

Positive subharmonic solutions to superlinear ODEs with indefinite weight *

Guglielmo Feltrin

*Département de Mathématique, Université de Mons
Place du Parc 20, B-7000 Mons, Belgium
e-mail: guglielmo.feltrin@umons.ac.be*

Abstract

We study the positive subharmonic solutions to the second order nonlinear ordinary differential equation

$$u'' + q(t)g(u) = 0,$$

where $g(u)$ has superlinear growth both at zero and at infinity, and $q(t)$ is a T -periodic sign-changing weight. Under the sharp mean value condition $\int_0^T q(t) dt < 0$, combining Mawhin's coincidence degree theory with the Poincaré-Birkhoff fixed point theorem, we prove that there exist positive subharmonic solutions of order k for any large integer k . Moreover, when the negative part of $q(t)$ is sufficiently large, using a topological approach still based on coincidence degree theory, we obtain the existence of positive subharmonics of order k for any integer $k \geq 2$.

1 Introduction

In this paper we deal with positive subharmonic solutions to nonlinear differential equations with indefinite weight and we gather some results recently obtained in [11, 25, 23].

Our investigation is devoted to the second order differential equation

$$u'' + f(t, u) = 0, \tag{1.1}$$

especially when the nonlinear vector field is of the form

$$f(t, u) := q(t)g(u),$$

*Work partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Progetto di Ricerca 2016: "Problemi differenziali non lineari: esistenza, molteplicità e proprietà qualitative delle soluzioni". It is also partially supported by the project "Existence and asymptotic behavior of solutions to systems of semilinear elliptic partial differential equations" (T.1110.14) of the *Fonds de la Recherche Fondamentale Collective*, Belgium.

AMS Subject Classification: Primary: 34C25, Secondary: 34B18, 37J10, 47H11.

Keywords: subharmonic solutions, superlinear indefinite problems, positive solutions, multiplicity results, Mawhin's coincidence degree, Poincaré-Birkhoff fixed point theorem.

thus covering the classical *superlinear indefinite case*, namely $g(u) = |u|^{p-1}u$, with $p > 1$, and $q(t)$ a sign-changing coefficient.

The investigation of boundary value problems associated with equation (1.1) when f is *superlinear at infinity* with respect to s , namely

$$\lim_{s \rightarrow \pm\infty} \frac{f(t, s)}{s} = \infty,$$

is a topic which has been widely studied, employing various different approaches. We refer to the introduction in [43] and the references therein for an interesting historical presentation on the subject.

In the present paper we deal with the *periodic indefinite problem* associated with (1.1), namely we suppose that $t \rightarrow f(t, s)$ is a T -periodic and sign-changing map. In this framework, starting with the pioneering work [17] by Butler, there are a lot of results for *oscillatory* solutions. In particular, they provide infinitely many periodic and subharmonic solutions with a large number of zeros, as well as globally bounded solutions defined on the real line and exhibiting a complex behavior (see for instance [18, 44, 48]). We remark that large sign-changing solutions to (1.1) have a strong oscillatory behavior and, in the indefinite case, some solutions blow up. Accordingly, the study of sign-changing solutions of indefinite problems has two main features: the absence of a priori bounds and the non-continuability of some solutions. For these reasons the analysis of sign-changing solutions is delicate and strong regularity assumptions on the nonlinearity are required, including that $f(t, s)$ is continuous, is locally Lipschitz in t and of locally bounded variation in s , the set of values of t for which $f(t, s) = 0$ is an isolated set (cf. [17] and subsequent contributions). However, we stress that no growth condition at $s = 0$ is required.

Our investigation is dedicated to the study of *positive* solutions of the *periodic indefinite problem* associated with (1.1) and provide a double contribution in this context. On one hand, we are going to present a topological approach that allows to avoid all the above regularity condition on $f(t, s)$: a minimal set of assumptions on the nonlinearity will be required, but including the superlinear growth at zero (cf. Remark 2.1). On the other hand, our paper is one of the fewer investigations on *positive* periodic solutions to equations like (\mathcal{E}) (see [5, 25], dealing with existence, multiplicity and chaotic dynamics of positive solutions). This latter aspect places in the investigation on indefinite equations of the form

$$-\Delta u = q(x)g(u), \quad u \in \Omega \subseteq \mathbb{R}^N,$$

that arise in many models concerning population dynamics, differential geometry and mathematical physics, and for which only non-negative solutions make sense. Concerning indefinite problems, we mention the contributions [2, 3, 6, 31] and we refer to the introductions in [1, 9, 22, 25, 47] for a more complete discussion and bibliography on the subject.

We can now illustrate our results. Let $\mathbb{R}^+ := [0, +\infty[$ denote the set of non-negative real numbers and let $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuously differentiable function such that

$$(g_*) \quad g(0) = 0, \quad g(s) > 0 \quad \text{for } s > 0.$$

Let $T > 0$ and let $q: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function.

In this survey we focus our attention on the second order ordinary differential equation

$$(\mathcal{E}) \quad u'' + q(t)g(u) = 0.$$

Our main goal is the investigation of positive subharmonic solutions to (\mathcal{E}) when $q(t)$ is a sign-changing function and $g(s)$ satisfies the following condition

$$(g_s) \quad g'(0) = 0 \quad \text{and} \quad \lim_{s \rightarrow +\infty} \frac{g(s)}{s} = +\infty,$$

namely when $g(s)$ has a superlinear growth at zero and at infinity, thus the classical case $g(s) = s^p$, with $p > 1$, is covered.

In this paper, following a standard definition, we use the terminology *subharmonic solution of order k* to (\mathcal{E}) (where $k \geq 2$ is an integer number) to indicate a kT -periodic solution to (\mathcal{E}) which is not ℓT -periodic for any $\ell = 1, \dots, k-1$, in other words, kT is the minimal period of $u(t)$ in the set of the integer multiples of T . It is worth noting that, assuming that T is the minimal period of $q(t)$, from hypothesis (g_*) we derive that kT is the minimal period of any positive subharmonic solution of order k (cf. the discussion in [25, § 4]). As a further remark, we underline that, if $u(t)$ is a positive subharmonic solution of order k to (\mathcal{E}) , then the $k-1$ time-translated functions $u(\cdot + \ell T)$, for $\ell = 1, \dots, k-1$, are positive subharmonic solutions of order k too. These solutions, though distinct, belong to the same periodicity class (in particular, they have to be considered equivalent when counting subharmonics).

From the above definition, it is clear that there are two main issues to face in the search of subharmonics to (\mathcal{E}) : the existence of positive kT -periodic solutions to (\mathcal{E}) and the proof that kT is the minimal period (in the sense described above) of some of these solutions (which is the most difficult point).

In this perspective, we now present two necessary conditions to the existence of positive kT -periodic solutions (where $k \geq 1$ an integer number). Indeed, if $u(t)$ is any positive kT -periodic solution to (\mathcal{E}) , then integrating equation (\mathcal{E}) on $[0, kT]$ we obtain

$$0 = - \int_0^{kT} u''(t) dt = \int_0^{kT} q(t)g(u(t)) dt.$$

Therefore, by (g_*) , $q(t)$ has to change its sign (if not identically zero). A second relation can be derived when $g'(s) > 0$ for $s > 0$, as in the case $g(s) = s^p$, with $p > 1$. Precisely, dividing equation (\mathcal{E}) by $g(u(t))$ and integrating by parts, we find

$$k \int_0^T q(t) dt = \int_0^{kT} q(t) dt = - \int_0^{kT} \left(\frac{u'(t)}{g(u(t))} \right)^2 g'(u(t)) dt < 0.$$

In the sequel we will show that this condition is also sufficient for the existence of subharmonics (cf. Theorem 2.1).

Since the main motivation for the present investigation is the superlinear equation $u'' + q(t)u^p = 0$ (with $p > 1$), as a natural hypothesis we suppose that $q: \mathbb{R} \rightarrow \mathbb{R}$ is a T -periodic locally integrable *sign-changing* function (i.e. an *indefinite* weight) satisfying the mean value condition

$$(q\#) \quad \int_0^T q(t) dt < 0.$$

Additionally, we assume that in a time-interval of length T there exists a finite number of closed pairwise disjoint subintervals where $q(t) \succ 0$ (i.e. $q(t) \geq 0$ almost everywhere and $q \not\equiv 0$ on each interval), separated by closed intervals where $q(t) \prec 0$ (i.e. $-q(t) \succ 0$). More precisely, thanks to the periodicity of $q(t)$ and for ease of notation, we assume that

(q_*) *there exist $m \geq 1$ closed and pairwise disjoint intervals I_1^+, \dots, I_m^+ separated by m closed intervals I_1^-, \dots, I_m^- such that*

$$q(t) \succ 0 \text{ on } I_i^+, \quad q(t) \prec 0 \text{ on } I_i^-,$$

and, moreover,

$$\bigcup_{i=1}^m I_i^+ \cup \bigcup_{i=1}^m I_i^- = [0, T].$$

In the manuscript we present two results of existence of infinitely many positive subharmonics.

The first one combines an application of the Poincaré-Birkhoff fixed point theorem, a smart trick used in [16] by Brown and Hess (that requires the strict convexity of $g(s)$) and coincidence degree theory. It states the following.

Theorem 1.1. *Let $q: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function satisfying ($q_{\#}$) and (q_*). Let $g \in \mathcal{C}^2(\mathbb{R}^+)$ satisfy (g_*), (g_s) and*

$$(g_{**}) \quad g''(s) > 0 \text{ for } s > 0.$$

Then there exists a positive T -periodic solution $u^(t)$ of equation (\mathcal{E}); moreover, there exists $k^* \geq 1$ such that for any integer $k \geq k^*$ there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation (\mathcal{E}) has two positive subharmonic solutions $u_{k,j}^{(i)}(t)$ ($i = 1, 2$) of order k (not belonging to the same periodicity class), such that $u_{k,j}^{(i)}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$.*

We underline that the subharmonic solutions obtained in Theorem 1.1 oscillate around a positive T -periodic solution $u^*(t)$ of (\mathcal{E}): this property is crucial in order to obtain the “minimality” of the period. We also stress that, taking $j = 1$ in the statement of Theorem 1.1, we have the existence of two subharmonic solutions of order k for any large integer k .

The second result follows a line of research initiated by Gómez-Reñasco and López-Gómez in [30], where the authors asserted that the Dirichlet problem associated with (\mathcal{E}) has at least $2^m - 1$ positive solutions when the negative part of the weight $q(t)$ is sufficiently large (and m is the number of positive humps of $q(t)$ separated by negative ones, as in hypothesis (q_*)). According to a standard notation adopted in the contributions that followed from [30] (see, for instance, [8, 10, 24, 27, 29]), it is convenient to introduce the parameter-dependent equation

$$(\mathcal{E}_\mu) \quad u'' + (a^+(t) - \mu a^-(t))g(u) = 0,$$

with $\mu > 0$ and

$$q(t) = a_\mu(t) := a^+(t) - \mu a^-(t), \quad t \in \mathbb{R}, \quad (1.2)$$

where $a^+(t)$ and $a^-(t)$ are, respectively, the positive and the negative part of a T -periodic locally integrable function $a: \mathbb{R} \rightarrow \mathbb{R}$.

In this setting, using a topological approach based on coincidence degree theory, we obtain the following.

Theorem 1.2. *Let $q: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function of the form (1.2) satisfying (q_*) . Let $g \in \mathcal{C}^1(\mathbb{R}^+)$ satisfy (g_*) and (g_s) . Then there exists $\mu^* > 0$ such that, for all $\mu > \mu^*$ and for every integer $k \geq 2$, equation (\mathcal{E}_μ) has a subharmonic solution of order k .*

Theorem 1.2 ensures the existence of infinitely many subharmonics taking $\mu > 0$ sufficiently large. We stress that in the statement we do not assume condition $(q_\#)$, since it is implicitly satisfied taking $\mu > \mu^*$ with

$$\mu^* \geq \mu^\# := \frac{\int_0^T a^+(t) dt}{\int_0^T a^-(t) dt}. \quad (1.3)$$

The plan of the paper is the following. In Section 2 we illustrate two preliminary results concerning existence and multiplicity of positive T -periodic solutions to equations (\mathcal{E}) and (\mathcal{E}_μ) , respectively; the proofs of both results are based on coincidence degree theory and are provided in [25, 23]. In Section 3 we give the proof of Theorem 1.1; some remarks on different possible generalizations are also given. Section 4 is devoted to Theorem 1.2: we prove the existence of infinitely many positive subharmonic solutions when the negative part of the weight is large enough and, in addition, we estimate the number of subharmonics of a given order. Finally, in Section 5 we compare the two main results and present some open questions. We conclude this paper with Appendix A where we discuss some basic facts about the coincidence degree defined in open and possibly unbounded sets and we state some lemmas for the computation of the degree, employed in Section 2.

2 Preliminary results: positive T -periodic solutions

In this section we present two theorems that ensure existence and, respectively, multiplicity of positive T -periodic solutions to (\mathcal{E}) . We take advantage of a topological approach introduced in [25, 23] based on Mawhin's coincidence degree theory (cf. [26, 36, 37]).

The first result states the existence of a positive T -periodic solution when the mean value condition $(q_\#)$ holds. Therefore, this theorem gives an answer to a question raised by Butler in [17]. In [17] the author proved that equation

$$u'' + q(t)|u|^{p-1}u = 0, \quad p > 1,$$

has infinitely many T -periodic solutions and all these solutions oscillate (have arbitrarily large zeros), by assuming that $q(t)$ is a continuous T -periodic function with only isolated zeros and such that

$$\int_0^T q(t) dt \geq 0.$$

Moreover, underlining that condition $(q_{\#})$ implies the existence of non-oscillatory solutions, Butler raised the question whether there can exist positive *periodic* solutions under hypothesis $(q_{\#})$. This is stated in the following theorem.

Theorem 2.1. *Let $q: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function satisfying $(q_{\#})$ and (q_*) . Let $g \in C^1(\mathbb{R}^+)$ satisfy (g_*) and (g_s) . Then there exists at least a positive T -periodic solution of equation (\mathcal{E}) .*

Proof. We give only a sketch of the proof, describing the main steps (which are developed in details in [23]).

Step 1. Mawhin's coincidence degree setting. First of all, using a standard procedure, we define the L^1 -Carathéodory function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$f(t, s) := \begin{cases} -s, & \text{if } s \leq 0; \\ q(t)g(s), & \text{if } s \geq 0; \end{cases}$$

and we deal with the T -periodic problem associated with

$$u'' + f(t, u) = 0. \quad (2.1)$$

Via a standard maximum principle, one can prove that every T -periodic solution $u(t)$ of (2.1) is non-negative and if $u(t)$ is a T -periodic solution of (2.1) with $u \not\equiv 0$, then $u(t) > 0$ for all $t \in \mathbb{R}$.

Secondly, we write the T -periodic problem associated with (2.1) as a *coincidence equation*

$$Lu = Nu, \quad u \in \text{dom } L. \quad (2.2)$$

Taking into account that solving the T -periodic problem associated with (2.1) is equivalent to solving equation (2.1) on $[0, T]$ together with the periodic boundary condition $u(0) = u(T)$ and $u'(0) = u'(T)$, we set $X := \mathcal{C}([0, T])$, the Banach space of continuous functions $u: [0, T] \rightarrow \mathbb{R}$ endowed with the sup-norm $\|u\|_{\infty} := \max_{t \in [0, T]} |u(t)|$, and $Z := L^1([0, T])$, the Banach space of integrable functions $v: [0, T] \rightarrow \mathbb{R}$ endowed with the norm $\|v\|_{L^1} := \int_0^T |v(t)| dt$. Next, on $\text{dom } L := \{u \in W^{2,1}([0, T]): u(0) = u(T), u'(0) = u'(T)\} \subseteq X$ we define the differential operator

$$L: u \mapsto -u'',$$

which is a linear Fredholm map of index zero. Moreover, in order to enter the coincidence degree setting, we introduce the projectors $P: X \rightarrow \ker L \cong \mathbb{R}$, $Q: Z \rightarrow \text{coker } L \cong Z/\text{Im } L \cong \mathbb{R}$, the right inverse $K_P: \text{Im } L \rightarrow \text{dom } L \cap \ker P$ of L , and the orientation-preserving isomorphism $J: \text{coker } L \rightarrow \ker L$. For the standard definition of these operators we refer to [13, § 2] and to [23, § 2]. Finally, let $N: X \rightarrow Z$ be the Nemytskii operator induced by the nonlinear function $f(t, s)$, that is

$$(Nu)(t) := f(t, u(t)), \quad t \in [0, T].$$

With this position, now we show how to reach the thesis using a topological approach based on Mawhin's coincidence degree. We refer to [26, 36, 37] for the classical definition and properties of the *coincidence degree* $D_L(L - N, \Omega)$ of L and N in Ω , where $\Omega \subseteq X$ is an open and bounded set (cf. also Appendix A).

Step 2. Degree on a small ball. Since $g'(0) = 0$, we can fix a (small) constant $r > 0$ such that the following property holds.

- If $\vartheta \in]0, 1]$ and $u(t)$ is any non-negative T -periodic solution of

$$u'' + \vartheta q(t)g(u) = 0,$$

then $\|u\|_\infty \neq r$.

Then, using condition $(q_\#)$, by Lemma A.1 we obtain that

$$D_L(L - N, B(0, r)) = 1. \quad (2.3)$$

Step 3. Degree on a large ball. Since $g(s)/s \rightarrow +\infty$ as $s \rightarrow +\infty$, we can fix a (large) constant $R > 0$ (with $R > r$) such that the following property holds.

- There exist a non-negative function $v \in L_T^1$ with $v \not\equiv 0$ and a constant $\nu_0 > 0$, such that every non-negative T -periodic solution $u(t)$ of

$$u'' + q(t)g(u) + \nu v(t) = 0, \quad (2.4)$$

for $\nu \in [0, \nu_0]$, satisfies $\|u\|_\infty \neq R$. Moreover, there are no T -periodic solutions $u(t)$ of (2.4) for $\nu = \nu_0$ with $0 \leq u(t) \leq R$, for all $t \in \mathbb{R}$.

Then, by Lemma A.2 we derive that

$$D_L(L - N, B(0, R)) = 0. \quad (2.5)$$

Step 4. Conclusion. From (2.3) and (2.5), using the additivity property of Mawhin's coincidence degree, we derive that

$$D_L(L - N, B(0, R) \setminus \overline{B(0, r)}) = -1.$$

Then, by the existence property of the degree, there exists at least a nontrivial solution $u^*(t)$ of (2.2) with $r < \|u^*\|_\infty < R$. Via a standard maximum principle, we conclude that $u^*(t)$ is a positive T -periodic solution of (2.1) and thus of (\mathcal{E}) . The theorem follows. \square

Now, we continue the investigation on Butler's open problem, by providing multiple positive T -periodic solutions to (\mathcal{E}) (depending on the nodal properties of the weight $q(t)$) when the negative part of $q(t)$ is sufficiently large. Accordingly, for this second part of the section, we deal with the parameter-dependent equation

$$(\mathcal{E}_\mu) \quad u'' + (a^+(t) - \mu a^-(t))g(u) = 0,$$

where $\mu > 0$, and we prove our result of multiplicity when μ is large enough. In the sequel, when dealing with the function $a(t)$, we denote with (a_*) the hypothesis about the existence of m intervals where $a(t) > 0$ separated by m intervals where $a(t) < 0$ (in $[0, T]$), which is the analogous of condition (q_*) referring to $q(t)$.

The following theorem holds (see also Figure 1).

Theorem 2.2. *Let $a: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function satisfying (a_*) . Let $g \in \mathcal{C}^1(\mathbb{R}^+)$ satisfy (g_*) and (g_s) . Then there exists $\mu^* > 0$ such that for all $\mu > \mu^*$ equation (\mathcal{E}_μ) has at least $2^m - 1$ positive T -periodic solutions.*

Proof. We give only a sketch of the proof, describing the main steps (which are developed in details in [25]).

Step 1. Notation and Mawhin's coincidence degree setting. For technical reasons, without loss of generality and consistently with assumption (a_*) , we can select the endpoints of each interval of positivity $I_i^+ = [\sigma, \tau]$ in such a manner that $a(t) \neq 0$ on all left neighborhoods of σ and on all right neighborhoods of τ . In [12] it is showed, in a different context, how this additional technical hypothesis can be avoided.

First of all, we fix the constants r and R , with $0 < r < R$, as in *Step 2* and *Step 3* of the proof of Theorem 2.1. Next, for every subset of indices $\mathcal{I} \subseteq \{1, \dots, m\}$ (possibly empty), we define two open unbounded sets

$$\Omega^{\mathcal{I}} := \left\{ u \in \mathcal{C}([0, T]) : \begin{aligned} &\max_{t \in I_i^+} |u(t)| < R, \quad i \in \mathcal{I}; \\ &\max_{t \in I_i^+} |u(t)| < r, \quad i \in \{1, \dots, m\} \setminus \mathcal{I} \end{aligned} \right\}$$

and

$$\Lambda^{\mathcal{I}} := \left\{ u \in \mathcal{C}([0, T]) : \begin{aligned} &r < \max_{t \in I_i^+} |u(t)| < R, \quad i \in \mathcal{I}; \\ &\max_{t \in I_i^+} |u(t)| < r, \quad i \in \{1, \dots, m\} \setminus \mathcal{I} \end{aligned} \right\}. \quad (2.6)$$

Now we enter the setting of Mawhin's coincidence degree in the same manner as we have done in *Step 1* of the proof of Theorem 2.1. Our goal is to compute the degree on the sets $\Omega^{\mathcal{I}}$ and $\Lambda^{\mathcal{I}}$, in particular we are going to prove that

$$D_L(L - N, \Lambda^{\mathcal{I}}) \neq 0, \quad \forall \mathcal{I} \subseteq \{1, \dots, m\}.$$

Since the sets $\Omega^{\mathcal{I}}$ and $\Lambda^{\mathcal{I}}$ are open and *unbounded*, we use the more general version of the coincidence degree for locally compact operators on open and possibly unbounded sets (see Appendix A and [22, Appendix B]).

Step 2. Degree on $\Omega^{\mathcal{I}}$. For any subset of indices $\mathcal{I} \subseteq \{1, \dots, m\}$ we compute $D_L(L - N, \Omega^{\mathcal{I}})$. First of all, we consider the case $\mathcal{I} = \emptyset$. As a consequence of a convexity argument, the excision property of the degree and formula (2.3), we obtain that

$$D_L(L - N, \Omega^{\emptyset}) = D_L(L - N, B(0, r)) = 1. \quad (2.7)$$

Secondly, let us consider a subset $\mathcal{I} \neq \emptyset$. Via a homotopic argument (by Lemma A.3), we can prove the existence of $\mu^* \geq \mu^\# > 0$ (where $\mu^\#$ is the constant defined in (1.3)) such that for all $\mu > \mu^*$ the degree $D_L(L - N, \Omega^{\mathcal{I}})$ is well-defined and the following formula holds

$$D_L(L - N, \Omega^{\mathcal{I}}) = 0, \quad \forall \emptyset \neq \mathcal{I} \subseteq \{1, \dots, m\}. \quad (2.8)$$

Step 3. Degree on $\Lambda^{\mathcal{I}}$. For any subset of indices $\mathcal{I} \subseteq \{1, \dots, m\}$ we compute $D_L(L - N, \Lambda^{\mathcal{I}})$. Via a purely combinatorial argument (done by induction on the cardinality $\#\mathcal{I}$ of the set \mathcal{I}), from (2.7) and (2.8), we deduce that for all $\mu > \mu^*$ the degree $D_L(L - N, \Lambda^{\mathcal{I}})$ is well-defined and the following formula holds

$$D_L(L - N, \Lambda^{\mathcal{I}}) = (-1)^{\#\mathcal{I}}, \quad \forall \mathcal{I} \subseteq \{1, \dots, m\}.$$

Step 4. Conclusion. Preliminarily, we underline that $0 \notin \Lambda^{\mathcal{I}}$ for all $\emptyset \neq \mathcal{I} \subseteq \{1, \dots, m\}$ and the sets $\Lambda^{\mathcal{I}}$ are pairwise disjoint. Since the number of nonempty subsets of a set with m elements is $2^m - 1$, there are $2^m - 1$ sets $\Lambda^{\mathcal{I}}$ not containing the null function. Since, from *Step 3*, in particular we have that

$$D_L(L - N, \Lambda^{\mathcal{I}}) \neq 0, \quad \forall \mathcal{I} \subseteq \{1, \dots, m\},$$

we conclude that there exist at least $2^m - 1$ nontrivial solution of (2.2). Via a standard maximum principle, we conclude that these nontrivial functions are positive T -periodic solutions of (2.1) and thus of (\mathcal{E}_μ) . The theorem follows. \square

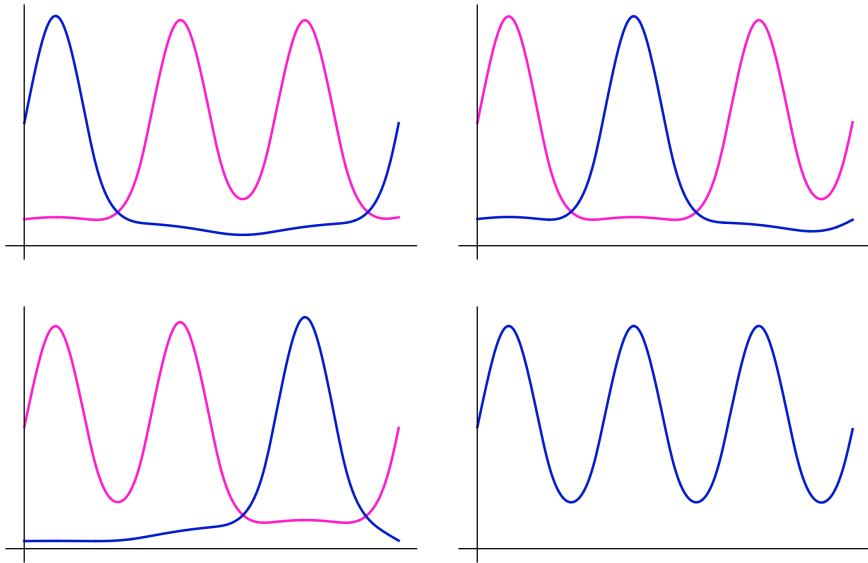


Figure 1: The figure shows an example of multiple positive solutions for the T -periodic boundary value problem associated with (\mathcal{E}_μ) . For this numerical simulation we have chosen $a(t) = \sin(6\pi t)$ for $t \in [0, 1]$, $\mu = 10$ and $g(s) = \max\{0, 400 s \arctan |s|\}$. Notice that the weight function $a(t)$ has 3 positive humps. We show the graphs of the 7 positive T -periodic solutions of (\mathcal{E}_μ) . We stress that $g(s)/s \not\rightarrow +\infty$ as $s \rightarrow +\infty$ (contrary to what is assumed in (g_s)), indeed Theorem 2.2 is also valid assuming only that $g(s)/s$ is sufficiently large as $s \rightarrow +\infty$, as observed in Remark 2.1.

Remark 2.1. We underline that one can employ a similar topological approach to obtain existence and multiplicity results (analogous to Theorem 2.1 and Theorem 2.2, respectively) dealing with an equation involving also a friction term as

$$(\mathcal{E}_{\mu,c}) \quad u'' + cu' + (a^+(t) - \mu a^-(t))g(u) = 0,$$

where $c \in \mathbb{R}$ is an arbitrary constant. In fact, in [25] it is showed how suitable monotonicity properties of the map $t \mapsto e^{ct}u'(t)$ replace the convexity/concavity of the solutions of (\mathcal{E}_μ) . This is the crucial point that allows to adapt the topological approach described above in this more general setting. However, even if both main results about subharmonic solutions, that will be proved in

the following sections, are based on these existence/multiplicity theorems, only one is valid in the non-variational context (cf. also Remark 4.2) and thus we prefer to skip the details of these general results.

Furthermore, we observe that Theorem 2.1 and Theorem 2.2 can be proved under less restrictive regularity hypotheses on $g(s)$. Concerning Theorem 2.1, the original version in [23] was stated and proved assuming $g(s)$ only continuous on \mathbb{R}^+ and continuously differentiable on a right neighborhood of zero (cf. [23, Theorem 3.2]), or assuming $g(s)$ continuous on \mathbb{R}^+ and regularly oscillating at zero (cf. [23, Theorem 3.1] and the references listed at the end of [23, § 1]). In fact, in Theorem 2.1 we assume $g(s)$ of class \mathcal{C}^1 but we stress that in the proof only the continuity of $g(s)$ and its continuous differentiability near zero is used (see the details of the proof in [23]).

In this perspective, also Theorem 2.2 is valid under weaker hypotheses and, in fact, in [25] the authors proved it assuming only that $g \in \mathcal{C}(\mathbb{R}^+)$ (with $\lim_{s \rightarrow 0^+} g(s)/s \rightarrow 0$ instead of $g'(0) = 0$) and thus are able to avoid all the additional regularity conditions on $g(s)$ (by taking μ large enough).

It is worth noting that, dealing with equation (\mathcal{E}_μ) , the more general version of Theorem 2.1 in [23] ensures the existence of a positive T -periodic solution assuming $\mu > \mu^\#$ (with $\mu^\#$ defined in (1.3)), the continuity of $g(s)$ and some regularity conditions near $s = 0$, while in the more general version of Theorem 2.2 in [25] we obtain multiplicity of positive T -periodic solutions assuming μ large (that is $\mu > \mu^* \geq \mu^\#$) and the continuity of $g(s)$. We stress that in the second result in order to avoid additional regularity assumptions on the nonlinearity $g(s)$, we need a stronger hypothesis on the weight (namely take μ large) and thus a worse estimate on μ^* .

Finally, we remark that in both theorems the growth condition at infinity can be weakened by dealing with nonlinearities $g(s)$ satisfying the less restrictive assumption at infinity

$$\liminf_{s \rightarrow +\infty} \frac{g(s)}{s} > \max_{i=1, \dots, m} \lambda_1^i,$$

where λ_1^i ($i = 1, \dots, m$) is the first eigenvalue of the eigenvalue problem on I_i^+

$$\varphi'' + \lambda q(t)\varphi = 0, \quad \varphi|_{\partial I_i^+} = 0,$$

(cf. hypothesis (q_*)). We refer to [25, 23] for the details (see also the example described in Figure 1). However, we avoid investigations in this direction since our main result presented in Section 3 needs that $g(s)$ is of class \mathcal{C}^2 and satisfying (g_s) . \triangleleft

3 First result: symplectic approach

In this section we present a symplectic approach, based on the Poincaré-Birkhoff fixed point theorem, which allows to obtain infinitely many subharmonics to

$$(\mathcal{E}) \quad u'' + q(t)g(u) = 0,$$

as stated in Theorem 1.1. We refer to [11] for the missing details.

As a first step, analogously as in *Step 3* of the proof of Theorem 2.1, we prove the existence of $R > 0$ (large enough) such that for every $\nu \geq 0$ and for every integer $k \geq 1$ any kT -periodic solution $u(t)$ to

$$u'' + q(t)g(u) + \nu \mathbb{1}_{\bigcup_{i=1}^m I_i^+}(t) = 0$$

satisfies $\max_{t \in \mathbb{R}} u(t) < R$ (where $\mathbb{1}_J$ denotes the indicator function of the interval J). For simplicity, we can choose $R > 0$ satisfying both the above property and the one in *Step 3* of the proof of Theorem 2.1.

As a second step, we introduce the extension $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as

$$h(t, s) := \begin{cases} 0, & \text{if } s \in]-\infty, 0[; \\ q(t)g(s), & \text{if } s \in [0, R]; \\ q(t)g(R) + q(t)g'(R)(s - R), & \text{if } s \in]R, +\infty[; \end{cases}$$

which is T -periodic in the first variable. Then, we consider the differential equation

$$u'' + h(t, u) = 0. \quad (3.1)$$

We observe that the global continuability for the solutions to (3.1) is guaranteed, since the map $s \mapsto h(t, s)$ has linear growth at infinity.

Finally, we introduce the following notation. For any $q \in L^1([0, T])$, we denote by $\lambda_0(q)$ the principal eigenvalue of the linear problem

$$v'' + (\lambda + q(t))v = 0, \quad (3.2)$$

with T -periodic boundary conditions.

The proof of Theorem 2.1 is grounded on the following result (see [15]).

Proposition 3.1. *Suppose that:*

(i) *there exists a T -periodic solution $u^*(t)$ of (3.1) satisfying*

$$\lambda_0(\partial_u h(t, u^*(t))) < 0; \quad (3.3)$$

(ii) *there exists a T -periodic function $\alpha \in W_{\text{loc}}^{2,1}(\mathbb{R})$ satisfying*

$$\alpha''(t) + h(t, \alpha(t)) \geq 0, \quad \text{a.e. } t \in \mathbb{R},$$

and

$$\alpha(t) < u^*(t), \quad \forall t \in \mathbb{R}.$$

Then there exists $k^ \geq 1$ such that for any integer $k \geq k^*$ there exists an integer $m_k \geq 1$ such that, for any integer j relatively prime with k and such that $1 \leq j \leq m_k$, equation (3.1) has two subharmonic solutions $u_{k,j}^{(1)}(t)$, $u_{k,j}^{(2)}(t)$ of order k (not belonging to the same periodicity class), such that, for $i = 1, 2$, $u_{k,j}^{(i)}(t) - u^*(t)$ has exactly $2j$ zeros in the interval $[0, kT[$ and*

$$\alpha(t) \leq u_{k,j}^{(i)}(t), \quad \forall t \in \mathbb{R}.$$

The proof of Proposition 3.1 consists of an application of the Poincaré-Birkhoff fixed point theorem in \mathbb{R}^2 . The crucial point is to prove that there exists a k -th iterate of the Poincaré map associated with (3.2) (which is an area-preserving homeomorphism) that satisfies the “twist condition” for an annular domain in the phase-plane. The main idea is to check that “smaller” solutions (departing sufficiently near the origin) make more than one revolution (around the origin) in $[0, kT]$ and “larger” solutions rotate very slowly and thus are unable to do a complete turn (around the origin). We highlight that the “minimality” of the period of the kT -periodic solutions is a direct consequence of this powerful fixed point theorem.

Furthermore, we introduce this crucial result about the computation of $\lambda_0(q(t)g'(u(t)))$ for every positive T -periodic solution $u(t)$ of (\mathcal{E}) (independently of its existence). The proof is based on the strict convexity assumption (g_{**}) and takes advantage of an algebraic trick adopted in [16].

Lemma 3.1. *Let $q: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function. Let $g \in C^2(\mathbb{R}^+)$ satisfy (g_*) and (g_{**}) . If $u(t)$ is a positive T -periodic solution of (\mathcal{E}) , then*

$$\lambda_0(q(t)g'(u(t))) < 0.$$

As a third step, we are going to check that conditions (i) and (ii) of Proposition 3.1 are satisfied.

Theorem 2.1 ensures the existence of a positive T -periodic function $u^*(t)$ of (\mathcal{E}) with $r < \|u^*\|_\infty < R$ and, by Lemma 3.1, $u^*(t)$ satisfies

$$\lambda_0(q(t)g'(u^*(t))) < 0.$$

Since $h(t, s) = q(t)g(s)$ for $0 \leq s \leq R$, we deduce that $u^*(t)$ is a (positive) T -periodic solution of (3.1) satisfying (3.3). Hence, we have verified condition (i) of Proposition 3.1.

Next, taking $\alpha(t) \equiv 0$, we immediately have condition (ii) verified, due to the fact that $\alpha(t)$ is a trivial solution of (3.1) and $0 = \alpha(t) < u^*(t)$ for every $t \in \mathbb{R}$.

As a final step, we apply Proposition 3.1. Clearly, due to the uniqueness for the Cauchy problems, we have that $u_{k,j}^{(i)}(t) > 0$ for any $t \in \mathbb{R}$. Moreover, from the choice of $R > 0$ we obtain that $u_{k,j}^{(i)}(t) < R$ for any $t \in \mathbb{R}$. Hence, the proof of Theorem 1.1 is concluded. \square

Remark 3.1. In [11], a more general version of Theorem 1.1 was proved and, as a consequence, the authors can also deal with nonlinearities $g(s)$ with a singularity, or nonlinearities $g(s)$ only with superlinear growth condition at zero (in this latter case, obtaining subharmonics to the parameter-dependent equation $u'' + \lambda q(t)g(u) = 0$ for $\lambda > 0$ large enough). \triangleleft

4 Second result: topological approach

In this section we present a topological approach, based on Mawhin’s coincidence degree theory, which allows to obtain infinitely many subharmonics to

$$(\mathcal{E}_\mu) \quad u'' + (a^+(t) - \mu a^-(t))g(u) = 0,$$

as stated in Theorem 1.2. We refer to [25] for the missing details and other discussion on this issue.

Let $a: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function satisfying condition (a_*) (on the interval $[0, T]$) with m intervals of positivity I_i^+ separated by m intervals of negativity I_i^- . In the following, $a(t)$ is treated as a kT -periodic function, for an integer $k \geq 2$. Accordingly, defining the intervals

$$J_{i,\ell}^\pm := I_i^\pm + \ell T, \quad \text{for } i = 1, \dots, m \text{ and } \ell \in \mathbb{Z},$$

in the periodicity interval $[0, kT]$ the weight $a(t)$ shows km positive humps $J_{i,\ell}^+$ separated by km negative ones $J_{i,\ell}^-$ (for $i = 1, \dots, m$ and $\ell = 0, \dots, k-1$).

As a first step, we apply Theorem 2.2 to the interval $[0, kT]$ and we obtain the following statement (see also Figure 2).

Theorem 4.1. *Let $a: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function satisfying (a_*) . Let $g \in C^1(\mathbb{R}^+)$ satisfy (g_*) and (g_s) . Then there exists $\mu^* > 0$ such that, for all $\mu > \mu^*$ and for every integer $k \geq 2$, equation (\mathcal{E}_μ) has at least $2^{km} - 1$ positive kT -periodic solutions.*

More precisely, there exist two constants r, R with $0 < r < R$ and $\mu^ > 0$ such that, for any $\mu > \mu^*$ and for any integer $k \geq 2$, given any km -tuple $\mathcal{S} = (s_1, \dots, s_{km}) \in \{0, 1\}^{km}$, with $\mathcal{S} \neq (0, \dots, 0)$, there exists at least a positive kT -periodic solution $u(t)$ of (\mathcal{E}_μ) such that*

- $\max_{t \in J_{i,\ell}^+} u(t) < r$, if $s_j = 0$ for $j = i + \ell m$;
- $r < \max_{t \in J_{i,\ell}^+} u(t) < R$, if $s_j = 1$ for $j = i + \ell m$.

The crucial point of the proof is to observe that the constant μ^* does not depend on the integer $k \geq 2$. Indeed, when applying Theorem 2.2 on the interval $[0, kT]$ we can choose the constant μ^* depending only on the local properties of $a(t)$ (in the interval $[0, T]$). The thesis follows from Theorem 2.2 and from the definition (2.6) of the sets $\Lambda^\mathcal{I}$ (with respect to the interval $[0, kT]$ and to the subintervals $J_{i,\ell}^+$, for $i = 1, \dots, m$ and $\ell = 0, \dots, k-1$).

As a second step, we prove that there is at least one subharmonic of order k among the $2^{km} - 1$ positive kT -periodic solutions of (\mathcal{E}_μ) provided by Theorem 4.1. It is sufficient to consider the string $(1, 0, \dots, 0)$ of length km (or equivalently the set of indices $\mathcal{I} := \{1\}$). Theorem 4.1 ensures the existence of a positive kT -periodic solution $u(t)$ to (\mathcal{E}_μ) such that $u \in \Lambda^{\{1\}}$. This implies that there exists $\hat{t}_1 \in J_{1,0}^+ = I_1^+$ such that $r < u(\hat{t}_1) < R$ and $0 < u(t) < r$ on $J_{i,\ell}^+$, for $i = 1, \dots, m$ and $\ell = 0, \dots, k-1$ with $(i, \ell) \neq (1, 0)$. Then $u(\hat{t}_1) \neq u(t)$ for all $t \in J_{i,\ell}^+$, for $i = 1, \dots, m$ and $\ell = 0, \dots, k-1$ with $(i, \ell) \neq (1, 0)$. Therefore $u(t)$ is not ℓT -periodic for every $\ell = 1, \dots, k-1$. We conclude that $u(t)$ is a subharmonic solution of order k . Hence, the proof of Theorem 1.2 is concluded. \square

We complete this section with a discussion on the number of subharmonics. Theorem 1.2 ensures the existence of at least a subharmonic solution of order k for every integer $k \geq 0$. Therefore, a natural question that arise is the investigation on the number of subharmonic solutions of order k . Thus now our goal is to select strings of length km which are minimal in some sense (in order to obtain the ‘‘minimality’’ of the period of kT -periodic solutions) and

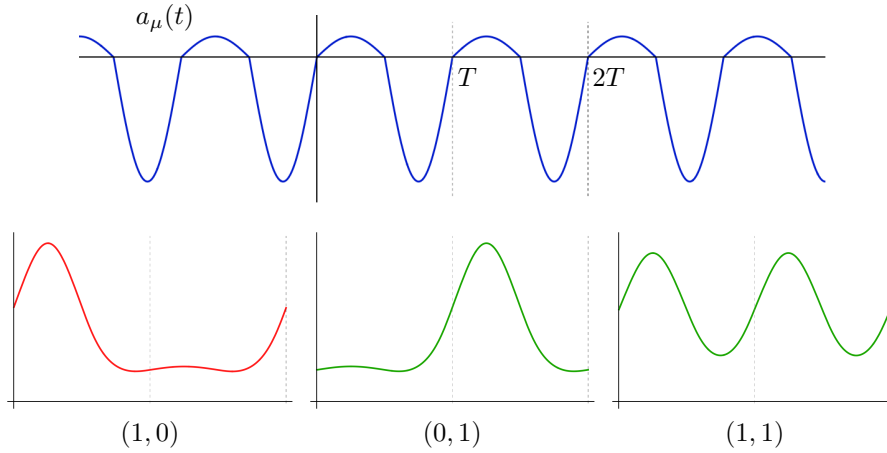


Figure 2: Using a numerical simulation we have studied the 2-periodic solutions of equation (\mathcal{E}_μ) where $a(t) = \sin(t)$, $T = 2\pi$, $\mu = 6$ and $g(s) = 100(s^2 + s^3)$. In the upper part, the figure shows the graph of the weight $q(t) = a_\mu(t)$; while in the lower part, it shows the graphs of three $2T$ -periodic positive solutions, in accordance with Theorem 4.1. The three solutions are in correspondence with the three strings $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively, as stated in Theorem 4.1. The first two solutions are subharmonic solutions of order 2 and the third one is a 1-periodic solution. As subharmonic solutions of order 2, we consider only the first one, since the second one is a translation by 1 of the first solution.

to count only once the strings corresponding to subharmonics belonging to the same periodicity class. To this end, we can take advantage of the combinatorial concept of *aperiodic necklaces of length k* which are made by arranging k beads whose color is chosen from a list of n colors (see Figure 3). Equivalently we can deal also with *n -ary Lyndon words of length k* (that is strings of k digits of an alphabet with n symbols which are strictly smaller in the lexicographic ordering than all of its nontrivial rotations).

It is easy to notice that there exists a one-to-one correspondence between the aperiodic necklaces of length k on n colors and the non-null strings of length k on n symbols. Consequently, in order to obtain an estimate on the number of positive subharmonic solution of order k to equation (\mathcal{E}_μ) , it is sufficient to compute the number $\mathcal{S}_n(k)$ of aperiodic necklaces of length k on n colors when $n = 2^m$. The number $\mathcal{S}_n(k)$ is given by *Witt's formula*:

$$\mathcal{S}_n(k) = \frac{1}{k} \sum_{l|k} \mu(l) n^{\frac{k}{l}}, \quad (4.1)$$

where $\mu(\cdot)$ is the Möbius function, defined on $\mathbb{N} \setminus \{0\}$ by $\mu(1) = 1$, $\mu(l) = (-1)^s$ if l is the product of s distinct primes and $\mu(l) = 0$ otherwise. For instance, the values of $\mathcal{S}_2(k)$ (number of binary Lyndon words of length k) for $k = 2, \dots, 10$ are 1, 2, 3, 6, 9, 18, 30, 56, 99.

Remark 4.1. Although in this context formula (4.1) is related to the concepts of aperiodic necklaces and Lyndon words, this formula come out in different areas of mathematics. Now we provide an overview of the several meanings of (4.1).

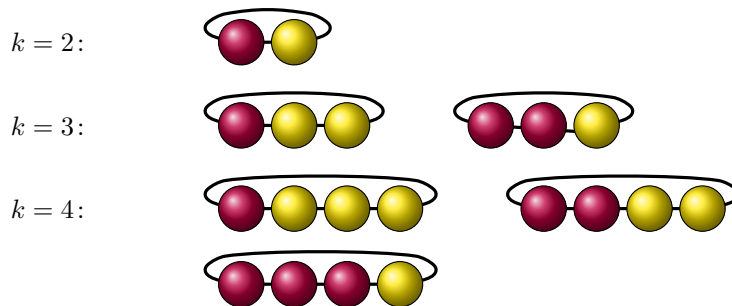


Figure 3: The figure shows the aperiodic necklaces made by arranging k beads whose color is chosen from a list of 2 colors, when $k \in \{2, 3, 4\}$. The aperiodic necklaces depicted in the figure correspond to the following Lyndon words on the alphabet $\{a, b\}$: ab for $k = 2$, abb and aab for $k = 3$, $abbb$, $aabb$ and $aaab$ for $k = 4$ (which are the minimal elements in the class of equivalence in the lexicographic ordering). For instance, for $k = 4$, note that the string $abab$ is not acceptable as it represents a sequence of period 2 and the string $bbaa$ is already counted as $aabb$.

Still in combinatorics, it is not difficult to see that $\mathcal{S}_n(k)$ is also the number of n -ary de Bruijn sequences of order k . A n -ary de Bruijn sequence of order k is a circular string of characters chosen in an alphabet of size n , for which every possible subsequence of length k appears as a substring of consecutive characters exactly once. For example $aaababbb$ and $bbbabaaa$ are the De Bruijn sequence of order 3 on the size-2 alphabet $\{a, b\}$. For more details about these concepts and other aspects of the formula in the context of combinatorics on words, we refer to [34, 35] and the very interesting historical survey [7, § 4].

The number $\mathcal{S}_n(k)$ has several meanings even outside combinatorics. For instance, the integer $\mathcal{S}_2(k)$ (of binary Lyndon words of length k) corresponds to the number of periodic points with minimal period k in the iteration of the tent map $f(x) := 2 \min\{x, 1 - x\}$ on the unit interval (cf. [20], also for more general formulas) and to the number of distinct cycles of minimal period k in a shift dynamical system associated with a totally disconnected hyperbolic iterated function system (cf. [4, Lemma 1, p. 171]). Concerning the more general formula for $\mathcal{S}_n(k)$, we just mention two other meanings. The classical *Witt's formula* (proved in 1937), which is still widely studied in algebra, gives the dimensions of the homogeneous components of degree k of the free Lie algebra over a finite set with n elements (cf. [34, Corollary 5.3.5]). Moreover, in Galois theory, $\mathcal{S}_n(k)$ is also the number of monic irreducible polynomials of degree k over the finite field \mathbb{F}_n , when n is a prime power (in this context (4.1) is also known as *Gauss formula*; we refer to [21, ch. 14, p. 588] for a possible proof).

It is not possible to mention here all the other several implications of formula (4.1), for example in symbolic dynamics, algebra, number theory and chaos theory. For this latter topic, we only recall the recent paper [32] where such numbers appear in connection with the study of period-doubling cascades.

Further information and references can be found in [28, 33, 45]. ◁

As a consequence, the following holds.

Theorem 4.2. *Let $a: \mathbb{R} \rightarrow \mathbb{R}$ be a T -periodic locally integrable function of the form (1.2) satisfying (a_*) . Let $g \in \mathcal{C}^1(\mathbb{R}^+)$ satisfy (g_*) and (g_s) . Then there*

exists $\mu^* > 0$ such that, for each $\mu > \mu^*$ and each integer $k \geq 2$, equation (\mathcal{E}_μ) has at least $\mathcal{S}_{2^m}(k)$ positive subharmonic solutions of order k .

Remark 4.2. In Remark 2.1 it is highlighted that Theorem 2.1 and Theorem 2.2 hold also dealing with equations of the form $(\mathcal{E}_{\mu,c})$ which involve a friction term cu' . Since the proof of Theorem 1.2, developed in the present section, is based on these preliminary results and on Theorem 4.1, we stress that Theorem 1.2 and Theorem 4.2 are also valid for equation $(\mathcal{E}_{\mu,c})$. We refer to [25] for the details. \triangleleft

Remark 4.3. In this section we have proved how to produce positive subharmonic solutions of arbitrary order k , which are “coded” by non-null strings of length k . Starting from Theorem 4.1, via a diagonal argument, in [25] the authors proved the existence of chaotic dynamics, precisely they show the existence of positive globally defined bounded solutions exhibiting a complex behavior given by a bi-infinite string (similarly as in Theorem 4.1). This is not the aim of the present manuscript and we refer also to [14, 46] for some interesting discussion on the subject. \triangleleft

5 Final comments and open problems

We complete the discussion on subharmonic solutions by proposing some open problems that arise from the comparison of the two main results and that will be subjects of future investigations.

First of all, we underline that both main theorems provide infinitely many positive subharmonic solutions to equation

$$(\mathcal{E}_\mu) \quad u'' + (a^+(t) - \mu a^-(t))g(u) = 0.$$

More precisely, we have the existence of positive subharmonic solution of order k when $\mu > \mu^\#$ and k is sufficiently large (by Theorem 1.1) and when μ is sufficiently large and $k \geq 2$ (by Theorem 1.2).

The first natural question is the following.

Open problem 1. Are the subharmonic solutions obtained in Theorem 1.1 and in Theorem 1.2 actually distinct?

It seems difficult to compare solutions obtained with different techniques. A possible way to handle this problem is to understand the behavior of the solutions, for example in the intervals where $a_\mu(t)$ is non-negative. In order to compare solutions, we raised the problem whether it is possible to describe the behavior of the subharmonic solutions obtained in Theorem 1.1 in the same manner as we describe the subharmonics in Theorem 1.2.

We recall that in Theorem 1.1 we obtain subharmonics oscillating around a T -periodic solution $u^*(t)$ of equation (\mathcal{E}_μ) (for $\mu > \mu^\#$). We also observe that Theorem 2.1 ensures the existence of (at least) $2^m - 1$ positive T -periodic solutions $u_i^*(t)$ of (\mathcal{E}_μ) for $\mu > \mu^*$ and any of these solutions is such that $\lambda_0(a_\mu(t)g'(u_i^*(t))) < 0$ (by Lemma 3.1). Therefore, we can apply $2^m - 1$ times Proposition 3.1, with respect to each periodic solution $u_i^*(t)$ and thus Theorem 1.1 is valid with respect to any $u_i^*(t)$.

Open problem 2. Are the subharmonic solutions obtained in the manner described above actually distinct?

Finally, as observed in Remark 4.2, Theorem 1.2 is valid also in the non-variational setting, that is dealing with equation

$$(\mathcal{E}_{\mu,c}) \quad u'' + cu' + (a^+(t) - \mu a^-(t))g(u) = 0,$$

where $c \in \mathbb{R}$ is an arbitrary constant.

Open problem 3. Can we provide positive subharmonic solutions in the non-variational setting when $\mu > \mu^\#$ and k is sufficiently large?

It is generally expected that the solutions obtained with the symplectic approach described in Section 3 disappear for (even small) perturbations that destroy the Hamiltonian structure. Dealing with equation $(\mathcal{E}_{\mu,c})$, we cannot apply the Poincaré-Birkhoff fixed point theorem as described in Proposition 3.1: a more general version of the Poincaré-Birkhoff theorem is necessary.

A Mawhin's coincidence degree

This appendix is devoted to recalling some basic facts about a version of Mawhin's coincidence degree for open and possibly unbounded sets that is used in the present paper. For more details about the coincidence degree, proofs and applications, we refer to [26, 36, 37] and the references therein.

Let X and Z be real Banach spaces and let

$$L: \text{dom } L (\subseteq X) \rightarrow Z$$

be a linear Fredholm mapping of index zero, i.e. $\text{Im } L$ is a closed subspace of Z and $\dim(\ker L) = \text{codim}(\text{Im } L)$ are finite. We denote by $\ker L = L^{-1}(0)$ the kernel of L , by $\text{Im } L \subseteq Z$ the image of L and by $\text{coker } L \cong Z/\text{Im } L$ the complementary subspace of $\text{Im } L$ in Z . Consider the linear continuous projections

$$P: X \rightarrow \ker L, \quad Q: Z \rightarrow \text{coker } L.$$

so that

$$X = \ker L \oplus \ker P, \quad Z = \text{Im } L \oplus \text{Im } Q.$$

We denote by

$$K_P: \text{Im } L \rightarrow \text{dom } L \cap \ker P$$

the right inverse of L , i.e. $LK_P(w) = w$ for each $w \in \text{Im } L$. Since $\ker L$ and $\text{coker } L$ are finite dimensional vector spaces of the same dimension, once an orientation on both spaces is fixed, we choose a linear orientation-preserving isomorphism $J: \text{coker } L \rightarrow \ker L$.

Let

$$N: X \rightarrow Z$$

be a nonlinear L -completely continuous operator, namely N and $K_P(\text{Id} - Q)N$ are continuous, and also $QN(B)$ and $K_P(\text{Id} - Q)N(B)$ are relatively compact sets, for each bounded set $B \subseteq X$. For example, N is L -completely continuous

when N is continuous, maps bounded sets to bounded sets and K_P is a compact linear operator. Consider the *coincidence equation*

$$Lu = Nu, \quad u \in \text{dom } L. \quad (\text{A.1})$$

One can easily prove that equation (A.1) is equivalent to the fixed point problem

$$u = \Phi(u) := Pu + JQNu + K_P(Id - Q)Nu, \quad u \in X. \quad (\text{A.2})$$

Notice that, under the above assumptions, $\Phi: X \rightarrow X$ is a completely continuous operator. Therefore, applying Leray-Schauder degree theory to the operator equation (A.2), it is possible to solve equation (A.1) when L is not invertible.

Let $\mathcal{O} \subseteq X$ be an open and *bounded* set such that

$$Lu \neq Nu, \quad \forall u \in \text{dom } L \cap \partial\mathcal{O}.$$

In this case, the *coincidence degree of L and N in \mathcal{O}* is defined as

$$D_L(L - N, \mathcal{O}) := \deg_{LS}(Id - \Phi, \mathcal{O}, 0),$$

where “ \deg_{LS} ” denotes the Leray-Schauder degree. We underline that D_L is independent on the choice of the linear orientation-preserving isomorphism J and of the projectors P and Q .

Now we present an extension of the coincidence degree to open (possibly unbounded) sets, following the standard approach used in the theory of fixed point index to define the Leray-Schauder degree for locally compact maps on arbitrary open sets (cf. [41, 42]). Extensions of coincidence degree to the case of general open sets have been already considered in previous articles (see for instance [19, 38, 39]).

Consider an open set $\Omega \subseteq X$ and suppose that the solution set

$$\text{Fix}(\Phi, \Omega) := \{u \in \Omega : u = \Phi u\} = \{u \in \Omega \cap \text{dom } L : Lu = Nu\}$$

is compact. According to the extension of Leray-Schauder degree, we can define

$$\deg_{LS}(Id - \Phi, \Omega, 0) := \deg_{LS}(Id - \Phi, \mathcal{V}, 0),$$

where \mathcal{V} is an open and bounded set with $\text{Fix}(\Phi, \Omega) \subseteq \mathcal{V} \subseteq \bar{\mathcal{V}} \subseteq \Omega$. The definition is independent of the choice of \mathcal{V} . In this case the *coincidence degree of L and N in Ω* is defined as

$$D_L(L - N, \Omega) := D_L(L - N, \mathcal{V}) = \deg_{LS}(Id - \Phi, \mathcal{V}, 0),$$

with \mathcal{V} as above. Using the excision property of the Leray-Schauder degree, it is easy to check that if Ω is an open and bounded set satisfying $Lu \neq Nu$, for all $u \in \partial\Omega \cap \text{dom } L$, this definition is the usual definition of coincidence degree described above.

The main properties of the coincidence degree are the following.

- *Additivity.* Let Ω_1, Ω_2 be open and disjoint subsets of Ω such that $\text{Fix}(\Phi, \Omega) \subseteq \Omega_1 \cup \Omega_2$. Then

$$D_L(L - N, \Omega) = D_L(L - N, \Omega_1) + D_L(L - N, \Omega_2).$$

- *Excision.* Let Ω_0 be an open subset of Ω such that $\text{Fix}(\Phi, \Omega) \subseteq \Omega_0$. Then

$$D_L(L - N, \Omega) = D_L(L - N, \Omega_0).$$

- *Existence theorem.* If $D_L(L - N, \Omega) \neq 0$, then $\text{Fix}(\Phi, \Omega) \neq \emptyset$, hence there exists $u \in \Omega \cap \text{dom } L$ such that $Lu = Nu$.
- *Homotopic invariance.* Let $H: [0, 1] \times \Omega \rightarrow X$, $H_\vartheta(u) := H(\vartheta, u)$, be a continuous homotopy such that

$$\mathcal{S} := \bigcup_{\vartheta \in [0, 1]} \{u \in \Omega \cap \text{dom } L: Lu = H_\vartheta u\}$$

is a compact set and there exists an open neighborhood \mathcal{W} of \mathcal{S} such that $\overline{\mathcal{W}} \subseteq \Omega$ and $(K_P(Id - Q)H)|_{[0, 1] \times \overline{\mathcal{W}}}$ is a compact map. Then the map $\vartheta \mapsto D_L(L - H_\vartheta, \Omega)$ is constant on $[0, 1]$.

The following two results are of crucial importance in the computation of the coincidence degree in open and bounded sets. The proofs are omitted. Lemma A.1 is taken from [26, Theorem IV.1] and [37, Theorem 2.4], while Lemma A.2 is a classical result (see [40]) adapted to the present setting as in [23, Lemma 2.2, Lemma 2.3]. By “ deg_B ” we denote the Brouwer degree.

Lemma A.1. *Let L and N be as above and let $\Omega \subseteq X$ be an open and bounded set. Suppose that*

$$Lu \neq \vartheta Nu, \quad \forall u \in \text{dom } L \cap \partial\Omega, \quad \forall \vartheta \in]0, 1],$$

and

$$QN(u) \neq 0, \quad \forall u \in \partial\Omega \cap \ker L.$$

Then

$$D_L(L - N, \Omega) = \text{deg}_B(-JQN|_{\ker L}, \Omega \cap \ker L, 0).$$

Lemma A.2. *Let L and N be as above and let $\Omega \subseteq X$ be an open and bounded set. Suppose that there exist a vector $v \neq 0$ and a constant $\alpha_0 > 0$ such that*

$$Lu \neq Nu + \alpha v, \quad \forall u \in \text{dom } L \cap \partial\Omega, \quad \forall \alpha \in [0, \alpha_0],$$

and

$$Lu \neq Nu + \alpha_0 v, \quad \forall u \in \text{dom } L \cap \Omega.$$

Then

$$D_L(L - N, \Omega) = 0.$$

Finally we state and prove a key lemma for the computation of the degree in open and unbounded sets. This result is a more general version of Lemma A.2.

Lemma A.3. *Let L and N be as above and let $\Omega \subseteq X$ be an open set. Suppose that there exist a vector $v \neq 0$ and a constant $\alpha_0 > 0$ such that*

- $Lu \neq Nu + \alpha v$, for all $u \in \text{dom } L \cap \partial\Omega$ and for all $\alpha \geq 0$;
- for all $\beta \geq 0$ there exists $R_\beta > 0$ such that if there exist $u \in \overline{\Omega} \cap \text{dom } L$ and $\alpha \in [0, \beta]$ with $Lu = Nu + \alpha v$, then $\|u\|_X \leq R_\beta$;

(c) there exists $\alpha_0 > 0$ such that $Lu \neq Nu + \alpha v$, for all $u \in \text{dom } L \cap \Omega$ and $\alpha \geq \alpha_0$.

Then

$$D_L(L - N, \Omega) = 0.$$

Proof. For $\alpha \geq 0$, let us consider the set

$$\mathcal{R}_\alpha := \{u \in \bar{\Omega} \cap \text{dom } L : Lu = Nu + \alpha v\} = \{u \in \bar{\Omega} : u = \Phi u + \alpha v^*\},$$

where $v^* := JQv + K_P(Id - Q)v$. Without loss of generality, we assume that $R_{\alpha'} < R_{\alpha''}$ for $\alpha' < \alpha''$. By conditions (a), for all $\alpha \geq 0$, the solution set \mathcal{R}_α is disjoint from $\partial\Omega$. Moreover, by conditions (b) and (c), \mathcal{R}_α is contained in $\Omega \cap B(0, R_{\alpha_0+1})$. So \mathcal{R}_α is bounded, and hence compact. In this manner we have proved that the coincidence degree $D_L(L - N - \alpha v, \Omega)$ is well defined for any $\alpha \geq 0$.

Now, condition (c), together with the property of existence of solutions when the degree D_L is nonzero, implies that there exists $\alpha_0 \geq 0$ such that

$$D_L(L - N - \alpha_0 v, \Omega) = 0.$$

On the other hand, from condition (b) applied to $\beta = \alpha_0$, repeating the same argument as above, we find that the set

$$\mathcal{S} := \bigcup_{\alpha \in [0, \alpha_0]} \mathcal{R}_\alpha = \bigcup_{\alpha \in [0, \alpha_0]} \{u \in \bar{\Omega} \cap \text{dom } L : Lu = Nu + \alpha v\}$$

is a compact subset of Ω . Hence, by the homotopic invariance of the coincidence degree, we have that

$$D_L(L - N, \Omega) = D_L(L - N - \alpha_0 v, \Omega) = 0.$$

This concludes the proof. \square

Acknowledgments

This work benefited from several fruitful discussions with Alberto Boscaggin and Fabio Zanolin, and from helpful suggestions of Elisa Sovrano. The author was also grateful to Pasquale Candito, Giuseppina D'Agui, Roberto Livrea, Salvatore A. Marano, Sunra Mosconi, for the invitation and the perfect hospitality both in the special session SS92 of the AIMS Conference in Orlando (July 1–5, 2016) and in the workshop in Catania (October 28–29, 2016). Indeed this survey paper is based on a talk given by the author in Catania.

The author wrote the present manuscript in his first month at University of Mons and he is grateful to the Department of Mathematics for the pleasant welcome, in particular to Christophe Troestler, and to Aline Goulard, Marie-Aurore Mainil, Monia Mestiri.

References

- [1] N. Ackermann, Long-time dynamics in semilinear parabolic problems with autocatalysis, in: Recent progress on reaction-diffusion systems and viscosity solutions, World Sci. Publ., Hackensack, NJ, 2009, pp. 1–30.

- [2] S. Alama, G. Tarantello, On semilinear elliptic equations with indefinite nonlinearities, *Calc. Var. Partial Differential Equations* 1 (1993) 439–475.
- [3] H. Amann, J. López-Gómez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, *J. Differential Equations* 146 (1998) 336–374.
- [4] M. Barnsley, *Fractals everywhere*, Academic Press, Inc., Boston, MA, 1988.
- [5] V. L. Barutello, A. Boscaggin, G. Verzini, Positive solutions with a complex behavior for superlinear indefinite ODEs on the real line, *J. Differential Equations* 259 (2015) 3448–3489.
- [6] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg, Superlinear indefinite elliptic problems and nonlinear Liouville theorems, *Topol. Methods Nonlinear Anal.* 4 (1994) 59–78.
- [7] J. Berstel, D. Perrin, The origins of combinatorics on words, *European J. Combin.* 28 (2007) 996–1022.
- [8] D. Bonheure, J. M. Gomes, P. Habets, Multiple positive solutions of superlinear elliptic problems with sign-changing weight, *J. Differential Equations* 214 (2005) 36–64.
- [9] A. Boscaggin, Positive periodic solutions to nonlinear ODEs with indefinite weight: an overview, *Rend. Semin. Mat. Univ. Politec. Torino*, to appear.
- [10] A. Boscaggin, W. Dambrosio, D. Papini, Multiple positive solutions to elliptic boundary blow-up problems, submitted.
- [11] A. Boscaggin, G. Feltrin, Positive subharmonic solutions to nonlinear ODEs with indefinite weight, *Commun. Contemp. Math.*, to appear.
- [12] A. Boscaggin, G. Feltrin, F. Zanolin, Positive solutions for super-sublinear indefinite problems: high multiplicity results via coincidence degree, *Trans. Amer. Math. Soc.*, to appear.
- [13] A. Boscaggin, G. Feltrin, F. Zanolin, Pairs of positive periodic solutions of nonlinear ODEs with indefinite weight: a topological degree approach for the super-sublinear case, *Proc. Roy. Soc. Edinburgh Sect. A* 146 (2016) 449–474.
- [14] A. Boscaggin, F. Zanolin, Positive periodic solutions of second order nonlinear equations with indefinite weight: multiplicity results and complex dynamics, *J. Differential Equations* 252 (2012) 2922–2950.
- [15] A. Boscaggin, F. Zanolin, Subharmonic solutions for nonlinear second order equations in presence of lower and upper solutions, *Discrete Contin. Dyn. Syst.* 33 (2013) 89–110.
- [16] K. J. Brown, P. Hess, Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem, *Differential Integral Equations* 3 (1990) 201–207.

- [17] G. J. Butler, Rapid oscillation, nonextendability, and the existence of periodic solutions to second order nonlinear ordinary differential equations, *J. Differential Equations* 22 (1976) 467–477.
- [18] A. Capietto, W. Dambrosio, D. Papini, Superlinear indefinite equations on the real line and chaotic dynamics, *J. Differential Equations* 181 (2002) 419–438.
- [19] A. Capietto, M. Henrard, J. Mawhin, F. Zanolin, A continuation approach to some forced superlinear Sturm-Liouville boundary value problems, *Topol. Methods Nonlinear Anal.* 3 (1994) 81–100.
- [20] B. S. Du, The minimal number of periodic orbits of periods guaranteed in Sharkovskii’s theorem, *Bull. Austral. Math. Soc.* 31 (1985) 89–103.
- [21] D. S. Dummit, R. M. Foote, *Abstract algebra*, 3rd ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004.
- [22] G. Feltrin, *Positive solutions to indefinite problems: a topological approach*, Ph.D. thesis, SISSA (Trieste), 2016.
- [23] G. Feltrin, F. Zanolin, Existence of positive solutions in the superlinear case via coincidence degree: the Neumann and the periodic boundary value problems, *Adv. Differential Equations* 20 (2015) 937–982.
- [24] G. Feltrin, F. Zanolin, Multiple positive solutions for a superlinear problem: a topological approach, *J. Differential Equations* 259 (2015) 925–963.
- [25] G. Feltrin, F. Zanolin, Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree, *J. Differential Equations*, to appear.
- [26] R. E. Gaines, J. Mawhin, *Coincidence degree, and nonlinear differential equations*, vol. 568 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin-New York, 1977.
- [27] M. Gaudenzi, P. Habets, F. Zanolin, An example of a superlinear problem with multiple positive solutions, *Atti Sem. Mat. Fis. Univ. Modena* 51 (2003) 259–272.
- [28] E. N. Gilbert, J. Riordan, Symmetry types of periodic sequences, *Illinois J. Math.* 5 (1961) 657–665.
- [29] P. M. Girão, J. M. Gomes, Multibump nodal solutions for an indefinite superlinear elliptic problem, *J. Differential Equations* 247 (2009) 1001–1012.
- [30] R. Gómez-Reñasco, J. López-Gómez, The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction-diffusion equations, *J. Differential Equations* 167 (2000) 36–72.
- [31] P. Hess, T. Kato, On some linear and nonlinear eigenvalue problems with an indefinite weight function, *Comm. Partial Differential Equations* 5 (1980) 999–1030.

- [32] M. R. Joglekar, E. Sander, J. A. Yorke, Fixed points indices and period-doubling cascades, *J. Fixed Point Theory Appl.* 8 (2010) 151–176.
- [33] T. Kociumaka, J. Radoszewski, W. Rytter, Computing k -th Lyndon word and decoding lexicographically minimal de Bruijn sequence, in: *Combinatorial pattern matching*, vol. 8486 of *Lecture Notes in Comput. Sci.*, Springer, Cham, 2014, pp. 202–211.
- [34] M. Lothaire, *Combinatorics on words*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1997.
- [35] P. A. MacMahon, Applications of a theory of permutations in circular procession to the theory of numbers, *Proc. London Math. Soc.* S1-23 (1891/92) 305–313.
- [36] J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, vol. 40 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, R.I., 1979.
- [37] J. Mawhin, *Topological degree and boundary value problems for nonlinear differential equations*, in: *Topological methods for ordinary differential equations (Montecatini Terme, 1991)*, vol. 1537 of *Lecture Notes in Mathematics*, Springer, Berlin, 1993, pp. 74–142.
- [38] J. Mawhin, C. Rebelo, F. Zanolin, Continuation theorems for Ambrosetti-Prodi type periodic problems, *Commun. Contemp. Math.* 2 (2000) 87–126.
- [39] P. Morassi, A note on the construction of coincidence degree, *Boll. Un. Mat. Ital. A* (7) 10 (1996) 421–433.
- [40] R. D. Nussbaum, Periodic solutions of some nonlinear, autonomous functional differential equations. II, *J. Differential Equations* 14 (1973) 360–394.
- [41] R. D. Nussbaum, The fixed point index and some applications, vol. 94 of *Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics]*, Presses de l'Université de Montréal, Montreal, QC, 1985.
- [42] R. D. Nussbaum, The fixed point index and fixed point theorems, in: *Topological methods for ordinary differential equations (Montecatini Terme, 1991)*, vol. 1537 of *Lecture Notes in Math.*, Springer, Berlin, 1993, pp. 143–205.
- [43] D. Papini, F. Zanolin, A topological approach to superlinear indefinite boundary value problems, *Topol. Methods Nonlinear Anal.* 15 (2000) 203–233.
- [44] D. Papini, F. Zanolin, On the periodic boundary value problem and chaotic-like dynamics for nonlinear Hill's equations, *Adv. Nonlinear Stud.* 4 (2004) 71–91.
- [45] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at <http://oeis.org>, 2010, Sequence A001037.
- [46] E. Sovrano, How to get complex dynamics? A note on a topological approach, submitted.

- [47] E. Sovrano, A negative answer to a conjecture arising in the study of selection-migration models in population genetics, submitted.
- [48] S. Terracini, G. Verzini, Oscillating solutions to second-order ODEs with indefinite superlinear nonlinearities, *Nonlinearity* 13 (2000) 1501–1514.

Preprint
January 2017