad (d = 1, 3, 5).$$

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + 2z^2 + 3w^2$ ($x, y, z, w \in \mathbb{Z}$) with $P(x, y, z, w) = 1$, whenever $P(x, y, z, w)$ is among the polynomials

$$x + 2y + w, y + z + w, y + 2z + w, y + 2z + 3w, x + 2y + 2z + w, x + 2y + 2z + 3w.$$

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + 3z^2 + 4w^2$ ($x, y, z, w \in \mathbb{Z}$) with $y + z + 2w = 1$. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + 2z^2 + 5w^2$ ($x, y, z, w \in \mathbb{Z}$) with $y + 2z + w = 1$.

Conjecture 4.13. (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $P_1(x, y, z, w) = 1$, whenever $P_1(x, y, z, w)$ is among the polynomials

$$\begin{aligned} &x + 2y + 3z, \quad x + 2y + 5z, \quad x + 3y + 4z, \quad y + 3z + 2w, \quad y + 3z + 4w, \quad 2y + z + w, \\ &\quad x + y + 2z + 2w, \quad x + 2y + 2z + 2w, \quad x + 2y + 3z + dw \quad (d = 1, 2, 4), \\ &x + 2y + 5z + 2w, \quad x + 2y + 5z + 6w, \quad x + 3y + 4z + 2w, \quad x + 3y + 4z + 4w. \end{aligned}$$

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $P_2(x, y, z, w) = 2$, whenever $P_2(x, y, z, w)$ is among the polynomials

$$\begin{aligned} &x + y + 2z + 2w, \quad x + y + 2z + 6w, \quad x + 2y + 3z + 2w, \\ &x + 2y + 3z + 6w, \quad x + 2y + 4z + 4w, \quad x + 2y + 5z + 2w, \quad 3x + 3y + 2z + 2w. \end{aligned}$$

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{Z}$) with $P_3(x, y, z, w) = 3$, whenever $P_3(x, y, z, w)$ is among the polynomials

$$x + 2y + 3z + 2w, \quad x + 2y + 3z + 4w, \quad x + 2y + 3z + 6w.$$

Conjecture 4.14. Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + 2y \in \{4^k : k \in \mathbb{N}\}$. Also, we may replace $x + 2y$ by $y - z + 3w$ (or $y + 2z - w$).

Remark 4.10. We can easily show that any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y = 1$.

Now we pose a conjecture similar to the 1-3-5 conjecture of Sun [S17b, Conjecture 4.3(i)].

Conjecture 4.15 (1-2-3 Conjecture). Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z$ is a square.

Remark 4.11. See [S, A275344] for related data. Each of the numbers

$$0, 1, 3, 5, 7, 14, 15, 16, 25, 30, 84, 169, 225$$

has a unique representation $x^2 + y^2 + z^2 + 2w^2$ ($x, y, z, w \in \mathbb{N}$) with $x + 2y + 3z$ a square. For example,

$$\begin{aligned} 33 &= 1^2 + 0^2 + 0^2 + 2 \times 4^2 \quad \text{with } 1 + 2 \times 0 + 3 \times 0 = 1^2, \\ 84 &= 4^2 + 6^2 + 0^2 + 2 \times 4^2 \quad \text{with } 4 + 2 \times 6 + 3 \times 0 = 4^2, \\ 169 &= 10^2 + 6^2 + 1^2 + 2 \times 4^2 \quad \text{with } 10 + 2 \times 6 + 3 \times 1 = 5^2, \\ 225 &= 10^2 + 6^2 + 9^2 + 2 \times 2^2 \quad \text{with } 10 + 2 \times 6 + 3 \times 9 = 7^2. \end{aligned}$$

We also have the following conjecture on restricted sums of three squares.

Conjecture 4.16. (i) Any $n \in \mathbb{Z}^+$ with $\text{ord}_2(n)$ odd can be written as $x^2 + y^2 + z^2$ ($x, y, z \in \mathbb{Z}$) with $x + 3y + 5z$ a square (or twice a square). Also, for any $n \in \mathbb{N} \setminus \{63\}$, we can write $4n + 1$ as $x^2 + y^2 + z^2$ ($x, y, z \in \mathbb{Z}$) with $x + 3y + 5z$ a square.

(ii) Any $n \in \mathbb{N}$ not of the form $4^k(8l+7)$ ($k, l \in \mathbb{N}$) can be written as $x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$ such that $x + 2y + 3z$ is a square or twice a square.

(iii) Let $n \in \mathbb{N}$. Then $8n + 1$ can be written as $x^2 + y^2 + z^2$ with $x, y \in \mathbb{Z}$ and $z \in \mathbb{Z}^+$ such that $x + 3y$ is a square. Also, we can write $8n + 6$ as $x^2 + y^2 + z^2$ with $x \in \mathbb{Z}$, $y, z \in \mathbb{N}$ and $2 \nmid z$ such that $x + 2y$ is a square.

(iv) Any positive odd integer can be written as $x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{Z}$ such that $2x + y + z$ is a square or a power of two. Also, we can write any positive odd integer as $x^2 + 2y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ such that $x + y + z$ is a square or twice a square.

(v) Any $n \in \mathbb{N}$ can be expressed as $x^2 + y^2 + z(3z - 1)/2$ with $x, y, z \in \mathbb{Z}$ such that $x + 2y$ is a square.

Remark 4.12. See [S, A283239, A283269, A283273, A283366 and A283299] for related data. It is known that any positive odd integer can be written as $x^2 + y^2 + 2z^2$ (or $x^2 + 2y^2 + 3z^2$) with $x, y, z \in \mathbb{Z}$ (cf. [D39, pp. 112-113]). By [S15, Theorem 1.7(ii)], we have $\{x^2 + y^2 + z(3z - 1)/2 : x, y, z \in \mathbb{Z}\} = \mathbb{N}$.

REFERENCES

- [D27] L. E. Dickson, *Quaternary quadratic forms representing all integers*, Amer. J. Math. **49** (1927), 39–56.
- [D39] L. E. Dickson, *Modern Elementary Theory of Numbers*, University of Chicago Press, Chicago, 1939.
- [K] I. Kaplansky, *The first nontrivial genus of positive definite ternary forms*, Math. Comp. **64** (1995), 341–345.
- [MW] C. J. Moreno and S. S. Wagstaff, *Sums of Squares of Integers*, Chapman & Hall/CRC, New York, 2005.
- [N] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Grad. Texts in Math., vol. 164, Springer, New York, 1996.
- [R] S. Ramanujan, *On the expression of a number in the form $ax^2 + by^2 + cz^2 + dw^2$* , Proc. Cambridge Philos. Soc. **19** (1917), 11–21.
- [SS] Y.-C. Sun and Z.-W. Sun, *Some refinements of Lagrange’s four-square theorem*, preprint, arXiv:1605.03074 [math.NT], 2016.
- [S] Z.-W. Sun, Sequences A275344, A279612, A279616, A280153, A280356, A280386, A280455, A280472, A281494, A281659, A281826, A281939, A281941, A281975, A281976, A281977, A281980, A282013, A282014, A282091, A282161, A282226, A282463, A282561, A282562, A282863, A282933, A282972, A283170, A283196, A283204, A283205, A283239, A283269, A283273, A283299, A283366, A283617, A284343 in OEIS, <http://oeis.org>.
- [S15] Z.-W. Sun, *On universal sums of polygonal numbers*, Sci. China Math. **58** (2015), 1367–1396.
- [S16] Z.-W. Sun, *A result similar to Lagrange’s theorem*, J. Number Theory **162** (2016), 190–211.
- [S17a] Z.-W. Sun, *On $x(ax + 1) + y(by + 1) + z(cz + 1)$ and $x(ax + b) + y(ay + c) + z(az + d)$* , J. Number Theory **171** (2017), 275–283.

- [S17b] Z.-W. Sun, *Refining Lagrange's four-square theorem*, J. Number Theory **175** (2017), 167–190.
- [W] K. S. Williams, *Number Theory in the Spirit of Liouville*, London Math. Soc. Student Texts, Vol. 76, Cambridge Univ. Press, Cambridge, 2011.