Q-analogues of the Fibo-Stirling numbers

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Abstract

Let F_n denote the n^{th} Fibonacci number relative to the initial conditions $F_0 = 0$ and $F_1 = 1$. In [2], we introduced Fibonacci analogues of the Stirling numbers called Fibo-Stirling numbers of the first and second kind. These numbers serve as the connection coefficients between the Fibo-falling factorial basis $\{(x)_{\downarrow_{F,n}} : n \geq 0\}$ and the Fibo-rising factorial basis $\{(x)_{\uparrow_{F,n}} : n \geq 0\}$ which are defined by $(x)_{\downarrow_{F,0}} = (x)_{\uparrow_{F,0}} = 1$ and for $k \geq 1$, $(x)_{\downarrow_{F,k}} = x(x-F_1)\cdots(x-F_{k-1})$ and $(x)_{\uparrow_{F,k}} = x(x+F_1)\cdots(x+F_{k-1})$. We gave a general rook theory model which allowed us to give combinatorial interpretations of the Fibo-Stirling numbers of the first and second kind.

There are two natural q-analogues of the falling and rising Fibo-factorial basis. That is, let $[x]_q = \frac{q^x-1}{q-1}$. Then we let $[x]_{\downarrow_{q,F,0}} = \overline{[x]}_{\downarrow_{q,F,0}} = [x]_{\uparrow_{q,F,0}} = \overline{[x]}_{\uparrow_{q,F,0}} = 1$ and, for $k > 0$, we let
$[x]_{\downarrow_{q,F,k}} = [x]_q[x-F_1]_q \cdots [x-F_{k-1}]_q$, $\overline{[x]}_{\downarrow_{q,F,k}} = [x]_q([x]_q - [F_1]_q)\cdots([x]_q - [F_{k-1}]_q)$,
$[x]_{\uparrow_{q,F,k}} = [x]_q[x+F_1]_q \cdots [x+F_{k-1}]_q$, and $\overline{[x]}_{\uparrow_{q,F,k}} = [x]_q([x]_q + [F_1]_q)\cdots([x]_q + [F_{k-1}]_q)$.

In this paper, we show we can modify the rook theory model of [2] to give combinatorial interpretations for the two different types q-analogues of the Fibo-Stirling numbers which arise as the connection coefficients between the two different q-analogues of the Fibonacci falling and rising factorial bases.

1 Introduction

Let $\mathbb{Q}$ denote the rational numbers and $\mathbb{Q}[x]$ denote the ring of polynomials over $\mathbb{Q}$. Many classical combinatorial sequences can be defined as connection coefficients between various basis of the polynomial ring $\mathbb{Q}[x]$. There are three very natural bases for $\mathbb{Q}[x]$. The usual power basis $\{x^n : n \geq 0\}$, the falling factorial basis $\{(x)_{\downarrow_n} : n \geq 0\}$, and the rising factorial basis $\{(x)_{\uparrow_n} : n \geq 0\}$. Here we let $(x)_{\downarrow_0} = (x)_{\uparrow_0} = 1$ and for $k \geq 1$, $(x)_{\downarrow_k} = x(x-1)\cdots(x-k+1)$ and $(x)_{\uparrow_k} = x(x+1)\cdots(x+k-1)$. Then the Stirling numbers of the first kind $s_{n,k}$, the Stirling numbers of the second kind $S_{n,k}$ and the Lah numbers $L_{n,k}$ are defined by specifying that for all $n \geq 0$,

$$(x)_{\downarrow_n} = \sum_{k=1}^{n} s_{n,k}\; x^k, \quad x^n = \sum_{k=1}^{n} S_{n,k}\; (x)_{\downarrow_k}, \text{ and } (x)_{\uparrow_n} = \sum_{k=1}^{n} L_{n,k}\; (x)_{\downarrow_k}.$$

The signless Stirling numbers of the first kind are defined by setting $c_{n,k} = (-1)^{n-k} s_{n,k}$. Then it is well known that $c_{n,k}$, $S_{n,k}$, and $L_{n,k}$ can also be defined by the recursions that $c_{0,0} = S_{0,0} = L_{0,0} = 1$, $c_{n,k} = S_{n,k} = L_{n,k} = 0$ if either $n < k$ or $k < 0$, and

$$c_{n+1,k} = c_{n,k-1} + n c_{n,k}, \quad S_{n+1,k} = S_{n,k-1} + k S_{n,k}, \text{ and } L_{n+1,k} = L_{n,k-1} + (n+k) L_{n,k}$$

for all $n, k \geq 0$. There are well known combinatorial interpretations of these connection coefficients. That is, $S_{n,k}$ is the number of set partitions of $[n] = \{1, \ldots, n\}$ into k parts, $c_{n,k}$ is the number of permutations in the symmetric group S_n with k cycles, and $L_{n,k}$ is the number of ways to place n labeled balls into k unlabeled tubes with at least one ball in each tube.

In [2], we introduced Fibonacci analogues of the number $s_{n,k}$, $S_{n,k}$, and $L_{n,k}$. We started with the tiling model of the F_n of [11]. That is, let $\mathcal{FT}_n$ denote the set of tilings a column of height n with tiles of height 1 or 2 such that bottom most tile is of height 1. For example, possible tiling configurations for $\mathcal{FT}_i$ for $i \leq 4$ are shown in

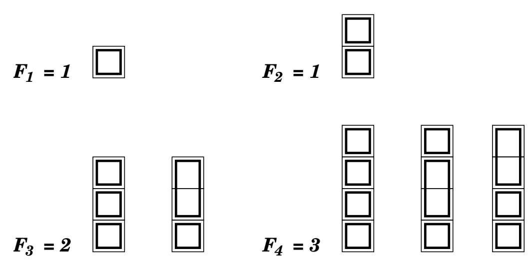

Figure 1: The tilings counted by F_i for $1 \leq i \leq 4$.

For each tiling $T \in \mathcal{FT}_n$, we let $\text{one}(T)$ is the number of tiles of height 1 in T and $\text{two}(T)$ is the number of tiles of height 2 in T and define

$$F_n(p,q) = \sum_{T \in \mathcal{FT}_n} q^{\text{one}(T)} p^{\text{two}(T)}.$$

It is easy to see that $F_1(p,q) = q$, $F_2(p,q) = q^2$ and $F_n(p,q) = qF_{n-1}(p,q) + pF_{n-2}(p,q)$ for $n \geq 2$ so that $F_n(1,1) = F_n$. We then defined the p,q-Fibo-falling factorial basis $\{(x)_{\downarrow_{F,p,q,n}} : n \geq 0\}$ and the p,q-Fibo-rising factorial basis $\{(x)_{\uparrow_{F,p,q,n}} : n \geq 0\}$ by setting $(x)_{\downarrow_{F,p,q,0}} = (x)_{\uparrow_{F,p,q,0}} = 1$ and setting

$$\begin{aligned}(x)_{\downarrow_{F,p,q,k}} &= x(x - F_1(p,q)) \cdots (x - F_{k-1}(p,q)) \text{ and} \\ (x)_{\uparrow_{F,p,q,k}} &= x(x + F_1(p,q)) \cdots (x + F_{k-1}(p,q))\end{aligned}$$

for $k \geq 1$.

Our idea to define p,q-Fibonacci analogues of the Stirling numbers of the first kind, $\mathbf{sf}_{n,k}(p,q)$, the Stirling numbers of the second kind, $\mathbf{Sf}_{n,k}(p,q)$, and the Lah numbers, $\mathbf{Lf}_{n,k}(p,q)$, is to define them to be the connection coefficients between the usual power basis $\{x^n : n \geq 0\}$ and the p,q-Fibo-rising factorial and p,q-Fibo-falling factorial bases. That is, we define $\mathbf{sf}_{n,k}(p,q)$, $\mathbf{Sf}_{n,k}(p,q)$, and $\mathbf{Lf}_{n,k}(p,q)$ by the equations

$$(x)_{\downarrow_{F,p,q,n}} = \sum_{k=1}^{n} \mathbf{sf}_{n,k}(p,q)\ x^k, \tag{1}$$

$$x^n = \sum_{k=1}^{n} \mathbf{Sf}_{n,k}(p,q)\ (x)_{\downarrow_{F,p,q,k}}, \text{ and} \tag{2}$$

$$(x)_{\uparrow_{F,p,q,n}} = \sum_{k=1}^{n} \mathbf{Lf}_{n,k}(p,q)\ (x)_{\downarrow_{F,p,q,k}} \tag{3}$$

for all $n \geq 0$.

It is easy to see that these equations imply simple recursions for the connection coefficients $\mathbf{sf}_{n,k}(p,q)$s, $\mathbf{Sf}_{n,k}(p,q)$s, and $\mathbf{Lf}_{n,k}(p,q)$s. That is, $\mathbf{sf}_{n,k}(p,q)$s, $\mathbf{Sf}_{n,k}(p,q)$s, and $\mathbf{Lf}_{n,k}(p,q)$s can be defined by the following recursions

$$\begin{aligned}
\mathbf{sf}_{n+1,k}(p,q) &= \mathbf{sf}_{n,k-1}(p,q) - F_n(p,q)\mathbf{sf}_{n,k}(p,q), \\
\mathbf{Sf}_{n+1,k}(p,q) &= \mathbf{Sf}_{n,k-1}(p,q) + F_k(p,q)\mathbf{Sf}_{n,k}(p,q), \text{ and} \\
\mathbf{Lf}_{n+1,k}(p,q) &= \mathbf{Lf}_{n,k-1}(p,q) + (F_k(p,q) + F_n(p,q))\mathbf{Lf}_{n,k}(p,q)
\end{aligned}$$

plus the boundary conditions

$$\mathbf{sf}_{0,0}(p,q) = \mathbf{Sf}_{0,0}(p,q) = \mathbf{Lf}_{0,0}(p,q) = 1$$

and

$$\mathbf{sf}_{n,k}(p,q) = \mathbf{Sf}_{n,k}(p,q) = \mathbf{Lf}_{n,k}(p,q) = 0$$

if $k > n$ or $k < 0$. If we define $\mathbf{cf}_{n,k}(p,q) := (-1)^{n-k}\mathbf{sf}_{n,k}(p,q)$, then $\mathbf{cf}_{n,k}(p,q)$s can be defined by the recursions

$$\mathbf{cf}_{n+1,k}(p,q) = \mathbf{cf}_{n,k-1}(p,q) + F_n(p,q)\mathbf{cf}_{n,k}(p,q) \tag{4}$$

plus the boundary conditions $\mathbf{cf}_{0,0}(p,q) = 1$ and $\mathbf{cf}_{n,k}(p,q) = 0$ if $k > n$ or $k < 0$. It also follows that

$$(x)_{\uparrow_{F,p,q,n}} = \sum_{k=1}^{n} \mathbf{cf}_{n,k}(p,q)\ x^k. \tag{5}$$

In [2], we developed a new rook theory model to give a combinatorial interpretation of the $\mathbf{cf}_{n,k}(p,q)$s and the $\mathbf{Sf}_{n,k}(p,q)$s and to give combinatorial proofs of their basic properties. This new rook theory model is a modification of the rook theory model for $S_{n,k}$ and $c_{n,k}$ except that we replace rooks by Fibonacci tilings.

The main goal of this paper is to show how that model can be modified to give combinatorial interpretations to two new q-analogues of the $\mathbf{cf}_{n,k}(1,1)$s and the $\mathbf{Sf}_{n,k}(1,1)$s. Let $[0]_q = 1$ and $[x]_q = \frac{1-q^x}{1-q}$. When n is a positive integer, then $[n]_q = 1 + q + \cdots + q^{n-1}$ is the usual q-analogue of n. Then there are two natural analogues of the falling and rising Fibo-factorial basis. First we let $[x]_{\downarrow_{q,F,0}} = \overline{[x]}_{\downarrow_{q,F,0}} = [x]_{\uparrow_{q,F,0}} = \overline{[x]}_{\uparrow_{q,F,0}} = 1$. For $k > 0$, we let $k > 0$,

$$\begin{aligned}
[x]_{\downarrow_{q,F,k}} &= [x]_q[x-F_1]_q \cdots [x-F_{k-1}]_q, \\
\overline{[x]}_{\downarrow_{q,F,k}} &= [x]_q([x]_q - [F_1]_q) \cdots ([x]_q - [F_{k-1}]_q), \\
[x]_{\uparrow_{q,F,k}} &= [x]_q[x+F_1]_q \cdots [x+F_{k-1}]_q, \text{ and} \\
\overline{[x]}_{\uparrow_{q,F,k}} &= [x]_q([x]_q + [F_1]_q) \cdots ([x]_q + [F_{k-1}]_q).
\end{aligned}$$

Then we define $\mathbf{cF}_{n,k}(q)$ and $\overline{\mathbf{cF}}_{n,k}(q)$ by the equations

$$[x]_{\uparrow_{q,F,n}} = \sum_{k=1}^{n} \mathbf{cF}_{n,k}(q)[x]_q^k \tag{6}$$

and

$$\overline{[x]}_{\uparrow_{q,F,n}} = \sum_{k=1}^{n} \overline{\mathbf{cF}}_{n,k}(q)[x]_q^k. \tag{7}$$

Similarly, we define $\mathbf{SF}_{n,k}(q)$ and $\overline{\mathbf{SF}}_{n,k}(q)$ by the equations

$$[x]_q^n = \sum_{k=1}^{n} \mathbf{SF}_{n,k}(q)[x]_{\downarrow_{q,F,k}} \tag{8}$$

and

$$[x]_q^n = \sum_{k=1}^{n} \overline{\mathbf{SF}}_{n,k}(q)\overline{[x]}_{\downarrow_{q,F,k}} \tag{9}$$

One can easily find recursions for these polynomials. For example,

$$\begin{aligned}
[x]_q^{n+1} &= \sum_{k=1}^{n+1} \mathbf{SF}_{n+1,k}(q)[x]_{\downarrow_{q,F,k}} = \sum_{k=1}^{n} \mathbf{SF}_{n,k}(q)[x]_{\downarrow_{q,F,k}}[x]_q \\
&= \sum_{k=1}^{n} \mathbf{SF}_{n,k}(q)[x]_{\downarrow_{q,F,k}}([F_k]_q + q^{F_k}[x-F_k]_q) \\
&= \sum_{k=1}^{n} [F_k]_q \mathbf{SF}_{n,k}(q)[x]_{\downarrow_{q,F,k}} + \sum_{k=1}^{n} q^{F_k}\mathbf{SF}_{n,k}(q)[x]_{\downarrow_{q,F,k+1}}.
\end{aligned}$$

Taking the coefficient of $[x]_{\downarrow_{q,F,k}}[x]_q$ on both sides shows that

$$\mathbf{SF}_{n+1,k}(q) = q^{F_{k-1}}\mathbf{SF}_{n,k-1}(q) + [F_k]_q\mathbf{SF}_{n,k}(q) \tag{10}$$

for $0 \leq k \leq n+1$. It is then easy to check that the $\mathbf{SF}_{n,k}(q)$s can be defined by the recursions (10) with the initial conditions that $\mathbf{SF}_{0,0}(q) = 1$ and $\mathbf{SF}_{n,k}(q) = 0$ if $k < 0$ or $n < k$. A similar argument will show that $\overline{\mathbf{SF}}_{n,k}(q)$ can be defined by the initial conditions that $\overline{\mathbf{SF}}_{0,0}(q) = 1$ and $\overline{\mathbf{SF}}_{n,k}(q) = 0$ if $k < 0$ or $n < k$ and the recursion

$$\overline{\mathbf{SF}}_{n+1,k}(q) = \overline{\mathbf{SF}}_{n,k-1}(q) + [F_k]_q\overline{\mathbf{SF}}_{n,k}(q). \tag{11}$$

for $0 \leq k \leq n+1$. Similarly, $\mathbf{cF}_{n,k}(q)$ can be defined by the initial conditions that $\mathbf{cF}_{0,0}(q) = 1$ and $\mathbf{cF}_{n,k}(q) = 0$ if $k < 0$ or $n < k$ and the recursion

$$\mathbf{cF}_{n+1,k}(q) = q^{F_{n-1}}\mathbf{cF}_{n,k-1}(q) + [F_n]_q\mathbf{cF}_{n,k}(q), \tag{12}$$

for $0 \leq k \leq n+1$, and $\overline{\mathbf{cF}}_{n,k}(q)$ can be defined by the initial conditions that $\overline{\mathbf{cF}}_{0,0}(q)$ and $\overline{\mathbf{cF}}_{n,k}(q) = 0$ if $k < 0$ or $n < k$ and the recursion

$$\overline{\mathbf{cF}}_{n+1,k}(q) = \overline{\mathbf{cF}}_{n,k-1}(q) + [F_n]_q\overline{\mathbf{cF}}_{n,k}(q) \tag{13}$$

for $0 \le k \le n+1$.

The main goal of this paper is to give a rook theory model for the polynomials $\mathbf{cF}_{n,k}(q)$, $\overline{\mathbf{cF}}_{n,k}(q)$, $\mathbf{SF}_{n,k}(q)$, and $\overline{\mathbf{SF}}_{n,k}(q)$. Our rook theory model will allow us to give combinatorial proofs of the defining equations (6), (7), (8), and (9) as well as combinatorial proofs of the recursions (10), (11), (12), and (13). We shall see that our rook theory model $\mathbf{cF}_{n,k}(q)$, $\overline{\mathbf{cF}}_{n,k}(q)$, $\mathbf{SF}_{n,k}(q)$, and $\overline{\mathbf{SF}}_{n,k}(q)$ is essentially the same as the the rook theory model used in [2] to interpret the $\mathbf{Sf}_{n,k}(p,q)$s and $\mathbf{Sf}_{n,k}(p,q)$s but with a different weighting scheme.

The outline of the paper is as follows. In Section 2, we describe a ranking and unranking theory for the set of Fibonacci tilings which will a crucial element in our weighting scheme for our rook theory model that we shall use to give combinatorial interpretations of the polynomials $\mathbf{cF}_{n,k}(q)$, $\overline{\mathbf{cF}}_{n,k}(q)$, $\mathbf{SF}_{n,k}(q)$, and $\overline{\mathbf{SF}}_{n,k}(q)$. In section 3, we shall review the rook theory model in [2] and show how it can be modified for our purposes. In Section 4, we shall prove general product formulas for Ferrers boards in our new model which will specialize (6), (7), (8), and (9) in the case where the Ferrers board is the staircase board whose column heights are $0, 1, \ldots, n-1$ reading from left to right. In Section 5, we shall prove various special properties of the polynomials $\mathbf{cF}_{n,k}(q)$, $\overline{\mathbf{cF}}_{n,k}(q)$, $\mathbf{SF}_{n,k}(q)$, and $\overline{\mathbf{SF}}_{n,k}(q)$.

2 Ranking and Unranking Fibonacci Tilings.

There is a well developed theory for ranking and unranking combinatorial objects. See for example, Williamson's book [14]. That is, give a collection of combinatorial objects $\mathcal{O}$ of cardinality n, one wants to define bijections rank : $\mathcal{O} \to \{0, \ldots, n-1\}$ and unrank : $\{0, \ldots, n-1\} \to \mathcal{O}$ which are inverses of each other. In our case, we let $\mathcal{F}_n$ denote the set of Fibonnaci tilings of height n. Then we construct a tree which we call the Fibonacci tree for F_n. That is, we start from the top of a Fibonacci tiling and branch left if we see a tile of height 1 and branch right if we see a tiling of height 2. For example, the Fibonacci tree for F_5 is pictured in Figure 2.

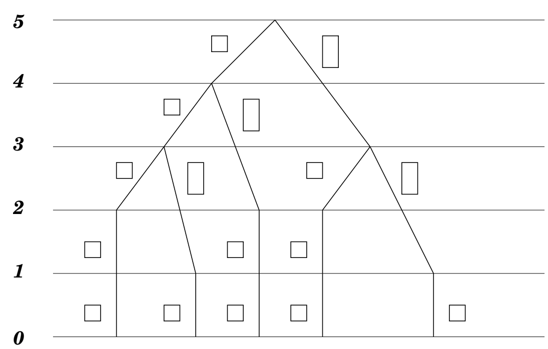

Figure 2: The tree for F_5

Then for any tiling $T \in \mathcal{F}_n$, we define the rank of T for F_n, $\text{rank}_n(T)$, to be the number of paths to the left of the path for T in the Fibonacci tree for F_n. Clearly

$$\{\text{rank}_n(T) : T \in \mathcal{F}_n\} = \{0, 1, 2, \ldots, F_n - 1\}$$

so that $\sum_{T \in \mathcal{F}_n} q^{\text{rank}_n(T)} = 1 + q + \cdots + q^{F_n - 1} = [F_n]_q$. It is, in fact, quite easy to see compute the functions rank_n and unrank_n in this situation. That is, suppose that we represent the tiling T as a sequence $seq(T) = (t_1, \ldots, t_n)$ where reading the tiles starting at the bottom, $t_i = 1$ if there is a tiling t_i of height 1 that ends at level i in T, $t_i = 2$ if there is t_i of height 2 that ends at level i in T, and $t_i = 0$ if there is no tile t_i that ends at level i in T. For example, the tiling of T height 9 pictured in Figure 3 would be represented by the sequence $seq(T) = (1, 0, 2, 1, 1, 1, 0, 2, 1)$.

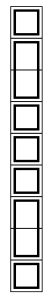

Figure 3: A tiling in $\mathcal{F}_9$.

For any statement A, we let $\chi(A) = 1$ is A is true and $\chi(A) = 0$ if A is false. Then we have the following lemma.

Lemma 1. *Suppose that $T \in \mathcal{F}_n$ is a Fibonacci tiling such that $seq(T) = (t_1, \ldots, t_n)$. Then* $\text{rank}_n(T) = \sum_{i=1}^{n} F_{i-1}\chi(t_i = 2)$.

Proof. The theorem is easy to prove by induction. It is clearly true for $n = 1$ and $n = 2$. Now suppose $n \geq 3$. Then it is easy to see from the Fibonacci tree for F_n that if $t_n = 2$ so that $t_{n-1} = 0$, then the tree that starts at level $n-1$ which represents taking the path to the left starting at level n is just the Fibonacci tree for F_{n-1} and hence this tree will contain F_{n-1} leaves which will all be to the left of path for the tiling T. Then the tree that starting at level $n-2$ which represents taking the path to the right starting at level n is just the Fibonacci tree for F_{n-2} and the number of paths in this tree which lie to the left of the path for T is just that the number of paths to the left of the tiling T' such that $seq(T') = (t_1, \ldots, t_{n-2})$ in the Fibonacci tree for F_{n-2}. Thus in this case

$$\text{rank}_n(T) = F_{n-1} + \text{rank}_{n-2}(T') = F_{n-1} + \text{rank}_{n-2}(t_1, \ldots, t_{n-2}). \tag{14}$$

On the other hand if $t_n = 1$, then we branch left at level n so that the number of paths to the left of the path for T in the Fibonacci tree for F_n will just be the number of paths to the left of the tiling T'' such that $seq(T'') = (t_1, \ldots, t_{n-1})$ in the Fibonacci tree for F_{n-1}. Thus in this case

$$\text{rank}_n(T) = \text{rank}_{n-1}(T'') = \text{rank}_{n-1}(t_1, \ldots, t_{n-1}). \tag{15}$$

□

For example, for the tiling T in Figure 3, $\text{rank}_9(T) = F_2 + F_8 = 1 + 21 = 22$.

For the unrank function, we must rely on Zeckendorf's theorem [15] which states that every positive integer n is uniquely represented as sum $n = \sum_{i=0}^{k} F_{c_i}$ where each $c_i \geq 2$ and $c_{i+1} > c_i + 1$. Indeed, Zeckendorf's theorem says that the greedy algorithm give us the proper representation. That is, given n, find k such that $F_k \leq n < F_{k+1}$, then the representation for n is gotten by taking the representation for $n - F_k$ and adding F_k. For example, suppose that we want to find T such that $\text{rank}_{13}(T) = 100$. Then

1. $F_{11} = 89 \leq 100 < F_{12} = 144$ so that we need to find the Fibonacci representation of $100 - 89 = 11$.

2. $F_6 = 8 \leq 11 < F_7 = 13$ so that we need to find the Fibonacci representation of $11 - 8 = 3$.

3. $F_4 = 3 \leq 3 < F_5 = 5$.

Thus we can represent $100 = F_4 + F_6 + F_{11} = 3 + 8 + 89$ so that

$$seq(T) = (1, 1, 1, 0, 2, 0, 2, 1, 1, 1, 0, 2, 1).$$

3 The rook theory model for the $\mathbf{SF}_{n,k}(q)$s and the $\mathbf{cF}_{n,k}(q)$s.

In this section, we shall give a rook theory model which will allow us to give combinatorial interpretations for the $\mathbf{SF}_{n,k}(q)$s and the $\mathbf{cF}_{n,k}(q)$s. This rook theory model is based on the one which Bach, Paudyal, and Remmel used in [2] to give combinatorial interpretations to the $\mathbf{Sf}_{n,k}(p,q)$s and the $\mathbf{cf}_{n,k}(p,q)$s. Thus, we shall briefly review the rook theory model in [2].

A Ferrers board $B = F(b_1, \ldots, b_n)$ is a board whose column heights are $b_1, \ldots, b_n$, reading from left to right, such that $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$. We shall let B_n denote the Ferrers board $F(0, 1, \ldots, n-1)$. For example, the Ferrers board $B = F(2, 2, 3, 5)$ is pictured on the left of Figure 4 and the Ferrers board B_4 is pictured on the right of Figure 4

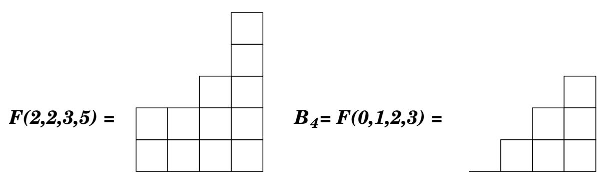

Figure 4: Ferrers boards.

Classically, there are two type of rook placements that we consider on a Ferrers board B. First we let $\mathcal{N}_k(B)$ be the set of all placements of k rooks in B such that no two rooks lie in the same row or column. We shall call an element of $\mathcal{N}_k(B)$ a placement of k non-attacking rooks in B or just a rook placement for short. We let $\mathcal{F}_k(B)$ be the set of all placements of k rooks in B such that no two rooks lie in the same column. We shall call an element of $\mathcal{F}_k(B)$ a file placement of k rooks in B. Thus file placements differ from rook placements in that file placements allow two rooks to be in the same row. For example, we exhibit a placement of 3 non-attacking rooks in $F(2, 2, 3, 5)$ on the left in Figure 5 and a file placement of 3 rooks on the right in Figure 5.

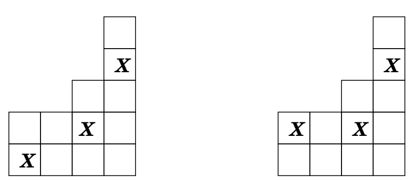

Figure 5: Examples of rook and file placements.

Given a Ferrers board $B = F(b_1, \ldots, b_n)$, we define the k-th rook number of B to be $r_k(B) = |\mathcal{N}_k(B)|$ and the k-th file number of B to be $f_k(B) = |\mathcal{F}_k(B)|$. Then the rook theory interpretation of the classical Stirling numbers is

$$\begin{aligned} S_{n,k} &= r_{n-k}(B_n) \text{ for all } 1 \leq k \leq n \text{ and} \\ c_{n,k} &= f_{n-k}(B_n) \text{ for all } 1 \leq k \leq n. \end{aligned}$$

The idea of [2] is to modify the sets $\mathcal{N}_k(B)$ and $\mathcal{F}_k(B)$ to replace rooks with Fibonacci tilings. The analogue of file placements is very straightforward. That is, if $B = F(b_1, \ldots, b_n)$, then we let $\mathcal{FT}_k(B)$ denote the set of all configurations such that there are k columns $(i_1, \ldots, i_k)$ of B where $1 \leq i_1 < \cdots < i_k \leq n$ such that in each column i_j, we have placed one of the tilings T_{i_j} for the Fibonacci number $F_{b_{i_j}}$. We shall call such a configuration a Fibonacci file placement and denote it by

$$P = ((i_1, T_{i_1}), \ldots, (i_k, T_{i_k})).$$

Let one(P) denote the number of tiles of height 1 that appear in P and two(P) denote the number of tiles of height 2 that appear in P. Then in [2], we defined the weight of P, $WF(P, p, q)$, to be $q^{\text{one}(P)} p^{\text{two}(P)}$. For example, we have pictured an element P of $\mathcal{FT}_3(F(2, 3, 4, 4, 5))$ in Figure 6 whose weight is $q^7 p^2$. Then we defined the k-th p, q-Fibonacci file polynomial of B, $\mathbf{fT}_k(B, p, q)$, by setting

$$\mathbf{fT}_k(B, p, q) = \sum_{P \in \mathcal{FT}_k(B)} WF(P, p, q).$$

If $k = 0$, then the only element of $\mathcal{FT}_k(B)$ is the empty placement whose weight by definition is 1.

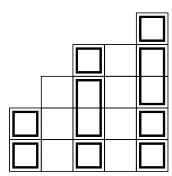

Figure 6: A Fibonacci file placement.

Then in [2], we proved the following theorem concerning Fibonacci file placements in Ferrers boards.

Theorem 2. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$. Let $B^- = F(b_1, \ldots, b_{n-1})$. Then for all $1 \leq k \leq n$,*

$$\mathbf{fT}_k(B, p, q) = \mathbf{fT}_k(B^-, p, q) + F_{b_n}(p, q) \mathbf{fT}_{k-1}(B^-, p, q). \tag{16}$$

To obtain the q-analogues that we desire for this paper, we define a new weight functions for elements of $\mathcal{FT}_k(B)$ where $B = F(b_1, \ldots, b_n)$ is Ferrers board. That is given a Fibonacci file placement $P = ((i_1, T_{i_1}), \ldots, (i_{n-k}, T_{i_{n-k}}))$ in $\mathcal{FT}_{n-k}(B)$, let $(j_1, \ldots, j_k)$ be the sequence of columns in B which have no tilings, reading from left to right. Then we define

$$\begin{aligned} \mathbf{w_{B,q}}(P) &= q^{\sum_{s=1}^{n-k} \text{rank}_{b_{i_s}}(T_{i_s}) + \sum_{t=1}^{k} F_{b_{j_t}}} \text{ and} \\ \overline{\mathbf{w_{B,q}}}(P) &= q^{\sum_{s=1}^{n-k} \text{rank}_{b_{i_s}}(T_{i_s})} \end{aligned}$$

Note that the only difference between these two weight functions is that if b_i is column that does not contain a tiling in P, then it contributes a factor of $q^{F_{b_i}}$ to $\mathbf{w_{B,q}}(P)$ and a factor of 1 to $\overline{\mathbf{w_{B,q}}}(P)$. We then define $\mathbf{FT}_k(B,q)$ and $\overline{\mathbf{FT}}_k(B,q)$, by setting

$$\begin{aligned}
\mathbf{FT}_k(B,q) &= \sum_{P\in\mathcal{FT}_k(B)} \mathbf{w_{B,q}}(P) \text{ and} \\
\overline{\mathbf{FT}}_k(B,q) &= \sum_{P\in\mathcal{FT}_k(B)} \overline{\mathbf{w_{B,q}}}(P).
\end{aligned}$$

If $k=0$, then the only element of $\mathcal{FT}_k(B)$ is the empty placement $\emptyset$ so that $\mathbf{w_{B,q}}(\emptyset) = q^{\sum_{i=1}^n F_{b_i}}$ and $\overline{\mathbf{w_{B,q}}}(\emptyset) = 1$.

Then we have the following analogue of Theorem 2.

Theorem 3. *Let $B = F(b_1,\ldots,b_n)$ be a Ferrers board where $0 \le b_1 \le \cdots \le b_n$ and $b_n > 0$. Let $B^- = F(b_1,\ldots,b_{n-1})$. Then for all $1 \le k \le n$,*

$$\mathbf{FT}_k(B,q) = q^{F_{b_n}}\mathbf{FT}_k(B^-,q) + [F_{b_n}]_q\mathbf{FT}_{k-1}(B^-,p,q) \tag{17}$$

and

$$\overline{\mathbf{FT}}_k(B,q) = \overline{\mathbf{FT}}_k(B^-,q) + [F_{b_n}]_q\mathbf{FT}_{k-1}(B^-,p,q). \tag{18}$$

Proof. We claim (17) results by classifying the Fibonacci file placements in $\mathcal{FT}_k(B)$ according to whether there is a tiling in the last column. If there is no tiling in the last column of P, then removing the last column of P produces an element of $\mathcal{FT}_k(B^-)$. Thus such placements contribute $q^{F_{b_n}}\mathbf{FT}_k(B^-,q)$ to $\mathbf{FT}_k(B,q)$ since the fact that the last column has no tiling means that it contributes a factor of $q^{F_{b_n}}$ to $\mathbf{w_{B,q}}(P)$. If there is a tiling in the last column, then the Fibonacci file placement that results by removing the last column is an element of $\mathcal{FT}_{k-1}(B^-)$ and the sum of the weights of the possible Fibonacci tilings of height b_n for the last column is $\sum_{T\in\mathcal{F}_{b_n}} q^{\mathrm{rank}_{b_n}(T)} = [F_{b_n}]_q$. Hence such placements contribute $[F_{b_n}]_q\mathbf{FT}_{k-1}(B^-,q)$ to $\mathbf{FT}_k(B,q)$. Thus

$$\mathbf{FT}_k(B,q) = q^{F_{b_n}}\mathbf{FT}_k(B^-,q) + [F_{b_n}]_q\mathbf{FT}_{k-1}(B^-,p,q).$$

A similar argument will prove (18). ☐

If $B = F(b_1,\ldots,b_n)$ is a Ferrers board, then we let B_x denote the board that results by adding x rows of length n below B. We label these rows from top to bottom with the numbers $1,2,\ldots,x$. We shall call the line that separates B from these x rows the *bar*. A mixed file placement P on the board B_x consists of picking for each column b_i either (i) a Fibonacci tiling T_i of height b_i above the bar or (ii) picking a row j below the bar to place a rook in the cell in row j and column i. Let $\mathcal{M}_n(B_x)$ denote set of all mixed rook placements on B. For any $P\in\mathcal{M}_n(B_x)$, we let one(P) denote the number of tiles of height 1 that appear in P and two(P) denote the set tiles of height 2 that appear in P. Then in [2], we defined the weight of P, $WF(P,p,q)$, to be $q^{\mathrm{one}(P)}p^{\mathrm{two}(P)}$. For example, Figure 7 pictures a mixed placement P in B_x where $B = F(2,3,4,4,5,5)$ and x is 9 such that $WF(P,p,q) = q^7p^2$.

Also in [2], we proved the following theorem by counting $\sum_{P\in\mathcal{M}_n(B_x)} WF(P,p,q)$ in two different ways.

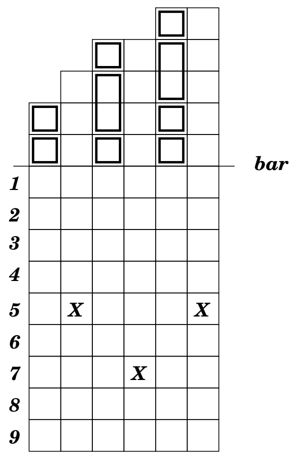

Figure 7: A mixed file placement.

Theorem 4. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$.*

$$(x + F_{b_1}(p,q))(x + F_{b_2}(p,q)) \cdots (x + F_{b_n}(p,q)) = \sum_{k=0}^{n} \mathbf{fT}_k(B,p,q)x^{n-k}. \tag{19}$$

To obtain the desired q-analogues for this paper, we must define new weight functions for mixed placements $P \in \mathcal{M}_n(B_x)$. That is, suppose that $P \cap B$ is the Fibonacci tile placement $Q = ((i_1, T_{i_1}), \ldots, (i_k, T_{i_{n-k}}))$, and suppose that, for the rooks below the bar in columns $1 \leq j_1 < \ldots j_k \leq n$, the rook in column j_s is in row d_{j_s} for $s = 1, \ldots, k$. Then we define

$$\begin{aligned}
\mathbf{w_{B_x,q}}(P) &= \mathbf{w_{B,q}}(P) q^{\sum_{t=1}^{k} d_{j_t} - 1} = q^{\sum_{s=1}^{n-k} \mathrm{rank}_{b_{i_s}}(T_{i_s}) + \sum_{t=1}^{k} F_{b_{j_t}} + d_{j_t} - 1} \text{ and} \\
\overline{\mathbf{w_{B_x,q}}}(P) &= \overline{\mathbf{w_{B,q}}}(P) q^{\sum_{t=1}^{k} d_{j_t} - 1} = q^{\sum_{s=1}^{n-k} \mathrm{rank}_{b_{i_s}}(T_{i_s}) + \sum_{t=1}^{k} d_{j_t} - 1}.
\end{aligned}$$

That is, for each column i the choice of a Fibonacci tiling T_i of height b_i above the bar contributes a factor of $q^{\mathrm{rank}_{b_i}(T_i)}$ to $\mathbf{w_{B_x,q}}(P)$ and the choice of picking a row j below the bar to place a rook in the cell in row j and column i contributes a factor of $q^{F_{b_i}+j-1}$ to $\mathbf{w_{B_x,q}}(P)$. Similarly, for each column b_i the choice of a Fibonacci tiling T_i of height b_i above the bar contributes a factor of $q^{\mathrm{rank}_{b_i}(T_i)}$ to $\overline{\mathbf{w_{B_x,q}}}(P)$ and the choice of picking a row j below the bar to place a rook in the cell in row j and column i contributes a factor of q^{j-1} to $\overline{\mathbf{w_{B_x,q}}}(P)$.

Then we have the following analogue of Theorem 4.

Theorem 5. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$. Then for all positive integers x,*

$$[x + F_{b_1}]_q [x + F_{b_2}]_q \cdots [x + F_{b_n}]_q = \sum_{k=0}^{n} \mathbf{FT}_k(B,q)[x]_q^{n-k} \tag{20}$$

and

$$([x]_q + [F_{b_1}]_q)([x]_q + [F_{b_2}]_q) \cdots ([x]_q + [F_{b_n}]_q) = \sum_{k=0}^{n} \overline{\mathbf{FT}}_k(B,q)[x]_q^{n-k} \tag{21}$$

Proof. To prove (20), fix x to be a positive integer and consider the sums

$$\begin{aligned} S &= \sum_{P \in \mathcal{M}_n(B_x)} \mathbf{w_{B_x,q}}(P) \text{ and} \\ \overline{S} &= \sum_{P \in \mathcal{M}_n(B_x)} \overline{\mathbf{w_{B_x,q}}}(P). \end{aligned}$$

For S, in a given column i, our choice of the Fibonacci tiling of height b_i will contribute a factor of $\sum_{T \in \mathcal{F}_n} q^{\mathrm{rank}_{b_i}(T)} = [F_{b_i}]_q$ to S. Our choice of placing a rook below the bar in column i contribute a factor of

$$\sum_{j=1}^{x} q^{F_{b_i}+j-1} = q^{F_{b_i}}(1 + q + q^2 + \cdots q^{x-1}) = q^{F_{b_i}}[x]_q$$

to S. As $[F_{b_i}]_q + q^{F_{b_i}}[x]_q = [x + F_{b_i}]_q$, each column of b_i of B contributes a factor of $[x + F_{b_i}]_q$ to S so that

$$S = \prod_{i=1}^{n} [x + F_{b_i}]_q.$$

For $\overline{S}$, in a given column i, our choice of the Fibonacci tiling of height b_i will contribute a factor of $\sum_{T \in \mathcal{F}_n} q^{\mathrm{rank}_{b_i}(T)} = [F_{b_i}]_q$ to S. Our choice of placing a rook below the bar in column i contribute a factor of

$$\sum_{j=1}^{x} q^{j-1} = [x]_q$$

to $\overline{S}$. Thus each column b_i contributes a factor of $[x]_q + [F_{b_i}]_q$ to S so that

$$S = \prod_{i=1}^{n} ([x]_q + [F_{b_i}]_q).$$

On the other hand, suppose that we fix a Fibonacci file placement $P \in \mathcal{FT}_k(B)$. Then we want to compute $S_P = \sum_{Q \in \mathcal{M}_n(B), Q \cap B = P} \mathbf{w_{B_x,q}}(Q)$ which is the sum of $\mathbf{w_{B_x,q}}(Q)$ over all mixed placements Q such that Q intersect B equals P. It it easy to see that such a Q arises by choosing a rook to be placed below the bar for each column that does not contain a tiling. Each such column contributes a factor of $1 + q + \cdots + q^{x-1} = [x]_q$ in addition to the weight $\mathbf{w_{B,q}}(P)$. Thus it follows that $S_P = \mathbf{w_{B,q}}(P)[x]_q^{n-k}$. Hence it follows that

$$\begin{aligned} S &= \sum_{k=0}^{n} \sum_{P \in \mathcal{FT}_k(B)} S_P \\ &= \sum_{k=0}^{n} [x]_q^{n-k} \sum_{P \in \mathcal{FT}_k(B)} \mathbf{w_{B,q}}(P) \\ &= \sum_{k=0}^{n} \mathbf{FT}_k(B,q)\ [x]_q^{n-k}. \end{aligned}$$

The same argument will show that

$$\overline{S} = \sum_{k=0}^{n} \overline{\mathbf{FT}}_k(B,q)\ [x]_q^{n-k}.$$

$\square$

Now consider the special case of the previous two theorems when $B_n = F(0,1,2,\ldots,n-1)$. Then (17) implies that

$$\mathbf{FT}_{n+1-k}(B_{n+1},q) = q^{F_n}\mathbf{FT}_{n+1-k}(B_n,p,q) + [F_n]_q\mathbf{FT}_{n-k}(B_n,q).$$

It then easily follows that for all $0 \leq k \leq n$,

$$\mathbf{cF}_{n,k}(q) = \mathbf{FT}_{n-k}(B_n,q). \tag{22}$$

Note that $\mathbf{cF}_{n,0}(q) = 0$ for all $n \geq 1$ since there are no Fibonacci file placements in $\mathcal{FT}_n(B_n)$ since there are only $n-1$ non-zero columns. Moreover such a situation, we see that (22) implies that

$$[x]_q[x+F_1]_q[x+F_2]_q\cdots[x+F_{n-1}]_q = \sum_{k=1}^{n}\mathbf{cF}_{n,k}(q)\ [x]_q^k.$$

Thus we have given a combinatorial proof of (6).

Similarly (18) implies that

$$\overline{\mathbf{FT}}_{n+1-k}(B_{n+1},q) = \overline{\mathbf{FT}}_{n+1-k}(B_n,p,q) + [F_n]_q\overline{\mathbf{FT}}_{n-k}(B_n,q).$$

It then easily follows that for all $0 \leq k \leq n$,

$$\overline{\mathbf{cF}}_{n,k}(q) = \overline{\mathbf{FT}}_{n-k}(B_n,q). \tag{23}$$

Moreover such a situation, we see that (23) implies that

$$[x]_q([x]_q+[F_1]_q)([x]_q+[F_2]_q)\cdots([x]_q+[F_{n-1}]_q) = \sum_{k=1}^{n}\overline{\mathbf{cF}}_{n,k}(q)\ [x]_q^k.$$

Thus we have given a combinatorial proof of (7).

The Fibonacci analogue of rook placements defined in [2] is a slight variation of Fibonacci file placements. The main difference is that each tiling will cancel some of the top most cells in each column to its right that has not been canceled by a tiling which is further to the left. Our goal is to ensure that if we start with a Ferrers board $B = F(b_1,\ldots,b_n)$, our cancellation scheme will ensure that the number of uncanceled cells in the empty columns are $b_1,\ldots,b_{n-k}$, reading from left to right. That is, if $B = F(b_1,\ldots,b_n)$, then we let $\mathcal{NT}_k(B)$ denote the set of all configurations such that that there are k columns $(i_1,\ldots,i_k)$ of B where $1 \leq i_1 < \cdots < i_k \leq n$ such that the following conditions hold.

1. In column i_1, we place a Fibonacci tiling $T_{i,1}$ of height b_{i_1} and for each $j > i_1$, this tiling cancels the top $b_j - b_{j-1}$ cells at the top of column j. This cancellation has the effect of ensuring that the number of uncanceled cells in the columns without tilings at this point is $b_1,\ldots,b_{n-1}$, reading from left to right.

2. In column i_2, our cancellation due to the tiling in column i_1 ensures that there are b_{i_2-1} uncanceled cells in column i_2. Then we place a Fibonacci tiling $T_{i,2}$ of height b_{i_2-1} and for each $j > i_2$, we cancel the top $b_{j-1} - b_{j-2}$ cells in column j that has not been canceled by the tiling in column i_1. This cancellation has the effect of ensuring that the number of uncanceled cells in columns without tilings at this point is $b_1,\ldots,b_{n-2}$, reading from left to right.

3. In general, when we reach column i_s, we assume that the cancellation due to the tilings in columns $i_1, \ldots, i_{j-1}$ ensure that the number of uncanceled cells in the columns without tilings is $b_1, \ldots, b_{n-(s-1)}$, reading from left to right. Thus there will be $b_{i_s-(s-1)}$ uncanceled cells in column i_s at this point. Then we place a Fibonacci tiling $T_{i,s}$ of height $b_{i_s-(s-1)}$ and for each $j > i_s$, this tiling will cancel the top $b_{j-(s-1)} - b_{j-s}$ cells in column j that has not been canceled by the tilings in columns $i_1, \ldots, i_{s-1}$. This cancellation has the effect of ensuring that the number of uncanceled cells in columns without tilings at this point is $b_1, \ldots, b_{n-s}$, reading from left to right.

We shall call such a configuration a Fibonacci rook placement and denote it by

$$P = ((i_1, T_{i_1}), \ldots, (i_k, T_{i_k})).$$

Let one(P) denote the number of tiles of height 1 that appear in P and two(P) denote the number of tiles of height 2 that appear in P. Then in [2], we defined the weight of P, $WF(P, p, q)$, to be $q^{\text{one}(P)} p^{\text{two}(P)}$. For example, on the left in Figure 8, we have pictured an element P of $\mathcal{NT}_3(F(2,3,4,4,6,6))$ whose weight is q^5p^2. In Figure 8, we have indicated the canceled cells by the tiling in column i by placing an i in the cell. We note in the special case where $B = F(0, k, 2k, \ldots, (n-1)k)$, then our cancellation scheme is quite simple. That is, each tiling just cancels the top k cells in each column to its right which has not been canceled by tilings to its left. For example, on the right in Figure 8, we have pictured an element P of $\mathcal{NT}_3(F(0,1,2,3,4,5))$ whose weight is q^6p. Again, we have indicated the canceled cells by the tiling in column i by placing an i in the cell.

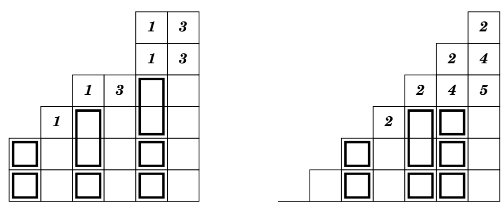

Figure 8: A Fibonacci rook placement.

We define the k-th p,q-Fibonacci rook polynomial of B, $\mathbf{rT}_k(B,p,q)$, by setting

$$\mathbf{rT}_k(B,p,q) = \sum_{P \in \mathcal{NT}_k(B)} WF(P,p,q).$$

If $k = 0$, then the only element of $\mathcal{FT}_k(B)$ is the empty placement whose weight by definition is 1.

Then in [2], we proved the following two theorems concerning Fibonacci rook placements in Ferrers boards.

Theorem 6. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$. Let $B^- = F(b_1, \ldots, b_{n-1})$. Then for all $1 \leq k \leq n$,*

$$\mathbf{rT}_k(B,p,q) = \mathbf{rT}_k(B^-,p,q) + F_{b_{n-(k-1)}}(p,q)\mathbf{rT}_{k-1}(B^-,p,q). \tag{24}$$

Theorem 7. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$.*

$$x^n = \sum_{k=0}^{n} \mathbf{rT}_{n-k}(B,p,q)(x - F_{b_1}(p,q))(x - F_{b_2}(p,q)) \cdots (x - F_{b_k}(p,q)). \tag{25}$$

To obtain the q-analogues that we want for this paper, we need to define two new weight functions on Fibonacci rook tilings. That is, suppose that $B = F(b_1, \ldots, b_n)$ is a Ferrers board and $P = ((i_1, T_{i_1}), \ldots, (i_k, T_{i_k}))$ is an Fibonacci rook tiling in $\mathcal{NT}_k(B)$. Then we know that the number of uncanceled cells in the $n-k$ columns which do not have tilings are $b_1, \ldots, b_{n-k}$ reading from left to right. Suppose that the number of uncanceled cells in the columns with tilings are $e_1, \ldots, e_k$ reading from left to right so that tiling T_{i_j} is of height e_j for $j = 1, \ldots, k$. The we define

$$\begin{aligned} \mathbf{W_{B,q}}(P) &= q^{\sum_{s=1}^{k} \mathrm{rank}_{e_s}(T_{i_s}) + \sum_{t=1}^{n-k} F_{b_t}} \text{ and} \\ \overline{\mathbf{W_{B,q}}}(P) &= q^{\sum_{s=1}^{k} \mathrm{rank}_{e_s}(T_{i_s})}. \end{aligned}$$

For example, if $B = (2,3,4,4,5,5)$ and $P = ((1,T_1),(3,T_3),(5,T_5))$ is the rook tiling pictured in Figure 8, then $e_1 = 2$, $e_2 = 3$ and $e_3 = 4$ and one can check that $\mathrm{rank}_2(T_1) = 0$, $\mathrm{rank}_3(T_3) = F_2 = 1$, and $\mathrm{rank}_4(T_5) = F_3 = 2$. Thus $\mathbf{W_{B,q}}(P) = q^{0+1+2+F_2+F_3+F_4} = q^9$ and $\overline{\mathbf{W_{B,q}}}(P) = q^{0+1+2} = q^3$. If $k = 0$, then the only element of $\mathcal{FT}_k(B)$ is the empty placement $\emptyset$ which means that $\mathbf{W_{B,q}}(\emptyset) = q^{\sum_{i=1}^{n} F_{b_i}}$ and $\mathbf{W_{B,q}}(\emptyset) = 1$.

Then we define $\mathbf{RT}_k(B,q)$ by setting

$$\mathbf{RT}_k(B,q) = \sum_{P \in \mathcal{NT}_k(B)} \mathbf{W_{B,q}}(P)$$

and

$$\overline{\mathbf{RT}}_k(B,q) = \sum_{P \in \mathcal{NT}_k(B)} \overline{\mathbf{W_{B,q}}}(P).$$

Note that because of our cancellation scheme, there is a very simple relationship between $\mathbf{RT}_k(B,q)$ and $\overline{\mathbf{RT}}_k(B,q)$ in the case where $B = F(b_1, \ldots, b_n)$. That is, in any placement $P \in \mathcal{NT}_k(B)$, the empty columns have $b_1, \ldots, b_{n-k}$ uncanceled cells, reading from left to right, so that

$$\mathbf{RT}_k(B,q) = q^{\sum_{i=1}^{n-k} F_{b_i}} \overline{\mathbf{RT}}_k(B,q). \tag{26}$$

Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board and x be a positive integer. Then we let $AugB_x$ denote the board where we start with B_x and add the flip of the board B about its baseline below the board. We shall call the the line that separates B from these x rows the *upper bar* and the line that separates the x rows from the flip of B added below the x rows the *lower bar*. We shall call the flipped version of B added below B_x the board $\overline{B}$. For example, if $B = F(2,3,4,4,5,5)$, then the board $AugB_7$ is pictured in Figure 9.

The analogue of mixed placements in $AugB_x$ are more complex than the mixed placements for B_x. We process the columns from left to right. If we are in column 1, then we can do one of the following three things.

i. We can put a Fibonacci tiling in cells in the first column in B. Then we must cancel the top-most cells in each of the columns in B to its right so that the number of uncanceled

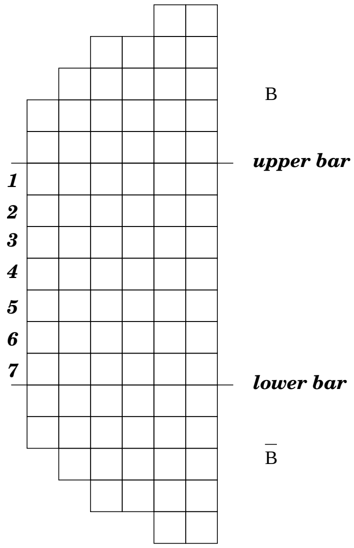

Figure 9: An example of an augmented board $AugB_x$.

cells in the columns to its right are $b_1, b_2, \ldots, b_{n-1}$, respectively, as we read from left to right. This means that we will cancel $b_i - b_{i-1}$ at the top of column i in B for $i = 2, \ldots, n$. We also cancel the same number of cells at the bottom of the corresponding columns of $\overline{B}$.

ii. We can place a rook in any row of column 1 that lies between the upper bar and lower bar. This rook will not cancel anything.

iii. We can put a flip of Fibonacci tiling in column 1 of $\overline{B}$. This tiling will not cancel anything.

Next assume that when we get to column j, the number of uncanceled cells in the columns that have no tilings in B and $\overline{B}$ are $b_1, \ldots, b_k$ for some k as we read from left to right. Suppose there are b_i uncanceled cells in B in column j. Then we can do one of three things.

i. We can put a Fibonacci tiling of height b_i in the uncanceled cells in column j in B. Then we must cancel top-most cells of the columns in B to its right so that the number of uncanceled cells in the columns which have no tilings up to this point are $b_1, b_2, \ldots, b_{k-1}$, We also cancel the same number of cells at the bottom of the corresponding columns of $\overline{B}$

ii. We can place a rook in any row of column j that lies between the upper bar and lower bar. This rook will not cancel anything.

iii. We can put a flip of Fibonacci tiling in the b_i uncanceled cells in column j of $\overline{B}$. This tiling will not cancel anything

We let $\mathcal{M}_n(AugB_x)$ denote set of all mixed rook placements on $AugB_x$. For any placement $P \in \mathcal{M}_n(AugB_x)$, we define $\mathbf{W_{AugB_x,q}}(P)$ and $\overline{\mathbf{W_{AugB_x,q}}}(P)$ as follows. For any column i, suppose that the number of uncanceled cells in B in column i is t_i. Then the factor $\mathbf{W_{i,AugB_x,q}}(P)$ that the placement in column i contributes to $\mathbf{W_{AugB_x,q}}(P)$ is

1. $q^{\mathrm{rank}_{t_i}(T_i)}$ if there is tiling T_i in B in column i,

2. $q^{F_{t_i}+s_i-1}$ if there is a rook in row s_i^{th} row from the top in the x rows that lie between the upper bar and lower bar, and

3. $-q^{\text{rank}_{t_i}(T_i)}$ if there is a flip of a tiling T_i in column i of $\overline{B}$.

Then we define

$$\mathbf{W_{AugB_x,q}}(P) = \prod_{i=1}^{n} \mathbf{W_{i,AugB_x,q}}(P).$$

Similarly, the factor $\overline{\mathbf{W_{i,AugB_x,q}}}(P)$ that the tile placement in column i contributes to $\overline{\mathbf{W_{AugB_x,q}}}(P)$ is

1. $q^{\text{rank}_{t_i}(T_i)}$ if there is tiling T_i in B in column i,

2. q^{s_i-1} if there is a rook in row s_i^{th} row from the top in the x rows that lie between the upper bar and lower bar, and

3. $-q^{\text{rank}_{t_i}(T_i)}$ if there is a flip of a tiling T_i in column i of $\overline{B}$.

Then we define

$$\overline{\mathbf{W_{AugB_x,q}}}(P) = \prod_{i=1}^{n} \overline{\mathbf{W_{i,AugB_x,q}}}(P).$$

For example, Figure 10 pictures a mixed placement P in $AugB_x$ where $B = F(2,3,4,4,5,5)$ and x is 7 where $\text{rank}_2(T_1) = 0$, $\text{rank}_4(T_4) = F_2 = 1$, and $\text{rank}_4(T_5) = F_3 = 2$ where T_i is the tiling in column i for $i \in \{1,4,5\}$. The rooks columns 2 and 6 are in row 5 and the rook in column 3 is in row 3 so that $s_2 = s_6 = 5$ and $s_3 = 3$. Thus

$$\begin{aligned}
\mathbf{W_{AugB_x,q}}(P) &= -q^{0+(4+F_2)+(2+F_3)+1+2+(4+F_4)} = -q^{19} \text{ and} \\
\overline{\mathbf{W_{AugB_x,q}}}(P) &= -q^{0+(4)+(2)+1+2+(4)} = -q^{13}
\end{aligned}$$

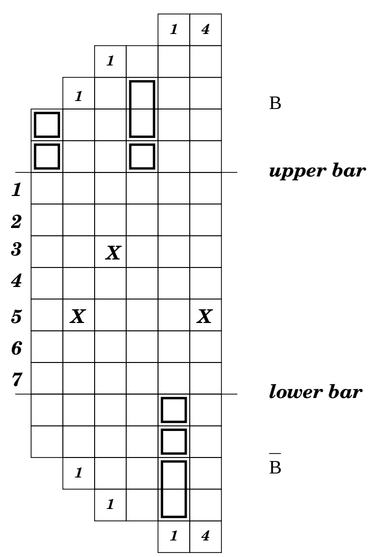

Figure 10: A mixed rook placement.

Our next theorem results from counting $\sum_{P\in\mathcal{M}_n(AugB_x)} \overline{\mathbf{W_{AugB_x,q}}}(P)$ in two different ways.

Theorem 8. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$ and $x \in \mathbb{P}$. Then*

$$[x]_q^n = \sum_{k=0}^{n} \overline{\mathbf{RT}}_{n-k}(B,q)([x]_q - [F_{b_1}]_q)([x]_q - [F_{b_2}]_q) \cdots ([x]_q - [F_{b_k}]_q). \tag{27}$$

Proof. Fix x to be a positive integer and consider the sum $S = \sum_{P \in \mathcal{M}_n(AugB_x)} \overline{\mathbf{W_{AugB_x,q}}}(P)$. First we consider the contribution of each column as we proceed from left to right. Given our three choices in column 1, the contribution of our choice of the tilings of height b_1 in column 1 of B is $[F_{b_1}]_q$, the choice of placing a rook in between the upper bar and the lower is $[x]_q$, and the contribution of our choice of the tilings of height b_1 in column 1 of $\overline{B}$ is $-[F_{b_1}]_q$. Thus the contribution of our choices in column 1 to S is $[F_{b_1}]_q + [x]_q - [F_{b_1}]_q = [x]_q$.

In general, after we have processed our choices in the first j columns, our cancellation scheme ensures that the number of uncanceled cells in B and $\overline{B}$ in the j-th column is b_i for some $i \leq j$. Thus given our three choices in column j, the contribution of our choice of the tilings of height b_i in column j of B is $[F_{b_i}]_q$, the choice of placing a rook in between the upper bar and the lower is $[x]_q$, and the contribution of our choice of the tilings of height b_i in column j of $\overline{B}$ is $-[F_{b_i}]_q$. Thus the contribution of our choices in column j to S is $[F_{b_i}]_q + [x]_q - [F_{b_i}]_q = [x]_q$. It follows that $S = [x]_q^n$.

On the other hand, suppose that we fix a Fibonacci rook placement $P \in \mathcal{NT}_{n-k}(B)$. Then we want to compute the $S_P = \sum_{Q \in \mathcal{M}_n(AugB_x), Q \cap B = P} \overline{\mathbf{W_{AugB_x,q}}}(P)$ which is the sum of $\overline{\mathbf{W_{AugB_x,q}}}(P)$ over all mixed placements Q such that Q intersect B equals P. Our cancellation scheme ensures that the number of uncanceled cells in B and $\overline{B}$ in the k columns that do not contain tilings in P is $b_1, \ldots, b_k$ as we read from right to left. For each such $1 \leq i \leq k$, the factor that arises from either choosing a rook to be placed in between the upper bar and lower bar or a flipped Fibonacci tiling of height b_i in $\overline{B}$ is $[x]_q - [F_{b_i}]_q$. It follows that

$$S_P = \overline{\mathbf{W_{B,q}}}(P) \prod_{i=1}^{k} [x]_q - [F_{b_i}]_q.$$

Hence it follows that

$$\begin{aligned} S &= \sum_{k=0}^{n} \sum_{P \in \mathcal{NT}_{n-k}(B)} S_P \\ &= \sum_{k=0}^{n} \left(\prod_{i=1}^{k} [x]_q - [F_{b_i}]_q \right) \sum_{P \in \mathcal{NT}_k(B)} \overline{\mathbf{W_{B,q}}}(P) \\ &= \sum_{k=0}^{n} \overline{\mathbf{RT}}_{n-k}(B,q) \left(\prod_{i=1}^{k} [x]_q - [F_{b_i}]_q \right). \end{aligned}$$

□

Theorem 9. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$ and $x \geq b_n$. Then*

$$[x]_q^n = \sum_{k=0}^{n} \mathbf{RT}_{n-k}(B,q)[x - F_{b_1}]_q [x - F_{b_2}]_q \cdots [x - F_{b_k}]_q. \tag{28}$$

Proof. It is easy to see from our cancellation scheme that

$$\mathbf{RT}_{n-k}(B,q) = q^{F_{b_1}+\cdots+F_{b_k}}\overline{\mathbf{RT}}_{n-k}(B,q).$$

Thus it follows from (27) that

$$[x]_q^n = \sum_{k=0}^{n} \mathbf{RT}_{n-k}(B,q) q^{-(F_{b_1}+\cdots+F_{b_k})}([x]_q - [F_{b_1}]_q)([x]_q - [F_{b_2}]_q)\cdots([x]_q - [F_{b_k}]_q).$$

However since $x \geq F_{b_i}$ for every i,

$$[x]_q - [F_{b_i}]_q = q^{F_{b_i}}[x - F_{b_i}]_q$$

so that

$$[x]_q^n = \sum_{k=0}^{n} \mathbf{RT}_{n-k}(B,q)[x - F_{b_1}]_q[x - F_{b_2}]_q\cdots[x - F_{b_k}]_q.$$

□

Now consider the special case of the previous three theorems when $B_n = F(0,1,2,\ldots,n-1)$. Then (17) implies that

$$\mathbf{RT}_{n+1-k}(B_{n+1},q) = q^{F_{k-1}}\mathbf{RT}_{n+1-k}(B_n,q) + [F_k]_q\mathbf{RT}_{n-k}(B_n,q).$$

Similarly (18) implies that

$$\overline{\mathbf{RT}}_{n+1-k}(B_{n+1},q) = \overline{\mathbf{RT}}_{n+1-k}(B_n,q) + [F_k]_q\overline{\mathbf{RT}}_{n-k}(B_n,q).$$

It then easily follows that for all $0 \leq k \leq n$,

$$\mathbf{SF}_{n,k}(q) = \mathbf{RT}_{n-k}(B_n,q) \tag{29}$$

and

$$\overline{\mathbf{SF}}_{n,k}(q) = \overline{\mathbf{RT}}_{n-k}(B_n,q). \tag{30}$$

Note that $\mathbf{SF}_{n,0}(q) = \overline{\mathbf{SF}}_{n,0}(q) = 0$ for all $n \geq 1$ since there are no Fibonacci rook placements in $\mathcal{NT}_n(B_n)$ since there are only $n-1$ non-zero columns. Moreover such a situation, we see that (29) implies that for $x \geq n$,

$$[x]_q^n = \sum_{k=1}^{n} \mathbf{SF}_{n,k}(q)[x]_q[x - F_1]_q[x - F_2]_q\cdots[x - F_{k-1}]_q$$

Thus we have given a combinatorial proof of (8). Similarly, (30) implies that for $x \geq n$,

$$[x]_q^n = \sum_{k=1}^{n} \overline{\mathbf{SF}}_{n,k}(q)[x]_q([x]_q - [F_1]_q)([x]_q - [F_2]_q)\cdots([x]_q - [F_{k-1}]_q)$$

Thus we have given a combinatorial proof of (9).

4 Identities for $\mathbf{SF}_{n,k}(q)$ and $\mathbf{cF}_{n,k}(q)$

In this section, we shall derive various identities and special values for the Fibonacci analogues of the Stirling numbers $\mathbf{SF}_{n,k}(q)$, $\overline{\mathbf{SF}}_{n,k}(q)$, $\mathbf{cF}_{n,k}(q)$, and $\overline{\mathbf{cF}}_{n,k}(q)$.

Note that by (26),

$$\mathbf{SF}_{n,k}(q) = q^{\sum_{i=1}^{k-1} F_i} \overline{\mathbf{SF}}_{n,k}(q). \tag{31}$$

Then we have the following theorem.

Theorem 10. *1.* $\overline{\mathbf{SF}}_{n,n}(q) = 1$ *and* $\mathbf{SF}_{n,n}(q) = q^{\sum_{i=1}^{n-1} F_i}$.

2. $\overline{\mathbf{SF}}_{n,n-1}(q) = \sum_{i=1}^{n-1} [F_i]_q$ *and* $\mathbf{SF}_{n,n-1}(q) = q^{\sum_{i=1}^{n-2} F_i} \sum_{i=1}^{n-1} [F_i]_q$.

3. $\overline{\mathbf{SF}}_{n,n-2}(q) = \sum_{i=1}^{n-2} [F_i]_q (\sum_{j=i}^{n-2} [F_j]_q)$ *and* $\mathbf{SF}_{n,n-2}(q) = q^{\sum_{i=1}^{n-3} F_i} \sum_{i=1}^{n-2} [F_i]_q (\sum_{j=i}^{n-2} [F_j]_q)$.

4. $\overline{\mathbf{SF}}_{n,1}(q) = 1$ *and* $\mathbf{SF}_{n,1}(q) = 1$.

5. $\overline{\mathbf{SF}}_{n,2}(q) = (n-1)$ *and* $\mathbf{SF}_{n,2}(q) = q(n-1)$.

6. $\overline{\mathbf{SF}}_{n,3}(q) = \frac{(1+q)^{n-1} - (q(n-1)+1)}{q^2}$ *and* $\mathbf{SF}_{n,3}(q) = (1+q)^{n-1} - (q(n-1)+1)$.

Proof. For (1), it is easy to see that $\overline{\mathbf{SF}}_{n,n}(q) = 1$ since the only placement in $\mathcal{FT}_{n-n}(B_n)$ is the empty placement. The fact that $\mathbf{SF}_{n,k}(q) = q^{\sum_{i=1}^{n-1} F_i}$ then follows from (31).

For (2), we can see that $\overline{\mathbf{SF}}_{n,n-1}(q) = \sum_{i=1}^{n-1} [F_i]_q$ because placements in $\mathcal{FT}_{n-(n-1)}(B_n) = \mathcal{FT}_1(B_n)$ have exactly one column which is filled with a Fibonacci tiling. If that column is column $i+1$, then $i \geq 1$ and the sum of the weights of the possible tilings in column i is $[F_i]_q$. The fact that $\mathbf{SF}_{n,n-1}(q) = q^{\sum_{i=1}^{n-2} F_i} \sum_{i=1}^{n-1} [F_i]_q$ then follows from (31).

For (3), we can classify the placements in $\mathcal{FT}_{n-(n-2)}(B_n) = \mathcal{FT}_2(B_n)$ by the left-most column which contains a tiling. If that column is column $i+1$, then $i \geq 1$ and the sum of the weights of the possible tilings in column i is $[F_i]_q$. Moreover, any tiling in column i cancels one cell in the remaining columns so that number of uncanceled cells in the columns to the right of column $i+1$ will be $i, \ldots, n-2$, reading from right to left. It then follows that

$$\overline{\mathbf{SF}}_{n,n-2}(q) = \sum_{i=1}^{n-2} [F_i]_q (\sum_{j=i}^{n-2} [F_j]_q).$$

The fact that

$$\mathbf{SF}_{n,n-1}(q) = q^{\sum_{i=1}^{n-3} F_i} \sum_{i=1}^{n-2} [F_i]_q (\sum_{j=i}^{n-2} [F_j]_q)$$

then follows from (31).

For (4), note that the elements in $\mathcal{FT}_{n-1}(B_n)$ have a tiling in every column. Given our cancellation scheme, there is exactly one such configuration. For example, the unique element of $\mathcal{FT}_5(B_6)$ is pictured in Figure 11 where we have placed is in the cells canceled by the tiling in column i. Thus the unique element of $\mathcal{FT}_{n-1}(B_n)$ is just the Fibonacci rook placement where there is tiling of height one in each column. Thus $\overline{\mathbf{SF}}_{n,1}(q) = \mathbf{SF}_{n,1}(q) = 1$ since the rank of each tiling height 1 is 0.

For (5), note that the elements in $\mathcal{FT}_{n-2}(B_n)$ have exactly one column $i \geq 2$ which does not have a tiling. Given our cancellation scheme, if the column with out a tiling is column $i \geq 2$,

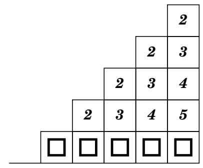

Figure 11: The Fibonacci rook tiling in $\mathcal{FT}_5(B_6)$.

then any non-empty column to the left of column i will be filled with a tiling of height 1 and every column to the right of column i will be filled with a tiling of height 2. For example, the unique element of $\mathcal{FT}_6(B_8)$ is pictured in Figure 12 where we have placed is in the cells canceled by the tiling in column i. Since the ranks of the tilings of heights 1 and 2 are 0, it follows that $\overline{\mathbf{SF}}_{n,2}(q) = n-1$. The fact that $\mathbf{SF}_{n,n-1}(q) = q(n-1)$ then follows from (31).

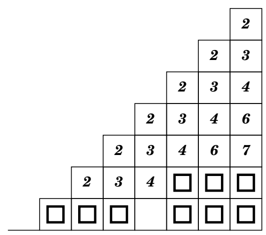

Figure 12: A Fibonacci rook tiling in $\mathcal{FT}_6(B_8)$.

For (6), we proceed by induction. Note that we have proved

$$\mathbf{SF}_{3,3}(q) = q^{F_1+F_2} = q^2 = (1+q)^2 - (2q+1).$$

Now assume that $n \geq 3$ and $\mathbf{SF}_{n,3}(q) = (1+q)^{n-1} - ((n-1)q+1)$. Then

$$\begin{aligned}
\mathbf{SF}_{n+1,3}(q) &= q^{F_2}\mathbf{SF}_{n,2}(q) + [F_3]_q \mathbf{SF}_{n,3}(q) \\
&= q\left(q(n-1)\right) + (1+q)\left((1+q)^{n-1} - ((n-1)q+1)\right) \\
&= q^2(n-1) + (1-q)^n - (n-1)q - (n-1)q^2 - q - 1 \\
&= (1-q)^n - (nq+1).
\end{aligned}$$

The fact that $\overline{\mathbf{SF}}_{n,3}(q) = \frac{(1+q)^{n-1} - ((n-1)q+1)}{q^2}$ then follows from (31). $\square$

Next we define

$$\overline{\mathbb{SF}}_k(q,t) := \sum_{n \geq k} \overline{\mathbf{SF}}_{n,k}(q) t^n$$

for $k \geq 1$ It follows from Theorem 10 that

$$\overline{\mathbb{SF}}_1(q,t) = \sum_{n\geq 1} \overline{\mathbf{SF}}_{n,1}(q) t^n = \sum_{n \geq 1} t^n = \frac{t}{1-t}. \tag{32}$$

Then for $k > 1$,

$$\begin{aligned}
\overline{\mathbb{SF}}_k(q,t) &= \sum_{n\geq k} \overline{\mathbf{SF}}_{n,k}(q)t^n \\
&= t^k + \sum_{n>k} \overline{\mathbf{SF}}_{n,k}(q)t^n \\
&= t^k + t\sum_{n>k} \left(\overline{\mathbf{SF}}_{n-1,k-1}(q) + [F_k]_q\overline{\mathbf{SF}}_{n-1,k-}(p,q)\right) t^{n-1} \\
&= t^k + t\left(\sum_{n>k} \overline{\mathbf{SF}}_{n-1,k-1}(q)t^{n-1}\right) + [F_k]_q t\left(\sum_{n>k} \overline{\mathbf{SF}}_{n-1,k}(q)t^{n-1}\right) \\
&= t^k + t(\overline{\mathbb{SF}}_{k-1}(q,t) - t^{k-1}) + [F_k]_q t\overline{\mathbb{SF}}_k(q,t).
\end{aligned}$$

It follows that

$$\overline{\mathbb{SF}}_k(q,t) = \frac{t}{(1-[F_k]_q t)}\overline{\mathbb{SF}}_{k-1}(q,t). \tag{33}$$

The following theorem easily follows from (32) and (33).

Theorem 11. *For all $k \geq 1$,*

$$\overline{\mathbb{SF}}_k(q,t) = \frac{t^k}{(1-[F_1]_q t)(1-[F_2]_q t)\cdots(1-[F_k]qt)}.$$

Note that it follows from (31) and Theorem 11 that

$$\mathbb{SF}_k(q,t) = \sum_{n\geq k} \mathbf{SF}_{n,k}(q)t^n = \frac{q^{\sum_{i=1}^{k-1} F_i} t^k}{(1-[F_1]_q t)(1-[F_2]_q t)\cdots(1-[F_k]qt)}.$$

For any formal power series in $f(x) = \sum_{n\geq 0} f_n x^n$, we let $f(x)|_{x^n} = f_n$ denote the coefficient of x^n in $f(x)$. Our next result will give formulas for $\overline{\mathbf{SF}}_{n,k}(q)|_{q^s}$ for $s = 0,1,2$.

Theorem 12. *1. For all $n \geq k \geq 1$, $\overline{\mathbf{SF}}_{n,k}(q)|_{q^0} = \binom{n-1}{k-1}$.*

2. For all $n > k \geq 2$, $\overline{\mathbf{SF}}_{n,k}(q)|_q = (k-2)\binom{n-1}{k}$.

3. For all $n \geq s$, $\overline{\mathbf{SF}}_{n,3}(q)|_{q^s} = \binom{n-1}{s+2}$.

4. For all $n \geq k \geq 3$, $\overline{\mathbf{SF}}_{n,k}(q)|_{q^2} = (k-3)\binom{n-1}{k} + \binom{k-1}{2}\binom{n-1}{k+1}$.

5. for all $n \geq k \geq 4$,

$$\overline{\mathbf{SF}}_{n,k}(q)|_{q^3} = (k-4)\binom{n-1}{k} + \left(\binom{k-1}{2} + \binom{k-2}{2} - 1\right)\binom{n-1}{k+1} + \binom{k}{3}\binom{n-1}{k+2}.$$

6. For all $n \geq k \geq 4$,

$$\begin{aligned}\overline{\mathbf{SF}}_{n,k}(q)|_{q^4} &= (k-4)\binom{n-1}{k} + \left(\binom{k-1}{2} + \binom{k-2}{2} + \binom{k-3}{2} - 3\right)\binom{n-1}{k+1} + \\ &\quad \left(2\binom{k}{3} + \binom{k-1}{3} - k + 1\right)\binom{n-1}{k+2} + \binom{k+1}{4}\binom{n-1}{k+3}.\end{aligned}$$

Proof. For (1), note that a placement P in $\mathcal{FT}_{n-k}(B_n)$ must have $k-1$ empty columns among columns $2, \ldots, n$. If $\overline{WF}(P) = 1$, then it must be the case that all the tilings in the columns which contain tilings in P must have rank 0 so that the tiling must contain only tiles of height 1. Thus P is completely determined by the choice of the $k-1$ empty columns among columns $2, \ldots, n$. Thus $\overline{\mathbf{SF}}_{n,k}(p,q)|_{q^0} = \binom{n-1}{k-1}$.

For (3), note that by part 6 of Theorem 10, we have that for any $s \geq 0$,

$$\begin{aligned}\overline{\mathbf{SF}}_{n,3}(q)|_{q^s} &= \mathbf{SF}_{n,3}(q)|_{q^{s+2}} = (1+q)^{n-1} - ((n-1)q+1)|_{q^{s+2}} \\ &= \binom{n-1}{s+2}.\end{aligned}$$

For (2), note that $\overline{\mathbf{SF}}_{n,2}(q)|_q = 0$ since $\overline{\mathbf{SF}}_{n,k}(q) = (n-1)$ by part 5 of Theorem 10. By (3), $\overline{\mathbf{SF}}_{n,3}(q)|_q = \binom{n-1}{3}$. Thus our formula holds for $n = 2$ and $n = 3$.

Next fix $k \geq 4$ and assume by induction that $\overline{\mathbf{SF}}_{n,k-1}(q)|_q = (k-3)\binom{n-1}{k-1}$ for all $n \geq k-1$. Then we shall prove by induction on n that $\overline{\mathbf{SF}}_{n,k}(q)|_q = (k-2)\binom{n-1}{k}$. The base case $n = k$ holds since $\overline{\mathbf{SF}}_{k,}(q) = 1$. But then assuming that $\overline{\mathbf{SF}}_{n,k}(q)|_q = (k-2)\binom{n-1}{k}$, we see that

$$\begin{aligned}\overline{\mathbf{SF}}_{n+1,k}(q)|_q &= \overline{\mathbf{SF}}_{n,k-1}(q)|_q + ((1+q+q^2+\cdots+q^{F_k-1})\overline{\mathbf{SF}}_{n,k}(q))|_q \\ &= (k-3)\binom{n-1}{k-1} + \overline{\mathbf{SF}}_{n,k}(q))|_{q^0} + \overline{\mathbf{SF}}_{n,k}(q))|_q \\ &= (k-3)\binom{n-1}{k-1} + \binom{n-1}{k-1} + (k-2)\binom{n-1}{k} \\ &= (k-2)\binom{n}{k}.\end{aligned}$$

Parts (4), (5), and (6) can easily be proved by induction.

For example, by (3),

$$\overline{\mathbf{SF}}_{n,3}(q)|_{q^2} = \binom{n-1}{4}$$

so that our formula holds for $k = 3$. Now suppose that $k \geq 4$ and our formula holds for $k-1$. That is,

$$\overline{\mathbf{SF}}_{n,k-1}(q)|_{q^2} = (k-4)\binom{n-1}{k-1} + \binom{k-2}{2}\binom{n-1}{k}.$$

Next observe that $\overline{\mathbf{SF}}_{k,k}(q)|_{q^2} = 0$ since $\overline{\mathbf{SF}}_{k,k}(q) = 1$ so that our formula holds for $n = k$. Note also that for $k \geq 4$, $F_k \geq 3$. But then for $n \geq k \geq 4$,

$$\begin{aligned}
\overline{\mathbf{SF}}_{n+1,k}(q)|_{q^2} &= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^2} + \left([F_k]_q \overline{\mathbf{SF}}_{n,k}(q)\right)|_{q^2} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^2} + \left((1+q+q^2)\overline{\mathbf{SF}}_{n,k}(q)\right)|_{q^2} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^2} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^0} + \overline{\mathbf{SF}}_{n,k}(q)|_{q} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^2} \\
&= (k-4)\binom{n-1}{k-1} + \binom{k-2}{2}\binom{n-1}{k} + \binom{n-1}{k-1} + (k-2)\binom{n-1}{k} \\
&\quad + \overline{\mathbf{SF}}_{n,k}(q)|_{q^2} \\
&= (k-3)\binom{n-1}{k-1} + \binom{k-1}{2}\binom{n-1}{k} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^2}.
\end{aligned}$$

This gives us a recursion for $\overline{\mathbf{SF}}_{n+1,k}(q)|_{q^2}$ in terms of $\overline{\mathbf{SF}}_{n,k}(q)|_{q^2}$ which we can iterate to prove that

$$\overline{\mathbf{SF}}_{n,k}(q)|_{q^2} = (k-3)\binom{n-1}{k} + \binom{k-1}{2}\binom{n-1}{k+1}.$$

For (5), we first have to establish the base case $k = 4$.

$$\begin{aligned}
\overline{\mathbf{SF}}_{n+1,4}(q)|_{q^3} &= \overline{\mathbf{SF}}_{n,3}(q)|_{q^3} + \left([F_4]_q \overline{\mathbf{SF}}_{n,4}(q)\right)|_{q^3} \\
&= \overline{\mathbf{SF}}_{n,3}(q)|_{q^3} + \left((1+q+q^2)\overline{\mathbf{SF}}_{n,4}(q)\right)|_{q^3} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^3} + \overline{\mathbf{SF}}_{n,k}(q)|_{q} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^2} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^3} \\
&= \binom{n-1}{5} + 2\binom{n-1}{4} + \left(\binom{n-1}{4} + 3\binom{n-1}{5}\right) + \overline{\mathbf{SF}}_{n,k}(q)|_{q^3} \\
&= 3\binom{n-1}{4} + 4\binom{n-1}{5} + \overline{\mathbf{SF}}_{n,4}(q)|_{q^3}.
\end{aligned}$$

This gives us a recursion for $\overline{\mathbf{SF}}_{n+1,4}(q)|_{q^3}$ in terms of $\overline{\mathbf{SF}}_{n,4}(q)|_{q^3}$ which we can iterate to prove that

$$\overline{\mathbf{SF}}_{n,4}(q)|_{q^3} = 3\binom{n-1}{5} + 4\binom{n-1}{6}.$$

Thus our formula for (5) holds for $k = 4$.

Next assume that $k \geq 5$. First we note that $\overline{\mathbf{SF}}_{k,k}(q)|_{q^3} = 0$ since $\overline{\mathbf{SF}}_{k,k}(q) = 1$ so that our formula holds for $n = k$. Note also that for $k \geq 5$, $F_k \geq 5$. Now suppose our formula holds for $k-1$. That is,

$$\overline{\mathbf{SF}}_{n,k-1}(q)|_{q^3} =$$
$$(k-5)\binom{n-1}{k-1} + \left(\binom{k-2}{2} + \binom{k-3}{2} - 1\right)\binom{n-1}{k} + \binom{k-1}{3}\binom{n-1}{k+1}.$$

Next observe that $\overline{\mathbf{SF}}_{k,k}(q)|_{q^3} = 0$ since $\overline{\mathbf{SF}}_{k,k}(q) = 1$ so that our formula holds for $n = k$. Note also that for $k \geq 4$, $F_k \geq 3$. But then for $n \geq k \geq 5$,

$$\begin{aligned}
\overline{\mathbf{SF}}_{n+1,k}(q)|_{q^3} &= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^3} + \left([F_k]_q \overline{\mathbf{SF}}_{n,k}(q)\right)|_{q^3} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^2} + \left((1+q+q^2+q^3)\overline{\mathbf{SF}}_{n,k}(q)\right)|_{q^3} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^3} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^0} + \overline{\mathbf{SF}}_{n,k}(q)|_{q} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^2} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^3} \\
&= (k-5)\binom{n-1}{k-1} + \left(\binom{k-2}{2} + \binom{k-3}{2} - 1\right)\binom{n-1}{k} + \binom{k-1}{3}\binom{n-1}{k+1} \\
&\quad + \binom{n-1}{k-1} + (k-2)\binom{n-1}{k} + (k-3)\binom{n-1}{k} + \binom{k-1}{2}\binom{n-1}{k+1} \\
&\quad + \overline{\mathbf{SF}}_{n,k}(q)|_{q^3} \\
&= (k-4)\binom{n-1}{k-1} + \left(\binom{k-1}{2} + \binom{k-2}{2} - 1\right)\binom{n-1}{k} + \binom{k}{3}\binom{n-1}{k+1} \\
&\quad + \overline{\mathbf{SF}}_{n,k}(q)|_{q^3}.
\end{aligned}$$

This gives us a recursion for $\overline{\mathbf{SF}}_{n+1,k}(q)|_{q^3}$ in terms of $\overline{\mathbf{SF}}_{n,k}(q)|_{q^3}$ which we can interate to prove that

$$\overline{\mathbf{SF}}_{n,k}(q)|_{q^3} = (k-4)\binom{n-1}{k} + \left(\binom{k-1}{2} + \binom{k-2}{2} - 1\right)\binom{n-1}{k+1} + \binom{k}{3}\binom{n-1}{k+2}.$$

For (6), again, we first have to establish the base case $k = 4$.

$$\begin{aligned}
\overline{\mathbf{SF}}_{n+1,4}(q)|_{q^4} &= \overline{\mathbf{SF}}_{n,3}(q)|_{q^4} + \left([F_4]_q \overline{\mathbf{SF}}_{n,4}(q)\right)|_{q^4} \\
&= \overline{\mathbf{SF}}_{n,3}(q)|_{q^4} + \left((1+q+q^2)\overline{\mathbf{SF}}_{n,4}(q)\right)|_{q^4} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^4} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^2} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^3} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^4} \\
&= \binom{n-1}{6} + \binom{n-1}{4} + 3\binom{n-1}{5} + 3\binom{n-1}{5} + 4\binom{n-1}{6} + \\
&\quad \overline{\mathbf{SF}}_{n,k}(q)|_{q^4} \\
&= \binom{n-1}{4} + 6\binom{n-1}{5} + 5\binom{n-1}{6} + \overline{\mathbf{SF}}_{n,4}(q)|_{q^3}.
\end{aligned}$$

This gives us a recursion for $\overline{\mathbf{SF}}_{n+1,4}(q)|_{q^4}$ in terms of $\overline{\mathbf{SF}}_{n,4}(q)|_{q^4}$ which we can iterate to prove that

$$\overline{\mathbf{SF}}_{n,4}(q)|_{q^4} = \binom{n-1}{5} + 6\binom{n-1}{6} + 5\binom{n-1}{7}.$$

Thus our formula for (6) holds for $k = 4$.

Next assume that $k \geq 5$. First we note that $\overline{\mathbf{SF}}_{k,k}(q)|_{q^4} = 0$ since $\overline{\mathbf{SF}}_{k,k}(q) = 1$ so that our formula holds for $n = k$. Note also that for $k \geq 5$, $F_k \geq 5$. Now suppose our formula holds for $k-1$. That is,

$$\begin{aligned}
\overline{\mathbf{SF}}_{n,k-1}(q)|_{q^4} &= (k-5)\binom{n-1}{k-1} + \left(\binom{k-2}{2} + \binom{k-3}{2} + \binom{k-3}{2} - 3\right)\binom{n-1}{k} + \\
&\quad \left(\binom{k-1}{3} + \binom{k-2}{3} - (k-1) + 1\right)\binom{n-1}{k+1} + \binom{k}{4}\binom{n-1}{k+2}.
\end{aligned}$$

Next observe that $\overline{\mathbf{SF}}_{k,k}(q)|_{q^5} = 0$ since $\overline{\mathbf{SF}}_{k,k}(q) = 1$ so that our formula holds for $n = k$. Note also that for $k \geq 5$, $F_k \geq 5$. But then for $n \geq k \geq 5$,

$$
\begin{aligned}
\overline{\mathbf{SF}}_{n+1,k}(q)|_{q^4} &= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^4} + \left([F_k]_q \overline{\mathbf{SF}}_{n,k}(q)\right)|_{q^4} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^4} + \left((1+q+q^2+q^3+q^4)\overline{\mathbf{SF}}_{n,k}(q)\right)|_{q^4} \\
&= \overline{\mathbf{SF}}_{n,k-1}(q)|_{q^3} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^0} + \overline{\mathbf{SF}}_{n,k}(q)|_{q} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^2} + \overline{\mathbf{SF}}_{n,k}(q)|_{q^3} \\
&\quad + \overline{\mathbf{SF}}_{n,k}(q)|_{q^4} \\
&= (k-5)\binom{n-1}{k-1} + \left(\binom{k-2}{2} + \binom{k-3}{2} + \binom{k-3}{2} - 3\right)\binom{n-1}{k} + \\
&\quad + \left(\binom{k-1}{3} + \binom{k-2}{3} - (k-1) + 1\right)\binom{n-1}{k+1} + \binom{k}{4}\binom{n-1}{k+2} \\
&\quad + \binom{n-1}{k-1} + (k-2)\binom{n-1}{k} + (k-3)\binom{n-1}{k} + \binom{k-1}{2}\binom{n-1}{k+1} \\
&\quad + (k-4)\binom{n-1}{k} + \left(\binom{k-1}{2} + \binom{k-2}{2} - 1\right)\binom{n-1}{k+1} + \binom{k}{3}\binom{n-1}{k+2} \\
&\quad + \overline{\mathbf{SF}}_{n,k}(q)|_{q^4} \\
&= (k-4)\binom{n-1}{k-1} + \left(\binom{k-1}{2} + \binom{k-2}{2} + \binom{k-3}{2} - 3\right)\binom{n-1}{k} \\
&\quad + \left(\binom{k}{3} + \binom{k-1}{3} - k + 1\right)\binom{n-1}{k+1} + \binom{k+1}{4}\binom{n-1}{k+2} \\
&\quad + \overline{\mathbf{SF}}_{n,k}(q)|_{q^4}.
\end{aligned}
$$

This gives us a recursion for $\overline{\mathbf{SF}}_{n+1,k}(q)|_{q^4}$ in terms of $\overline{\mathbf{SF}}_{n,k}(q)|_{q^4}$ which we can iterate to prove that

$$
\begin{aligned}
\overline{\mathbf{SF}}_{n,k}(q)|_{q^4} &= (k-4)\binom{n-1}{k} + \left(\binom{k-1}{2} + \binom{k-2}{2} + \binom{k-2}{2} - 3\right)\binom{n-1}{k+1} + \\
&\quad \left(2\binom{k}{3} + \binom{k-1}{2} - k + 1\right)\binom{n-1}{k+2} + \binom{k+1}{4}\binom{n-1}{k+3}.
\end{aligned}
$$

□

A sequence of real numbers $a_0, \ldots, a_n$ is is said to be *unimodal* if there is a $0 \leq j \leq n$ such that $a_0 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$ and is said to be *log-concave* if for $0 \leq i \leq n$, $a_i^2 - a_{i-1}a_{i+1} \geq 0$ where we set $a_{-1} = a_{n+1} = 0$. If a sequence is log-concave, then it is unimodal. A polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ is said to be *unimodal* if $a_0, \ldots, a_n$ is a unimodal sequence and is said to be log-concave if $a_0, \ldots, a_n$ is log concave.

It is easy to see from Theorem 10 that $\overline{\mathbf{SF}}_{n,k}(q)$ is unimodal for all $n \geq k$ when $k \in \{1, 2, 3\}$. Computational evidence suggests that $\overline{\mathbf{SF}}_{n,4}(q)$ is unimodal for all $n \geq 4$ and that $\overline{\mathbf{SF}}_{n,5}(q)$ is unimodal for all $n \geq 5$. However, it is not the case that $\overline{\mathbf{SF}}_{n,6}(q)$ is unimodal for all $n \geq 6$. For example, one can use part 3 of Theorem 10 to compute

$$
\begin{aligned}
&\overline{\mathbf{SF}}_{8,6}(q) = \\
&21 + 28q + 31q^2 + 29q^3 + 30q^4 + 25q^5 + 23q^2 + 22q^7 + 15q^8 + 10q^9 + 7q^{10} + 5q^{11} + 3q^{12} + 2q^{13} + q^{14}.
\end{aligned}
$$

It is not difficult to see that for any Ferrers board $B = F(b_1, \ldots, b_n)$, the coefficients that appear in the polynomials $\mathbf{FT}_k(B,q)$ and $\overline{\mathbf{FT}}_k(B,q)$ are essentially the same. That is, we have the following theorem.

Theorem 13. *Let $B = F(b_1, \ldots, b_n)$ be a skyline board. Then*

$$\overline{\mathbf{FT}}_k(B,q) = \left(\prod_{i=1}^{n}(1+[F_{b_i}]_q z)\right)|_{z^k} \tag{34}$$

and

$$\mathbf{FT}_k(B,q) = \left(\prod_{i=1}^{n}(q^{F_{b_i}}+[F_{b_i}]_q z)\right)|_{z^k} = q^{\sum_{i=1}^{n}F_{b_i}}\left(\prod_{i=1}^{n}(1+\frac{1}{q}[F_{b_i}]_{\frac{1}{q}} z)\right)|_{z^k} \tag{35}$$

Proof. It is easy to see that if we are creating a Fibonacci file tiling in $\mathcal{FT}_k(B)$, then in column i, we have two choices, namely, we can leave the column empty or put a Fibonacci tiling of height b_i. For $\overline{\mathbf{FT}}_k(B,q)$, the weight of an empty column is 1 and the sum of weights of the Fibonacci tilings of height b_i is $[F_{b_i}]_q$. Thus $(\prod_{i=1}^{n}(1+[F_{b_i}]_q z))\,|_{z^k}$ is equal to the sum over all Fibonacci file tilings where exactly k columns have tiling which is equal to $\overline{\mathbf{FT}}_k(B,q)$.

Similarly, For $\mathbf{FT}_k(B,q)$, the weight of an empty column i when it is empty is $q^{F_{b_i}}$ and the sum of weights of the Fibonacci tilings of height b_i is $[F_{b_i}]_q$. Thus $\left(\prod_{i=1}^{n}(q^{F_{b_i}}+[F_{b_i}]_q z)\right)|_{z^k}$ is equal to the sum over all Fibonacci file tilings where exactly k columns have tiling which is equal to $\mathbf{FT}_k(B,q)$. ☐

It follow that for any n, the coefficient of q^n in $(\prod_{i=1}^{n}(1+[F_{b_i}]_q z))\,|_{z^k}$ is equal to the coefficient of $\frac{1}{q^{n+k}}$ in $\left(\prod_{i=1}^{n}(1+\frac{1}{q}[F_{b_i}]_{\frac{1}{q}} z)\right)|_{z^k}$. It follows that

$$\overline{\mathbf{FT}}_k(B,q)_{q^n} = \mathbf{FT}_k(B,q)\big|_{q^{-n-k+\sum_{i=1}^{n}F_{b_i}}}.$$

It is easy to see from (34) that

$$\overline{\mathbf{cF}}_{n,n-1}(q) = \sum_{i=1}^{n-1}[F_i]_q$$

so that coefficient of q^k in $\overline{\mathbf{cF}}_{n,n-1}(q)$ weakly decreases as k goes from 0 to $F_{n-1}-1$. It follows that the coefficient of q^k in $\mathbf{cF})_{n,n-1}(q)$ weakly increase. Similarly, it is easy to see that

$$\overline{\mathbf{cF}}_{n,1}(q) = \prod_{i=1}^{n-1}[F_i]_q$$

so that $\overline{\mathbf{cF}}_{n,1}(q)$ is just the rank generating function of a product of chains which is know to be symmetric and unimodal, see [3].

From our computational evidence, it seems that the polynomials $\overline{\mathbf{cF}}_{n,2}(q)$ are unimodal. However, it is not the case $\overline{\mathbf{cF}}_{n,k}(q)$ are unimodal for all n and k. For example, $\overline{\mathbf{cF}}_{9,7}(q)$ starts out

$$28 + 42q + 50q^2 + 53q^3 + 58q^4 + 57q^5 + 58q^6 + 60q^7 + \ldots.$$

Finally, our results show that the matrices $||(-1)^{n-k}\overline{\mathbf{cF}}_{n,k}(q)||$ and $||\overline{\mathbf{SF}}_{n,k}(q)||$ are inverses of each other. One can give a combinatorial proof of this fact. Indeed, the combinatorial proof of [2] which shows that matrices $||(-1)^{n-k}\mathbf{cf}_{n,k}(q)||$ and $||\mathbf{Sf}_{n,k}(q)||$ are inverses of each other can also be applied to show that the matrices $||(-1)^{n-k}\overline{\mathbf{cF}}_{n,k}(q)||$ and $||\overline{\mathbf{SF}}_{n,k}(q)||$ are inverses of each other.

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