

OPERADS OF DECORATED CLIQUES

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ABSTRACT. The vector space of all polygons with configurations of diagonals is endowed with an operad structure. This is the consequence of a functorial construction C introduced here, which takes unitary magmas \mathcal{M} as input and produces operads. The obtained operads involve regular polygons with configurations of arcs labeled on \mathcal{M} , called \mathcal{M} -decorated cliques and generalizing usual polygons with configurations of diagonals. We provide here a complete study of the operads $C\mathcal{M}$. By considering combinatorial subfamilies of \mathcal{M} -decorated cliques defined, for instance, by limiting the maximal number of crossing diagonals or the maximal degree of the vertices, we obtain suboperads and quotients of $C\mathcal{M}$. This leads to a new hierarchy of operads containing, among others, operads on noncrossing configurations, Motzkin configurations, forests, dissections of polygons, and involutions. We show that the suboperad of noncrossing configurations is Koszul and exhibit its presentation by generators and relations. Besides, the construction C leads to alternative definitions of several operads, like the operad of bicolored noncrossing configurations and the operads of simple and double multi-tildes.

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INTRODUCTION

Regular polygons endowed with configurations of diagonals are very classical combinatorial objects. Up to some restrictions or enrichments, sets of these polygons can be put in bijection with several combinatorial families. For instance, it is well-known that triangulations [DLRS10], forming a particular subset of the set of all polygons, are in one-to-one correspondence with binary trees, and a lot of structures and operations on binary trees translate nicely on triangulations. Indeed, among others, the rotation operation on binary trees [Knu98] is the covering relation of the Tamari order [HT72] and this operation translates as a diagonal flip in triangulations. Also, noncrossing configurations [FN99] form an other interesting subfamily of such polygons. Natural generalizations of noncrossing configurations consist in allowing, with more or less restrictions, some crossing diagonals. One of these families is formed by the multi-triangulations [CP92], that are polygons wherein the number of mutually crossing diagonal is bounded. Besides, let us remark that the class of combinatorial objects in bijection with sets of polygons with configurations of diagonals is large enough in order to contain, among others, dissections of polygons, noncrossing partitions, permutations, and involutions.

On the other hand, coming historically from algebraic topology [May72, BV73], operads provide an abstraction of the notion of operators (of any arities) and their compositions. In more concrete terms, operads are algebraic structures abstracting the notion of planar rooted trees and their grafting operations (see [LV12] for a complete exposition of the theory and [Mén15] for an exposition focused on symmetric set-operads). The modern treatment of operads in algebraic combinatorics consists in regarding combinatorial objects like operators endowed with gluing operations mimicking the composition of operators. In the last years, a lot of combinatorial sets and combinatorial spaces have been endowed fruitfully with a structure of an operad (see for instance [Cha08] for an exposition of known interactions between operads and combinatorics, focused on trees, [LMN13, GLMN16] where operads abstracting operations in language theory are introduced, [CG14] for the study of an operad involving particular noncrossing configurations, [Gir15] for a general construction of operads on many combinatorial sets, [Gir16a] where operads are constructed from posets, and [CHN16] where operads on various species of trees are introduced). In most of the cases, this approach brings results about enumeration, helps to discover new statistics, and leads to establish new links (by morphisms) between different combinatorial sets or spaces. We can observe that most of the subfamilies of polygons endowed with configurations of diagonals discussed above are stable for several natural composition operations. Even better, some of these can be described as the cloture with respect to these composition operations of small sets of polygons. For this reason, operads are very promising candidates, among the modern algebraic structures, to study such objects under an algebraic and combinatorial flavor.

The purpose of this work is twofold. First, we are concerned in endowing the linear span of the polygons with configurations of arcs with a structure of an operad. This leads to see these objects under a new light, stressing some of their combinatorial and algebraic properties. Second, we would provide a general construction of operads of polygons rich

enough so that it includes some already known operads. As a consequence, we obtain alternative definitions of existing operads and new interpretations of these. For this aim, we work here with \mathcal{M} -decorated cliques (or \mathcal{M} -cliques for short), that are complete graphs whose arcs are labeled on \mathcal{M} , where \mathcal{M} is a unitary magma. These objects are natural generalizations of polygons with configurations of arcs since the arcs of any \mathcal{M} -clique labeled by the unit of \mathcal{M} are considered as missing. The elements of \mathcal{M} different from the unit allow moreover to handle polygons with arcs of different colors. For instance, each usual noncrossing configuration c can be encoded by an \mathbb{N}_2 -clique p , where \mathbb{N}_2 is the cyclic additive unitary magma $\mathbb{Z}/2\mathbb{Z}$, wherein each arc labeled by $1 \in \mathbb{N}_2$ in p denotes the presence of the same arc in c , and each arc labeled by $0 \in \mathbb{N}_2$ in p denotes its absence in c . Our construction is materialized by a functor C from the category of unitary magmas to the category of operads. It builds, from any unitary magma \mathcal{M} , an operad $C\mathcal{M}$ on \mathcal{M} -cliques. The partial composition $p \circ_i q$ of two \mathcal{M} -cliques p and q of $C\mathcal{M}$ consists in gluing the i th edge of p (with respect to a precise indexation) and a special arc of q , called the base, together to form a new \mathcal{M} -clique. The magmatic operation of \mathcal{M} explains how to relabel the two overlapping arcs.

This operad $C\mathcal{M}$ has a lot of properties, which can be apprehended both under a combinatorial and an algebraic point of view. First, many families of particular polygons with configurations of arcs form quotients or suboperads of $C\mathcal{M}$. We can for instance control the degrees of the vertices or the crossings between diagonals to obtain new operads. We can also forbid all diagonals, or some labels for the diagonals or the edges, or all inclusions of diagonals, or even all cycles formed by arcs. All these combinatorial particularities and restrictions on \mathcal{M} -cliques behave well algebraically. Moreover, by using the fact that the direct sum of two ideals of an operad \mathcal{O} is still an ideal of \mathcal{O} , these constructions can be mixed to get even more operads. For instance, it is well-known that Motzkin configurations, that are polygons with disjoint noncrossing diagonals, are enumerated by Motzkin numbers [Mot48]. Since a Motzkin configuration can be encoded by an \mathcal{M} -clique where all vertices are of degrees at most 1 and no diagonal crosses another one, we obtain an operad $\text{Mot}\mathcal{M}$ on colored Motzkin configurations which is both a quotient of $\text{Deg}_1\mathcal{M}$, the quotient of $C\mathcal{M}$ consisting in all \mathcal{M} -cliques such that all vertices are of degrees at most 1, and of $\text{NC}\mathcal{M}$, the quotient (and suboperad) of $C\mathcal{M}$ consisting in all noncrossing \mathcal{M} -cliques. We also get quotients of $C\mathcal{M}$ involving, among others, Schröder trees, forests of paths, forests of trees, dissections of polygons, Lucas configurations, with colored versions for each of these. This leads to a new hierarchy of operads, wherein links between its components appear as surjective or injective operad morphisms. Table 1 lists the main operads constructed in this work and gathers some information about these. One of the most notable of these is built by considering the \mathbb{D}_0 -cliques that have vertices of degrees at most 1, where \mathbb{D}_0 is the multiplicative unitary magma on $\{0, 1\}$. This is in fact the quotient $\text{Deg}_1\mathbb{D}_0$ of $C\mathbb{D}_0$ and involves involutions (or equivalently, standard Young tableaux by the Robinson-Schensted correspondence [Lot02]). To the best of our knowledge, $\text{Deg}_1\mathbb{D}_0$ is the first nontrivial operad on these objects. As an important remark at this stage, let us highlight that when \mathcal{M} is nontrivial, $C\mathcal{M}$ is not a binary operad. Indeed, all its minimal generating sets are infinite and its generators have

Operad	Objects	Status with respect to $C\mathcal{M}$	Place
$C\mathcal{M}$	\mathcal{M} -cliques	—	Section 2
$\text{Lab}_{B,E,D}\mathcal{M}$	\mathcal{M} -cliques with restricted labels	Suboperad	Section 3.1.1
$\text{Whi}\mathcal{M}$	White \mathcal{M} -cliques	Suboperad	Section 3.1.2
$\text{Cro}_k\mathcal{M}$	\mathcal{M} -cliques of crossings at most k	Suboperad and quotient	Section 3.1.3
$\text{Bub}\mathcal{M}$	\mathcal{M} -bubbles	Quotient	Section 3.1.4
$\text{Deg}_k\mathcal{M}$	\mathcal{M} -cliques of degrees at most k	Quotient	Section 3.1.5
$\text{Inf}\mathcal{M}$	Inclusion-free \mathcal{M} -cliques	Quotient	Section 3.1.6
$\text{Acy}\mathcal{M}$	Acyclic \mathcal{M} -cliques	Quotient	Section 3.1.7
$\text{NC}\mathcal{M}$	noncrossing \mathcal{M} -cliques	Suboperad and quotient	Section 4

TABLE 1. The main operads defined in this paper. All these operads depend on a unitary magma \mathcal{M} which has, in some cases, to satisfy some precise conditions. Some of these operads depend also on a nonnegative integer k or subsets B , E , and D of \mathcal{M} .

arbitrary high arities. Nevertheless, the biggest binary suboperad of $C\mathcal{M}$ is the operad $\text{NC}\mathcal{M}$ of noncrossing configurations and this operad is quadratic and Koszul, regardless of \mathcal{M} . Furthermore, the construction C maintains some links with the operad RatFct of rational functions introduced by Loday [Lod10]. In fact, provided that \mathcal{M} satisfies some conditions, each \mathcal{M} -clique encodes a rational function. This defines an operad morphism from $C\mathcal{M}$ to RatFct . Moreover, the construction C allows to construct already known operads in original ways. For instance, for well-chosen unitary magmas \mathcal{M} , the operads $C\mathcal{M}$ contain \mathcal{FF}_k , a suboperad of the operad of formal fractions \mathcal{FF} [CHN16], BNC , the operad of bicolored noncrossing configurations [CG14], and MT and DMT , two operads respectively defined in [LMN13] and [GLMN16] that involve multi-tildes and double multi-tildes, operators coming from formal language theory [CCM11].

This text is organized as follows. Section 1 sets our notations, general definitions, and tools about nonsymmetric operads (since we deal only with nonsymmetric operads here, we call these simply operads).

In Section 2, we introduce \mathcal{M} -cliques and some definitions about these. Then, the construction C is described and the fact that C is a functor from the category of unitary magmas to the category of set-operads (treated here as operads in the category of vector spaces) is established (Theorem 2.1.1). We show that the Hadamard product of two operads obtained as images of C is isomorphic to an operad in the image of C (Proposition 2.1.2). We then investigate the general properties of the operads $C\mathcal{M}$. We compute their dimensions (Proposition 2.2.1), describe one of their minimal generating sets (Proposition 2.2.3), and describe all their associative elements (Proposition 2.2.4). Notice that the description of the associative elements of $C\mathcal{M}$ relies on its linear structure. Indeed, even if $C\mathcal{M}$ is well-defined in the category of sets, some of its associative elements are nontrivial sums of \mathcal{M} -cliques. We also explicit some symmetries of $C\mathcal{M}$ (Proposition 2.2.5), show that, as

a set-operad, $C\mathcal{M}$ is basic [Val07] if and only if \mathcal{M} is right cancellable (Proposition 2.2.6), and show that $C\mathcal{M}$ is a cyclic operad [GK95] (Proposition 2.2.7). Next, two additional bases of $C\mathcal{M}$ are defined, the H-basis and the K-basis, defined from two natural partial order relations on \mathcal{M} -cliques and their Möbius functions. We give expressions for the partial composition of $C\mathcal{M}$ over these two bases (Propositions 2.2.8 and 2.2.9). This section ends by explaining how any \mathcal{M} -clique p encodes a rational function $F_\theta(p)$ of $\mathbb{K}(\mathbb{U})$, \mathbb{U} being a commutative alphabet, when \mathcal{M} is \mathbb{Z} -graded. We show that F_θ is an operad morphism from $C\mathcal{M}$ to RatFct , the operad of rational functions [Lod10] (Theorem 2.2.10). This morphism is not injective but its image contains all Laurent polynomials on \mathbb{U} (Proposition 2.2.11).

Then, Section 3 is devoted to define several suboperads and quotients of $C\mathcal{M}$. All the quotients we consider are of the form $C\mathcal{M}/\mathfrak{A}$, where \mathfrak{A} is an operad ideal of $C\mathcal{M}$ generated by a family of \mathcal{M} -cliques we want to discard. For instance, the quotient $\text{Bub}\mathcal{M}$ of $C\mathcal{M}$ on \mathcal{M} -bubbles (that are \mathcal{M} -cliques without diagonals) is defined from the operad ideal $\mathfrak{A}_{\text{Bub}}$ generated by all the \mathcal{M} -cliques having at least one diagonal (Proposition 3.1.4). Some of the constructions presented here require that \mathcal{M} has no nontrivial unit divisors. This is the case for instance for the quotient $\text{Acy}\mathcal{M}$ involving acyclic \mathcal{M} -cliques (Proposition 3.1.9). We also construct and briefly study the suboperad $\text{Lab}_{B,E,D}\mathcal{M}$ involving \mathcal{M} -cliques with restrictions for its labels (Proposition 3.1.1), the suboperad $\text{Whi}\mathcal{M}$ involving \mathcal{M} -cliques with unlabeled edges, the quotient and suboperad $\text{Cro}_k\mathcal{M}$ involving \mathcal{M} -cliques such that each diagonal crosses at most k other diagonals (Proposition 3.1.3), the quotient $\text{Deg}_k\mathcal{M}$ involving \mathcal{M} -cliques such that each vertex is of degree at most k (Proposition 3.1.5), and $\text{Inf}\mathcal{M}$ involving \mathcal{M} -cliques without inclusions between arcs (Proposition 3.1.6).

We focus next, in Section 4, on the study of the suboperad $\text{NC}\mathcal{M}$ of $C\mathcal{M}$ on the noncrossing \mathcal{M} -cliques. We first show that $\text{NC}\mathcal{M}$ inherits from a lot of properties of $C\mathcal{M}$, as the same description for its associative elements and the fact that it is a cyclic operad (Proposition 4.1.1). The operad $\text{NC}\mathcal{M}$ admits an alternative realization in terms of \mathcal{M} -Schröder trees, that are Schröder trees with edges labeled on \mathcal{M} satisfying some conditions. This is the consequence of the fact that the set of all noncrossing \mathcal{M} -cliques of a given arity is in one-to-one correspondence with a set of trees whose internal nodes are labeled by \mathcal{M} -bubbles (Proposition 4.1.4). The partial composition of $\text{NC}\mathcal{M}$ on \mathcal{M} -Schröder trees is a grafting of trees together with a relabeling of edges or a contraction of edges. To continue the study of $\text{NC}\mathcal{M}$, we describe one of its minimal generating families (Proposition 4.1.5), provide an algebraic equation for its Hilbert series (Proposition 4.1.6), and give a formula for its dimensions involving Narayana numbers [Nar55] (Proposition 4.1.7). In order to compute the space of the relations of $\text{NC}\mathcal{M}$, we use techniques of rewrite systems of trees [BN98]. Thus, we define a convergent rewrite rule \rightarrow and show that the space induced by \rightarrow is the space of relations of $\text{NC}\mathcal{M}$, leading to a presentation by generators and relations of $\text{NC}\mathcal{M}$ (Theorem 4.2.8). The existence of a convergent orientation of the space of the relations of $\text{NC}\mathcal{M}$ implies by [Hof10] that this operad is Koszul (Theorem 4.2.9). Then, we turn our attention on suboperads of $\text{NC}\mathcal{M}$ generated by some finite families of bubbles. Under some conditions on the considered sets of bubbles, we can describe the Hilbert series of these suboperads of $\text{NC}\mathcal{M}$ by a system of algebraic equations (Proposition 4.3.2). We give two examples of suboperads of $\text{NC}\mathcal{M}$ generated by some subsets

of triangles, including one which is a suboperad of NCD_0 isomorphic to the operad Motz of Motzkin paths defined in [Gir15]. From the presentation of $\text{NC}\mathcal{M}$, we list the relations between the operations of the algebras over $\text{NC}\mathcal{M}$. We describe the free algebra over one generator over $\text{NC}\mathcal{M}$, and a general way to construct algebras over $\text{NC}\mathcal{M}$ from associative algebras endowed with some linear maps (Theorem 4.4.1). Moreover, when \mathcal{M} is a monoid, there is a simple way to endow the space $\mathbb{K}\langle \mathcal{M}^* \rangle$ of all noncommutative polynomials on \mathcal{M} with the structure of an algebra over $\text{NC}\mathcal{M}$ (Proposition 4.4.2). We end this section by considering the Koszul dual $\text{NC}\mathcal{M}^\perp$ of $\text{NC}\mathcal{M}$ (which is well-defined since $\text{NC}\mathcal{M}$ is a binary and quadratic operad). We compute its presentation (Proposition 4.5.1), express an algebraic equation for its Hilbert series (Proposition 4.5.3), give a formula for its dimensions (Proposition 4.5.4), and establish a combinatorial realization of $\text{NC}\mathcal{M}^\perp$ as a graded space involving dual \mathcal{M} -cliques (Proposition 4.5.5), that are \mathcal{M}^2 -cliques with some constraints for the labels of their arcs.

This work ends with Section 5, where we use the construction C to provide alternative definitions of some known operads. We hence construct the operad NCP of based noncrossing trees [Cha07, Ler11] (Proposition 5.1.1), the suboperad \mathcal{FF}_4 of the operad of formal fractions \mathcal{FF} [CHN16] (Proposition 5.1.2), and the operad of bicolored noncrossing configurations BNC [CG14] (Proposition 5.2.1). For this reason, in particular, all the suboperads of BNC can be obtained from the construction C. This includes for example the dipterous operad [LR03, Zin12]. We also construct some versions of the operads MT of multi-tildes [CCM11, LMN13] (Proposition 5.3.1) and DMT of double multi-tildes [GLMN16] (Proposition 5.3.2) which are trivial in arity 1.

General notations and conventions. All the algebraic structures of this article have a field of characteristic zero \mathbb{K} as ground field. For any set S , $\text{Vect}(S)$ denotes the linear span of the elements of S . For any integers a and c , $[a, c]$ denotes the set $\{b \in \mathbb{N} : a \leq b \leq c\}$ and $[n]$, the set $[1, n]$. The cardinality of a finite set S is denoted by $\#S$. For any set A , A^* denotes the set of all finite sequences, called words, of elements of A . For any $n \geq 0$, A^n (resp. $A^{\geq n}$) is the set of all words on A of length n (resp. at least n). The word of length 0 is the empty word denoted by ϵ . If u is a word, its letters are indexed from left to right from 1 to its length $|u|$. For any $i \in [|u|]$, u_i is the letter of u at position i . If a is a letter and n is a nonnegative integer, a^n denotes the word consisting in n occurrences of a . For any letter a , $|u|_a$ denotes the number of occurrences of a in u .

1. ELEMENTARY DEFINITIONS AND TOOLS

We set here our notations and recall some definitions about operads and the related structures. The main purposes are to provide tools to compute presentations and prove Koszulity of operads. For this, it is important to handle precise definitions about free operads, trees, and rewrite rules on trees. This is the starting point of this section.

1.1. Trees and rewrite rules. Unless otherwise specified, we use in the sequel the standard terminology (i.e., *node*, *edge*, *root*, *child*, etc.) about planar rooted trees [Knu97]. For the sake of completeness, let us recall the most important definitions and set our notations.

1.1.1. Trees. Let t be a planar rooted tree. The *arity* of a node of t is its number of children. An *internal node* (resp. a *leaf*) of t is a node with a nonzero (resp. null) arity. Internal nodes can be *labeled*, that is, each internal node of a tree is associated with an element of a certain set. Given an internal node x of t , due to the planarity of t , the children of x are totally ordered from left to right and are thus indexed from 1 to the arity ℓ of x . For any $i \in [\ell]$, the *i th subtree* of t is the tree rooted at the i th child of t . Similarly, the leaves of t are totally ordered from left to right and thus are indexed from 1 to the number of its leaves. A tree s is a *subtree* of t if it is possible to fit s at a certain place of t , by possibly superimposing leaves of s and internal nodes of t . In this case, we say that t *admits an occurrence* of (the *pattern*) s . Conversely, we say that t *avoids* s if there is no occurrence of s in t . In our graphical representations, each planar rooted tree is depicted so that its root is the uppermost node. Since we consider in the sequel only planar rooted trees, we shall call these simply *trees*.

1.1.2. Rewrite rules. Let S be a set of trees. A *rewrite rule* on S is a binary relation \rightarrow on S such that for all trees s and s' of S , $s \rightarrow s'$ only if s and s' have the same number of leaves. We say that a tree t is *rewritable in one step* into t' by \rightarrow if there exist two trees s and s' satisfying $s \rightarrow s'$ and t has a subtree s such that, by replacing s by s' in t , we obtain t' . We denote by $t \Rightarrow t'$ this property, so that \Rightarrow is a binary relation on S . When $t = t'$ or when there exists a sequence of trees (t_1, \dots, t_{k-1}) with $k \geq 1$ such that $t \Rightarrow t_1 \Rightarrow \dots \Rightarrow t_{k-1} \Rightarrow t'$, we say that t is *rewritable* by \Rightarrow into t' and we denote this property by $t \xRightarrow{*} t'$. In other words, $\xRightarrow{*}$ is the reflexive and transitive closure of \Rightarrow . We denote by $\xrightarrow{*}$ the reflexive and transitive closure of \rightarrow and by $\overset{*}{\leftrightarrow}$ (resp. $\overset{*}{\rightleftarrows}$) the reflexive, transitive, and symmetric closure of \rightarrow (resp. \Rightarrow). The *vector space induced* by \rightarrow is the subspace of the linear span $\text{Vect}(S)$ of all trees of S generated by the family of all $t - t'$ such that $t \overset{*}{\leftrightarrow} t'$.

For instance, let S be the set of all trees where internal nodes are labeled on $\{a, b, c\}$ and consider the rewrite rule \rightarrow on S satisfying

$$\begin{array}{c} \cdot \\ | \\ \text{b} \\ / \quad \backslash \\ \cdot \quad \cdot \end{array} \rightarrow \begin{array}{c} \cdot \\ | \\ \text{a} \\ / \quad \backslash \\ \cdot \quad \text{a} \\ \cdot \quad \cdot \end{array}, \quad (1.1.1a) \quad \begin{array}{c} \cdot \\ | \\ \text{a} \\ / \quad \backslash \\ \cdot \quad \text{c} \\ \cdot \quad \cdot \end{array} \rightarrow \begin{array}{c} \cdot \\ | \\ \text{a} \\ / \quad \backslash \\ \cdot \quad \text{c} \\ \cdot \quad \cdot \end{array}. \quad (1.1.1b)$$

We then have the following steps of rewritings by \rightarrow :

$$\begin{array}{c} \cdot \\ | \\ \text{b} \\ / \quad \backslash \\ \text{c} \quad \text{a} \\ / \quad \backslash \\ \cdot \quad \cdot \end{array} \Rightarrow \begin{array}{c} \cdot \\ | \\ \text{a} \\ / \quad \backslash \\ \text{c} \quad \text{b} \\ / \quad \backslash \\ \cdot \quad \cdot \end{array} \Rightarrow \begin{array}{c} \cdot \\ | \\ \text{a} \\ / \quad \backslash \\ \text{c} \quad \text{c} \\ / \quad \backslash \\ \cdot \quad \text{b} \\ \cdot \quad \cdot \end{array} \Rightarrow \begin{array}{c} \cdot \\ | \\ \text{a} \\ / \quad \backslash \\ \text{c} \quad \text{a} \\ / \quad \backslash \\ \text{c} \quad \text{a} \\ / \quad \backslash \\ \cdot \quad \cdot \end{array}. \quad (1.1.2)$$

We shall use the standard terminology (*terminating, normal form, confluent, convergent, etc.*) about rewrite rules [BN98]. Let us recall the most important definitions. Let \rightarrow be a rewrite rule on a set S of trees. We say that \rightarrow is *terminating* if there is no infinite chain $t \Rightarrow t_1 \Rightarrow t_2 \Rightarrow \dots$. In this case, any tree t of S that cannot be rewritten by \rightarrow is a *normal form* for \rightarrow . We say that \rightarrow is *confluent* if for any trees t , τ_1 , and τ_2 such that $t \xRightarrow{*} \tau_1$ and $t \xRightarrow{*} \tau_2$, there exists a tree t' such that $\tau_1 \xRightarrow{*} t'$ and $\tau_2 \xRightarrow{*} t'$. When \rightarrow is both terminating and confluent, \rightarrow is *convergent*.

1.2. Operads and Koszulity. We adopt most of notations and conventions of [LV12] about operads. For the sake of completeness, we recall here the elementary notions about operads employed thereafter.

1.2.1. Nonsymmetric operads. A *nonsymmetric operad in the category of vector spaces*, or a *nonsymmetric operad* for short, is a graded vector space

$$\mathcal{O} := \bigoplus_{n \geq 1} \mathcal{O}(n) \quad (1.2.1)$$

together with linear maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \rightarrow \mathcal{O}(n + m - 1), \quad n, m \geq 1, i \in [n], \quad (1.2.2)$$

called *partial compositions*, and a distinguished element $\mathbb{1} \in \mathcal{O}(1)$, the *unit* of \mathcal{O} . This data has to satisfy the three relations

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \quad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i \in [n], j \in [m], \quad (1.2.3a)$$

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \quad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i < j \in [n], \quad (1.2.3b)$$

$$\mathbb{1} \circ_1 x = x = x \circ_1 \mathbb{1}, \quad x \in \mathcal{O}(n), i \in [n]. \quad (1.2.3c)$$

Since we consider in this paper only nonsymmetric operads, we shall call these simply *operads*. Moreover, in this work, we shall only consider operads \mathcal{O} for which $\mathcal{O}(1)$ has dimension 1.

When \mathcal{O} is such that all $\mathcal{O}(n)$ have finite dimensions for all $n \geq 1$, the *Hilbert series* of \mathcal{O} is the series $\mathcal{H}_{\mathcal{O}}(t)$ defined by

$$\mathcal{H}_{\mathcal{O}}(t) := \sum_{n \geq 1} \dim \mathcal{O}(n) t^n. \quad (1.2.4)$$

If x is an element of \mathcal{O} such that $x \in \mathcal{O}(n)$ for a $n \geq 1$, we say that n is the *arity* of x and we denote it by $|x|$. The *complete composition map* of \mathcal{O} is the linear map

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \dots \otimes \mathcal{O}(m_n) \rightarrow \mathcal{O}(m_1 + \dots + m_n), \quad (1.2.5)$$

defined, for any $x \in \mathcal{O}(n)$ and $y_1, \dots, y_n \in \mathcal{O}$, by

$$x \circ [y_1, \dots, y_n] := (\dots ((x \circ_n y_n) \circ_{n-1} y_{n-1}) \dots) \circ_1 y_1. \quad (1.2.6)$$

If \mathcal{O}_1 and \mathcal{O}_2 are two operads, a linear map $\phi : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is an *operad morphism* if it respects arities, sends the unit of \mathcal{O}_1 to the unit of \mathcal{O}_2 , and commutes with partial composition maps. We say that \mathcal{O}_2 is a *suboperad* of \mathcal{O}_1 if \mathcal{O}_2 is a graded subspace of \mathcal{O}_1 , \mathcal{O}_1 and \mathcal{O}_2 have the

same unit, and the partial compositions of \mathcal{O}_2 are the ones of \mathcal{O}_1 restricted on \mathcal{O}_2 . For any subset G of \mathcal{O} , the *operad generated* by G is the smallest suboperad \mathcal{O}^G of \mathcal{O} containing G . When $\mathcal{O}^G = \mathcal{O}$ and G is minimal with respect to the inclusion among the subsets of G satisfying this property, G is a *minimal generating set* of \mathcal{O} and its elements are *generators* of \mathcal{O} . An *operad ideal* of \mathcal{O} is a graded subspace I of \mathcal{O} such that, for any $x \in \mathcal{O}$ and $y \in I$, $x \circ_i y$ and $y \circ_j x$ are in I for all valid integers i and j . Given an operad ideal I of \mathcal{O} , one can define the *quotient operad* \mathcal{O}/I of \mathcal{O} by I in the usual way.

Let us recall and set some more definitions about operads. The *Hadamard product* between the two operads \mathcal{O}_1 and \mathcal{O}_2 is the operad $\mathcal{O}_1 * \mathcal{O}_2$ satisfying $(\mathcal{O}_1 * \mathcal{O}_2)(n) = \mathcal{O}_1(n) \otimes \mathcal{O}_2(n)$, and its partial composition is defined component-wise from the partial compositions of \mathcal{O}_1 and \mathcal{O}_2 . An element x of $\mathcal{O}(2)$ is *associative* if $x \circ_1 x = x \circ_2 x$. A *symmetry* of \mathcal{O} is either an automorphism or an antiautomorphism of \mathcal{O} . The set of all symmetries of \mathcal{O} forms a group for the map composition, called the *group of symmetries* of \mathcal{O} . A basis $B := \sqcup_{n \geq 1} B(n)$ of \mathcal{O} is a *set-operad basis* if all partial compositions of elements of B belong to B . In this case, we say that \mathcal{O} is a *set-operad* with respect to the basis B . Moreover, when all the maps

$$\circ_i^y : B(n) \rightarrow B(n + m - 1), \quad n, m \geq 1, i \in [n], y \in B(m), \quad (1.2.7)$$

defined by

$$\circ_i^y(x) = x \circ_i y, \quad x \in B(n), \quad (1.2.8)$$

are injective, we say that B is a *basic set-operad basis* of \mathcal{O} . This notion is a slightly modified version of the original notion of basic set-operads introduced by Vallette [Val07]. Finally, \mathcal{O} is *cyclic* (see [GK95]) if there is a map

$$\rho : \mathcal{O}(n) \rightarrow \mathcal{O}(n), \quad n \geq 1, \quad (1.2.9)$$

satisfying, for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $i \in [n]$,

$$\rho(\mathbb{1}) = \mathbb{1}, \quad (1.2.10a)$$

$$\rho^{n+1}(x) = x, \quad (1.2.10b)$$

$$\rho(x \circ_i y) = \begin{cases} \rho(y) \circ_m \rho(x) & \text{if } i = 1, \\ \rho(x) \circ_{i-1} y & \text{otherwise.} \end{cases} \quad (1.2.10c)$$

We call such a map ρ a *rotation map*.

1.2.2. Syntax trees and free operads. Let $G := \sqcup_{n \geq 1} G(n)$ be a graded set. The *arity* of an element x of G is n provided that $x \in G(n)$. A *syntax tree* on G is a planar rooted tree such that its internal nodes of arity n are labeled by elements of arity n of G . The *degree* (resp. *arity*) of a syntax tree is its number of internal nodes (resp. leaves). For instance, if $G := G(2) \sqcup G(3)$ with $G(2) := \{a, c\}$ and $G(3) := \{b\}$,

$$\begin{array}{c} \cdot \\ | \\ \text{b} \\ / \quad \backslash \\ \text{c} \quad \text{a} \\ / \quad \backslash \quad / \quad \backslash \\ \text{c} \quad \text{b} \quad \text{a} \\ / \quad \backslash \quad / \quad \backslash \\ \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \quad (1.2.11)$$

is a syntax tree on G of degree 5 and arity 8. Its root is labeled by b and has arity 3.

Let $\mathfrak{G} := \bigoplus_{n \geq 1} \mathfrak{G}(n)$ be a graded vector space. In particular, \mathfrak{G} is a graded set so that we can consider syntax trees on \mathfrak{G} . The *free operad* over \mathfrak{G} is the operad $\text{Free}(\mathfrak{G})$ wherein for any $n \geq 1$, $\text{Free}(\mathfrak{G})(n)$ is the linear span of the syntax trees on \mathfrak{G} of arity n . The labeling of the internal nodes of the trees of $\text{Free}(\mathfrak{G})$ is linear in the sense that if t is a syntax tree on \mathfrak{G} having an internal node labeled by $x + \lambda y \in \mathfrak{G}$, $\lambda \in \mathbb{K}$, then, in $\text{Free}(\mathfrak{G})$, we have $t = t_x + \lambda t_y$, where t_x (resp. t_y) is the tree obtained by labeling by x (resp. y) the considered node labeled by $x + \lambda y$ in t . The partial composition $s \circ_i t$ of $\text{Free}(\mathfrak{G})$ of two syntax trees s and t on \mathfrak{G} consists in grafting the root of t on the i th leaf of s . The unit \perp of $\text{Free}(\mathfrak{G})$ is the tree consisting in one leaf. For instance, by setting $\mathfrak{G} := \text{Vect}(G)$ where G is the graded set defined in the previous example, one has in $\text{Free}(\mathfrak{G})$,

$$\begin{array}{c} \text{a} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{b} \end{array} \circ_3 \begin{array}{c} \text{c} \\ \diagup \quad \diagdown \\ \text{a} + \text{c} \end{array} = \begin{array}{c} \text{a} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{b} \\ \quad \quad \diagup \quad \diagdown \\ \quad \quad \text{c} \quad \text{a} \end{array} + \begin{array}{c} \text{a} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{b} \\ \quad \quad \diagup \quad \diagdown \\ \quad \quad \text{c} \quad \text{c} \end{array} . \quad (1.2.12)$$

We denote by $c : \mathfrak{G} \rightarrow \text{Free}(\mathfrak{G})$ the inclusion map, sending any x of \mathfrak{G} to the *corolla* labeled by x , that is the syntax tree consisting in a single internal node labeled by x attached to a required number of leaves. In the sequel, if required by the context, we shall implicitly see any element x of \mathfrak{G} as the corolla $c(x)$ of $\text{Free}(\mathfrak{G})$. For instance, when x and y are two elements of \mathfrak{G} , we shall simply denote by $x \circ_i y$ the syntax tree $c(x) \circ_i c(y)$ for all valid integers i .

1.2.3. Evaluations and treelike expressions. For any operad \mathcal{O} , by seeing \mathcal{O} as a graded vector space, $\text{Free}(\mathcal{O})$ is by definition the free operad on \mathcal{O} . The *evaluation map* of \mathcal{O} is the map

$$\text{ev} : \text{Free}(\mathcal{O}) \rightarrow \mathcal{O}, \quad (1.2.13)$$

defined linearly by induction, for any syntax tree t on \mathcal{O} , by

$$\text{ev}(t) := \begin{cases} \mathbb{1} \in \mathcal{O} & \text{if } t = \perp, \\ x \circ [\text{ev}(t_1), \dots, \text{ev}(t_k)] & \text{otherwise,} \end{cases} \quad (1.2.14)$$

where x is the label of the root of t and t_1, \dots, t_k are, from left to right, the subtrees of t . This map is the unique surjective operad morphism from $\text{Free}(\mathcal{O})$ to \mathcal{O} satisfying $\text{ev}(c(x)) = x$ for all $x \in \mathcal{O}$. If S is a subspace of \mathcal{O} , a *treelike expression* on S of $x \in \mathcal{O}$ is a tree t of $\text{Free}(\mathcal{O})$ such that $\text{ev}(t) = x$ and all internal nodes of t are labeled on S .

1.2.4. Presentations by generators and relations. A *presentation* of an operad \mathcal{O} consists in a pair (G, \mathfrak{R}) such that $G := \bigcup_{n \geq 1} G(n)$ is a graded set, \mathfrak{R} is a subspace of $\text{Free}(\mathfrak{G})$, where $\mathfrak{G} := \text{Vect}(G)$, and \mathcal{O} is isomorphic to $\text{Free}(\mathfrak{G}) / \langle \mathfrak{R} \rangle$, where $\langle \mathfrak{R} \rangle$ is the operad ideal of $\text{Free}(\mathfrak{G})$ generated by \mathfrak{R} . We call G the *space of generators* and \mathfrak{R} the *space of relations* of \mathcal{O} . We say that \mathcal{O} is *quadratic* if there is a presentation (G, \mathfrak{R}) of \mathcal{O} such that \mathfrak{R} is a homogeneous subspace of $\text{Free}(\mathfrak{G})$ consisting in syntax trees of degree 2. Besides, we say that \mathcal{O} is *binary* if there is a presentation (G, \mathfrak{R}) of \mathcal{O} such that \mathfrak{G} is concentrated in arity 2. Furthermore,

if \mathcal{O} admits a presentation (G, \mathfrak{R}) and \rightarrow is a rewrite rule on $\text{Free}(\mathfrak{G})$ such that the space induced by \rightarrow is \mathfrak{R} , we say that \rightarrow is an *orientation* of \mathfrak{R} .

1.2.5. Koszul duality and Koszulity. In [GK94], Ginzburg and Kapranov extended the notion of Koszul duality of quadratic associative algebras to quadratic operads. Starting with an operad \mathcal{O} admitting a binary and quadratic presentation (G, \mathfrak{R}) where G is finite, the *Koszul dual* of \mathcal{O} is the operad $\mathcal{O}^!$, isomorphic to the operad admitting the presentation (G, \mathfrak{R}^\perp) where \mathfrak{R}^\perp is the annihilator of \mathfrak{R} in $\text{Free}(\mathfrak{G})$, \mathfrak{G} being the space $\text{Vect}(G)$, with respect to the scalar product

$$\langle -, - \rangle : \text{Free}(\mathfrak{G})(3) \otimes \text{Free}(\mathfrak{G})(3) \rightarrow \mathbb{K} \quad (1.2.15)$$

linearly defined, for all $x, x', y, y' \in \mathfrak{G}(2)$, by

$$\langle x \circ_i y, x' \circ_{i'} y' \rangle := \begin{cases} 1 & \text{if } x = x', y = y', \text{ and } i = i' = 1, \\ -1 & \text{if } x = x', y = y', \text{ and } i = i' = 2, \\ 0 & \text{otherwise.} \end{cases} \quad (1.2.16)$$

Then, with knowledge of a presentation of \mathcal{O} , one can compute a presentation of $\mathcal{O}^!$.

Recall that a quadratic operad \mathcal{O} is *Koszul* if its Koszul complex is acyclic [GK94, LV12]. Furthermore, when \mathcal{O} is Koszul and admits an Hilbert series, the Hilbert series of \mathcal{O} and of its Koszul dual $\mathcal{O}^!$ are related [GK94] by

$$\mathcal{H}_{\mathcal{O}}(-\mathcal{H}_{\mathcal{O}^!}(-t)) = t. \quad (1.2.17)$$

Relation (1.2.17) can be used either to prove that an operad is not Koszul (it is the case when the coefficients of the hypothetical Hilbert series of the Koszul dual admits coefficients that are not negative integers) or to compute the Hilbert series of the Koszul dual of a Koszul operad.

In this work, to prove the Koszulity of an operad \mathcal{O} , we shall make use of a tool introduced by Dotsenko and Khoroshkin [DK10] in the context of Gröbner bases for operads, which reformulates in our context, by using rewrite rules on syntax trees, in the following way.

Lemma 1.2.1. *Let \mathcal{O} be an operad admitting a quadratic presentation (G, \mathfrak{R}) . If there exists an orientation \rightarrow of \mathfrak{R} such that \rightarrow is a convergent rewrite rule, then \mathcal{O} is Koszul.*

When \rightarrow satisfies the conditions contained in the statement of Lemma 1.2.1, the set of the normal forms of \rightarrow forms a basis of \mathcal{O} , called *Poincaré-Birkhoff-Witt basis*. These bases arise from the work of Hoffbeck [Hof10] (see also [LV12]).

1.2.6. *Algebras over operads.* Any operad \mathcal{O} encodes a category of algebras whose objects are called \mathcal{O} -*algebras*. An \mathcal{O} -algebra $\mathcal{A}_\mathcal{O}$ is a vector space endowed with a linear left action

$$\cdot : \mathcal{O}(n) \otimes \mathcal{A}_\mathcal{O}^{\otimes n} \rightarrow \mathcal{A}_\mathcal{O}, \quad n \geq 1, \quad (1.2.18)$$

satisfying the relations imposed by the structure of \mathcal{O} , that are

$$(x \circ_i y) \cdot (a_1 \otimes \cdots \otimes a_{n+m-1}) = x \cdot (a_1 \otimes \cdots \otimes a_{i-1} \otimes y \cdot (a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}), \quad (1.2.19)$$

for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, $i \in [n]$, and $a_1 \otimes \cdots \otimes a_{n+m-1} \in \mathcal{A}_\mathcal{O}^{\otimes n+m-1}$.

Notice that, by (1.2.19), if G is a generating set of \mathcal{O} , it is enough to define the action of each $x \in G$ on $\mathcal{A}_\mathcal{O}^{\otimes |x|}$ to wholly define \cdot . In other words, any element x of \mathcal{O} of arity n plays the role of a linear operation

$$x : \mathcal{A}_\mathcal{O}^{\otimes n} \rightarrow \mathcal{A}_\mathcal{O}, \quad (1.2.20)$$

taking n elements of $\mathcal{A}_\mathcal{O}$ as inputs and computing an element of $\mathcal{A}_\mathcal{O}$. By a slight but convenient abuse of notation, for any $x \in \mathcal{O}(n)$, we shall denote by $x(a_1, \dots, a_n)$, or by $a_1 x a_2$ if x has arity 2, the element $x \cdot (a_1 \otimes \cdots \otimes a_n)$ of $\mathcal{A}_\mathcal{O}$, for any $a_1 \otimes \cdots \otimes a_n \in \mathcal{A}_\mathcal{O}^{\otimes n}$. Observe that by (1.2.19), any associative element of \mathcal{O} gives rise to an associative operation on $\mathcal{A}_\mathcal{O}$.

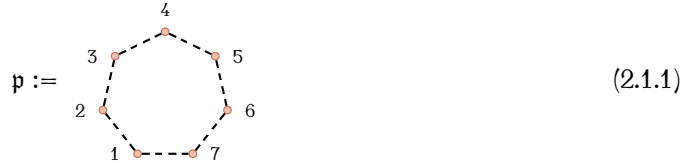
2. FROM UNITARY MAGMAS TO OPERADS

We describe in this section our construction from unitary magmas to operads and study its main algebraic and combinatorial properties.

2.1. **Cliques, unitary magmas, and operads.** We present here our main combinatorial objects, the decorated cliques. The construction \mathbb{C} , which takes a unitary magma as input and produces an operad, is defined.

2.1.1. *Cliques.* A *clique* of *size* $n \geq 1$ is a complete graph \mathfrak{p} on the set of vertices $[n + 1]$. An *arc* of \mathfrak{p} is a pair of integers (x, y) with $1 \leq x < y \leq n + 1$, a *diagonal* is an arc (x, y) different from $(x, x + 1)$ and $(1, n + 1)$, and an *edge* is an arc of the form $(x, x + 1)$ and different from $(1, n + 1)$. We denote by $\mathcal{A}_\mathfrak{p}$ (resp. $\mathcal{D}_\mathfrak{p}$, $\mathcal{E}_\mathfrak{p}$) the set of all arcs (resp. diagonals, edges) of \mathfrak{p} . For any $i \in [n]$, the *i th edge* of \mathfrak{p} is the edge $(i, i + 1)$, and the arc $(1, n + 1)$ is the *base* of \mathfrak{p} .

In our graphical representations, each clique is depicted so that its base is the bottom-most segment, vertices are implicitly numbered from 1 to $n + 1$ in the clockwise direction, and the diagonals are not drawn. For example,



is a clique of size 6. Its set of all diagonals satisfies

$$\mathcal{D}_p = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 6), (4, 7), (5, 7)\}, \quad (2.1.2)$$

its set of all edges satisfies

$$\mathcal{E}_p = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)\}, \quad (2.1.3)$$

and its set of all arcs satisfies

$$\mathcal{A}_p = \mathcal{D}_p \sqcup \mathcal{E}_p \sqcup \{(1, 7)\}. \quad (2.1.4)$$

2.1.2. Unitary magmas and decorated cliques. Recall first that a unitary magma is a set endowed with a binary operation \star admitting a left and right unit $\mathbb{1}_{\mathcal{M}}$. For convenience, we denote by $\bar{\mathcal{M}}$ the set $\mathcal{M} \setminus \{\mathbb{1}_{\mathcal{M}}\}$. To explore some examples in this article, we shall mostly consider four sorts of unitary magmas: the additive unitary magma on all integers denoted by \mathbb{Z} , the cyclic additive unitary magma on $\mathbb{Z}/\ell\mathbb{Z}$ denoted by \mathbb{N}_ℓ , the unitary magma

$$\mathbb{D}_\ell := \{\mathbb{1}, 0, d_1, \dots, d_\ell\} \quad (2.1.5)$$

where $\mathbb{1}$ is the unit of \mathbb{D}_ℓ , 0 is absorbing, and $d_i \star d_j = 0$ for all $i, j \in [\ell]$, and the unitary magma

$$\mathbb{E}_\ell := \{\mathbb{1}, e_1, \dots, e_\ell\} \quad (2.1.6)$$

where $\mathbb{1}$ is the unit of \mathbb{E}_ℓ and $e_i \star e_j = \mathbb{1}$ for all $i, j \in [\ell]$. Observe that since

$$e_1 \star (e_1 \star e_2) = e_1 \star \mathbb{1} = e_1 \neq e_2 = \mathbb{1} \star e_2 = (e_1 \star e_1) \star e_2, \quad (2.1.7)$$

all unitary magmas \mathbb{E}_ℓ , $\ell \geq 2$, are not monoids.

An *\mathcal{M} -decorated clique* (or an *\mathcal{M} -clique* for short) is a clique p endowed with a map

$$\phi_p : \mathcal{A}_p \rightarrow \mathcal{M}. \quad (2.1.8)$$

For convenience, for any arc (x, y) of p , we shall denote by $p(x, y)$ the value $\phi_p((x, y))$. Moreover, we say that the arc (x, y) is *labeled* by $p(x, y)$. When the arc (x, y) is labeled by an element different from $\mathbb{1}_{\mathcal{M}}$, we say that the arc (x, y) is *solid*. Again for convenience, we denote by p_0 the label $p(1, n + 1)$ of the base of p , where n is the size of p . Moreover, we denote by p_i , $i \in [n]$, the label $p(i, i + 1)$ of the i th edge of p . By convention, we require that the \mathcal{M} -clique of size 1 having its base labeled by $\mathbb{1}_{\mathcal{M}}$ is the only such object of size 1. The set of all \mathcal{M} -cliques is denoted by $\mathcal{G}_{\mathcal{M}}$.

In our graphical representations, we shall represent any \mathcal{M} -clique p by drawing a clique of the same size as the one of p following the conventions explained before, and by labeling some of its arcs in the following way. If (x, y) is a solid arc of p , we represent it by a line decorated by $p(x, y)$. If (x, y) is not solid and is an edge or the base of p , we represent it

as a dashed line. In the remaining case, when (x, y) is a diagonal of p and is not solid, we do not draw it. For instance,

$$p := \begin{array}{c} \text{Diagram of an } \mathcal{M}\text{-clique } p \text{ with 6 vertices and arcs labeled } -1, 1, 2, 3. \end{array} \quad (2.1.9)$$

is a \mathbb{Z} -clique such that, among others $p(1, 2) = -1$, $p(1, 5) = 2$, $p(3, 7) = -1$, $p(5, 7) = 1$, $p(2, 3) = 0$ (because 0 is the unit of \mathbb{Z}), and $p(2, 6) = 0$ (for the same reason).

Let us now provide some definitions and statistics on \mathcal{M} -cliques. Let p be an \mathcal{M} -clique of size n . The *skeleton* of p is the undirected graph $\text{skel}(p)$ on the set of vertices $[n + 1]$ and such that for any $x < y \in [n + 1]$, there is an arc $\{x, y\}$ in $\text{skel}(p)$ if (x, y) is a solid arc of p . The *degree* of a vertex x of p is the number of solid arcs adjacent to x . The *degree* $\text{degr}(p)$ of p is the maximal degree among its vertices. Two (non-necessarily solid) diagonals (x, y) and (x', y') of p are *crossing* if $x < x' < y < y'$ or $x' < x < y' < y$. The *crossing* of a solid diagonal (x, y) of p is the number of solid diagonals (x', y') such that (x, y) and (x', y') are crossing. The *crossing* $\text{cros}(p)$ of p is the maximal crossing among its solid diagonals. When $\text{cros}(p) = 0$, there are no crossing diagonals in p and in this case, p is *noncrossing*. A (non-necessarily solid) arc (x, y) *includes* a (non-necessarily solid) arc (x', y') of p if $x \leq x' < y' \leq y$. We say that p is *inclusion-free* if for any solid arcs (x, y) and (x', y') of p such that (x, y) includes (x', y') , $(x, y) = (x', y')$. Besides, p is *acyclic* if the subgraph of p consisting in all its solid arcs is acyclic. When p has no solid edges nor solid base, p is *white*. The *border* of p is the word $\text{bor}(p)$ of length n such that for any $i \in [n]$, $\text{bor}(p)_i = p_i$. If p has no solid diagonals, p is an *\mathcal{M} -bubble*. An \mathcal{M} -bubble is hence specified by a pair (b, u) where b is the label of its base and u is its border. Obviously, all \mathcal{M} -bubbles are noncrossing \mathcal{M} -cliques. The set of all \mathcal{M} -bubbles is denoted by $\mathcal{B}_{\mathcal{M}}$. We denote by $\mathcal{T}_{\mathcal{M}}$ the set of all *\mathcal{M} -triangles*, that are \mathcal{M} -cliques of size 2. Notice that any \mathcal{M} -triangle is also an \mathcal{M} -bubble.

2.1.3. Partial composition of \mathcal{M} -cliques. From now, the *arity* of an \mathcal{M} -clique p is its size and is denoted by $|p|$. For any unitary magma \mathcal{M} , we define the vector space

$$\mathcal{C}\mathcal{M} := \bigoplus_{n \geq 1} \mathcal{C}\mathcal{M}(n), \quad (2.1.10)$$

where $\mathcal{C}\mathcal{M}(n)$ is the linear span of all \mathcal{M} -cliques of arity n , $n \geq 1$. The set $\mathcal{B}_{\mathcal{M}}$ forms hence a basis of $\mathcal{C}\mathcal{M}$ called *fundamental basis*. Observe that the space $\mathcal{C}\mathcal{M}(1)$ has dimension 1 since it is the linear span of the \mathcal{M} -clique $\circ - \circ$. We endow $\mathcal{C}\mathcal{M}$ with partial composition maps

$$\circ_i : \mathcal{C}\mathcal{M}(n) \otimes \mathcal{C}\mathcal{M}(m) \rightarrow \mathcal{C}\mathcal{M}(n + m - 1), \quad n, m \geq 1, i \in [n], \quad (2.1.11)$$

defined linearly, in the fundamental basis, in the following way. Let p and q be two \mathcal{M} -cliques of respective arities n and m , and $i \in [n]$ be an integer. We set $p \circ_i q$ as the \mathcal{M} -clique

of arity $n + m - 1$ such that, for any arc (x, y) where $1 \leq x < y \leq n + m$,

$$(\mathfrak{p} \circ_i \mathfrak{q})(x, y) := \begin{cases} \mathfrak{p}(x, y) & \text{if } y \leq i, \\ \mathfrak{p}(x, y - m + 1) & \text{if } x \leq i < i + m \leq y \text{ and } (x, y) \neq (i, i + m), \\ \mathfrak{p}(x - m + 1, y - m + 1) & \text{if } i + m \leq x, \\ \mathfrak{q}(x - i + 1, y - i + 1) & \text{if } i \leq x < y \leq i + m \text{ and } (x, y) \neq (i, i + m), \\ \mathfrak{p}_i \star \mathfrak{q}_0 & \text{if } (x, y) = (i, i + m), \\ \mathbb{1}_{\mathcal{M}} & \text{otherwise.} \end{cases} \quad (2.1.12)$$

We recall that \star denotes the operation of \mathcal{M} and $\mathbb{1}_{\mathcal{M}}$ its unit. In a geometric way, $\mathfrak{p} \circ_i \mathfrak{q}$ is obtained by gluing the base of \mathfrak{q} onto the i th edge of \mathfrak{p} , by relabeling the common arcs between \mathfrak{p} and \mathfrak{q} , respectively the arcs $(i, i + 1)$ and $(1, m + 1)$, by $\mathfrak{p}_i \star \mathfrak{q}_0$, and by adding all required non solid diagonals on the graph thus obtained to become a clique (see Figure 1). For example, in $\mathbb{C}\mathbb{Z}$, one has the two partial compositions

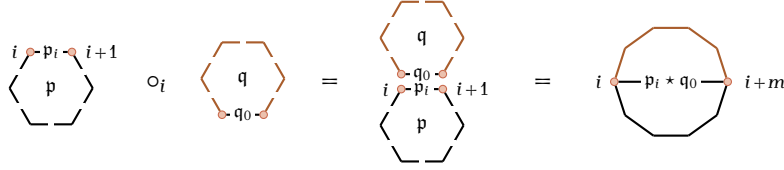


FIGURE 1. The partial composition of $\mathbb{C}\mathcal{M}$, described in geometric terms. Here, \mathfrak{p} and \mathfrak{q} are two \mathcal{M} -cliques. The arity of \mathfrak{q} is m and i is an integer between 1 and $|\mathfrak{p}|$.

(2.1.13a)

(2.1.13b)

2.1.4. Functorial construction from unitary magmas to operads. If \mathcal{M}_1 and \mathcal{M}_2 are two unitary magmas and $\theta : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a unitary magma morphism, we define

$$\mathbb{C}\theta : \mathbb{C}\mathcal{M}_1 \rightarrow \mathbb{C}\mathcal{M}_2 \quad (2.1.14)$$

as the linear map sending any \mathcal{M}_1 -clique \mathfrak{p} of arity n to the \mathcal{M}_2 -clique $(\mathbb{C}\theta)(\mathfrak{p})$ of the same arity such that, for any arc (x, y) where $1 \leq x < y \leq n + 1$,

$$((\mathbb{C}\theta)(\mathfrak{p}))(x, y) := \theta(\mathfrak{p}(x, y)). \quad (2.1.15)$$

In a geometric way, $(C\theta)(p)$ is the \mathcal{M}_2 -clique obtained by relabeling each arc of p by the image of its label by θ .

Theorem 2.1.1. *The construction C is a functor from the category of unitary magmas to the category of operads. Moreover, C respects injections and surjections.*

Proof. Let \mathcal{M} be a unitary magma. The fact that $C\mathcal{M}$ endowed with the partial composition (2.1.12) is an operad can be established by showing that the two associativity relations (1.2.3a) and (1.2.3b) of operads are satisfied. This is a technical but a simple verification. Since $C\mathcal{M}(1)$ contains $\circ - \circ$ and this element is the unit for this partial composition, (1.2.3c) holds. Moreover, let \mathcal{M}_1 and \mathcal{M}_2 be two unitary magmas and $\theta : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a unitary magma morphism. The fact that the map $C\theta$ defined in (2.1.15) is an operad morphism is a straightforward checking. All this imply that C is a functor. Finally, the fact that C respects injections and surjections is also a straightforward verification. \square

We name the construction C as the *clique construction* and $C\mathcal{M}$ as the *\mathcal{M} -clique operad*. Observe that the fundamental basis of $C\mathcal{M}$ is a set-operad basis of $C\mathcal{M}$. Besides, when \mathcal{M} is the trivial unitary magma $\{1_{\mathcal{M}}\}$, $C\mathcal{M}$ is the linear span of all decorated cliques having only non-solid arcs. Thus, each space $C\mathcal{M}(n)$, $n \geq 1$, is of dimension 1 and it follows from the definition of the partial composition of $C\mathcal{M}$ that this operad is isomorphic to the associative operad As . The next result shows that the clique construction is compatible with the Cartesian product of unitary magmas.

Proposition 2.1.2. *Let \mathcal{M}_1 and \mathcal{M}_2 be two unitary magmas. Then, the operads $(C\mathcal{M}_1) * (C\mathcal{M}_2)$ and $C(\mathcal{M}_1 \times \mathcal{M}_2)$ are isomorphic.*

Proof. Let $\phi : (C\mathcal{M}_1) * (C\mathcal{M}_2) \rightarrow C(\mathcal{M}_1 \times \mathcal{M}_2)$ be the map defined linearly as follows. For any \mathcal{M}_1 -clique p of $C\mathcal{M}_1$ and any \mathcal{M}_2 -clique q of $C\mathcal{M}_2$ both of arity n , $\phi(p \otimes q)$ is the $\mathcal{M}_1 \times \mathcal{M}_2$ -clique defined, for any $1 \leq x < y \leq n + 1$, by

$$(\phi(p \otimes q))(x, y) := (p(x, y), q(x, y)). \quad (2.1.16)$$

Let the map $\psi : C(\mathcal{M}_1 \times \mathcal{M}_2) \rightarrow (C\mathcal{M}_1) * (C\mathcal{M}_2)$ defined linearly, for any $\mathcal{M}_1 \times \mathcal{M}_2$ -clique r of $C(\mathcal{M}_1 \times \mathcal{M}_2)$ of arity n , as follows. The \mathcal{M}_1 -clique p and the \mathcal{M}_2 -clique q of arity n of the tensor $p \otimes q := \psi(r)$ are defined, for any $1 \leq x < y \leq n + 1$, by $p(x, y) := a$ and $q(x, y) := b$ where $(a, b) = r(x, y)$. Since we observe immediately that ψ is the inverse of ϕ , ϕ is a bijection. Moreover, it follows from the definition of the partial composition of clique operads that ϕ is an operad morphism. The statement of the proposition follows. \square

2.2. General properties. We investigate here some properties of clique operads, as their dimensions, their minimal generating sets, the fact that they admit a cyclic operad structure, and describe their partial compositions over two alternative bases.

2.2.1. *Binary relations.* Let us start by remarking that, depending on the cardinality m of \mathcal{M} , the set of all \mathcal{M} -cliques can be interpreted as particular binary relations. When $m \geq 4$, let us set $\mathcal{M} = \{1_{\mathcal{M}}, a, b, c, \dots\}$ so that a , b , and c are distinguished pairwise distinct elements of \mathcal{M} different from $1_{\mathcal{M}}$. Given an \mathcal{M} -clique p of arity $n \geq 2$, we build a binary relation \mathfrak{R} on $[n + 1]$ satisfying, for all $x < y \in [n + 1]$,

$$\begin{aligned} x \mathfrak{R} y & \quad \text{if } p(x, y) = a, \\ y \mathfrak{R} x & \quad \text{if } p(x, y) = b, \\ x \mathfrak{R} y \text{ and } y \mathfrak{R} x & \quad \text{if } p(x, y) = c. \end{aligned} \tag{2.2.1}$$

In particular, when $m = 2$ (resp. $m = 3$, $m = 4$), $\mathcal{M} = \{1, c\}$ (resp. $\mathcal{M} = \{1, a, b\}$, $\mathcal{M} = \{1, a, b, c\}$) and the set of all \mathcal{M} -cliques of arities $n \geq 2$ is in one-to-one correspondence with the set of all irreflexive and symmetric (resp. irreflexive and antisymmetric, irreflexive) binary relations on $[n + 1]$. Therefore, the operads $C\mathcal{M}$ can be interpreted as operads involving binary relations with more or less properties.

2.2.2. *Dimensions and minimal generating set.*

Proposition 2.2.1. *Let \mathcal{M} be a finite unitary magma. For all $n \geq 2$,*

$$\dim C\mathcal{M}(n) = m^{\binom{n+1}{2}}, \tag{2.2.2}$$

where $m := \#\mathcal{M}$.

Proof. By definition of the clique construction and of \mathcal{M} -cliques, the dimension of $C\mathcal{M}(n)$ is the number of maps from the set $\{(x, y) \in [n + 1]^2 : x < y\}$ to \mathcal{M} . Therefore, when $n \geq 2$, this implies (2.2.2). \square

From Proposition 2.2.1, the first dimensions of $C\mathcal{M}$ depending on $m := \#\mathcal{M}$ are

$$1, 1, 1, 1, 1, 1, 1, 1, \quad m = 1, \tag{2.2.3a}$$

$$1, 8, 64, 1024, 32768, 2097152, 268435456, 68719476736, \quad m = 2, \tag{2.2.3b}$$

$$1, 27, 729, 59049, 14348907, 10460353203, 22876792454961, \\ 150094635296999121, \quad m = 3, \tag{2.2.3c}$$

$$1, 64, 4096, 1048576, 1073741824, 4398046511104, 72057594037927936, \\ 4722366482869645213696, \quad m = 4. \tag{2.2.3d}$$

Except for the first terms, the second one forms Sequence [A006125](#), the third one forms Sequence [A047656](#), and the last one forms Sequence [A053763](#) of [Slo].

Lemma 2.2.2. *Let \mathcal{M} be a unitary magma, p be an \mathcal{M} -clique of arity $n \geq 2$, and (x, y) be a diagonal of p . Then, the following two assertions are equivalent:*

- (i) *the crossing of (x, y) is 0;*
- (ii) *the \mathcal{M} -clique p expresses as $p = q \circ_x \tau$, where q is an \mathcal{M} -clique of arity $n + x - y + 1$ and τ is an \mathcal{M} -clique of arity $y - x$.*

Proof. Assume first that (i) holds. Set q as the \mathcal{M} -clique of arity $n + x - y + 1$ defined, for any arc (z, t) where $1 \leq z < t \leq n + x - y + 2$, by

$$q(z, t) := \begin{cases} p(z, t) & \text{if } t \leq x, \\ p(z, t + y - x - 1) & \text{if } x + 1 \leq t, \\ p(z + y - x - 1, t + y - x - 1) & \text{otherwise,} \end{cases} \quad (2.2.4)$$

and τ as the \mathcal{M} -clique of arity $y - x$ defined, for any arc (z, t) where $1 \leq z < t \leq y - x + 1$, by

$$\tau(z, t) := \begin{cases} p(z + x - 1, t + x - 1) & \text{if } (z, t) \neq (1, y - x + 1), \\ \mathbb{1}_{\mathcal{M}} & \text{otherwise.} \end{cases} \quad (2.2.5)$$

By following the definition of the partial composition of $C\mathcal{M}$, one obtains $p = q \circ_x \tau$, whence (ii) holds.

Assume conversely that (ii) holds. By definition of the partial composition of $C\mathcal{M}$, the fact that $p = q \circ_x \tau$ implies that $p(x', y') = \mathbb{1}_{\mathcal{M}}$ for any arc (x', y') such that (x, y) and (x', y') are crossing. Therefore, (i) holds. \square

Let $\mathcal{P}_{\mathcal{M}}$ be the set of all \mathcal{M} -cliques p of arity $n \geq 2$ that do not satisfy the property of the statement of Lemma 2.2.2. In other words, $\mathcal{P}_{\mathcal{M}}$ is the set of all \mathcal{M} -cliques such that, for any (non-necessarily solid) diagonal (x, y) of p , there is at least one solid diagonal (x', y') of p such that (x, y) and (x', y') are crossing. We call $\mathcal{P}_{\mathcal{M}}$ the set of all **prime \mathcal{M} -cliques**. Observe that, according to this description, all \mathcal{M} -triangles are prime.

Proposition 2.2.3. *Let \mathcal{M} be a unitary magma. The set $\mathcal{P}_{\mathcal{M}}$ is a minimal generating set of $C\mathcal{M}$.*

Proof. We show by induction on the arity that $\mathcal{P}_{\mathcal{M}}$ is a generating set of $C\mathcal{M}$. Let p be an \mathcal{M} -clique. If p is of arity 1, $p = \bullet - \bullet$ and hence p trivially belongs to $(C\mathcal{M})^{\mathcal{P}_{\mathcal{M}}}$. Let us assume that p is of arity $n \geq 2$. First, if $p \in \mathcal{P}_{\mathcal{M}}$, then $p \in (C\mathcal{M})^{\mathcal{P}_{\mathcal{M}}}$. Otherwise, p is an \mathcal{M} -clique which satisfies the description of the statement of Lemma 2.2.2. Therefore, by this lemma, there are two \mathcal{M} -cliques q and τ and an integer $x \in [|p|]$ such that $|q| < |p|$, $|\tau| < |p|$, and $p = q \circ_x \tau$. By induction hypothesis, q and τ belong to $(C\mathcal{M})^{\mathcal{P}_{\mathcal{M}}}$ and hence, p also belongs to $(C\mathcal{M})^{\mathcal{P}_{\mathcal{M}}}$.

Finally, by Lemma 2.2.2, if p is a prime \mathcal{M} -clique, p cannot be expressed as a partial composition of prime \mathcal{M} -cliques. Moreover, since the space $C\mathcal{M}(1)$ is trivial, these arguments imply that $\mathcal{P}_{\mathcal{M}}$ is a minimal generating set of $C\mathcal{M}$. \square

2.2.3. Associative elements.

Proposition 2.2.4. *Let \mathcal{M} be a unitary magma and f be an element of $C\mathcal{M}(2)$ of the form*

$$f := \sum_{p \in \mathcal{T}_{\mathcal{M}}} \lambda_p p, \quad (2.2.6)$$

where the $\lambda_p, p \in \mathcal{T}_{\mathcal{M}}$, are coefficients of \mathbb{K} . Then, f is associative if and only if

$$\sum_{\substack{p_1, q_0 \in \mathcal{M} \\ \delta = p_1 * q_0}} \lambda \begin{array}{c} \wedge \\ p_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_1 \quad q_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} = 0, \quad p_0, p_2, q_1, q_2 \in \mathcal{M}, \delta \in \bar{\mathcal{M}}, \quad (2.2.7a)$$

$$\sum_{\substack{p_1, q_0 \in \mathcal{M} \\ p_1 * q_0 = \mathbb{1}_{\mathcal{M}}}} \lambda \begin{array}{c} \wedge \\ p_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_1 \quad q_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} - \lambda \begin{array}{c} \wedge \\ q_1 \quad p_1 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_2 \quad p_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} = 0, \quad p_0, p_2, q_1, q_2 \in \mathcal{M}, \quad (2.2.7b)$$

$$\sum_{\substack{p_2, q_0 \in \mathcal{M} \\ \delta = p_2 * q_0}} \lambda \begin{array}{c} \wedge \\ p_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_1 \quad q_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} = 0, \quad p_0, p_1, q_1, q_2 \in \mathcal{M}, \delta \in \bar{\mathcal{M}}. \quad (2.2.7c)$$

Proof. The element f defined in (2.2.6) is associative if and only if $f \circ_1 f - f \circ_2 f = 0$. Therefore, this property is equivalent to the fact that

$$\begin{aligned} f \circ_1 f - f \circ_2 f &= \left(\sum_{\substack{p, q \in \mathcal{T}_{\mathcal{M}} \\ \delta = p_1 * q_0 \neq \mathbb{1}_{\mathcal{M}}}} \lambda_p \lambda_q \begin{array}{c} \wedge \\ q_1 \quad \delta \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \begin{array}{c} \wedge \\ q_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) + \left(\sum_{\substack{p, q \in \mathcal{T}_{\mathcal{M}} \\ p_1 * q_0 = \mathbb{1}_{\mathcal{M}}}} \lambda_p \lambda_q \begin{array}{c} \wedge \\ q_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \begin{array}{c} \wedge \\ q_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \\ &\quad - \left(\sum_{\substack{p, q \in \mathcal{T}_{\mathcal{M}} \\ \delta = p_2 * q_0 \neq \mathbb{1}_{\mathcal{M}}}} \lambda_p \lambda_q \begin{array}{c} \wedge \\ p_1 \quad \delta \quad q_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \begin{array}{c} \wedge \\ q_1 \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) - \left(\sum_{\substack{p, q \in \mathcal{T}_{\mathcal{M}} \\ p_2 * q_0 = \mathbb{1}_{\mathcal{M}}}} \lambda_p \lambda_q \begin{array}{c} \wedge \\ p_1 \quad q_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \begin{array}{c} \wedge \\ q_1 \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \\ &= \left(\sum_{\substack{p_0, p_2, q_1, q_2 \in \mathcal{M} \\ \delta \in \bar{\mathcal{M}}}} \left(\sum_{\substack{p_1, q_0 \in \mathcal{M} \\ \delta = p_1 * q_0}} \lambda \begin{array}{c} \wedge \\ p_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_1 \quad q_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} \right) \begin{array}{c} \wedge \\ q_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \\ &\quad + \left(\sum_{p_0, p_2, q_1, q_2 \in \mathcal{M}} \left(\sum_{\substack{p_1, q_0 \in \mathcal{M} \\ p_1 * q_0 = \mathbb{1}_{\mathcal{M}}}} \lambda \begin{array}{c} \wedge \\ p_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_1 \quad q_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} - \lambda \begin{array}{c} \wedge \\ q_1 \quad p_1 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_2 \quad p_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} \right) \begin{array}{c} \wedge \\ q_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \\ &\quad - \left(\sum_{\substack{p_0, p_1, q_1, q_2 \in \mathcal{M} \\ \delta \in \bar{\mathcal{M}}}} \left(\sum_{\substack{p_2, q_0 \in \mathcal{M} \\ \delta = p_2 * q_0}} \lambda \begin{array}{c} \wedge \\ p_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \lambda \begin{array}{c} \wedge \\ q_1 \quad q_2 \\ \swarrow \quad \searrow \\ q_0 \end{array} \right) \begin{array}{c} \wedge \\ q_1 \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \\ &= 0, \end{aligned} \quad (2.2.8)$$

and hence, is equivalent to the fact that (2.2.7a), (2.2.7b), and (2.2.7c) hold. \square

For instance, by Proposition 2.2.4, the binary elements

$$\begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array}, \quad (2.2.9a)$$

$$\begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} + \begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} - \begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} + \begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} - \begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} + \begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} - \begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} - \begin{array}{c} \wedge \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ -1 \end{array} \quad (2.2.9b)$$

of CN_2 are associative, and the binary elements

$$\begin{array}{c} \wedge \\ 0 \quad 0 \\ \swarrow \quad \searrow \\ -0 \end{array} - \begin{array}{c} \wedge \\ 0 \quad 0 \\ \swarrow \quad \searrow \\ -0 \end{array}, \quad (2.2.10a)$$

$$\begin{array}{c} \triangle \\ \leftarrow 0 \rightarrow \end{array} - \begin{array}{c} \triangle \\ \leftarrow 0 \rightarrow \end{array} - \begin{array}{c} \triangle \\ \leftarrow 0 \rightarrow \end{array} + \begin{array}{c} \triangle \\ \leftarrow 0 \rightarrow \end{array} \quad (2.2.10b)$$

of $\mathbb{C}\mathbb{D}_0$ are associative.

2.2.4. Symmetries. Let $\text{ret} : \mathbb{C}\mathcal{M} \rightarrow \mathbb{C}\mathcal{M}$ be the linear map sending any \mathcal{M} -clique p of arity n to the \mathcal{M} -clique $\text{ret}(p)$ of the same arity such that, for any arc (x, y) where $1 \leq x < y \leq n + 1$,

$$(\text{ret}(p))(x, y) := p(n - y + 2, n - x + 2). \quad (2.2.11)$$

In a geometric way, $\text{ret}(p)$ is the \mathcal{M} -clique obtained by applying on p a reflection trough the vertical line passing by its base. For instance, one has in $\mathbb{C}\mathbb{Z}$,

$$\text{ret} \left(\begin{array}{c} \triangle \\ \leftarrow 1 \rightarrow \end{array} \right) = \begin{array}{c} \triangle \\ \leftarrow 1 \rightarrow \end{array} \quad (2.2.12)$$

Proposition 2.2.5. *Let \mathcal{M} be a unitary magma. Then, the group of symmetries of $\mathbb{C}\mathcal{M}$ contains the map ret and all the maps $C\theta$ where θ are unitary magma automorphisms of \mathcal{M} .*

Proof. When θ is a unitary magma automorphism of \mathcal{M} , since by Theorem 2.1.1 C is a functor respecting bijections, $C\theta$ is an operad automorphism of $\mathbb{C}\mathcal{M}$. Hence, $C\theta$ belongs to the group of symmetries of $\mathbb{C}\mathcal{M}$. Moreover, the fact that ret belongs to the group of symmetries of $\mathbb{C}\mathcal{M}$ can be established by showing that this map is an antiautomorphism of $\mathbb{C}\mathcal{M}$, directly from the definition of the partial composition of $\mathbb{C}\mathcal{M}$ and that of ret . \square

2.2.5. Basic set-operad basis. A unitary magma \mathcal{M} is *right cancellable* if for any $x, y, z \in \mathcal{M}$, $y \star x = z \star x$ implies $y = z$.

Proposition 2.2.6. *Let \mathcal{M} be a unitary magma. The fundamental basis of $\mathbb{C}\mathcal{M}$ is a basic set-operad basis if and only if \mathcal{M} is right cancellable.*

Proof. Assume first that \mathcal{M} is right cancellable. Let $n \geq 1$, $i \in [n]$, and p, p' , and q be three \mathcal{M} -cliques such that p and p' are of arity n . If $\circ_i^q(p) = \circ_i^q(p')$, we have $p \circ_i q = p' \circ_i q$. By definition of the partial composition map of $\mathbb{C}\mathcal{M}$, any same arc of p and p' have the same label, unless possibly the edge $(i, i + 1)$. Moreover, we have $p_i \star q_0 = p'_i \star q_0$. Since \mathcal{M} is right cancellable, this implies that $p_i = p'_i$, and hence, $p = p'$. This shows that the maps \circ_i^q are injective and thus, that the fundamental basis of $\mathbb{C}\mathcal{M}$ is a basic set-operad basis.

Conversely, assume that the fundamental basis of $\mathbb{C}\mathcal{M}$ is a basic set-operad basis. Then, in particular, for all $n \geq 1$ and all \mathcal{M} -cliques p, p' , and q such that p and p' are of arity n , $\circ_1^q(p) = \circ_1^q(p')$ implies $p = p'$. This is equivalent to state that $p_1 \star q_0 = p'_1 \star q_0$ implies $p_1 = p'_1$. This amount exactly to state that \mathcal{M} is right cancellable. \square

2.2.6. *Cyclic operad structure.* Let $\rho : \mathcal{CM} \rightarrow \mathcal{CM}$ be the linear map sending any \mathcal{M} -clique p of arity n to the \mathcal{M} -clique $\rho(p)$ of the same arity such that, for any arc (x, y) where $1 \leq x < y \leq n + 1$,

$$(\rho(p))(x, y) := \begin{cases} p(x + 1, y + 1) & \text{if } y \leq n, \\ p(1, x + 1) & \text{otherwise } (y = n + 1). \end{cases} \quad (2.2.13)$$

In a geometric way, $\rho(p)$ is the \mathcal{M} -clique obtained by applying a rotation of one step of p in the counterclockwise direction. For instance, one has in \mathcal{CZ} ,

$$\rho \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = \begin{array}{c} \text{Diagram 2} \end{array}. \quad (2.2.14)$$

Proposition 2.2.7. *Let \mathcal{M} be a unitary magma. The map ρ is a rotation map of \mathcal{CM} , endowing this operad with a cyclic operad structure.*

Proof. The fact that ρ is a rotation map for \mathcal{CM} follows from a technical but straightforward verification of the fact that Relations (1.2.10a), (1.2.10b), and (1.2.10c) hold. \square

2.2.7. *Alternative bases.* If p and q are two \mathcal{M} -cliques of the same arity, the *Hamming distance* $h(p, q)$ between p and q is the number of arcs (x, y) such that $p(x, y) \neq q(x, y)$. Let \leq_{be} be the partial order relation on the set of all \mathcal{M} -cliques, where, for any \mathcal{M} -cliques p and q , one has $p \leq_{be} q$ if q can be obtained from p by replacing some labels $1_{\mathcal{M}}$ of its edges or its base by other labels of \mathcal{M} . In the same way, let \leq_d be the partial order on the same set where $p \leq_d q$ if q can be obtained from p by replacing some labels $1_{\mathcal{M}}$ of its diagonals by other labels of \mathcal{M} .

For all \mathcal{M} -cliques p , let the elements of \mathcal{CM} defined by

$$H_p := \sum_{\substack{p' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{be} p}} p', \quad (2.2.15a)$$

and

$$K_p := \sum_{\substack{p' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_d p}} (-1)^{h(p', p)} p'. \quad (2.2.15b)$$

For instance, in \mathcal{CZ} ,

$$H = \begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} + \begin{array}{c} \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \end{array}, \quad (2.2.16a)$$

$$K = \begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \end{array} - \begin{array}{c} \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \end{array}. \quad (2.2.16b)$$

Since by Möbius inversion, one has for any \mathcal{M} -clique p ,

$$\sum_{\substack{p' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{\text{be}} p}} (-1)^{h(p', p)} H_{p'} = p = \sum_{\substack{p' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{\text{d}} p}} K_{p'}, \quad (2.2.17)$$

by triangularity, the family of all the H_p (resp. K_p) forms a basis of $C\mathcal{M}$ called the **H-basis** (resp. the **K-basis**).

If p is an \mathcal{M} -clique, $d_0(p)$ (resp. $d_i(p)$) is the \mathcal{M} -clique obtained by replacing the label of the base (resp. i th edge) of p by $\mathbb{1}_{\mathcal{M}}$.

Proposition 2.2.8. *Let \mathcal{M} be a unitary magma. The partial composition of $C\mathcal{M}$ expresses over the H-basis, for any \mathcal{M} -cliques p and q different from $\circ - \circ$ and any valid integer i , as*

$$H_p \circ_i H_q = \begin{cases} H_{p \circ_i q} + H_{d_i(p) \circ_i q} + H_{p \circ_i d_0(q)} + H_{d_i(p) \circ_i d_0(q)} & \text{if } p_i \neq \mathbb{1}_{\mathcal{M}} \text{ and } q_0 \neq \mathbb{1}_{\mathcal{M}}, \\ H_{p \circ_i q} + H_{d_i(p) \circ_i q} & \text{if } p_i \neq \mathbb{1}_{\mathcal{M}}, \\ H_{p \circ_i q} + H_{p \circ_i d_0(q)} & \text{if } q_0 \neq \mathbb{1}_{\mathcal{M}}, \\ H_{p \circ_i q} & \text{otherwise.} \end{cases} \quad (2.2.18)$$

Proof. From the definition of the H-basis, we have

$$\begin{aligned} H_p \circ_i H_q &= \sum_{\substack{p', q' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{\text{be}} p \\ q' \leq_{\text{be}} q}} p' \circ_i q' \\ &= \sum_{\substack{p', q' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{\text{be}} p \\ q' \leq_{\text{be}} q \\ p_i \neq \mathbb{1}_{\mathcal{M}} \\ q_0 \neq \mathbb{1}_{\mathcal{M}}}} p' \circ_i q' + \sum_{\substack{p', q' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{\text{be}} p \\ q' \leq_{\text{be}} q \\ p_i \neq \mathbb{1}_{\mathcal{M}} \\ q_0 = \mathbb{1}_{\mathcal{M}}}} p' \circ_i q' + \sum_{\substack{p', q' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{\text{be}} p \\ q' \leq_{\text{be}} q \\ p_i = \mathbb{1}_{\mathcal{M}} \\ q_0 \neq \mathbb{1}_{\mathcal{M}}}} p' \circ_i q' + \sum_{\substack{p', q' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_{\text{be}} p \\ q' \leq_{\text{be}} q \\ p_i = \mathbb{1}_{\mathcal{M}} \\ q_0 = \mathbb{1}_{\mathcal{M}}}} p' \circ_i q'. \end{aligned} \quad (2.2.19)$$

Let s_1 (resp. s_2, s_3, s_4) be the first (resp. second, third, fourth) summand of the last member of (2.2.19). There are four cases to explore depending on whether the i th edge of p and the base of q are solid or not. From the definition of the H-basis and of the partial order relation \leq_{be} , we have that

- (a) when $p_i \neq \mathbb{1}_{\mathcal{M}}$ and $q_0 \neq \mathbb{1}_{\mathcal{M}}$, $s_1 = H_{p \circ_i q}$, $s_2 = H_{p \circ_i d_0(q)}$, $s_3 = H_{d_i(p) \circ_i q}$, and $s_4 = H_{d_i(p) \circ_i d_0(q)}$;
- (b) when $p_i \neq \mathbb{1}_{\mathcal{M}}$ and $q_0 = \mathbb{1}_{\mathcal{M}}$, $s_1 = 0$, $s_2 = H_{p \circ_i q}$, $s_3 = 0$, and $s_4 = H_{d_i(p) \circ_i q}$;
- (c) when $p_i = \mathbb{1}_{\mathcal{M}}$ and $q_0 \neq \mathbb{1}_{\mathcal{M}}$, $s_1 = 0$, $s_2 = 0$, $s_3 = H_{p \circ_i q}$, and $s_4 = H_{p \circ_i d_0(q)}$;
- (d) and when $p_i = \mathbb{1}_{\mathcal{M}}$ and $q_0 = \mathbb{1}_{\mathcal{M}}$, $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, and $s_4 = H_{p \circ_i q}$.

By assembling these cases together, we retrieve the stated result. \square

Proposition 2.2.9. *Let \mathcal{M} be a unitary magma. The partial composition of $C\mathcal{M}$ expresses over the K-basis, for any \mathcal{M} -cliques p and q different from $\circ - \circ$ and any valid integer i , as*

$$K_p \circ_i K_q = \begin{cases} K_{p \circ_i q} & \text{if } p_i \star q_0 = \mathbb{1}_{\mathcal{M}}, \\ K_{p \circ_i q} + K_{d_i(p) \circ_i d_0(q)} & \text{otherwise.} \end{cases} \quad (2.2.20)$$

Proof. Let m be the arity of q . From the definition of the K -basis and of the partial order relation \leq_d , we have

$$\begin{aligned}
 K_p \circ_i K_q &= \sum_{\substack{p', q' \in \mathcal{G}_{\mathcal{M}} \\ p' \leq_d p \\ q' \leq_d q}} (-1)^{h(p', p) + h(q', q)} p' \circ_i q' \\
 &= \sum_{\substack{p', q' \in \mathcal{G}_{\mathcal{M}} \\ p' \circ_i q' \leq_d p \circ_i q \\ p'_i = p_i \\ q'_0 = q_0}} (-1)^{h(p', p) + h(q', q)} p' \circ_i q' \\
 &= \sum_{\substack{r \in \mathcal{G}_{\mathcal{M}} \\ r \leq_d p \circ_i q \\ r(i, i+m-1) = p_i \star q_0}} (-1)^{h(r, p \circ_i q)} r.
 \end{aligned} \tag{2.2.21}$$

When $p_i \star q_0 = \mathbb{1}_{\mathcal{M}}$, (2.2.21) is equal to $K_{p \circ_i q}$. Otherwise, when $p_i \star q_0 \neq \mathbb{1}_{\mathcal{M}}$, we have

$$\begin{aligned}
 \sum_{\substack{r \in \mathcal{G}_{\mathcal{M}} \\ r \leq_d p \circ_i q \\ r(i, i+m-1) = p_i \star q_0}} (-1)^{h(r, p \circ_i q)} r &= \sum_{\substack{r \in \mathcal{G}_{\mathcal{M}} \\ r \leq_d p \circ_i q}} (-1)^{h(r, p \circ_i q)} r - \sum_{\substack{r \in \mathcal{G}_{\mathcal{M}} \\ r \leq_d p \circ_i q \\ r(i, i+m-1) \neq p_i \star q_0}} (-1)^{h(r, p \circ_i q)} r \\
 &= K_{p \circ_i q} - \sum_{\substack{r \in \mathcal{G}_{\mathcal{M}} \\ r \leq_d d_i(p) \circ_i d_0(q)}} (-1)^{h(r, p \circ_i q)} r \\
 &= K_{p \circ_i q} - \sum_{\substack{r \in \mathcal{G}_{\mathcal{M}} \\ r \leq_d d_i(p) \circ_i d_0(q)}} (-1)^{1+h(r, d_i(p) \circ_i d_0(q))} r \\
 &= K_{p \circ_i q} + K_{d_i(p) \circ_i d_0(q)}.
 \end{aligned} \tag{2.2.22}$$

This proves the claimed formula for the partial composition of $C_{\mathcal{M}}$ over the K -basis. \square

For instance, in $C\mathbb{Z}$,

$$H \circ_2 H = H + 2H + H, \tag{2.2.23a}$$

$$K \circ_2 K = K + K, \tag{2.2.23b}$$

$$H \circ_3 H = H + H + H + H, \tag{2.2.23c}$$

$$K \circ_3 K = K + K. \tag{2.2.23d}$$

$$H \circ_2 H = H + 2H + H, \tag{2.2.23e}$$

$$\begin{array}{c} \text{K} \\ \diagdown \quad \diagup \\ -1 \quad 1 \\ \diagup \quad \diagdown \\ 2 \quad 1 \\ \diagdown \quad \diagup \\ \text{K} \end{array} \circ_2 \begin{array}{c} \text{K} \\ \diagdown \quad \diagup \\ -1 \quad 1 \\ \diagup \quad \diagdown \\ -1 \quad 1 \\ \diagdown \quad \diagup \\ \text{K} \end{array} = \text{K} \begin{array}{c} \diagdown \quad \diagup \\ -1 \quad 1 \\ \diagup \quad \diagdown \\ -1 \quad 1 \\ \diagdown \quad \diagup \\ -1 \quad 1 \\ \diagup \quad \diagdown \\ 2 \quad 1 \\ \diagdown \quad \diagup \\ \text{K} \end{array}, \quad (2.2.23f)$$

and in \mathbb{D}_1 ,

$$\begin{array}{c} \text{H} \\ \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ d_1 \quad 0 \\ \diagdown \quad \diagup \\ \text{H} \end{array} \circ_2 \begin{array}{c} \text{H} \\ \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ \text{H} \end{array} = 3 \text{H} \begin{array}{c} \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ d_1 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ \text{H} \end{array} + \text{H} \begin{array}{c} \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ d_1 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ \text{H} \end{array}, \quad (2.2.24a)$$

$$\begin{array}{c} \text{K} \\ \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ d_1 \quad 0 \\ \diagdown \quad \diagup \\ \text{K} \end{array} \circ_2 \begin{array}{c} \text{K} \\ \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ \text{K} \end{array} = \text{K} \begin{array}{c} \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ d_1 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ \text{K} \end{array} + \text{K} \begin{array}{c} \diagdown \quad \diagup \\ 0 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ d_1 \quad 0 \\ \diagup \quad \diagdown \\ 0 \quad 0 \\ \diagdown \quad \diagup \\ \text{K} \end{array}. \quad (2.2.24b)$$

2.2.8. Rational functions. The graded vector space of all commutative rational functions $\mathbb{K}(\mathbb{U})$, where \mathbb{U} is the infinite commutative alphabet $\{u_1, u_2, \dots\}$, has the structure of an operad RatFct introduced by Loday [Lod10] and defined as follows. Let $\text{RatFct}(n)$ be the subspace $\mathbb{K}(u_1, \dots, u_n)$ of $\mathbb{K}(\mathbb{U})$ and

$$\text{RatFct} := \bigoplus_{n \geq 1} \text{RatFct}(n). \quad (2.2.25)$$

Observe that since RatFct is a graded space, each rational function has an arity. Hence, by setting $f_1(u_1) := 1$ and $f_2(u_1, u_2) := 1$, f_1 is of arity 1 while f_2 is of arity 2, so that f_1 and f_2 are considered as different rational functions. The partial composition of two rational functions $f \in \text{RatFct}(n)$ and $g \in \text{RatFct}(m)$ satisfies

$$f \circ_i g := f(u_1, \dots, u_{i-1}, u_i + \dots + u_{i+m-1}, u_{i+m}, \dots, u_{n+m-1}) g(u_i, \dots, u_{i+m-1}). \quad (2.2.26)$$

The rational function f of $\text{RatFct}(1)$ defined by $f(u_1) := 1$ is the unit of RatFct . As shown by Loday, this operad is (nontrivially) isomorphic to the operad Mould introduced by Chapoton [Cha07].

Let us assume that \mathcal{M} is a \mathbb{Z} -graded unitary magma, that is a unitary magma such that there exists a unitary magma morphism $\theta : \mathcal{M} \rightarrow \mathbb{Z}$. We say that θ is a *rank function* of \mathcal{M} . In this context, let

$$F_\theta : \mathbb{C}\mathcal{M} \rightarrow \text{RatFct} \quad (2.2.27)$$

be the linear map defined, for any \mathcal{M} -clique \mathfrak{p} , by

$$F_\theta(\mathfrak{p}) := \prod_{(x,y) \in \mathcal{A}_\mathfrak{p}} (u_x + \dots + u_{y-1})^{\theta(\mathfrak{p}(x,y))}. \quad (2.2.28)$$

For instance, by considering the unitary magma \mathbb{Z} together with its identity map Id as rank function, one has

$$F_{\text{Id}} \left(\begin{array}{c} \diagdown \quad \diagup \\ -2 \quad 3 \\ \diagup \quad \diagdown \\ -1 \quad 2 \\ \diagdown \quad \diagup \\ -1 \quad 1 \\ \diagup \quad \diagdown \\ \text{K} \end{array} \right) = \frac{(u_1 + u_2 + u_3 + u_4)^2 (u_1 + u_2 + u_3 + u_4 + u_5 + u_6) u_4^3}{u_1 (u_3 + u_4 + u_5 + u_6)^2 (u_5 + u_6)}. \quad (2.2.29)$$

Theorem 2.2.10. *Let \mathcal{M} be a \mathbb{Z} -graded unitary magma and θ be a rank function of \mathcal{M} . The map F_θ is an operad morphism from $C\mathcal{M}$ to RatFct .*

Proof. To gain concision, for all positive integers $x < y$, we denote by $\mathbb{U}_{x,y}$ the sums $u_x + \dots + u_{y-1}$. Let \mathfrak{p} and \mathfrak{q} be two \mathcal{M} -cliques of respective arities n and m , and $i \in [n]$ be an integer. From the definition of the partial composition of $C\mathcal{M}$, the one (2.2.26) of RatFct , and the fact that θ is a unitary magma morphism, we have

$$\begin{aligned}
F_\theta(\mathfrak{p}) \circ_i F_\theta(\mathfrak{q}) &= (F_\theta(\mathfrak{p})) (u_1, \dots, u_{i-1}, \mathbb{U}_{i,i+m}, u_{i+m}, \dots, u_{n+m-1}) (F_\theta(\mathfrak{q})) (u_i, \dots, u_{i+m-1}) \\
&= \left(\prod_{1 \leq x < y \leq i-1} \mathbb{U}_{x,y}^{\theta(\mathfrak{p}(x,y))} \right) \left(\prod_{i+1 \leq x < y \leq n+1} \mathbb{U}_{x+m-1,y+m-1}^{\theta(\mathfrak{p}(x,y))} \right) \mathbb{U}_{i,i+m}^{\theta(\mathfrak{p}_i)} \\
&\quad \left(\prod_{1 \leq x < y \leq m+1} \mathbb{U}_{x+i-1,y+i-1}^{\theta(\mathfrak{q}(x,y))} \right) \\
&= \left(\prod_{1 \leq x < y \leq i-1} \mathbb{U}_{x,y}^{\theta(\mathfrak{p}(x,y))} \right) \left(\prod_{i+1 \leq x < y \leq n+1} \mathbb{U}_{x+m-1,y+m-1}^{\theta(\mathfrak{p}(x,y))} \right) \mathbb{U}_{i,i+m}^{\theta(\mathfrak{p}_i) + \theta(\mathfrak{q}_0)} \\
&\quad \left(\prod_{\substack{1 \leq x < y \leq m+1 \\ (x,y) \neq (1,m+1)}} \mathbb{U}_{x+i-1,y+i-1}^{\theta(\mathfrak{q}(x,y))} \right) \\
&= \left(\prod_{1 \leq x < y \leq i-1} \mathbb{U}_{x,y}^{\theta(\mathfrak{p}(x,y))} \right) \left(\prod_{i+1 \leq x < y \leq n+1} \mathbb{U}_{x+m-1,y+m-1}^{\theta(\mathfrak{p}(x,y))} \right) \mathbb{U}_{i,i+m}^{\theta(\mathfrak{p}_i * \mathfrak{q}_0)} \\
&\quad \left(\prod_{\substack{1 \leq x < y \leq m+1 \\ (x,y) \neq (1,m+1)}} \mathbb{U}_{x+i-1,y+i-1}^{\theta(\mathfrak{q}(x,y))} \right) \\
&= \prod_{(x,y) \in \mathcal{A}_{\mathfrak{p} \circ_i \mathfrak{q}}} \mathbb{U}_{x,y}^{\theta((\mathfrak{p} \circ_i \mathfrak{q})(x,y))} \\
&= F_\theta(\mathfrak{p} \circ_i \mathfrak{q}).
\end{aligned} \tag{2.2.30}$$

Moreover, since $\theta(\mathbb{1}_{\mathcal{M}}) = 0$, one has $F_\theta(\bullet \dashv \bullet) = 1$, so that F_θ sends the unit of $C\mathcal{M}$ to the unit of RatFct . Therefore, F_θ is an operad morphism. \square

The operad morphism F_θ is not injective. Indeed, by considering the magma \mathbb{Z} together with its identity map Id as rank function, one one has for instance

$$F_{\text{Id}} \left(\begin{array}{c} \triangle \\ \dashv \\ \triangle \\ \dashv \\ \triangle \end{array} \right) = (u_1 + u_2) - u_1 - u_2 = 0, \tag{2.2.31a}$$

$$F_{\text{Id}} \left(\begin{array}{c} \square \\ \dashv \\ \square \\ \dashv \\ \square \end{array} \right) = \frac{1}{u_2 u_3} - \frac{1}{(u_2 + u_3) u_3} - \frac{1}{u_2 (u_2 + u_3)} = 0. \tag{2.2.31b}$$

Proposition 2.2.11. *The subspace of RatFct of all Laurent polynomials on \mathbb{U} is the image by $F_{\text{Id}} : \text{C}\mathbb{Z} \rightarrow \text{RatFct}$ of the subspace of $\text{C}\mathbb{Z}$ consisting in the linear span of all \mathbb{Z} -bubbles.*

Proof. First, by Theorem 2.2.10, F_{Id} is a well-defined operad morphism from $\text{C}\mathbb{Z}$ to RatFct. Let $u_1^{\alpha_1} \dots u_n^{\alpha_n}$ be a Laurent monomial, where $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ and $n \geq 1$. Consider also the \mathbb{Z} -clique p_α of arity $n + 1$ satisfying

$$p_\alpha(x, y) := \begin{cases} \alpha_x & \text{if } y = x + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2.32)$$

Observe that p_α is a \mathbb{Z} -bubble. By definition of F_{Id} , we have $F_{\text{Id}}(p_\alpha) = u_1^{\alpha_1} \dots u_n^{\alpha_n}$. Now, since a Laurent polynomial is a linear combination of some Laurent monomials, by the linearity of F_{Id} , the statement of the proposition follows. \square

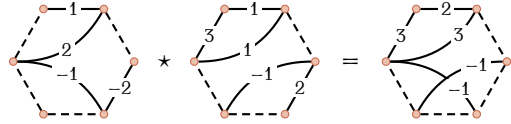
On each homogeneous subspace $\text{C}\mathcal{M}(n)$ of the elements of arity $n \geq 1$ of $\text{C}\mathcal{M}$, let the product

$$\star : \text{C}\mathcal{M}(n) \otimes \text{C}\mathcal{M}(n) \rightarrow \text{C}\mathcal{M}(n) \quad (2.2.33)$$

defined linearly, for each \mathcal{M} -cliques p and q of $\text{C}\mathcal{M}(n)$, by

$$(p \star q)(x, y) := p(x, y) \star q(x, y), \quad (2.2.34)$$

where (x, y) is any arc such that $1 \leq x < y \leq n + 1$. For instance, in $\text{C}\mathbb{Z}$,



$$\quad \quad \quad \star \quad \quad \quad = \quad \quad \quad (2.2.35)$$

Proposition 2.2.12. *Let \mathcal{M} be a \mathbb{Z} -graded unitary magma and θ be a rank function of \mathcal{M} . For any homogeneous elements f and g of $\text{C}\mathcal{M}$ of the same arity,*

$$F_\theta(f)F_\theta(g) = F_\theta(f \star g). \quad (2.2.36)$$

Proof. Let p and q be two \mathcal{M} -cliques of $\text{C}\mathcal{M}$ of arity n . By definition of the operation \star on $\text{C}\mathcal{M}(n)$ and the fact that θ is a unitary magma morphism,

$$\begin{aligned} F_\theta(p)F_\theta(q) &= \left(\prod_{(x,y) \in \mathcal{A}_p} (u_x + \dots + u_{y-1})^{\theta(p(x,y))} \right) \left(\prod_{(x,y) \in \mathcal{A}_q} (u_x + \dots + u_{y-1})^{\theta(q(x,y))} \right) \\ &= \prod_{1 \leq x < y \leq n+1} (u_x + \dots + u_{y-1})^{\theta(p(x,y)) + \theta(q(x,y))} \\ &= \prod_{1 \leq x < y \leq n+1} (u_x + \dots + u_{y-1})^{\theta(p(x,y) \star q(x,y))} \\ &= F_\theta(p \star q). \end{aligned} \quad (2.2.37)$$

By the linearity of F_θ and of \star , (2.2.36) follows. \square

Proposition 2.2.13. *Let \mathfrak{p} be an \mathcal{M} -clique of $C\mathbb{Z}$. Then,*

$$\frac{1}{F_{\text{Id}}(\mathfrak{p})} = F_{\text{Id}}((C\eta)(\mathfrak{p})), \quad (2.2.38)$$

where $\eta : \mathbb{Z} \rightarrow \mathbb{Z}$ is the unitary magma morphism defined by $\eta(x) := -x$ for all $x \in \mathbb{Z}$.

Proof. Observe that $(C\eta)(\mathfrak{p})$ is the \mathcal{M} -clique obtained by relabeling each arc (x, y) of \mathfrak{p} by $-\mathfrak{p}(x, y)$. Hence, since η is a unitary magma morphism, we have

$$\begin{aligned} F_{\text{Id}}((C\eta)(\mathfrak{p})) &= \prod_{(x,y) \in \mathcal{A}_{\mathfrak{p}}} (\mathbf{u}_x + \cdots + \mathbf{u}_{y-1})^{\theta(-\mathfrak{p}(x,y))} \\ &= \prod_{(x,y) \in \mathcal{A}_{\mathfrak{p}}} (\mathbf{u}_x + \cdots + \mathbf{u}_{y-1})^{-\theta(\mathfrak{p}(x,y))} \\ &= \frac{1}{F_{\text{Id}}(\mathfrak{p})} \end{aligned} \quad (2.2.39)$$

as expected. \square

3. QUOTIENTS AND SUBOPERADS

We define here quotients and suboperads of $C\mathcal{M}$, leading to the construction of some new operads involving various combinatorial objects which are, basically, \mathcal{M} -cliques with some restrictions.

3.1. Main substructures. Most of the natural subfamilies of \mathcal{M} -cliques that can be described by simple combinatorial properties as \mathcal{M} -cliques with restrained labels for the bases, edges, and diagonals, white \mathcal{M} -cliques, \mathcal{M} -cliques with a fixed maximal value for their crossings, \mathcal{M} -bubbles, \mathcal{M} -cliques with a fixed maximal value for their degrees, inclusion-free \mathcal{M} -cliques, and acyclic \mathcal{M} -cliques inherit from the algebraic structure of operad of $C\mathcal{M}$ and form quotients and suboperads of $C\mathcal{M}$. We construct and briefly study here these main substructures of $C\mathcal{M}$.

3.1.1. Restricting the labels. In what follows, if X and Y are two subsets of \mathcal{M} , $X \star Y$ denotes the set $\{x \star y : x \in X \text{ and } y \in Y\}$.

Let B, E , and D be three subsets of \mathcal{M} and $\text{Lab}_{B,E,D}\mathcal{M}$ be the subspace of $C\mathcal{M}$ generated by all \mathcal{M} -cliques \mathfrak{p} such that the bases of \mathfrak{p} are labeled on B , all edges of \mathfrak{p} are labeled on E , and all diagonals of \mathfrak{p} are labeled on D .

Proposition 3.1.1. *Let \mathcal{M} be a unitary magma and B, E , and D be three subsets of \mathcal{M} . If $\mathbf{1}_{\mathcal{M}} \in B$, $\mathbf{1}_{\mathcal{M}} \in D$, and $E \star B \subseteq D$, $\text{Lab}_{B,E,D}\mathcal{M}$ is a suboperad of $C\mathcal{M}$.*

Proof. First, since $\mathbf{1}_{\mathcal{M}} \in B$, the unit $\circ \dashv \circ$ of $C\mathcal{M}$ belongs to $\text{Lab}_{B,E,D}\mathcal{M}$. Consider now two \mathcal{M} -cliques \mathfrak{p} and \mathfrak{q} of $\text{Lab}_{B,E,D}\mathcal{M}$ and a partial composition $\tau := \mathfrak{p} \circ_i \mathfrak{q}$ for a valid integer i . By the definition of the partial composition of $C\mathcal{M}$, the base of τ has the same label as the base of \mathfrak{p} , and all edges of τ have labels coming from the ones of \mathfrak{p} and \mathfrak{q} . Moreover, all diagonals of τ are either non-solid, or come from diagonals of \mathfrak{p} and \mathfrak{q} , or are the diagonal $\tau(i, i + |\mathfrak{q}|)$ which is labeled by $\mathfrak{p}_i \star \mathfrak{q}_0$. Since $\mathbf{1}_{\mathcal{M}} \in D$, $\mathfrak{p}_i \in E$, $\mathfrak{q}_0 \in B$, and $E \star B \subseteq D$, all

the labels of these diagonals are in D . For these reasons, τ is in $\text{Lab}_{B,E,D}\mathcal{M}$. Whence the statement of the proposition. \square

Proposition 3.1.2. *Let \mathcal{M} be a unitary magma and B, E , and D be three finite subsets of \mathcal{M} . For all $n \geq 2$,*

$$\dim \text{Lab}_{B,E,D}\mathcal{M}(n) = be^n d^{(n+1)(n-2)/2}, \quad (3.1.1)$$

where $b := \#B$, $e := \#E$, and $d := \#D$.

Proof. By Proposition 2.2.1, there are $m^{\binom{n+1}{2}}$ \mathcal{M} -cliques of arity n , where $m := \#\mathcal{M}$. Hence, there are $m^{\binom{n+1}{2}}/m^{n+1}$ \mathcal{M} -cliques of arity n with all edges and the base labeled by $\mathbb{1}_{\mathcal{M}}$. This also says that there are $d^{\binom{n+1}{2}}/d^{n+1}$ \mathcal{M} -cliques of arity n with all diagonals labeled on D and all edges and the base labeled by $\mathbb{1}_{\mathcal{M}}$. Since an \mathcal{M} -clique of $\text{Lab}_{B,E,D}\mathcal{M}(n)$ have its n edges labeled on E and its base labeled on B , (3.1.1) follows. \square

3.1.2. *White cliques.* Let $\text{Whi}\mathcal{M}$ be the subspace of $\text{C}\mathcal{M}$ generated by all white \mathcal{M} -cliques. Since, by definition of white \mathcal{M} -cliques,

$$\text{Whi}\mathcal{M} = \text{Lab}_{\{\mathbb{1}_{\mathcal{M}}\}, \{\mathbb{1}_{\mathcal{M}}\}, \mathcal{M}\mathcal{M}}, \quad (3.1.2)$$

by Proposition 3.1.1, $\text{Whi}\mathcal{M}$ is a suboperad of $\text{C}\mathcal{M}$. It follows from Proposition 3.1.2 that when \mathcal{M} is finite, the dimensions of $\text{Whi}\mathcal{M}$ satisfy, for any $n \geq 2$,

$$\dim \text{Whi}\mathcal{M}(n) = m^{(n+1)(n-2)/2}, \quad (3.1.3)$$

where $m := \#\mathcal{M}$.

3.1.3. *Restricting the crossing.* Let $k \geq 0$ be an integer and $\mathfrak{R}_{\text{Cro}_k\mathcal{M}}$ be the subspace of $\text{C}\mathcal{M}$ generated by all \mathcal{M} -cliques p such that $\text{cros}(p) \geq k + 1$. As a quotient of graded vector spaces,

$$\text{Cro}_k\mathcal{M} := \text{C}\mathcal{M} / \mathfrak{R}_{\text{Cro}_k\mathcal{M}} \quad (3.1.4)$$

is the linear span of all \mathcal{M} -cliques p such that $\text{cros}(p) \leq k$.

Proposition 3.1.3. *Let \mathcal{M} be a unitary magma and $k \geq 0$ be an integer. Then, the space $\text{Cro}_k\mathcal{M}$ is both a quotient and a suboperad of $\text{C}\mathcal{M}$.*

Proof. Let us first prove that $\text{Cro}_k\mathcal{M}$ is a quotient of $\text{C}\mathcal{M}$. For this, observe that if p and q are two \mathcal{M} -cliques,

$$\text{cros}(p \circ_i q) = \max\{\text{cros}(p), \text{cros}(q)\} \quad (3.1.5)$$

for any valid integer i . For this reason, if p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Cro}_k\mathcal{M}}$, each clique obtained by a partial composition involving p and other \mathcal{M} -cliques is still in $\mathfrak{R}_{\text{Cro}_k\mathcal{M}}$. This proves that $\mathfrak{R}_{\text{Cro}_k\mathcal{M}}$ is an operad ideal of $\text{C}\mathcal{M}$ and hence, that $\text{Cro}_k\mathcal{M}$ is a quotient of $\text{C}\mathcal{M}$.

To prove that $\text{Cro}_k\mathcal{M}$ is also a suboperad of $\text{C}\mathcal{M}$, consider two \mathcal{M} -cliques p and q of $\text{Cro}_k\mathcal{M}$. By (3.1.5), all \mathcal{M} -cliques $p \circ_i q$ are still in $\text{Cro}_k\mathcal{M}$, for all valid integer i . Moreover, the unit $\bullet \circ \bullet$ of $\text{C}\mathcal{M}$ belongs to $\text{Cro}_k\mathcal{M}$. This imply that $\text{Cro}_k\mathcal{M}$ is a suboperad of $\text{C}\mathcal{M}$. \square

For instance, in the operad $\text{Cro}_2\mathbb{Z}$, we have

$$(3.1.6)$$

When $0 \leq k' \leq k$ are integers, by Proposition 3.1.3, $\text{Cro}_k\mathcal{M}$ and $\text{Cro}_{k'}\mathcal{M}$ are both quotients and suboperads of $\text{C}\mathcal{M}$. First, since any \mathcal{M} -clique of $\text{Cro}_{k'}\mathcal{M}$ is also an \mathcal{M} -clique of $\text{Cro}_k\mathcal{M}$, $\text{Cro}_{k'}\mathcal{M}$ is a suboperad of $\text{Cro}_k\mathcal{M}$. Second, since $\mathfrak{R}_{\text{Cro}_k\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\text{Cro}_{k'}\mathcal{M}}$, $\text{Cro}_{k'}\mathcal{M}$ is a quotient of $\text{Cro}_k\mathcal{M}$.

Remark that $\text{Cro}_0\mathcal{M}$ is the linear span of all noncrossing \mathcal{M} -cliques. We can see these objects as noncrossing configurations [FN99] where the edges and bases are colored by elements of \mathcal{M} and the diagonals, by elements of $\bar{\mathcal{M}}$. The operad $\text{Cro}_0\mathcal{M}$ has a lot of properties and will be studied in details in Section 4.

3.1.4. *Bubbles.* Let $\mathfrak{R}_{\text{Bub}\mathcal{M}}$ be the subspace of $\text{C}\mathcal{M}$ generated by all \mathcal{M} -cliques that are not bubbles. As a quotient of graded vector spaces,

$$\text{Bub}\mathcal{M} := \text{C}\mathcal{M} / \mathfrak{R}_{\text{Bub}\mathcal{M}} \quad (3.1.7)$$

is the linear span of all \mathcal{M} -bubbles.

Proposition 3.1.4. *Let \mathcal{M} be a unitary magma. Then, the space $\text{Bub}\mathcal{M}$ is a quotient operad of $\text{C}\mathcal{M}$.*

Proof. If p and q are two \mathcal{M} -cliques, all solid diagonals of p and q appears in $p \circ_i q$, for any valid integer i . For this reason, if p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Bub}\mathcal{M}}$, each \mathcal{M} -clique obtained by a partial composition involving p and other \mathcal{M} -cliques is still in $\mathfrak{R}_{\text{Bub}\mathcal{M}}$. This proves that $\mathfrak{R}_{\text{Bub}\mathcal{M}}$ is an operad ideal of $\text{C}\mathcal{M}$ and implies the statement of the proposition. \square

For instance, in the operad $\text{Bub}\mathbb{Z}$, we have

$$(3.1.8a)$$

$$(3.1.8c)$$

$$(3.1.8b)$$

$$(3.1.8d)$$

When \mathcal{M} is finite, the dimensions of $\text{Bub}\mathcal{M}$ satisfy, for any $n \geq 2$,

$$\dim \text{Bub}\mathcal{M}(n) = m^{n+1}, \quad (3.1.9)$$

where $m := \#\mathcal{M}$.

3.1.5. *Restricting the degrees.* Let $k \geq 0$ be an integer and $\mathfrak{R}_{\text{Deg}_k \mathcal{M}}$ be the subspace of $C\mathcal{M}$ generated by all \mathcal{M} -cliques p such that $\text{degr}(p) \geq k + 1$. As a quotient of graded vector spaces,

$$\text{Deg}_k \mathcal{M} := C\mathcal{M} / \mathfrak{R}_{\text{Deg}_k \mathcal{M}} \quad (3.1.10)$$

is the linear span of all \mathcal{M} -cliques p such that $\text{degr}(p) \leq k$.

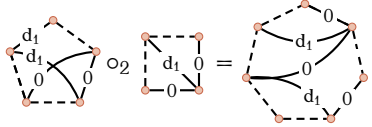
Proposition 3.1.5. *Let \mathcal{M} be a unitary magma without nontrivial unit divisors and $k \geq 0$ be an integer. Then, the space $\text{Deg}_k \mathcal{M}$ is a quotient operad of $C\mathcal{M}$.*

Proof. Since \mathcal{M} has no nontrivial unit divisors, for any \mathcal{M} -cliques p and q of $C\mathcal{M}$, each solid arc of p (resp. q) gives rise to a solid arc in $p \circ_i q$, for any valid integer i . Hence,

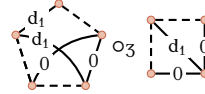
$$\text{degr}(p \circ_i q) \geq \max\{\text{degr}(p), \text{degr}(q)\}, \quad (3.1.11)$$

and then, if p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Deg}_k \mathcal{M}}$, each \mathcal{M} -clique obtained by a partial composition involving p and other \mathcal{M} -cliques is still in $\mathfrak{R}_{\text{Deg}_k \mathcal{M}}$. This proves that $\mathfrak{R}_{\text{Deg}_k \mathcal{M}}$ is an operad ideal of $C\mathcal{M}$ and implies the statement of the proposition. \square

For instance, in the operad $\text{Deg}_3 \mathbb{D}_2$ (observe that \mathbb{D}_2 is a unitary magma without nontrivial unit divisors), we have



$$(3.1.12a)$$

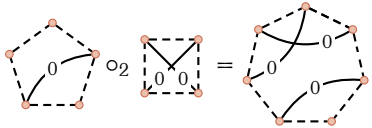


$$(3.1.12b)$$


When $0 \leq k' \leq k$ are integers, by Proposition 3.1.5, $\text{Deg}_k \mathcal{M}$ and $\text{Deg}_{k'} \mathcal{M}$ are both quotient operads of $C\mathcal{M}$. Moreover, since $\mathfrak{R}_{\text{Deg}_k \mathcal{M}}$ is a subspace of $\mathfrak{R}_{\text{Deg}_{k'} \mathcal{M}}$, $\text{Deg}_{k'} \mathcal{M}$ is a quotient operad of $\text{Deg}_k \mathcal{M}$.

Observe that $\text{Deg}_0 \mathcal{M}$ is the linear span of all \mathcal{M} -cliques without solid arcs. If p and q are such \mathcal{M} -cliques, all partial compositions $p \circ_i q$ are equal to the unique \mathcal{M} -clique without solid arcs of arity $|p| + |q| - 1$. For this reason, $\text{Deg}_0 \mathcal{M}$ is the associative operad As .

Any skeleton of an \mathcal{M} -clique of arity n of $\text{Deg}_1 \mathcal{M}$ can be seen as a partition of the set $[n + 1]$ in singletons or pairs. Therefore, $\text{Deg}_1 \mathcal{M}$ can be seen as an operad on such colored partitions, where each pair of the partitions have one color among the set \mathcal{M} . In the operad $\text{Deg}_1 \mathbb{D}_0$ (observe that \mathbb{D}_0 is the only unitary magma without nontrivial unit divisors on two elements), one has for instance



$$(3.1.13a)$$



$$(3.1.13b)$$

By seeing each solid arc (x, y) of an \mathcal{M} -clique p of $\text{Deg}_1 \mathbb{D}_0$ of arity n as the transposition exchanging the letter x and the letter y , we can interpret p as an involution of \mathfrak{S}_{n+1} made of the product of these transpositions. Hence, $\text{Deg}_1 \mathbb{D}_0$ can be seen as an operad

on involutions. Under this point of view, the partial compositions (3.1.13a) and (3.1.13b) translate on permutations as

$$42315 \circ_2 3412 = 6452317, \quad (3.1.14a)$$

$$42315 \circ_3 3412 = 0. \quad (3.1.14b)$$

Equivalently, by the Robinson-Schensted correspondence (see for instance [Lot02]), $\text{Deg}_1 \mathbb{D}_0$ is an operad of standard Young tableaux. The dimensions of $\text{Deg}_1 \mathbb{D}_0$ operad begin by

$$1, 4, 10, 26, 76, 232, 764, 2620, \quad (3.1.15)$$

and form, except for the first terms, Sequence **A000085** of [Slo]. Moreover, when $\#\mathcal{M} = 3$, the dimensions of $\text{Deg}_1 \mathcal{M}$ begin by

$$1, 7, 25, 81, 331, 1303, 5937, 26785, \quad (3.1.16)$$

and form, except for the first terms, Sequence **A047974** of [Slo].

Besides, any skeleton of an \mathcal{M} -clique of $\text{Deg}_2 \mathcal{M}$ can be seen as a *thunderstorm graph*, i.e., a graph where connected components are cycles or paths. Therefore, $\text{Deg}_2 \mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color among the set $\bar{\mathcal{M}}$. When $\#\mathcal{M} = 2$, the dimensions of this operad begin by

$$1, 8, 41, 253, 1858, 15796, 152219, 1638323, \quad (3.1.17)$$

and form, except for the first terms, Sequence **A136281** of [Slo].

3.1.6. Inclusion-free cliques. Let $\mathfrak{R}_{\text{Inf} \mathcal{M}}$ be the subspace of $C\mathcal{M}$ generated by all \mathcal{M} -cliques that are not inclusion-free. As a quotient of graded vector spaces,

$$\text{Inf} \mathcal{M} := C\mathcal{M} / \mathfrak{R}_{\text{Inf} \mathcal{M}} \quad (3.1.18)$$

is the linear span of all inclusion-free \mathcal{M} -cliques.

Proposition 3.1.6. *Let \mathcal{M} be a unitary magma without nontrivial unit divisors. Then, the space $\text{Inf} \mathcal{M}$ is a quotient operad of $C\mathcal{M}$.*

Proof. Since \mathcal{M} has no nontrivial unit divisors, for any \mathcal{M} -cliques p and q of $C\mathcal{M}$, each solid arc of p (resp. q) gives rise to a solid arc in $p \circ_i q$, for any valid integer i . For this reason, if p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Inf} \mathcal{M}}$, p is not inclusion-free and each \mathcal{M} -clique obtained by a partial composition involving p and other \mathcal{M} -cliques is still not inclusion-free and thus, belongs to $\mathfrak{R}_{\text{Inf} \mathcal{M}}$. This proves that $\mathfrak{R}_{\text{Inf} \mathcal{M}}$ is an operad ideal of $C\mathcal{M}$ and implies the statement of the proposition. \square

For instance, in the operad $\text{Inf} \mathbb{D}_2$,

$$(3.1.19a)$$

$$(3.1.19b)$$

Recall that a *Dyck path* of *size* n is a word u of $\{a, b\}^{2n}$ such that $|u|_a = |u|_b$ and, for each prefix v of u , $|v|_a \geq |v|_b$.

Lemma 3.1.7. *Let \mathcal{M} be a finite unitary magma without nontrivial unit divisors. For all $n \geq 2$, the set of all \mathcal{M} -cliques of $\text{Inf } \mathcal{M}(n)$ is in one-to-one correspondence with the set of all Dyck paths of size $n + 1$ wherein letters a at even positions are colored on $\bar{\mathcal{M}}$. Moreover, there is a correspondence between these two sets that sends any \mathcal{M} -clique of $\text{Inf } \mathcal{M}(n)$ with k solid edges to a Dyck path with exactly k letters a at even positions, for any $0 \leq k \leq n$.*

Proof. In this proof, we denote by a_c the letter a of a Dyck path colored by $c \in \bar{\mathcal{M}}$. Given an \mathcal{M} -clique p of $\text{Inf } \mathcal{M}(n)$, we decorate each vertex x of p by

- (1) aa_c if x has one outgoing arc and no incoming arc, where c is the label of the outgoing arc from x ;
- (2) bb if x has no outgoing arc and one incoming arc;
- (3) ba_c if x has both one outgoing arc and one incoming arc, where c is the label of the outgoing arc from x ;
- (4) ab otherwise.

Let ϕ be the map sending p to the word obtained by concatenating the decorations of the vertices of p thus described, read from 1 to $n + 1$.

We show that ϕ is a bijection between the two sets of the statement of the lemma. First, observe that since p is inclusion-free, for each vertex y of p , there is at most one incoming arc to y and one outgoing arc from y . For this reason, for any vertex y of p , the total number of incoming arcs to vertices $x \leq y$ of p is smaller than or equal to the total number of outgoing arcs to vertices $x \leq y$ of p , and the total number of vertices having an incoming arc is equal to the total number of vertices having an outgoing arc in p . Thus, by forgetting the colorations of its letters, the word $\phi(p)$ is a Dyck path.

Besides, given a Dyck path u of size $n + 1$ wherein letters a at even positions are colored on $\bar{\mathcal{M}}$, one can build a unique \mathcal{M} -clique p of $\text{Inf } \mathcal{M}(n)$ such that $\phi(p) = u$. Indeed, by reading the letters of u two by two, one knows the number of outgoing and incoming arcs for each vertex of p . Since p is inclusion-free, there is one unique way to connect these vertices by solid diagonals without creating inclusions of arcs. Moreover, by (1), (2), (3), and (4), the colors of the letters a at even positions allow to label the solid arcs of p . Hence ϕ is a bijection as claimed.

Finally, by definition of ϕ , we observe that if p has exactly k solid arcs, the Dyck path $\phi(p)$ has exactly k occurrences of the letter a at even positions, whence the whole statement of the lemma. \square

Let $\text{nar}(n, k)$ be the *Narayana number* [Nar55] defined for all $0 \leq k \leq n - 2$ by

$$\text{nar}(n, k) := \frac{1}{k+1} \binom{n-2}{k} \binom{n-1}{k}. \quad (3.1.20)$$

The number of Dyck paths of size $n - 1$ and exactly k occurrences of the factor ab is $\text{nar}(n, k)$. Equivalently, this is also the number of binary trees with n leaves and exactly k internal nodes having an internal node as a left child.

Proposition 3.1.8. *Let \mathcal{M} be a finite unitary magma without nontrivial unit divisors. For all $n \geq 2$,*

$$\dim \text{Inf } \mathcal{M}(n) = \sum_{0 \leq k \leq n} (m - 1)^k \text{nar}(n + 2, k), \quad (3.1.21)$$

where $m := \#\mathcal{M}$.

Proof. It is known from [Sul98] that the number of Dyck paths of size $n + 1$ with k occurrences of the letter a at even positions is the Narayana number $\text{nar}(n + 2, k)$. Hence, by using this property together with Lemma 3.1.7, we obtain that the number of inclusion-free \mathcal{M} -cliques of size n with k solid arcs is $(m - 1)^k \text{nar}(n + 2, k)$. Therefore, since an inclusion-free \mathcal{M} -clique of arity n can have at most n solid arcs, (3.1.21) holds. \square

The skeletons of the \mathcal{M} -cliques of $\text{Inf } \mathcal{M}$ of arities greater than 1 are the graphs such that, if $\{x, y\}$ and $\{x', y'\}$ are two arcs such that $x \leq x' < y' \leq y$, then $x = x'$ and $y = y'$. Therefore, $\text{Inf } \mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color among the set $\bar{\mathcal{M}}$. Equivalently, as Lemma 3.1.7 shows, $\text{Inf } \mathcal{M}$ can be seen as an operad of Dyck paths where letters a at even positions are colored on $\bar{\mathcal{M}}$.

By Proposition 3.1.8, when $\#\mathcal{M} = 2$, the dimensions of $\text{Inf } \mathcal{M}$ begin by

$$1, 5, 14, 42, 132, 429, 1430, 4862, \quad (3.1.22)$$

and form, except for the first terms, Sequence **A000108** of [Slo]. When $\#\mathcal{M} = 3$, the dimensions of $\text{Inf } \mathcal{M}$ begin by

$$1, 11, 45, 197, 903, 4279, 20793, 103049, \quad (3.1.23)$$

and form, except for the first terms, Sequence **A001003** of [Slo]. When $\#\mathcal{M} = 4$, the dimensions of $\text{Inf } \mathcal{M}$ begin by

$$1, 19, 100, 562, 3304, 20071, 124996, 793774, \quad (3.1.24)$$

and form, except for the first terms, Sequence **A007564** of [Slo].

3.1.7. Acyclic decorated cliques. Let $\mathfrak{A}_{\text{Acy } \mathcal{M}}$ be the subspace of $C\mathcal{M}$ generated by all \mathcal{M} -cliques that are not acyclic. As a quotient of graded vector spaces,

$$\text{Acy } \mathcal{M} := C\mathcal{M} / \mathfrak{A}_{\text{Acy } \mathcal{M}} \quad (3.1.25)$$

is the linear span of all acyclic \mathcal{M} -cliques.

Proposition 3.1.9. *Let \mathcal{M} be a unitary magma without nontrivial unit divisors. Then, the space $\text{Acy } \mathcal{M}$ is a quotient operad of $C\mathcal{M}$.*

Proof. Since \mathcal{M} has no nontrivial unit divisors, for any \mathcal{M} -cliques p and q of $C\mathcal{M}$, each solid arc of p (resp. q) gives rise to a solid arc in $p \circ_i q$, for any valid integer i . For this reason, if p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Acy}\mathcal{M}}$, p is not acyclic and each \mathcal{M} -clique obtained by a partial composition involving p and other \mathcal{M} -cliques is still not acyclic and thus, belongs to $\mathfrak{R}_{\text{Acy}\mathcal{M}}$. This proves that $\mathfrak{R}_{\text{Acy}\mathcal{M}}$ is an operad ideal of $C\mathcal{M}$ and implies the statement of the proposition. \square

For instance, in the operad $\text{Acy}\mathbb{D}_2$,

$$(3.1.26a)$$

$$(3.1.26b)$$

The skeletons of the \mathcal{M} -cliques of $\text{Acy}\mathcal{M}$ of arities greater than 1 are acyclic graphs or equivalently, forest of non-rooted trees. Therefore, $\text{Acy}\mathcal{M}$ can be seen as an operad on colored forests of trees, where the edges of the trees of the forests have one color among the set $\bar{\mathcal{M}}$. When $\#\mathcal{M} = 2$, the dimensions of $\text{Acy}\mathcal{M}$ begin by

$$1, 7, 38, 291, 2932, 36961, 561948, 10026505, \quad (3.1.27)$$

and form, except for the first terms, Sequence [A001858](#) of [Slo].

3.2. Secondary substructures. Some more substructures of $C\mathcal{M}$ are constructed and briefly studied here. They are constructed by mixing some of the constructions of the seven main substructures of $C\mathcal{M}$ defined in Section 3.1 in the following sense.

For any operad \mathcal{O} and operad ideals \mathfrak{R}_1 and \mathfrak{R}_2 of \mathcal{O} , the space $\mathfrak{R}_1 + \mathfrak{R}_2$ is still an operad ideal of \mathcal{O} , and $\mathcal{O}/_{\mathfrak{R}_1 + \mathfrak{R}_2}$ is a quotient of both $\mathcal{O}/_{\mathfrak{R}_1}$ and $\mathcal{O}/_{\mathfrak{R}_2}$. Moreover, if \mathcal{O}' is a suboperad of \mathcal{O} and \mathfrak{R} is an operad ideal of \mathcal{O} , the space $\mathfrak{R} \cap \mathcal{O}'$ is an operad ideal of \mathcal{O}' , and $\mathcal{O}'/_{\mathfrak{R} \cap \mathcal{O}'}$ is a quotient of \mathcal{O}' and a suboperad of $\mathcal{O}/_{\mathfrak{R}}$. For these reasons (straightforwardly provable), we can combine the constructions of the previous section to build a bunch of new suboperads and quotients of $C\mathcal{M}$.

3.2.1. Colored white noncrossing configurations. When \mathcal{M} is a unitary magma, let

$$\text{WNC}\mathcal{M} := \text{Whi}\mathcal{M}/_{\mathfrak{R}_{\text{Croc}}\mathcal{M} \cap \text{Whi}\mathcal{M}}. \quad (3.2.1)$$

The \mathcal{M} -cliques of $\text{WNC}\mathcal{M}$ are white noncrossing \mathcal{M} -cliques.

Proposition 3.2.1. *Let \mathcal{M} be a finite unitary magma. For all $n \geq 2$,*

$$\dim \text{WNC}\mathcal{M}(n) = \sum_{0 \leq k \leq n-2} m^k (m-1)^{n-k-2} \text{nar}(n, k), \quad (3.2.2)$$

where $m := \#\mathcal{M}$.

Proof. From its definition, $\text{WNC}\mathcal{M}$ can be seen as the suboperad of $\text{Cro}_0\mathcal{M}$ restricted on the linear span of all white noncrossing \mathcal{M} -cliques. For this reason,

$$\dim \text{WNC}\mathcal{M}(n) = \frac{1}{m^{n+1}} \dim \text{Cro}_0\mathcal{M}(n). \quad (3.2.3)$$

By using the upcoming Proposition 4.1.7 for an expression for $\dim \text{Cro}_0\mathcal{M}(n)$, we obtain the stated result. \square

When $\#\mathcal{M} = 2$, the dimensions of $\text{WNC}\mathcal{M}$ begin by

$$1, 1, 3, 11, 45, 197, 903, 4279, \quad (3.2.4)$$

and form Sequence **A001003** of [Slo]. When $\#\mathcal{M} = 3$, the dimensions of $\text{WNC}\mathcal{M}$ begin by

$$1, 1, 5, 31, 215, 1597, 12425, 99955, \quad (3.2.5)$$

and form Sequence **A269730** of [Slo]. Observe that these dimensions are shifted versions the ones of the γ -polytridendriform operads TDendr_γ [Gir16b] with $\gamma := \#\mathcal{M} - 1$.

3.2.2. Colored forests of paths. When \mathcal{M} is a unitary magma without nontrivial unit divisors, let

$$\text{Pat}\mathcal{M} := \mathcal{C}\mathcal{M} / \mathfrak{R}_{\text{Deg}_2\mathcal{M}} + \mathfrak{R}_{\text{Acy}\mathcal{M}}. \quad (3.2.6)$$

The skeletons of the \mathcal{M} -cliques of $\text{Pat}\mathcal{M}$ are forests of non-rooted trees that are paths. Therefore, $\text{Pat}\mathcal{M}$ can be seen as an operad on colored such graphs, where the arcs of the graphs have one color among the set $\bar{\mathcal{M}}$.

When $\#\mathcal{M} = 2$, the dimensions of $\text{Pat}\mathcal{M}$ begin by

$$1, 7, 34, 206, 1486, 12412, 117692, 1248004, \quad (3.2.7)$$

an form, except for the first terms, Sequence **A011800** of [Slo].

3.2.3. Colored forests. When \mathcal{M} is a unitary magma without nontrivial unit divisors, let

$$\text{For}\mathcal{M} := \mathcal{C}\mathcal{M} / \mathfrak{R}_{\text{Cro}_0\mathcal{M}} + \mathfrak{R}_{\text{Acy}\mathcal{M}}. \quad (3.2.8)$$

The skeletons of the \mathcal{M} -cliques of $\text{For}\mathcal{M}$ are forests of rooted trees having no arcs $\{x, y\}$ and $\{x', y'\}$ satisfying $x < x' < y < y'$. Therefore, $\text{For}\mathcal{M}$ can be seen as an operad on such colored forests, where the edges of the forests have one color among the set $\bar{\mathcal{M}}$. When $\#\mathcal{M} = 2$, the dimensions of $\text{For}\mathcal{M}$ begin by

$$1, 7, 33, 81, 1083, 6854, 45111, 305629, \quad (3.2.9)$$

and form, except for the first terms, Sequence **A054727**, of [Slo].

3.2.4. *Colored Motzkin configurations.* When \mathcal{M} is a unitary magma without nontrivial unit divisors, let

$$\text{Mot}\mathcal{M} := \mathcal{C}\mathcal{M} / (\mathfrak{R}_{\text{Croq}}\mathcal{M} + \mathfrak{R}_{\text{Deg}_1}\mathcal{M}). \quad (3.2.10)$$

The skeletons of the \mathcal{M} -cliques of $\text{Mot}\mathcal{M}$ are configurations of non-intersecting chords on a circle. Equivalently, these objects are graphs of involutions (see Section 3.1.5) having no arcs $\{x, y\}$ and $\{x', y'\}$ satisfying $x < x' < y < y'$. These objects are enumerated by Motzkin numbers [Mot48]. Therefore, $\text{Mot}\mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color among the set $\bar{\mathcal{M}}$. When $\#\mathcal{M} = 2$, the dimensions of $\text{Mot}\mathcal{M}$ begin by

$$1, 4, 9, 21, 51, 127, 323, 835, \quad (3.2.11)$$

and form, except for the first terms, Sequence **A001006**, of [Slo].

3.2.5. *Colored dissections of polygons.* When \mathcal{M} is a unitary magma without nontrivial unit divisors, let

$$\text{Dis}\mathcal{M} := \text{Whi}\mathcal{M} / (\mathfrak{R}_{\text{Croq}}\mathcal{M} + \mathfrak{R}_{\text{Deg}_1}\mathcal{M}) \cap \text{Whi}\mathcal{M}. \quad (3.2.12)$$

The skeletons of the \mathcal{M} -cliques of $\text{Dis}\mathcal{M}$ are *strict dissections of polygons*, that are graphs of Motzkin configurations with no arcs of the form $\{x, x + 1\}$ or $\{1, n + 1\}$, where $n + 1$ is the number of vertices of the graphs. Therefore, $\text{Dis}\mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color among the set $\bar{\mathcal{M}}$. When $\#\mathcal{M} = 2$, the dimensions of $\text{Dis}\mathcal{M}$ begin by

$$1, 1, 3, 6, 13, 29, 65, 148, \quad (3.2.13)$$

and form, except for the first terms, Sequence **A093128** of [Slo].

3.2.6. *Colored Lucas configurations.* When \mathcal{M} is a unitary magma without nontrivial unit divisors, let

$$\text{Luc}\mathcal{M} := \mathcal{C}\mathcal{M} / (\mathfrak{R}_{\text{Bub}}\mathcal{M} + \mathfrak{R}_{\text{Deg}_1}\mathcal{M}). \quad (3.2.14)$$

The skeletons of the \mathcal{M} -cliques of $\text{Luc}\mathcal{M}$ are graphs such that all vertices are of degrees at most 1 and all arcs are of the form $\{x, x + 1\}$ or $\{1, n + 1\}$, where $n + 1$ is the number of vertices of the graphs. Therefore, $\text{Luc}\mathcal{M}$ can be seen as an operad on such colored graphs, where the arcs of the graphs have one color among the set $\bar{\mathcal{M}}$. When $\#\mathcal{M} = 2$, the dimensions of $\text{Luc}\mathcal{M}$ begin by

$$1, 4, 7, 11, 18, 29, 47, 76, \quad (3.2.15)$$

and form, except for the first terms, Sequence **A000032** of [Slo].

3.3. Relations between substructures. The suboperads and quotients of $C\mathcal{M}$ constructed in Sections 3.1 and 3.2 are linked by injective or surjective operad morphisms. To establish these, we begin with the following lemma.

Lemma 3.3.1. *Let \mathcal{M} be a unitary magma. Then,*

- (i) *the space $\mathfrak{R}_{\text{Acy}\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\text{Deg}_1\mathcal{M}}$;*
- (ii) *the spaces $\mathfrak{R}_{\text{Inf}\mathcal{M}}$ and $\mathfrak{R}_{\text{Bub}\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\text{Deg}_0\mathcal{M}}$;*
- (iii) *the spaces $\mathfrak{R}_{\text{Cro}_0\mathcal{M}}$ and $\mathfrak{R}_{\text{Deg}_2\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\text{Bub}\mathcal{M}}$;*
- (iv) *the spaces $\mathfrak{R}_{\text{Deg}_2\mathcal{M}}$ and $\mathfrak{R}_{\text{Acy}\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\text{Inf}\mathcal{M}}$.*

Proof. All the spaces appearing in the statement of the lemma are subspaces of $C\mathcal{M}$ generated by some subfamilies of \mathcal{M} -cliques. Therefore, to prove the assertions of the lemma, we shall prove inclusions of adequate subfamilies of such objects.

If p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Acy}\mathcal{M}}$, by definition, p has a cycle formed by solid arcs. Hence, p has in particular a solid arc and a vertex of degree 2 or more. For this reason, since $\mathfrak{R}_{\text{Deg}_1\mathcal{M}}$ is the linear span of all \mathcal{M} -cliques of degrees 2 or more, p is in $\mathfrak{R}_{\text{Deg}_1\mathcal{M}}$. This implies (i).

If p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Inf}\mathcal{M}}$ or $\mathfrak{R}_{\text{Bub}\mathcal{M}}$, by definition, p has in particular a solid arc. Hence, since $\mathfrak{R}_{\text{Deg}_0\mathcal{M}}$ is the linear span of all \mathcal{M} -cliques with at least one vertex with a positive degree, p is in $\mathfrak{R}_{\text{Deg}_0\mathcal{M}}$. This implies (ii).

If p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Cro}_0\mathcal{M}}$ or $\mathfrak{R}_{\text{Deg}_2\mathcal{M}}$, p has in particular a solid diagonal. Indeed, when p is in $\mathfrak{R}_{\text{Cro}_0\mathcal{M}}$ this property is immediate. When p is in $\mathfrak{R}_{\text{Deg}_2\mathcal{M}}$, since p has a vertex x of degree 3 or more, the skeleton of p has three arcs $\{x, y_1\}$, $\{x, y_2\}$, and $\{x, y_3\}$ with $y_i \neq x - 1$, $y_i \neq x + 1$, and $y_i \neq |p| + 1$ for at least one $i \in [3]$, so that the arc $(\min\{x, y_i\}, \max\{x, y_i\})$ is a solid diagonal of p . For this reason, since $\mathfrak{R}_{\text{Bub}\mathcal{M}}$ is the linear span of all \mathcal{M} -cliques with at least one solid diagonal, p is in $\mathfrak{R}_{\text{Bub}\mathcal{M}}$. This implies (iii).

If p is an \mathcal{M} -clique of $\mathfrak{R}_{\text{Deg}_2\mathcal{M}}$ or $\mathfrak{R}_{\text{Acy}\mathcal{M}}$, p has in particular a solid arc included in another one. Indeed, when p is in $\mathfrak{R}_{\text{Deg}_2\mathcal{M}}$, since p has a vertex x of a degree 3 or more, the skeleton of p has three arcs $\{x, y_1\}$, $\{x, y_2\}$, and $\{x, y_3\}$. One can check that for all relative orders between the vertices x , y_1 , y_2 , and y_3 , one of these arcs includes another one, so that p is not inclusion-free. When p is in $\mathfrak{R}_{\text{Acy}\mathcal{M}}$, p contains a cycle formed by solid arcs. Let $x_1, x_2, \dots, x_k, k \geq 3$, be the vertices of p that form this cycle. We can assume without loss of generality that $x_1 \leq x_i$ for all $i \in [k]$ and thus, that (x_1, x_2) and (x_1, x_k) are solid arcs of p being part of the cycle. Then, when $x_2 < x_k$, since $x_1 \leq x_1 < x_2 \leq x_k$, the arc (x_1, x_k) includes (x_1, x_2) . Otherwise, $x_k < x_2$, and since $x_1 \leq x_1 < x_k \leq x_2$, the arc (x_1, x_2) includes (x_1, x_k) . For these reasons, since $\mathfrak{R}_{\text{Inf}\mathcal{M}}$ is the linear span of all \mathcal{M} -cliques that are non inclusion-free, p is in $\mathfrak{R}_{\text{Inf}\mathcal{M}}$. This implies (iv). \square

3.3.1. Relations between the main substructures. Let us list and explain the morphisms between the main substructures of $C\mathcal{M}$. First, Lemma 3.3.1 implies that there are surjective operad morphisms from $\text{Acy}\mathcal{M}$ to $\text{Deg}_1\mathcal{M}$, from $\text{Inf}\mathcal{M}$ to $\text{Deg}_0\mathcal{M}$, from $\text{Bub}\mathcal{M}$ to $\text{Deg}_0\mathcal{M}$, from $\text{Cro}_0\mathcal{M}$ to $\text{Bub}\mathcal{M}$, from $\text{Deg}_2\mathcal{M}$ to $\text{Bub}\mathcal{M}$, from $\text{Deg}_2\mathcal{M}$ to $\text{Inf}\mathcal{M}$, and from $\text{Acy}\mathcal{M}$ to $\text{Inf}\mathcal{M}$. Second, when B, E , and D are subsets of \mathcal{M} such that $\mathbb{1}_{\mathcal{M}} \in B$, $\mathbb{1}_{\mathcal{M}} \in E$,

and $E \star B \subseteq D$, $\text{Whi}\mathcal{M}$ is a suboperad of $\text{Lab}_{B,E,D}\mathcal{M}$. Finally, there is a surjective operad morphism from $\text{Whi}\mathcal{M}$ to the associative operad As sending any \mathcal{M} -clique p of $\text{Whi}\mathcal{M}$ to the unique basis element of As of the same arity as the one of p . The relations between the main suboperads and quotients of $\text{C}\mathcal{M}$ built here are summarized in the diagram of operad morphisms of Figure 2.

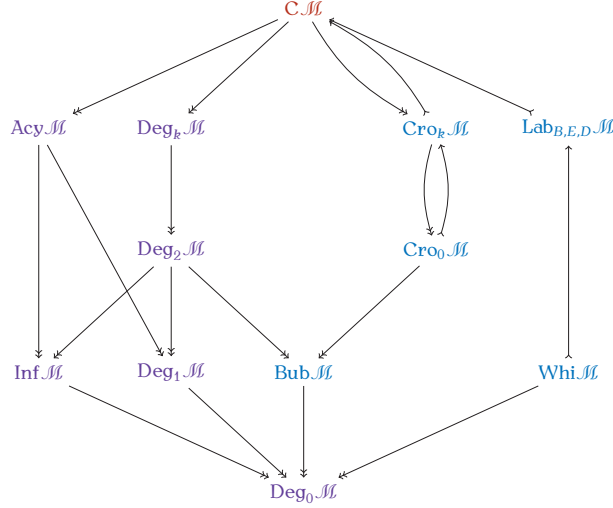


FIGURE 2. The diagram of the main suboperads and quotients of $\text{C}\mathcal{M}$. Arrows \dashrightarrow (resp. \rightarrow) are injective (resp. surjective) operad morphisms. Here, \mathcal{M} is a unitary magma without nontrivial unit divisors, k is a positive integer, and B , E , and D are subsets of \mathcal{M} such that $1_{\mathcal{M}} \in B$, $1_{\mathcal{M}} \in E$, and $E \star B \subseteq D$.

3.3.2. *Relations between the secondary and main substructures.* Let us now list and explain the morphisms between the secondary and main substructures of $\text{C}\mathcal{M}$. First, immediately from their definitions, $\text{WNC}\mathcal{M}$ is a suboperad of $\text{Cro}_0\mathcal{M}$ and a quotient of $\text{Whi}\mathcal{M}$, $\text{Pat}\mathcal{M}$ is both a quotient of $\text{Deg}_2\mathcal{M}$ and $\text{Acy}\mathcal{M}$, $\text{For}\mathcal{M}$ is both a quotient of $\text{Cro}_0\mathcal{M}$ and $\text{Acy}\mathcal{M}$, $\text{Mot}\mathcal{M}$ is both a quotient of $\text{Cro}_0\mathcal{M}$ and $\text{Deg}_1\mathcal{M}$, $\text{Dis}\mathcal{M}$ is a suboperad of $\text{Mot}\mathcal{M}$ and a quotient of $\text{WNC}\mathcal{M}$, and $\text{Luc}\mathcal{M}$ is both a quotient of $\text{Bub}\mathcal{M}$ and $\text{Deg}_1\mathcal{M}$. Moreover, since by Lemma 3.3.1, $\mathfrak{R}_{\text{Acy}\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\text{Deg}_1\mathcal{M}}$, $\mathfrak{R}_{\text{Deg}_2\mathcal{M}}$ and $\mathfrak{R}_{\text{Acy}\mathcal{M}}$ are subspaces of $\mathfrak{R}_{\text{Inf}\mathcal{M}}$, and $\mathfrak{R}_{\text{Cro}_0\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\text{Bub}\mathcal{M}}$, we respectively have that $\mathfrak{R}_{\text{Deg}_2\mathcal{M}} + \mathfrak{R}_{\text{Acy}\mathcal{M}}$ is a subspace of both $\mathfrak{R}_{\text{Deg}_1\mathcal{M}}$ and $\mathfrak{R}_{\text{Inf}\mathcal{M}}$, $\mathfrak{R}_{\text{Cro}_0\mathcal{M}} + \mathfrak{R}_{\text{Acy}\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\text{Cro}_0\mathcal{M}} + \mathfrak{R}_{\text{Deg}_1\mathcal{M}}$, and $\mathfrak{R}_{\text{Cro}_0\mathcal{M}} + \mathfrak{R}_{\text{Deg}_1\mathcal{M}}$ is a subspace of $\mathfrak{R}_{\text{Bub}\mathcal{M}} + \mathfrak{R}_{\text{Deg}_1\mathcal{M}}$. For these reasons, there are surjective operad morphisms from $\text{Pat}\mathcal{M}$ to $\text{Deg}_1\mathcal{M}$, from $\text{Pat}\mathcal{M}$ to $\text{Inf}\mathcal{M}$, from $\text{For}\mathcal{M}$ to $\text{Mot}\mathcal{M}$, and from $\text{Mot}\mathcal{M}$ to $\text{Luc}\mathcal{M}$. The relations between the secondary suboperads and quotients of $\text{C}\mathcal{M}$ built here are summarized in the diagram of operad morphisms of Figure 3.

4. OPERADS OF NONCROSSING DECORATED CLIQUES

We perform here a complete study of the suboperad $\text{Cro}_0\mathcal{M}$ of noncrossing \mathcal{M} -cliques defined in Section 3.1.3. For simplicity, this operad is denoted in the sequel as $\text{NC}\mathcal{M}$ and

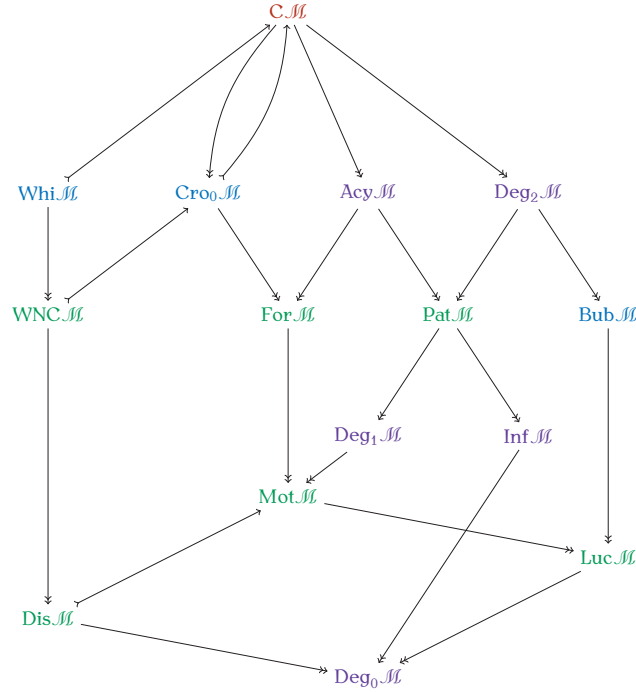


FIGURE 3. The diagram of the secondary suboperads and quotients of $C.M.$ together with some of their related main suboperads and quotients of $C.M.$. Arrows \rightarrow (resp. \twoheadrightarrow) are injective (resp. surjective) operad morphisms. Here, M is a unitary magma without nontrivial unit divisors.

named as the *noncrossing M -clique operad*. The process giving from any unitary magma M the operad $NC.M$ is called the *noncrossing clique construction*.

4.1. General properties. To study $NC.M$, we begin by establishing the fact that $NC.M$ inherits from some properties of $C.M$. Then, we shall describe a realization of $NC.M$ in terms of decorated Schröder trees, compute a minimal generating set of $NC.M$, and compute its dimensions.

First of all, we call *fundamental basis* of $NC.M$ the fundamental basis of $C.M$ restricted on noncrossing M -cliques. By definition of $NC.M$ and by Proposition 3.1.3, the partial composition $p \circ_i q$ of two noncrossing M -cliques p and q in $NC.M$ is equal to the partial composition $p \circ_i q$ in $C.M$. Therefore, the fundamental basis of $NC.M$ is a set-operad basis.

4.1.1. *First properties.*

Proposition 4.1.1. *Let M be a unitary magma. Then,*

- (i) *the associative elements of $NC.M$ are the ones of $C.M$;*
- (ii) *the group of symmetries of $NC.M$ contains the map ret (defined by (2.2.11)) and all the maps $C\theta$ where θ are unitary magma automorphisms of M ;*

(iii) the fundamental basis of $\text{NC}\mathcal{M}$ is a basic set-operad basis if and only if \mathcal{M} is right cancellable;

(iv) the map ρ (defined by (2.2.13)) is a rotation map of $\text{NC}\mathcal{M}$ endowing it with a cyclic operad structure.

Proof. First, since by Proposition 3.1.3 $\text{NC}\mathcal{M}$ is a suboperad of $\text{C}\mathcal{M}$, each associative element of $\text{NC}\mathcal{M}$ is an associative element of $\text{C}\mathcal{M}$. Moreover, since all \mathcal{M} -bubbles are in $\text{NC}\mathcal{M}$ and, by Proposition 2.2.4, all associative elements of $\text{C}\mathcal{M}$ are linear combinations of \mathcal{M} -bubbles, each associative element of $\text{C}\mathcal{M}$ belongs to $\text{NC}\mathcal{M}$. Whence (i). Besides, since for any noncrossing \mathcal{M} -clique p , $\text{ret}(p)$ (resp. $\rho(p)$) is still noncrossing, by Proposition 2.2.5 (resp. Proposition 2.2.7), (ii) (resp. (iv)) holds. Finally, since again by Proposition 3.1.3, $\text{NC}\mathcal{M}$ is a suboperad of $\text{C}\mathcal{M}$, Proposition 2.2.6 and the fact that $\text{NC}\mathcal{M}(2) = \text{C}\mathcal{M}(2)$ imply (iii). \square

4.1.2. *Treelike expressions on bubbles.* Let p be a noncrossing \mathcal{M} -clique of arity $n \geq 2$, and (x, y) be a diagonal or the base of p . Consider the path $(x = z_1, z_2, \dots, z_k, z_{k+1} = y)$ in p such that $k \geq 2$, for all $i \in [k+1]$, $x \leq z_i \leq y$, and for all $i \in [k]$, z_{i+1} is the greatest vertex of p so that (z_i, z_{i+1}) is a solid diagonal or a (non-necessarily solid) edge of p . The area of p adjacent to (x, y) is the \mathcal{M} -bubble q of arity k whose base is labeled by $p(x, y)$ and $q_i = p(z_i, z_{i+1})$ for all $i \in [k]$. From a geometric point of view, q is the unique maximal component of p adjacent to the arc (x, y) , without solid diagonals, and bounded by solid diagonals or edges of p . For instance, for the noncrossing \mathbb{Z} -clique



the path associated with the diagonal $(4, 9)$ of p is $(4, 5, 6, 8, 9)$. For this reason, the area of p adjacent to $(4, 9)$ is the \mathbb{Z} -bubble



Proposition 4.1.2. *Let \mathcal{M} be a unitary magma and p be a noncrossing \mathcal{M} -clique of arity greater than 1. Then, there is a unique \mathcal{M} -bubble q with a maximal arity $k \geq 2$ such that $p = q \circ [\tau_1, \dots, \tau_k]$, where each τ_i , $i \in [k]$, is a noncrossing \mathcal{M} -clique with a base labeled by $\mathbb{1}_{\mathcal{M}}$.*

Proof. Let q' be the area of p adjacent to its base and k' be the arity of q' . By definition of the partial composition of $\text{NC}\mathcal{M}$, for all \mathcal{M} -cliques u, u', u_1 , and u_2 , if $u = u' \circ_i u_1 = u' \circ_i u_2$ and u_1 and u_2 have bases labeled by $\mathbb{1}_{\mathcal{M}}$, then $u_1 = u_2$. This implies in particular that there are unique noncrossing \mathcal{M} -cliques τ'_i , $i \in [k']$, with bases labeled by $\mathbb{1}_{\mathcal{M}}$ such that $p = q' \circ [\tau'_1, \dots, \tau'_{k'}]$. Finally, the fact that q' is the area of p adjacent to its base implies the maximality for the arity of q' . The statement of the proposition follows. \square

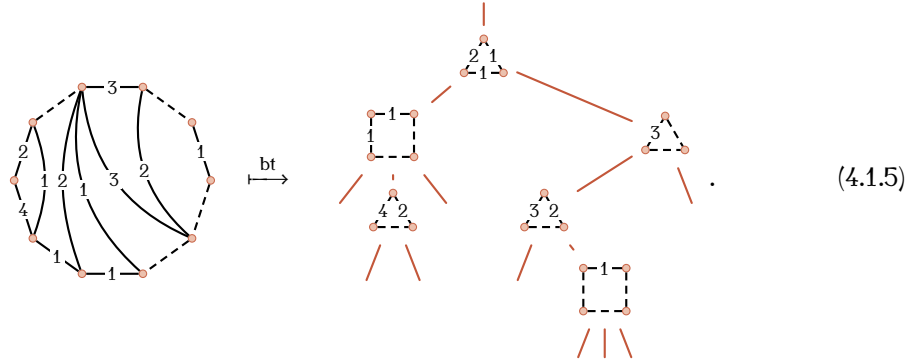
Consider the map

$$\text{bt} : \text{NC}\mathcal{M} \rightarrow \text{Free}(\text{Vect}(\mathcal{B}_{\mathcal{M}})) \quad (4.1.3)$$

defined linearly and recursively by $\text{bt}(\circ - \circ) := \perp$ and, for any noncrossing \mathcal{M} -clique p of arity greater than 1, by

$$\text{bt}(p) := c(q) \circ [\text{bt}(\tau_1), \dots, \text{bt}(\tau_k)], \quad (4.1.4)$$

where $p = q \circ [\tau_1, \dots, \tau_k]$ is the unique decomposition of p stated in Proposition 4.1.2. We call $\text{bt}(p)$ the *bubble tree* of p . For instance, in $\text{NC}\mathbb{Z}$,



Lemma 4.1.3. *Let \mathcal{M} be a unitary magma. For any noncrossing \mathcal{M} -clique p , $\text{bt}(p)$ is a treelike expression on $\mathcal{B}_{\mathcal{M}}$ of p .*

Proof. We proceed by induction on the arity n of p . If $n = 1$, since $p = \circ - \circ$ and $\text{bt}(\circ - \circ) = \perp$, the statement of the lemma immediately holds. Otherwise, one has $\text{bt}(p) = c(q) \circ [\text{bt}(\tau_1), \dots, \text{bt}(\tau_k)]$ where p uniquely decomposes as $p = q \circ [\tau_1, \dots, \tau_k]$ under the conditions stated by Proposition 4.1.2. By definition of areas and of the map bt , q is an \mathcal{M} -bubble. Moreover, by induction hypothesis, any $\text{bt}(\tau_i)$, $i \in [k]$, is a treelike expression on $\mathcal{B}_{\mathcal{M}}$ of τ_i . Hence, $\text{bt}(p)$ is a treelike expression on $\mathcal{B}_{\mathcal{M}}$ of p . \square

Proposition 4.1.4. *Let \mathcal{M} be a unitary magma. Then, the map bt is injective and the image of bt is the linear span of all syntax trees t on $\mathcal{B}_{\mathcal{M}}$ such that*

- (i) *the root of t is labeled by an \mathcal{M} -bubble;*
- (ii) *the internal nodes of t different from the root are labeled by \mathcal{M} -bubbles whose bases are labeled by $\mathbb{1}_{\mathcal{M}}$;*
- (iii) *if x and y are two internal nodes of t such that y is the i th child of x , the i th edge of the bubble labeling x is solid.*

Proof. First of all, since by definition bt sends a basis element of $\text{NC}\mathcal{M}$ to a basis element of $\text{Free}(\text{Vect}(\mathcal{B}_{\mathcal{M}}))$, it is sufficient to show that bt is injective as a map from $\mathcal{C}_{\mathcal{M}}$ to the set of syntax trees on $\mathcal{B}_{\mathcal{M}}$ to establish that it is an injective linear map. For this, we proceed by induction on the arity n . If $n = 1$, since $\text{bt}(\circ - \circ) = \perp$ and $\text{NC}\mathcal{M}(1)$ is of dimension 1, bt is injective. Assume now that p and p' are two noncrossing \mathcal{M} -cliques of arity n such

that $\text{bt}(p) = \text{bt}(p')$. Hence, p (resp. p') uniquely decompose as $p = q \circ [\tau_1, \dots, \tau_k]$ (resp. $p' = q' \circ [\tau'_1, \dots, \tau'_k]$) as stated by Proposition 4.1.2 and

$$\text{bt}(p) = c(q) \circ [\text{bt}(\tau_1), \dots, \text{bt}(\tau_k)] = c(q') \circ [\text{bt}(\tau'_1), \dots, \text{bt}(\tau'_k)] = \text{bt}(p'). \quad (4.1.6)$$

Now, because by definition of areas, all bases of the τ_i and τ'_i , $i \in [k]$, are labeled by $\mathbb{1}_{\mathcal{M}}$, this implies that $q = q'$. Therefore, we have $\text{bt}(\tau_i) = \text{bt}(\tau'_i)$ for all $i \in [k]$, so that, by induction hypothesis, $\tau_i = \tau'_i$ for all $i \in [k]$. Hence, bt is injective.

The definition of bt together with Proposition 4.1.2 lead to the fact that for any noncrossing \mathcal{M} -clique p , the syntax tree $\text{bt}(p)$ satisfies (i), (ii), and (iii). Conversely, let t be a syntax tree satisfying (i), (ii), and (iii). Let us show by structural induction on t that there is a noncrossing \mathcal{M} -clique p such that $\text{bt}(p) = t$. If $t = \perp$, the property holds because $\text{bt}(\circ \dashv \circ) = \perp$. Otherwise, one has $t = s \circ [u_1, \dots, u_k]$ where s is a syntax tree of degree 1 and the u_i , $i \in [k]$, are syntax trees. Since t satisfies (i), (ii), and (iii), the trees s and u_i , $i \in [k]$, satisfy the same three properties. Therefore, by induction hypothesis, there are noncrossing \mathcal{M} -cliques q and τ_i , $i \in [k]$, such that $\text{bt}(q) = s$ and $\text{bt}(\tau_i) = u_i$. Set now p as the noncrossing \mathcal{M} -clique $q \circ [\tau_1, \dots, \tau_k]$. By definition of the map bt and the unique decomposition stated in Proposition 4.1.2 for p , one obtains that $\text{bt}(p) = t$. \square

Observe that bt is not an operad morphism. Indeed,

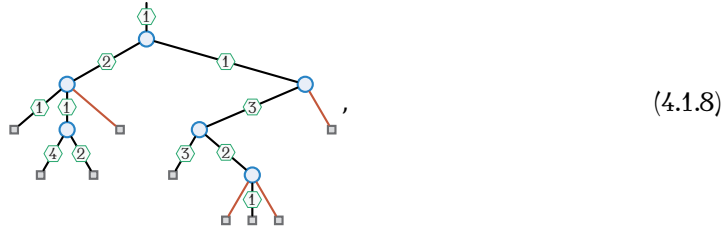
$$\text{bt} \left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} \circ_1 \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} \right) = \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} \neq \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} = \text{bt} \left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} \right) \circ_1 \text{bt} \left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \end{array} \right). \quad (4.1.7)$$

Observe that (4.1.7) holds for all unitary magmas \mathcal{M} since $\mathbb{1}_{\mathcal{M}}$ is always idempotent.

4.1.3. Realization in terms of decorated Schröder trees. Recall that a *Schröder tree* is a rooted planar tree such that all internal nodes have at least two children. An *\mathcal{M} -Schröder tree* t is a Schröder tree such that each edge connecting two internal nodes is labeled on $\bar{\mathcal{M}}$, each edge connecting an internal node and a leaf is labeled on \mathcal{M} , and the outgoing edge from the root of t is labeled on \mathcal{M} (see (4.1.8) for an example of a \mathbb{Z} -Schröder tree).

From the description of the image of the map bt provided by Proposition 4.1.4, any bubble tree t of a noncrossing \mathcal{M} -clique p of arity n can be encoded by an \mathcal{M} -Schröder tree s with n leaves. Indeed, this \mathcal{M} -Schröder tree is obtained by considering each internal node x of t and by labeling the edge connecting x and its i th child by the label of the i th edge of the \mathcal{M} -bubble labeling x . The outgoing edge from the root of s is labeled by the label of the base of the \mathcal{M} -bubble labeling the root of t . For instance, the bubble tree

of (4.1.5) is encoded by the \mathbb{Z} -Schröder tree



where the labels of the edges are drawn in the hexagons and where unlabeled edges are implicitly labeled by $\mathbb{1}_{\mathcal{M}}$. We shall use these drawing conventions in the sequel. As a side remark, observe that the \mathcal{M} -Schröder tree encoding a noncrossing \mathcal{M} -clique p and the dual tree of p (in the usual meaning) have the same underlying unlabeled tree.

This encoding of noncrossing \mathcal{M} -cliques by bubble trees is reversible and hence, one can interpret $\text{NC}\mathcal{M}$ as an operad on the linear span of all \mathcal{M} -Schröder trees. Hence, through this interpretation, if s and t are two \mathcal{M} -Schröder trees and i is a valid integer, the tree $s \circ_i t$ is computed by grafting the root of t to the i th leaf of s . Then, by denoting by b the label of the edge adjacent to the root of t and by a the label of the edge adjacent to the i th leaf of s , we have two cases to consider, depending on the value of $c := a \star b$. If $c \neq \mathbb{1}_{\mathcal{M}}$, we label the edge connecting s and t by c . Otherwise, when $c = \mathbb{1}_{\mathcal{M}}$, we contract the edge connecting s and t by merging the root of t and the father of the i th leaf of s (see Figure 4). For instance, in NCN_3 , one has the two partial compositions

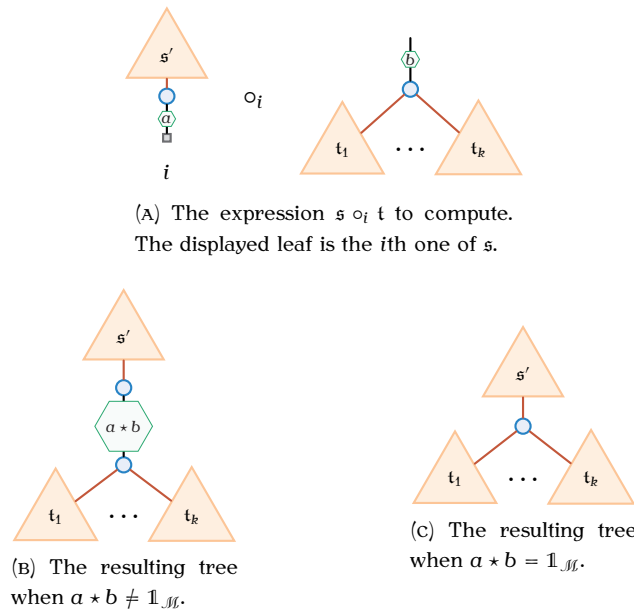


FIGURE 4. The partial composition of $\text{NC}\mathcal{M}$ realized on \mathcal{M} -Schröder trees. Here, the two cases (b) and (c) for the computation of $s \circ_i t$ are shown, where s and t are two \mathcal{M} -Schröder trees. In these drawings, the triangles denote subtrees.

$$(4.1.9a)$$

$$(4.1.9b)$$

In the sequel, we shall indifferently see $\text{NC}\mathcal{M}$ as an operad on noncrossing \mathcal{M} -cliques or on \mathcal{M} -Schröder trees.

4.1.4. Minimal generating set.

Proposition 4.1.5. *Let \mathcal{M} be a unitary magma. The set $\mathcal{T}_{\mathcal{M}}$ of all \mathcal{M} -triangles is a minimal generating set of $\text{NC}\mathcal{M}$.*

Proof. We start by showing by induction on the arity that the suboperad $(\text{NC}\mathcal{M})^{\mathcal{T}_{\mathcal{M}}}$ of $\text{NC}\mathcal{M}$ generated by $\mathcal{T}_{\mathcal{M}}$ is $\text{NC}\mathcal{M}$. It is immediately true in arity 1. Let p be a noncrossing \mathcal{M} -clique of arity $n \geq 2$. Proposition 4.1.2 says in particular that we can express p as $p = q \circ [\tau_1, \dots, \tau_k]$ where q is an \mathcal{M} -bubble of arity $k \geq 2$ and the τ_i , $i \in [k]$, are noncrossing \mathcal{M} -cliques. Since q is an \mathcal{M} -bubble, it can be expressed as

$$q = \begin{array}{c} \triangle \\ \text{---} q_k \\ \triangle \end{array} \circ_1 \begin{array}{c} \triangle \\ \text{---} q_{k-1} \\ \triangle \end{array} \circ_1 \cdots \circ_1 \begin{array}{c} \triangle \\ \text{---} q_3 \\ \triangle \end{array} \circ_1 \begin{array}{c} \triangle \\ \text{---} q_1 \\ \triangle \end{array} \circ_1 \begin{array}{c} \triangle \\ \text{---} q_2 \\ \triangle \end{array} . \quad (4.1.10)$$

Observe that in (4.1.10), brackets are not necessary since \circ_1 is associative. Since $k \geq 2$, the arities of each τ_i , $i \in [k]$, are smaller than the one of p . For this reason, by induction hypothesis, each τ_i belongs to $(\text{NC}\mathcal{M})^{\mathcal{T}_{\mathcal{M}}}$. Moreover, since (4.1.10) shows an expression of q by partial compositions of \mathcal{M} -triangles, q also belongs to $(\text{NC}\mathcal{M})^{\mathcal{T}_{\mathcal{M}}}$. This implies that it is also the case for p . Hence, $\text{NC}\mathcal{M}$ is generated by $\mathcal{T}_{\mathcal{M}}$.

Finally, due to the fact that the partial composition of two \mathcal{M} -triangles is an \mathcal{M} -clique of arity 3, if p is an \mathcal{M} -triangle, p cannot be expressed as a partial composition of \mathcal{M} -triangles. Moreover, since the space $\text{NC}\mathcal{M}(1)$ is trivial, these arguments imply that $\mathcal{T}_{\mathcal{M}}$ is a minimal generating set of $\text{NC}\mathcal{M}$. \square

Proposition 4.1.5 also says that $\text{NC}\mathcal{M}$ is the smallest suboperad of $\text{C}\mathcal{M}$ that contains all \mathcal{M} -triangles and that $\text{NC}\mathcal{M}$ is the biggest binary suboperad of $\text{C}\mathcal{M}$.

4.1.5. *Dimensions.* We now use the notion of bubble trees introduced in Section 4.1.2 to compute the dimensions of $\text{NC}\mathcal{M}$.

Proposition 4.1.6. *Let \mathcal{M} be a finite unitary magma. The Hilbert series $\mathcal{H}_{\text{NC}\mathcal{M}}(t)$ of $\text{NC}\mathcal{M}$ satisfies*

$$t + (m^3 - 2m^2 + 2m - 1)t^2 + (2m^2t - 3mt + 2t - 1)\mathcal{H}_{\text{NC}\mathcal{M}}(t) + (m - 1)\mathcal{H}_{\text{NC}\mathcal{M}}(t)^2 = 0, \quad (4.1.11)$$

where $m := \#\mathcal{M}$.

Proof. By Proposition 4.1.4, the set of noncrossing \mathcal{M} -cliques is in one-to-one correspondence with the set of the syntax trees on $\mathcal{B}_{\mathcal{M}}$ that satisfy (i), (ii), and (iii). Let us call $T(t)$ the generating series of these trees and $S(t)$ the generating series of these trees with the extra condition that the roots are labeled by \mathcal{M} -bubbles whose bases are labeled by $\mathbf{1}_{\mathcal{M}}$. Immediately from its description, $S(t)$ satisfies

$$S(t) = t + \sum_{n \geq 2} ((m - 1)S(t) + t)^n, \quad (4.1.12)$$

and $T(t)$ satisfies

$$T(t) = t + m(S(t) - t). \quad (4.1.13)$$

As the set of all noncrossing \mathcal{M} -cliques forms the fundamental basis of $\text{NC}\mathcal{M}$, one has $\mathcal{H}_{\text{NC}\mathcal{M}}(t) = T(t)$. We eventually obtain (4.1.11) from (4.1.12) and (4.1.13) by a direct computation. \square

We deduce from Proposition 4.1.6 that the Hilbert series of $\text{NC}\mathcal{M}$ satisfies

$$\mathcal{H}_{\text{NC}\mathcal{M}}(t) = \frac{1 - (2m^2 - 3m + 2)t - \sqrt{1 - 2(2m^2 - m)t + m^2t^2}}{2(m - 1)}, \quad (4.1.14)$$

where $m := \#\mathcal{M} \neq 1$.

By using Narayana numbers, whose definition is recalled in Section 3.1.6, one can state the following result.

Proposition 4.1.7. *Let \mathcal{M} be a finite unitary magma. For all $n \geq 2$,*

$$\dim \text{NC}\mathcal{M}(n) = \sum_{0 \leq k \leq n-2} m^{n+k+1} (m - 1)^{n-k-2} \text{nar}(n, k), \quad (4.1.15)$$

where $m := \#\mathcal{M}$.

Proof. As shown by Proposition 4.1.4, each noncrossing \mathcal{M} -clique \mathfrak{p} of $\text{NC}\mathcal{M}(n)$ can be encoded by a unique syntax tree $\text{bt}(\mathfrak{p})$ on $\mathcal{B}_{\mathcal{M}}$ satisfying some conditions. Moreover, Proposition 4.1.5 shows that any noncrossing \mathcal{M} -clique can be expressed (not necessarily in a unique way) as partial compositions of several \mathcal{M} -triangles. By combining these two results, we obtain that any noncrossing \mathcal{M} -clique \mathfrak{p} can be encoded by a syntax tree on $\mathcal{T}_{\mathcal{M}}$ obtained from $\text{bt}(\mathfrak{p})$ by replacing any of its nodes \mathfrak{s} of arity $\ell \geq 3$ by left comb binary syntax trees \mathfrak{s}' on $\mathcal{T}_{\mathcal{M}}$ satisfying

$$\mathfrak{s}' := \mathfrak{c}(\mathfrak{q}^1) \circ_1 \mathfrak{c}(\mathfrak{q}^2) \circ_1 \cdots \circ_1 \mathfrak{c}(\mathfrak{q}^{\ell-1}), \quad (4.1.16)$$

where the q^i , $i \in [\ell - 1]$, are the unique \mathcal{M} -triangles such that for any $i \in [2, \ell - 1]$, the base of q^i is labeled by $\mathbb{1}_{\mathcal{M}}$, for any $j \in [\ell - 2]$, the first edge of q^j is labeled by $\mathbb{1}_{\mathcal{M}}$, and $\text{ev}(s') = \text{ev}(s)$. Observe that in (4.1.16), brackets are not necessary since \circ_1 is associative. Therefore, p can be encoded in a unique way as a binary syntax tree t on $\mathcal{T}_{\mathcal{M}}$ satisfying the following restrictions:

- (i) the \mathcal{M} -triangles labeling the internal nodes of t which are not the root have bases labeled by $\mathbb{1}_{\mathcal{M}}$;
- (ii) if x and y are two internal nodes of t such that y is the right child of x , the second edge of the bubble labeling x is solid.

To establish (4.1.15), since the set of all noncrossing \mathcal{M} -cliques forms the fundamental basis of $\text{NC}\mathcal{M}$, we now have to count these binary trees. Consider a binary tree t of arity $n \geq 2$ with exactly $k \in [0, n - 2]$ internal nodes having an internal node as a left child. There are m ways to label the base of the \mathcal{M} -triangle labeling the root of t , m^k ways to label the first edges of the \mathcal{M} -triangles labeling the internal nodes of t that have an internal node as left child, m^n ways to label the first (resp. second) edges of the \mathcal{M} -triangles labeling the internal nodes of t having a leaf as left (resp. right) child, and, since there are exactly $n - k - 2$ internal nodes of t having an internal node as a right child, there are $(m - 1)^{n-k-2}$ ways to label the second edges of the \mathcal{M} -triangles labeling these internal nodes. Now, since $\text{nar}(n, k)$ counts the binary trees with n leaves and exactly k internal nodes having an internal node as a left child, and a binary tree with n leaves can have at most $n - 2$ internal nodes having an internal node as left child, (4.1.15) follows. \square

We can use Proposition 4.1.7 to compute the first dimensions of $\text{NC}\mathcal{M}$. For instance, depending on $m := \#\mathcal{M}$, we have the following sequences of dimensions:

$$1, 1, 1, 1, 1, 1, 1, 1, \quad m = 1, \quad (4.1.17a)$$

$$1, 8, 48, 352, 2880, 25216, 231168, 2190848, \quad m = 2, \quad (4.1.17b)$$

$$1, 27, 405, 7533, 156735, 349263, 81520425, 1967414265, \quad m = 3, \quad (4.1.17c)$$

$$1, 64, 1792, 62464, 2437120, 101859328, 4459528192, 201889939456. \quad m = 4, \quad (4.1.17d)$$

The second one forms, except for the first terms, Sequence [A054726](#) of [Slo]. The last two sequences are not listed in [Slo] at this time.

4.2. Presentation and Koszulity. The aim of this section is to establish a presentation by generators and relations of $\text{NC}\mathcal{M}$. For this, we will define an adequate rewrite rule on the set of the syntax trees on $\mathcal{T}_{\mathcal{M}}$ and prove that it admits the required properties.

4.2.1. *Space of relations.* Let $\mathfrak{R}_{\text{NC}\mathcal{M}}$ be the subspace of $\text{Free}(\text{Vect}(\mathcal{T}_{\mathcal{M}}))$ (3) generated by the elements

$$c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{q}_0 \end{array} \right) - c \left(\begin{array}{c} \text{r}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{r}_0 \end{array} \right), \quad \text{if } \text{p}_1 \star \text{q}_0 = \text{r}_1 \star \text{r}_0 \neq \mathbb{1}_{\mathcal{M}}, \quad (4.2.1a)$$

$$c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{q}_0 \end{array} \right) - c \left(\begin{array}{c} \text{q}_1 \quad \text{r}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_2 \quad \text{p}_2 \\ \wedge \\ \text{r}_0 \end{array} \right), \quad \text{if } \text{p}_1 \star \text{q}_0 = \text{r}_2 \star \text{r}_0 = \mathbb{1}_{\mathcal{M}}, \quad (4.2.1b)$$

$$c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{q}_0 \end{array} \right) - c \left(\begin{array}{c} \text{p}_1 \quad \text{r}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{r}_0 \end{array} \right), \quad \text{if } \text{p}_2 \star \text{q}_0 = \text{r}_2 \star \text{r}_0 \neq \mathbb{1}_{\mathcal{M}}, \quad (4.2.1c)$$

where p , q , and r are \mathcal{M} -triangles.

Lemma 4.2.1. *Let \mathcal{M} be a unitary magma, and \mathfrak{s} and \mathfrak{t} be two syntax trees of arity 3 on $\mathcal{T}_{\mathcal{M}}$. Then, $\mathfrak{s} - \mathfrak{t}$ belongs to $\mathfrak{R}_{\text{NC}\mathcal{M}}$ if and only if $\text{ev}(\mathfrak{s}) = \text{ev}(\mathfrak{t})$.*

Proof. Assume first that $\mathfrak{s} - \mathfrak{t}$ belongs to $\mathfrak{R}_{\text{NC}\mathcal{M}}$. Then, $\mathfrak{s} - \mathfrak{t}$ is a linear combination of elements of the form (4.2.1a), (4.2.1b), and (4.2.1c). Now, observe that if p , q , and r are three \mathcal{M} -triangles,

(a) when $\delta := \text{p}_1 \star \text{q}_0 = \text{r}_1 \star \text{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, we have

$$\text{ev} \left(c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{q}_0 \end{array} \right) \right) = \begin{array}{c} \text{q}_2 \\ \text{---} \\ \text{q}_1 \quad \delta \quad \text{p}_2 \\ \diagup \quad \diagdown \\ \text{p}_0 \end{array} = \text{ev} \left(c \left(\begin{array}{c} \text{r}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{r}_0 \end{array} \right) \right), \quad (4.2.2)$$

(b) when $\text{p}_1 \star \text{q}_0 = \text{r}_2 \star \text{r}_0 = \mathbb{1}_{\mathcal{M}}$, we have

$$\text{ev} \left(c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{q}_0 \end{array} \right) \right) = \begin{array}{c} \text{q}_2 \\ \text{---} \\ \text{q}_1 \quad \text{p}_2 \\ \diagup \quad \diagdown \\ \text{p}_0 \end{array} = \text{ev} \left(c \left(\begin{array}{c} \text{q}_1 \quad \text{r}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_2 \quad \text{p}_2 \\ \wedge \\ \text{r}_0 \end{array} \right) \right), \quad (4.2.3)$$

(c) when $\delta := \text{p}_2 \star \text{q}_0 = \text{r}_2 \star \text{r}_0 \neq \mathbb{1}_{\mathcal{M}}$, we have

$$\text{ev} \left(c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{q}_0 \end{array} \right) \right) = \begin{array}{c} \text{q}_1 \\ \text{---} \\ \text{p}_1 \quad \delta \quad \text{q}_2 \\ \diagdown \quad \diagup \\ \text{p}_0 \end{array} = \text{ev} \left(c \left(\begin{array}{c} \text{p}_1 \quad \text{r}_2 \\ \wedge \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \wedge \\ \text{r}_0 \end{array} \right) \right). \quad (4.2.4)$$

This shows that all evaluations in $\text{NC}\mathcal{M}$ of (4.2.1a), (4.2.1b), and (4.2.1c) are equal to zero. Therefore, $\text{ev}(\mathfrak{s} - \mathfrak{t}) = 0$ and hence, one has $\text{ev}(\mathfrak{s}) - \text{ev}(\mathfrak{t}) = 0$ and, as expected, $\text{ev}(\mathfrak{s}) = \text{ev}(\mathfrak{t})$.

Let us now assume that $\text{ev}(\mathfrak{s}) = \text{ev}(\mathfrak{t})$ and let $\mathfrak{r} := \text{ev}(\mathfrak{s})$. As \mathfrak{s} is of arity 3, \mathfrak{r} also is of arity 3 and thus,

$$\mathfrak{r} \in \left\{ \begin{array}{c} \begin{array}{c} \text{q}_2 \\ \text{---} \\ \text{q}_1 \quad \delta \quad \text{p}_2 \\ \diagup \quad \diagdown \\ \text{p}_0 \end{array}, \begin{array}{c} \text{q}_2 \\ \text{---} \\ \text{q}_1 \quad \text{p}_2 \\ \diagup \quad \diagdown \\ \text{p}_0 \end{array}, \begin{array}{c} \text{q}_1 \\ \text{---} \\ \text{p}_1 \quad \delta \quad \text{q}_2 \\ \diagdown \quad \diagup \\ \text{p}_0 \end{array} : \text{p}, \text{q} \in \mathcal{T}_{\mathcal{M}}, \delta \in \overline{\mathcal{M}} \end{array} \right\}. \quad (4.2.5)$$

Now, by definition of the partial composition of $\text{NC}\mathcal{M}$, if \mathfrak{r} has the form of the first (resp. second, third) noncrossing \mathcal{M} -clique appearing in (4.2.5), \mathfrak{s} and \mathfrak{t} are respectively of the form of the first and second syntax trees of (4.2.1a) (resp. (4.2.1b), (4.2.1c)). Hence, in all cases, $\mathfrak{s} - \mathfrak{t}$ is in $\mathfrak{R}_{\text{NC}\mathcal{M}}$. \square

Proposition 4.2.2. *Let \mathcal{M} be a finite unitary magma. Then, the dimension of the space $\mathfrak{R}_{\text{NC},\mathcal{M}}$ satisfies*

$$\dim \mathfrak{R}_{\text{NC},\mathcal{M}} = 2m^6 - 2m^5 + m^4, \quad (4.2.6)$$

where $m := \#\mathcal{M}$.

Proof. For any $x \in \mathcal{M}$, let $f(x)$ be the number of ordered pairs $(y, z) \in \mathcal{M}^2$ such that $x = y \star z$. Since \mathcal{M} is finite, $f : \mathcal{M} \rightarrow \mathbb{N}$ is a well-defined map.

Let \equiv be the equivalence relation on the set of the syntax trees on $\mathcal{T}_{\mathcal{M}}$ of arity 3 satisfying $\mathfrak{s} \equiv \mathfrak{t}$ if \mathfrak{s} and \mathfrak{t} are two such syntax trees satisfying $\text{ev}(\mathfrak{s}) = \text{ev}(\mathfrak{t})$. Let also C be the set of all noncrossing \mathcal{M} -cliques of arity 3. For any $\tau \in C$, we denote by $[\tau]_{\equiv}$ the set of all syntax trees \mathfrak{s} satisfying $\text{ev}(\mathfrak{s}) = \tau$. Proposition 4.1.5 says in particular that any $\tau \in C$ can be obtained by a partial composition of two \mathcal{M} -triangles, and hence, all $[\tau]_{\equiv}$ are nonempty sets and thus, are \equiv -equivalence classes.

Moreover, by Lemma 4.2.1, for any syntax trees \mathfrak{s} and \mathfrak{t} , one has $\mathfrak{s} \equiv \mathfrak{t}$ if and only if $\mathfrak{s} - \mathfrak{t}$ is in $\mathfrak{R}_{\text{NC},\mathcal{M}}$. For this reason, the dimension of $\mathfrak{R}_{\text{NC},\mathcal{M}}$ is linked with the cardinalities of all \equiv -equivalence classes by

$$\dim \mathfrak{R}_{\text{NC},\mathcal{M}} = \sum_{\tau \in C} \#[\tau]_{\equiv} - 1. \quad (4.2.7)$$

Let us compute (4.2.7) by enumerating each \equiv -equivalence class $[\tau]_{\equiv}$.

Observe that since τ is of arity 3, it can be of three different forms according to the presence of a solid diagonal.

(a) If

$$\tau = \begin{array}{ccc} \circ & \xrightarrow{q_2} & \circ \\ \swarrow & \delta & \searrow \\ \circ & & \circ \\ \nwarrow & & \nearrow \\ \circ & \xrightarrow{p_0} & \circ \end{array} \quad (4.2.8)$$

for some $p_0, p_2, q_1, q_2 \in \mathcal{M}$ and $\delta \in \bar{\mathcal{M}}$, to have $\mathfrak{s} \in [\tau]_{\equiv}$, we necessarily have

$$\mathfrak{s} = \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{p_0} & \circ \end{array} \right) \circ_1 \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{q_0} & \circ \end{array} \right) \quad (4.2.9)$$

where $p_1, q_0 \in \mathcal{M}$ and $p_1 \star q_0 = \delta$. Hence, $\#[\tau]_{\equiv} = f(\delta)$.

(b) If

$$\tau = \begin{array}{ccc} \circ & \xrightarrow{q_2} & \circ \\ \swarrow & & \searrow \\ \circ & & \circ \\ \nwarrow & & \nearrow \\ \circ & \xrightarrow{p_0} & \circ \end{array} \quad (4.2.10)$$

for some $p_0, p_2, q_1, q_2 \in \mathcal{M}$, to have $\mathfrak{s} \in [\tau]_{\equiv}$, we necessarily have

$$\mathfrak{s} \in \left\{ \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{p_0} & \circ \end{array} \right) \circ_1 \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{q_0} & \circ \end{array} \right), \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{p_0} & \circ \end{array} \right) \circ_2 \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{r_0} & \circ \end{array} \right) \right\} \quad (4.2.11)$$

where $p_1, q_0, r_0, r_2 \in \mathcal{M}$, $p_1 \star q_0 = \mathbb{1}_{\mathcal{M}}$, and $r_2 \star r_0 = \mathbb{1}_{\mathcal{M}}$. Hence, $\#[\tau]_{\equiv} = 2f(\mathbb{1}_{\mathcal{M}})$.

(c) Otherwise,

$$\tau = \begin{array}{ccc} \circ & \xrightarrow{q_1} & \circ \\ \swarrow & \delta & \searrow \\ \circ & & \circ \\ \nwarrow & & \nearrow \\ \circ & \xrightarrow{p_0} & \circ \end{array} \quad (4.2.12)$$

for some $p_0, p_1, q_1, q_2 \in \mathcal{M}$ and $\delta \in \bar{\mathcal{M}}$, and to have $\mathfrak{s} \in [\tau]_{\equiv}$, we necessarily have

$$\mathfrak{s} = \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{p_0} & \circ \end{array} \right) \circ_2 \mathfrak{c} \left(\begin{array}{ccc} \circ & \wedge & \circ \\ \swarrow & & \searrow \\ \circ & \xrightarrow{q_0} & \circ \end{array} \right) \quad (4.2.13)$$

where $p_2, q_0 \in \mathcal{M}$ and $p_2 \star q_0 = \delta$. Hence, $\#[\tau]_{\equiv} = f(\delta)$.

Therefore, by using the fact that

$$\sum_{\delta \in \mathcal{M}} f(\delta) = m^2, \quad (4.2.14)$$

from (4.2.7) we obtain

$$\begin{aligned} \dim \mathfrak{R}_{\text{NC}, \mathcal{M}} &= \left(\sum_{\substack{p_0, p_2, q_1, q_2 \in \mathcal{M} \\ \delta \in \mathcal{M}}} f(\delta) - 1 \right) + \left(\sum_{p_0, p_2, q_1, q_2 \in \mathcal{M}} 2f(\mathbb{1}_{\mathcal{M}}) - 1 \right) + \left(\sum_{\substack{p_0, p_1, q_1, q_2 \in \mathcal{M} \\ \delta \in \mathcal{M}}} f(\delta) - 1 \right) \\ &= m^4 \left(2 \left(\sum_{\delta \in \mathcal{M}} f(\delta) - 1 \right) + 2f(\mathbb{1}_{\mathcal{M}}) - 1 \right) \\ &= 2m^6 - 2m^5 + m^4, \end{aligned} \quad (4.2.15)$$

establishing the statement of the proposition. \square

Observe that, by Proposition 4.2.2, the dimension of $\mathfrak{R}_{\text{NC}, \mathcal{M}}$ only depends on the cardinality of \mathcal{M} and not on its operation \star .

4.2.2. Rewrite rule. Let \rightarrow be the rewrite rule on the set of the syntax trees on $\mathcal{T}_{\mathcal{M}}$ satisfying

$$c \left(\begin{array}{c} p_1 \quad p_2 \\ \wedge \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} q_1 \quad q_2 \\ \wedge \\ \swarrow \quad \searrow \\ q_0 \end{array} \right) \rightarrow c \left(\begin{array}{c} \delta \quad p_2 \\ \wedge \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} q_1 \quad q_2 \\ \wedge \\ \swarrow \quad \searrow \\ \text{---} \end{array} \right), \quad \text{if } q_0 \neq \mathbb{1}_{\mathcal{M}}, \text{ where } \delta := p_1 \star q_0, \quad (4.2.16a)$$

$$c \left(\begin{array}{c} p_1 \quad p_2 \\ \wedge \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} q_1 \quad q_2 \\ \wedge \\ \swarrow \quad \searrow \\ q_0 \end{array} \right) \rightarrow c \left(\begin{array}{c} q_1 \quad \delta \\ \wedge \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} q_2 \quad p_2 \\ \wedge \\ \swarrow \quad \searrow \\ \text{---} \end{array} \right), \quad \text{if } p_1 \star q_0 = \mathbb{1}_{\mathcal{M}}, \quad (4.2.16b)$$

$$c \left(\begin{array}{c} p_1 \quad p_2 \\ \wedge \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} q_1 \quad q_2 \\ \wedge \\ \swarrow \quad \searrow \\ q_0 \end{array} \right) \rightarrow c \left(\begin{array}{c} p_1 \quad \delta \\ \wedge \\ \swarrow \quad \searrow \\ p_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} q_1 \quad q_2 \\ \wedge \\ \swarrow \quad \searrow \\ \text{---} \end{array} \right), \quad \text{if } q_0 \neq \mathbb{1}_{\mathcal{M}}, \text{ where } \delta := p_2 \star q_0, \quad (4.2.16c)$$

where p and q are \mathcal{M} -triangles.

Lemma 4.2.3. *Let \mathcal{M} be a unitary magma. Then, the vector space induced by the rewrite rule \rightarrow is $\mathfrak{R}_{\text{NC}, \mathcal{M}}$.*

Proof. Let s and t be two syntax trees on $\mathcal{T}_{\mathcal{M}}$ such that $s \rightarrow t$. We have three cases to consider depending on the form of s and t .

(a) if s (resp. t) is of the form described by the left (resp. right) member of (4.2.16a), we have

$$\text{ev}(s) = \begin{array}{c} \begin{array}{ccc} & q_2 & \\ \swarrow & & \searrow \\ q_1 & \delta & p_2 \\ \swarrow & & \searrow \\ & p_0 & \end{array} \\ = \text{ev}(t), \end{array} \quad (4.2.17)$$

where $\delta := p_1 \star q_0$.

(b) If \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.16b), we have

$$\text{ev}(\mathfrak{s}) = \begin{array}{c} \circ \text{---} q_2 \text{---} \circ \\ | \quad | \\ \circ \text{---} p_0 \text{---} \circ \end{array} = \text{ev}(\mathfrak{t}). \quad (4.2.18)$$

(c) Otherwise, \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.16c). We have

$$\text{ev}(\mathfrak{s}) = \begin{array}{c} \circ \text{---} q_1 \text{---} \circ \\ | \quad \delta \quad | \\ \circ \text{---} p_0 \text{---} \circ \end{array} = \text{ev}(\mathfrak{t}), \quad (4.2.19)$$

where $\delta := p_2 \star q_0$.

Therefore, by Lemma 4.2.1 we have $\mathfrak{s} - \mathfrak{t} \in \mathfrak{R}_{\text{NC}\mathcal{M}}$ for each case. This leads to the fact that $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$ implies $\mathfrak{s} - \mathfrak{t} \in \mathfrak{R}_{\text{NC}\mathcal{M}}$, and shows that the space induced by \rightarrow is a subspace of $\mathfrak{R}_{\text{NC}\mathcal{M}}$.

Let us now assume that \mathfrak{s} and \mathfrak{t} are two syntax trees on $\mathcal{T}_{\mathcal{M}}$ such that $\mathfrak{s} - \mathfrak{t}$ is a generator of $\mathfrak{R}_{\text{NC}\mathcal{M}}$ among (4.2.1a), (4.2.1b), and (4.2.1c).

(a) If \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.1a), we have by (4.2.16a),

$$\mathfrak{s} \xrightarrow{*} c \left(\begin{array}{c} \circ \text{---} p_2 \text{---} \circ \\ | \quad | \\ \circ \text{---} p_0 \text{---} \circ \end{array} \right) \circ_1 c \left(\begin{array}{c} \circ \text{---} q_1 \text{---} \circ \\ | \quad | \\ \circ \text{---} q_2 \text{---} \circ \end{array} \right) \quad \text{and} \quad \mathfrak{t} \xrightarrow{*} c \left(\begin{array}{c} \circ \text{---} p_2 \text{---} \circ \\ | \quad | \\ \circ \text{---} p_0 \text{---} \circ \end{array} \right) \circ_1 c \left(\begin{array}{c} \circ \text{---} q_1 \text{---} \circ \\ | \quad | \\ \circ \text{---} q_2 \text{---} \circ \end{array} \right), \quad (4.2.20)$$

where $\delta := p_1 \star q_0$ and $\delta' := r_1 \star r_0$. Since by (4.2.1a), $\delta = \delta'$, we obtain that $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$.

(b) If \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.1b), we have by (4.2.16b) and by (4.2.16c),

$$\mathfrak{s} \rightarrow c \left(\begin{array}{c} \circ \text{---} q_1 \text{---} \circ \\ | \quad | \\ \circ \text{---} p_0 \text{---} \circ \end{array} \right) \circ_2 c \left(\begin{array}{c} \circ \text{---} q_2 \text{---} \circ \\ | \quad | \\ \circ \text{---} p_2 \text{---} \circ \end{array} \right) \quad \text{and} \quad \mathfrak{t} \xrightarrow{*} c \left(\begin{array}{c} \circ \text{---} q_1 \text{---} \circ \\ | \quad | \\ \circ \text{---} p_0 \text{---} \circ \end{array} \right) \circ_2 c \left(\begin{array}{c} \circ \text{---} q_2 \text{---} \circ \\ | \quad | \\ \circ \text{---} p_2 \text{---} \circ \end{array} \right). \quad (4.2.21)$$

We obtain that $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$.

(c) Otherwise, \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.1c). We have by (4.2.16c),

$$\mathfrak{s} \xrightarrow{*} c \left(\begin{array}{c} \circ \text{---} p_1 \text{---} \circ \\ | \quad \delta \\ \circ \text{---} p_0 \text{---} \circ \end{array} \right) \circ_2 c \left(\begin{array}{c} \circ \text{---} q_1 \text{---} \circ \\ | \quad | \\ \circ \text{---} q_2 \text{---} \circ \end{array} \right) \quad \text{and} \quad \mathfrak{t} \xrightarrow{*} c \left(\begin{array}{c} \circ \text{---} p_1 \text{---} \circ \\ | \quad \delta' \\ \circ \text{---} p_0 \text{---} \circ \end{array} \right) \circ_2 c \left(\begin{array}{c} \circ \text{---} q_1 \text{---} \circ \\ | \quad | \\ \circ \text{---} q_2 \text{---} \circ \end{array} \right), \quad (4.2.22)$$

where $\delta := p_2 \star q_0$ and $\delta' := r_2 \star r_0$. Since by (4.2.1c), $\delta = \delta'$, we obtain that $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$.

Hence, for each case, we have $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$. This shows that $\mathfrak{R}_{\text{NC}\mathcal{M}}$ is a subspace of the space induced by \rightarrow . The statement of the lemma follows. \square

Lemma 4.2.4. *For any unitary magma \mathcal{M} , the rewrite rule \rightarrow is terminating.*

Proof. By denoting by T_n the set of all syntax trees on $\mathcal{T}_{\mathcal{M}}$ of arity n , let $\phi : T_n \rightarrow \mathbb{N}^2$ be the map defined in the following way. For any syntax tree \mathfrak{t} of T_n , $\phi(\mathfrak{t}) := (\alpha, \beta)$ where α is the sum, for all internal nodes x of \mathfrak{t} , of the number of internal nodes in the left subtree of x , and β is the number of internal nodes of \mathfrak{t} labeled by an \mathcal{M} -triangle whose base is not labeled by $\mathbb{1}_{\mathcal{M}}$. Let \mathfrak{s} and \mathfrak{t} be two syntax trees of T_3 such that $\mathfrak{s} \rightarrow \mathfrak{t}$. Due to the definition of \rightarrow , we have three configurations to explore. In what follows, $\eta : \mathcal{M} \rightarrow \mathbb{N}$ is the map satisfying $\eta(a) := 0$ if $a = \mathbb{1}_{\mathcal{M}}$ and $\eta(a) := 1$ otherwise.

(a) If \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.16a), we have, by denoting by \leq the lexicographic order on \mathbb{N}^2 ,

$$\begin{aligned} \phi \left(\mathfrak{c} \left(\begin{array}{c} \mathfrak{p}_1 \quad \mathfrak{p}_2 \\ \mathfrak{p}_0 \end{array} \right) \circ_1 \mathfrak{c} \left(\begin{array}{c} \mathfrak{q}_1 \quad \mathfrak{q}_2 \\ \mathfrak{q}_0 \end{array} \right) \right) &= (1, \eta(\mathfrak{p}_0) + 1) \\ &> (1, \eta(\mathfrak{p}_0)) = \phi \left(\mathfrak{c} \left(\begin{array}{c} \delta \quad \mathfrak{p}_2 \\ \mathfrak{p}_0 \end{array} \right) \circ_1 \mathfrak{c} \left(\begin{array}{c} \mathfrak{q}_1 \quad \mathfrak{q}_2 \\ \mathfrak{q}_0 \end{array} \right) \right), \end{aligned} \quad (4.2.23)$$

where $\delta := \mathfrak{p}_1 \star \mathfrak{q}_0$.

(b) If \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.16b), we have

$$\begin{aligned} \phi \left(\mathfrak{c} \left(\begin{array}{c} \mathfrak{p}_1 \quad \mathfrak{p}_2 \\ \mathfrak{p}_0 \end{array} \right) \circ_1 \mathfrak{c} \left(\begin{array}{c} \mathfrak{q}_1 \quad \mathfrak{q}_2 \\ \mathfrak{q}_0 \end{array} \right) \right) &= (1, \eta(\mathfrak{p}_0) + \eta(\mathfrak{q}_0)) \\ &> (0, \eta(\mathfrak{p}_0)) = \phi \left(\mathfrak{c} \left(\begin{array}{c} \mathfrak{q}_1 \quad \mathfrak{p}_2 \\ \mathfrak{p}_0 \end{array} \right) \circ_2 \mathfrak{c} \left(\begin{array}{c} \mathfrak{q}_1 \quad \mathfrak{q}_2 \\ \mathfrak{q}_0 \end{array} \right) \right). \end{aligned} \quad (4.2.24)$$

(c) Otherwise, \mathfrak{s} (resp. \mathfrak{t}) is of the form described by the left (resp. right) member of (4.2.16c). We have

$$\begin{aligned} \phi \left(\mathfrak{c} \left(\begin{array}{c} \mathfrak{p}_1 \quad \mathfrak{p}_2 \\ \mathfrak{p}_0 \end{array} \right) \circ_2 \mathfrak{c} \left(\begin{array}{c} \mathfrak{q}_1 \quad \mathfrak{q}_2 \\ \mathfrak{q}_0 \end{array} \right) \right) &= (0, \eta(\mathfrak{p}_0) + 1) \\ &> (0, \eta(\mathfrak{p}_0)) = \phi \left(\mathfrak{c} \left(\begin{array}{c} \mathfrak{p}_1 \quad \delta \\ \mathfrak{p}_0 \end{array} \right) \circ_2 \mathfrak{c} \left(\begin{array}{c} \mathfrak{q}_1 \quad \mathfrak{q}_2 \\ \mathfrak{q}_0 \end{array} \right) \right), \end{aligned} \quad (4.2.25)$$

where $\delta := \mathfrak{p}_2 \star \mathfrak{q}_0$.

Therefore, for all syntax trees \mathfrak{s} and \mathfrak{t} such that $\mathfrak{s} \rightarrow \mathfrak{t}$, $\phi(\mathfrak{s}) > \phi(\mathfrak{t})$. This implies that for all syntax trees \mathfrak{s} and \mathfrak{t} such that $\mathfrak{s} \neq \mathfrak{t}$ and $\mathfrak{s} \xrightarrow{*} \mathfrak{t}$, $\phi(\mathfrak{s}) > \phi(\mathfrak{t})$. Since $(0, 0)$ is the smallest element of \mathbb{N}^2 with respect to the lexicographic order \leq , the statement of the lemma follows. \square

Lemma 4.2.5. *Let \mathcal{M} be a unitary magma. The set of the normal forms of the rewrite rule \rightarrow is the set of the syntax trees \mathfrak{t} on $\mathcal{T}_{\mathcal{M}}$ such that, for any internal nodes x and y of \mathfrak{t} where y is a child of x ,*

- (i) *the base of the \mathcal{M} -triangle labeling y is labeled by $\mathbb{1}_{\mathcal{M}}$;*
- (ii) *if y is a left child of x , the first edge of the \mathcal{M} -triangle labeling x is not labeled by $\mathbb{1}_{\mathcal{M}}$.*

Proof. By Lemma 4.2.4, \rightarrow is terminating. Therefore, \rightarrow admits normal forms, which are by definition the syntax trees on $\mathcal{T}_{\mathcal{M}}$ that cannot be rewritten by \rightarrow .

Let \mathfrak{t} be a normal form of \rightarrow . The fact that \mathfrak{t} satisfies (i) is an immediate consequence of the fact that \mathfrak{t} avoids the patterns appearing as left members of (4.2.16a) and (4.2.16c). Moreover, since \mathfrak{t} avoids the patterns appearing as left members of (4.2.16b), one cannot have $\mathfrak{p}_1 \star \mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, where \mathfrak{p} (resp. \mathfrak{q}) is the label of x (resp. y). Since by (i), $\mathfrak{q}_0 = \mathbb{1}_{\mathcal{M}}$, we necessarily have $\mathfrak{p}_1 \neq \mathbb{1}_{\mathcal{M}}$. Hence, \mathfrak{t} satisfies (ii).

Conversely, if \mathfrak{t} is a syntax tree on $\mathcal{T}_{\mathcal{M}}$ satisfying (i) and (ii), a direct inspection shows that one cannot rewrite \mathfrak{t} by \rightarrow . Therefore, \mathfrak{t} is a normal form of \rightarrow . \square

Lemma 4.2.6. *Let \mathcal{M} be a finite unitary magma. The generating series of the normal forms of the rewrite rule \rightarrow is the Hilbert series $\mathcal{H}_{\text{NC}\mathcal{M}(t)}$ of $\text{NC}\mathcal{M}$.*

Proof. First, since by Lemma 4.2.4, \rightarrow is terminating, and since for any $n \geq 1$, due to the finiteness of \mathcal{M} , there are finitely many syntax trees on $\mathcal{T}_{\mathcal{M}}$ of arity n , the generating series $T(t)$ of the normal forms of \rightarrow is well-defined.

Let $S(t)$ be the generating series of the normal forms of \rightarrow such that the bases of the \mathcal{M} -triangles labeling the roots are labeled by $\mathbf{1}_{\mathcal{M}}$. Immediately from the description of the normal forms of \rightarrow provided by Lemma 4.2.5, we obtain that $S(t)$ satisfies

$$S(t) = t + mtS(t) + (m-1)mS(t)^2. \quad (4.2.26)$$

Again by Lemma 4.2.5, we have

$$T(t) = t + m(S(t) - t). \quad (4.2.27)$$

A direct computation shows that $T(t)$ satisfies the algebraic equation

$$t + (m^3 - 2m^2 + 2m - 1)t^2 + (2m^2t - 3mt + 2t - 1)T(t) + (m-1)T(t)^2 = 0. \quad (4.2.28)$$

Hence, by Proposition 4.1.6, we observe that $T(t) = \mathcal{H}_{\text{NC}\mathcal{M}(t)}$. \square

Lemma 4.2.7. *For any finite unitary magma \mathcal{M} , the rewrite rule \rightarrow is confluent.*

Proof. By contradiction, assume that \rightarrow is not confluent. Since by Lemma 4.2.4, \rightarrow is terminating, there is an integer $n \geq 1$ and two normal forms t and t' of \rightarrow of arity n such that $t \neq t'$ and $t \stackrel{*}{\rightarrow} t'$. Now, Lemma 4.2.1 together with Lemma 4.2.3 imply that $\text{ev}(t) = \text{ev}(t')$. By Proposition 4.1.5, the map $\text{ev} : \text{Free}(\text{Vect}(\mathcal{T}_{\mathcal{M}})) \rightarrow \text{NC}\mathcal{M}$ is surjective, leading to the fact that the number of normal forms of \rightarrow of arity n is greater than the number of noncrossing \mathcal{M} -cliques of arity n . However, by Lemma 4.2.6, there are as many normal forms of \rightarrow of arity n as noncrossing \mathcal{M} -cliques of arity n . This raises a contradiction and proves the statement of the lemma. \square

4.2.3. *Presentation and Koszulity.* The results of Sections 4.2.1 and 4.2.2 are finally used here to state a presentation of $\text{NC}\mathcal{M}$ and the fact that $\text{NC}\mathcal{M}$ is a Koszul operad.

Theorem 4.2.8. *Let \mathcal{M} be a finite unitary magma. Then, $\text{NC}\mathcal{M}$ admits the presentation $(\mathcal{T}_{\mathcal{M}}, \mathfrak{R}_{\text{NC}\mathcal{M}})$.*

Proof. First, since by Lemmas 4.2.4 and 4.2.7, \rightarrow is a convergent rewrite rule, and since by Lemma 4.2.3, the space induced by \rightarrow is $\mathfrak{R}_{\text{NC}\mathcal{M}}$, we can regard the underlying space of the quotient operad

$$\mathcal{O} := \text{Free}(\text{Vect}(\mathcal{T}_{\mathcal{M}})) / \langle \mathfrak{R}_{\text{NC}\mathcal{M}} \rangle \quad (4.2.29)$$

as the linear span of all normal forms of \rightarrow . Moreover, as a consequence of Lemma 4.2.1, the map $\phi : \mathcal{O} \rightarrow \text{NC}\mathcal{M}$ defined linearly for any normal form t of \rightarrow by $\phi(t) := \text{ev}(t)$ is an operad morphism. Now, by Proposition 4.1.5, ϕ is surjective. Moreover, by Lemma 4.2.6, we obtain that the dimensions of the spaces $\mathcal{O}(n)$, $n \geq 1$, are the ones of $\text{NC}\mathcal{M}(n)$. Hence, ϕ is an operad isomorphism and the statement of the theorem follows. \square

4.3.1. Treelike expressions on bubbles. Let B and E be two subsets of \mathcal{M} . We denote by $\mathcal{G}_{\mathcal{M}}^{B,E}$ the set of all \mathcal{M} -bubbles p such that the bases of p are labeled on B and all edges of p are labeled on E . Moreover, we say that \mathcal{M} is (E, B) -quasi-injective if for all $x, x' \in E$ and $y, y' \in B$, $x \star y = x' \star y' \neq \mathbb{1}_{\mathcal{M}}$ implies $x = x'$ and $y = y'$.

Lemma 4.3.1. *Let \mathcal{M} be a unitary magma, and B and E be two subsets of \mathcal{M} . If \mathcal{M} is (E, B) -quasi-injective, then any \mathcal{M} -clique admits at most one treelike expression on $\mathcal{G}_{\mathcal{M}}^{B,E}$ of a minimal degree.*

Proof. Assume that p is an \mathcal{M} -clique admitting a treelike expression on $\mathcal{G}_{\mathcal{M}}^{B,E}$. This implies that the base of p is labeled on B , all solid diagonals of p are labeled on $B \star E$, and all edges of p are labeled on E . By Proposition 4.1.2 and Lemma 4.1.3, the tree $t := \text{bt}(p)$ is a treelike expression of p on $\mathcal{B}_{\mathcal{M}}$ of a minimal degree. Now, observe that t is not necessarily a syntax tree on $\mathcal{G}_{\mathcal{M}}^{B,E}$ as required since some of its internal nodes can be labeled by bubbles that do not belong to $\mathcal{G}_{\mathcal{M}}^{B,E}$. Since \mathcal{M} is (E, B) -quasi-injective, there is one unique way to relabel the internal nodes of t by bubbles of $\mathcal{G}_{\mathcal{M}}^{B,E}$ to obtain a syntax tree on $\mathcal{G}_{\mathcal{M}}^{B,E}$ such that $\text{ev}(t') = \text{ev}(t)$. By construction, t' satisfies the properties of the statement of the lemma. \square

4.3.2. Dimensions. Let G be a set of \mathcal{M} -bubbles and $\Xi := \{\xi_{x_0}, \xi_{x_1}, \dots\}$ be a set of non-commutative variables. Given $x_i \in \mathcal{M}$, let B_{x_i} be the series of $\mathbb{N}\langle\langle\Xi\rangle\rangle$ defined by

$$B_{x_i}(\xi_{x_0}, \xi_{x_1}, \dots) := \sum_{\substack{p \in \mathcal{G}_{\mathcal{M}}^G \\ p \neq \text{---} \circ \text{---} \circ}} \prod_{i \in [|p|]} \xi_{p_i}, \quad (4.3.1)$$

where $\mathcal{G}_{\mathcal{M}}^G$ is the set of all \mathcal{M} -bubbles that can be obtained by partial compositions of elements of G . Observe from (4.3.1) that a noncommutative monomial $u \in \Xi^{\geq 2}$ appears in B_{x_i} with 1 as coefficient if and only if there is in the suboperad of $\text{NC}\mathcal{M}$ generated by G an \mathcal{M} -bubble with a base labeled by x_i and with u as border.

Let also for any $x_i \in \mathcal{M}$, the series F_{x_i} of $\mathbb{N}\langle\langle t \rangle\rangle$ defined by

$$F_{x_i}(t) := B_{x_i}\left(t + \bar{F}_{x_0}(t), t + \bar{F}_{x_1}(t), \dots\right), \quad (4.3.2)$$

where for any $x_i \in \mathcal{M}$,

$$\bar{F}_{x_i}(t) := \sum_{\substack{x_j \in \mathcal{M} \\ x_i \star x_j \neq \mathbb{1}_{\mathcal{M}}}} F_{x_j}(t). \quad (4.3.3)$$

Proposition 4.3.2. *Let \mathcal{M} be a unitary magma and G be a finite set of \mathcal{M} -bubbles such that, by denoting by B (resp. E) the set of the labels of the bases (resp. edges) of the elements of G , \mathcal{M} is (E, B) -quasi-injective. Then, the Hilbert series $\mathcal{H}_{(\text{NC}\mathcal{M})^G}(t)$ of the suboperad of $\text{NC}\mathcal{M}$ generated by G satisfies*

$$\mathcal{H}_{(\text{NC}\mathcal{M})^G}(t) = t + \sum_{x_i \in \mathcal{M}} F_{x_i}(t). \quad (4.3.4)$$

Proof. By Lemma 4.3.1, any \mathcal{M} -clique of $(\text{NC}\mathcal{M})^G$ admits exactly one treelike expression on \mathcal{M} -bubbles of $(\text{NC}\mathcal{M})^G$ of a minimal degree. For this reason, and as a consequence of the definition (4.3.3) of the series $\bar{F}_{x_i}(t)$, $x_i \in \mathcal{M}$, the series $F_{x_i}(t)$ is the generating series of all \mathcal{M} -cliques of $(\text{NC}\mathcal{M})^G$ different from $\circ - \circ$ and with a base labeled by $x_i \in \mathcal{M}$. Therefore, the expression (4.3.4) for the Hilbert series of $(\text{NC}\mathcal{M})^G$ follows. \square

As a side remark, Proposition 4.3.2 can be proved by using the notion of bubble decompositions of operads developed in [CG14]. This result provides a practical method to compute the dimensions of some suboperads $(\text{NC}\mathcal{M})^G$ of $\text{NC}\mathcal{M}$ by describing the series (4.3.1) of the bubbles of $\mathcal{B}_{\mathcal{M}}^G$. This result implies also, when G satisfies the requirement of Proposition 4.3.2, that the Hilbert series of $(\text{NC}\mathcal{M})^G$ is algebraic.

4.3.3. *First example : a cubic suboperad.* Consider the suboperad of NCE_2 generated by

$$G := \left\{ \begin{array}{c} \triangle_{e_1} \\ \triangle_{e_2} \end{array} \right\}. \quad (4.3.5)$$

Computer experiments show that the generators of $(\text{NCE}_2)^G$ are not subjected to any quadratic relation but are subjected to the four cubic nontrivial relations

$$\begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \circ_2 \begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \right) = \begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \right), \quad (4.3.6a)$$

$$\begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_1} \\ \triangle_{e_2} \end{array} \circ_2 \begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \right) = \begin{array}{c} \triangle_{e_1} \\ \triangle_{e_2} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \right), \quad (4.3.6b)$$

$$\begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \circ_2 \begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \right) = \begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \begin{array}{c} \triangle_{e_1} \\ \triangle_{e_1} \end{array} \right), \quad (4.3.6c)$$

$$\begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_1} \\ \triangle_{e_2} \end{array} \circ_2 \begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \right) = \begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \left(\begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \circ_2 \begin{array}{c} \triangle_{e_2} \\ \triangle_{e_2} \end{array} \right). \quad (4.3.6d)$$

Hence, $(\text{NCE}_2)^G$ is not a quadratic operad. Moreover, it is possible to prove that this operad does not admit any other nontrivial relations between its generators. This can be performed by defining a rewrite rule on the syntax trees on G , consisting in rewriting the left patterns of (4.3.6a), (4.3.6b), (4.3.6c), and (4.3.6d) into their respective right patterns, and by checking that this rewrite rule admits the required properties (like the ones establishing the presentation of $\text{NC}\mathcal{M}$ by Theorem 4.2.8). The existence of this nonquadratic operad shows that $\text{NC}\mathcal{M}$ contains nonquadratic suboperads even if it is quadratic.

One can prove by induction on the arity that the set of bubbles of $(\text{NCE}_2)^G$ is the set $B_1 \sqcup B_2$ where B_1 (resp. B_2) is the set of all bubbles whose bases are labeled by e_1 (resp. e_2) and the border is $1e_1$ (resp. $1e_2$), or 111^*e_1 , or 111^*e_2 . Hence, we obtain that

$$B_{\mathbb{1}}(\xi_{\mathbb{1}}, \xi_{e_1}, \xi_{e_2}) = 0, \quad (4.3.7a)$$

$$B_{e_1}(\xi_{\mathbb{1}}, \xi_{e_1}, \xi_{e_2}) = \frac{\xi_{\mathbb{1}}}{1 - \xi_{\mathbb{1}}} (\xi_{e_1} + \xi_{\mathbb{1}} \xi_{e_2}) = B_{e_2}(\xi_{\mathbb{1}}, \xi_{e_2}, \xi_{e_1}), \quad (4.3.7b)$$

Moreover, one can check that G satisfies the conditions required by Proposition 4.3.2. We hence have

$$\bar{F}_{\mathbb{1}}(t) = F_{e_1}(t) + F_{e_2}(t), \quad (4.3.8a) \quad \bar{F}_{e_1}(t) = F_{\mathbb{1}}(t) = \bar{F}_{e_2}(t), \quad (4.3.8b)$$

and

$$F_1(t) = 0, \quad (4.3.9a)$$

$$F_{e_1}(t) = B_{e_1}(t + F_{e_1}(t) + F_{e_2}(t), t, t) = B_{e_2}(t + F_{e_1}(t) + F_{e_2}(t), t, t) = F_{e_2}(t), \quad (4.3.9b)$$

By Proposition 4.3.2, the Hilbert series of $(\text{NC}\mathbb{E}_2)^G$ satisfies

$$\mathcal{H}_{(\text{NC}\mathbb{E}_2)^G}(t) = t + F_1(t) + F_{e_1}(t) + F_{e_2}(t) = t + 2F_{e_1}(t), \quad (4.3.10)$$

and, by a straightforward computation, we obtain that this series satisfies the algebraic equation

$$t + (t - 1)\mathcal{H}_{(\text{NC}\mathbb{E}_2)^G}(t) + (2t + 1)\mathcal{H}_{(\text{NC}\mathbb{E}_2)^G}(t)^2 = 0. \quad (4.3.11)$$

The first dimensions of $(\text{NC}\mathbb{E}_2)^G$ are

$$1, 2, 8, 36, 180, 956, 5300, 30316, \quad (4.3.12)$$

and form Sequence [A129148](#) of [Slo].

4.3.4. Second example : a suboperad of Motzkin paths. Consider the suboperad of $\text{NC}\mathbb{D}_0$ generated by

$$G := \left\{ \begin{array}{c} \text{triangle} \\ \text{square} \end{array} \right\}. \quad (4.3.13)$$

Computer experiments show that the generators of $(\text{NC}\mathbb{D}_0)^G$ are subjected to four quadratic nontrivial relations

$$\begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} \circ_1 \begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} = \begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} \circ_2 \begin{array}{c} \text{triangle} \\ \text{triangle} \end{array}, \quad (4.3.14a)$$

$$\begin{array}{c} \text{square} \\ \text{square} \end{array} \circ_1 \begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} = \begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} \circ_2 \begin{array}{c} \text{square} \\ \text{square} \end{array}, \quad (4.3.14b)$$

$$\begin{array}{c} \text{triangle} \\ \text{square} \end{array} \circ_1 \begin{array}{c} \text{square} \\ \text{square} \end{array} = \begin{array}{c} \text{square} \\ \text{square} \end{array} \circ_2 \begin{array}{c} \text{triangle} \\ \text{square} \end{array}, \quad (4.3.14c)$$

$$\begin{array}{c} \text{square} \\ \text{square} \end{array} \circ_1 \begin{array}{c} \text{square} \\ \text{square} \end{array} = \begin{array}{c} \text{square} \\ \text{square} \end{array} \circ_3 \begin{array}{c} \text{square} \\ \text{square} \end{array}. \quad (4.3.14d)$$

It is possible to prove that this operad does not admit any other nontrivial relations between its generators. This can be performed by defining a rewrite rule on the syntax trees on G , consisting in rewriting the left patterns of (4.3.14a), (4.3.14b), (4.3.14c), and (4.3.14d) into their respective right patterns, and by checking that this rewrite rule admits the required properties (like the ones establishing the presentation of $\text{NC}\mathcal{M}$ by Theorem 4.2.8).

One can prove by induction on the arity that the set of bubbles of $(\text{NC}\mathbb{D}_0)^G$ is the set of all bubbles whose bases are labeled by $\mathbb{1}$ and borders are words of $\{1, 0\}^{\geq 2}$ such that each occurrence of 0 has a $\mathbb{1}$ immediately at its left and a $\mathbb{1}$ immediately at its right. Hence, we obtain that

$$B_1(\xi_1, \xi_0) = \frac{1}{1 - \xi_1 - \xi_1 \xi_0} \xi_1 - \xi_1, \quad (4.3.15a)$$

$$B_0(\xi_1, \xi_0) = 0. \quad (4.3.15b)$$

Moreover, one can check that G satisfies the conditions required by Proposition 4.3.2. We hence have

$$\bar{F}_1(t) = F_0(t), \quad (4.3.16a) \quad \bar{F}_0(t) = F_1(t) + F_0(t), \quad (4.3.16b)$$

and

$$F_1(t) = B_1(t, t + F_1(t)), \quad (4.3.17a) \quad F_0(t) = 0. \quad (4.3.17b)$$

By Proposition 4.3.2, the Hilbert series of $(\text{NC}\mathbb{D}_0)^G$ satisfies

$$\mathcal{H}_{(\text{NC}\mathbb{D}_0)^G}(t) = t + F_1(t), \quad (4.3.18)$$

and, by a straightforward computation, we obtain that this series satisfies the algebraic equation

$$t + (t - 1)\mathcal{H}_{(\text{NC}\mathbb{D}_0)^G}(t) + t\mathcal{H}_{(\text{NC}\mathbb{D}_0)^G}(t)^2 = 0. \quad (4.3.19)$$

The first dimensions of $(\text{NC}\mathbb{D}_0)^G$ are

$$1, 1, 2, 4, 9, 21, 51, 127, \quad (4.3.20)$$

and form Sequence **A001006** of [Slo]. The operad $(\text{NC}\mathbb{D}_0)^G$ has the same presentation by generators and relations (and thus, the same Hilbert series) as the operad Motz defined in [Gir15], involving Motzkin paths. Hence, $(\text{NC}\mathbb{D}_0)^G$ and Motz are two isomorphic operads. Note in passing that these two operads are not isomorphic to the operad $\text{Mot}\mathbb{D}_0$ constructed in Section 3.2.4 and involving Motzkin configurations. Indeed, the sequence of the dimensions of this last operad is a shifted version of the one of $(\text{NC}\mathbb{D}_0)^G$ and Motz.

4.4. Algebras over the noncrossing clique operads. We begin by briefly describing $\text{NC}\mathcal{M}$ -algebras in terms of relations between their operations and the free $\text{NC}\mathcal{M}$ -algebras over one generator. We continue this section by providing two ways to construct (non-necessarily free) $\text{NC}\mathcal{M}$ -algebras. The first one takes as input an associative algebra endowed with endofunctions satisfying some conditions, and the second one takes as input a monoid.

4.4.1. Relations. From the presentation of $\text{NC}\mathcal{M}$ established by Theorem 4.2.8, any $\text{NC}\mathcal{M}$ -algebra is a vector space \mathcal{A} endowed with binary linear operations

$$\hat{\underset{\Delta}{\underset{p_0}{\underset{\Delta}{p_1 p_2}}}} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad \mathbf{p} \in \mathcal{T}_{\mathcal{M}}, \quad (4.4.1)$$

satisfying, for all $a_1, a_2, a_3 \in \mathcal{A}$, the relations

$$\left(a_1 \hat{\underset{\Delta}{\underset{q_0}{\underset{\Delta}{q_1 q_2}}}} a_2 \right) \hat{\underset{\Delta}{\underset{p_0}{\underset{\Delta}{p_1 p_2}}}} a_3 = \left(a_1 \hat{\underset{\Delta}{\underset{r_0}{\underset{\Delta}{q_1 q_2}}}} a_2 \right) \hat{\underset{\Delta}{\underset{p_0}{\underset{\Delta}{r_1 p_2}}}} a_3, \quad \text{if } \mathbf{p}_1 \star \mathbf{q}_0 = \mathbf{r}_1 \star \mathbf{r}_0 \neq \mathbf{1}_{\mathcal{M}}, \quad (4.4.2a)$$

$$\left(a_1 \hat{\underset{\Delta}{\underset{q_0}{\underset{\Delta}{q_1 q_2}}}} a_2 \right) \hat{\underset{\Delta}{\underset{p_0}{\underset{\Delta}{p_1 p_2}}}} a_3 = a_1 \hat{\underset{\Delta}{\underset{p_0}{\underset{\Delta}{q_1 r_2}}}} \left(a_2 \hat{\underset{\Delta}{\underset{r_0}{\underset{\Delta}{q_2 p_2}}}} a_3 \right), \quad \text{if } \mathbf{p}_1 \star \mathbf{q}_0 = \mathbf{r}_2 \star \mathbf{r}_0 = \mathbf{1}_{\mathcal{M}}, \quad (4.4.2b)$$

$$a_1 \hat{\underset{\Delta}{\underset{p_0}{\underset{\Delta}{p_1 p_2}}}} \left(a_2 \hat{\underset{\Delta}{\underset{q_0}{\underset{\Delta}{q_1 q_2}}}} a_3 \right) = a_1 \hat{\underset{\Delta}{\underset{p_0}{\underset{\Delta}{p_1 r_2}}}} \left(a_2 \hat{\underset{\Delta}{\underset{r_0}{\underset{\Delta}{q_1 q_2}}}} a_3 \right), \quad \text{if } \mathbf{p}_2 \star \mathbf{q}_0 = \mathbf{r}_2 \star \mathbf{r}_0 \neq \mathbf{1}_{\mathcal{M}}, \quad (4.4.2c)$$

where \mathbf{p} , \mathbf{q} , and \mathbf{r} are \mathcal{M} -triangles. Remark that \mathcal{M} has to be finite because Theorem 4.2.8 requires this property as premise.

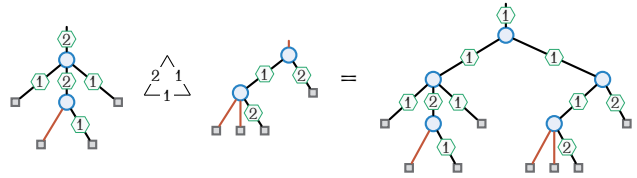
4.4.2. *Free algebras over one generator.* From the realization of $\text{NC}\mathcal{M}$ coming from its definition as a suboperad of $\text{C}\mathcal{M}$, the free $\text{NC}\mathcal{M}$ -algebra over one generator is the linear span $\text{NC}\mathcal{M}$ of all noncrossing \mathcal{M} -cliques endowed with the linear operations

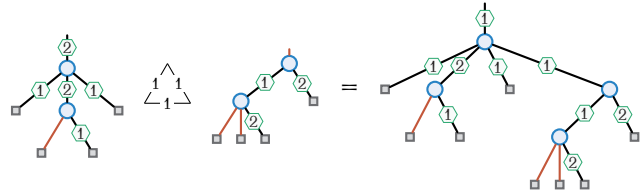
$$\hat{\Delta}_{p_0}^{p_1 p_2} : \text{NC}\mathcal{M}(n) \otimes \text{NC}\mathcal{M}(m) \rightarrow \text{NC}\mathcal{M}(n+m), \quad p \in \mathcal{T}_{\mathcal{M}}, n, m \geq 1, \quad (4.4.3)$$

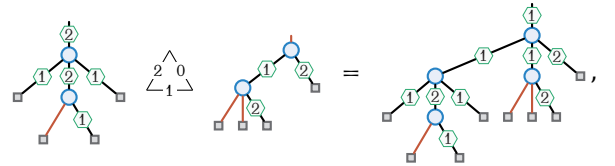
defined, for any noncrossing \mathcal{M} -cliques q and τ , by

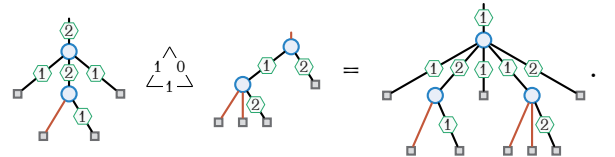
$$q \hat{\Delta}_{p_0}^{p_1 p_2} \tau := \left(\begin{array}{c} p_1 \quad p_2 \\ \swarrow \quad \searrow \\ p_0 \end{array} \circ_2 \tau \right) \circ_1 q. \quad (4.4.4)$$

In terms of \mathcal{M} -Schröder trees (see Section 4.1.3), (4.4.4) is the \mathcal{M} -Schröder tree obtained by grafting the \mathcal{M} -Schröder trees of q and τ respectively as left and right children of a binary corolla having its edge adjacent to the root labeled by p_0 , its first edge labeled by $p_1 \star q_0$, and second edge labeled by $p_2 \star \tau_0$, and by contracting each of these two edges when labeled by $1_{\mathcal{M}}$. For instance, in the free NCN_3 -algebra, we have


(4.4.5a)


(4.4.5b)


(4.4.5c)


(4.4.5d)

4.4.3. *From associative algebras.* Let \mathcal{A} be an associative algebra with associative product denoted by \odot , and

$$\omega_x : \mathcal{A} \rightarrow \mathcal{A}, \quad x \in \mathcal{M}, \quad (4.4.6)$$

be a family of linear maps, not necessarily associative algebra morphisms, indexed by the elements of \mathcal{M} . We say that \mathcal{A} together with this family (4.4.6) of maps is \mathcal{M} -compatible if

$$\omega_{1_{\mathcal{M}}} = \text{Id}_{\mathcal{A}} \quad (4.4.7)$$

where $\text{Id}_{\mathcal{A}}$ is the identity map on \mathcal{A} , and

$$\omega_x \circ \omega_y = \omega_{x \star y}, \quad (4.4.8)$$

for all $x, y \in \mathcal{M}$. Let us now use \mathcal{M} -compatible associative algebras to construct NC \mathcal{M} -algebras.

Theorem 4.4.1. *Let \mathcal{M} be a finite unitary magma and \mathcal{A} be an \mathcal{M} -compatible associative algebra. The vector space \mathcal{A} endowed with the binary linear operations*

$$\widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad \mathbf{p} \in \mathcal{T}_{\mathcal{M}}, \quad (4.4.9)$$

defined for each \mathcal{M} -triangle \mathbf{p} and any $a_1, a_2 \in \mathcal{A}$ by

$$a_1 \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} a_2 := \omega_{p_0}(\omega_{p_1}(a_1) \odot \omega_{p_2}(a_2)), \quad (4.4.10)$$

is an NC \mathcal{M} -algebra.

Proof. Let us prove that the operations (4.4.9) satisfy Relations (4.4.2a), (4.4.2b), and (4.4.2c) of NC \mathcal{M} -algebras. Since \mathcal{M} is finite, this amounts to show that these operations endow \mathcal{A} with an NC \mathcal{M} -algebra structure. For this, let a_1, a_2 , and a_3 be three elements of \mathcal{A} , and \mathbf{p}, \mathbf{q} , and \mathbf{r} be three \mathcal{M} -triangles.

(a) When $p_1 \star q_0 = r_1 \star r_0 \neq \mathbf{1}_{\mathcal{M}}$, since by (4.4.8), $\omega_{p_1} \circ \omega_{q_0} = \omega_{r_1} \circ \omega_{r_0}$,

$$\begin{aligned} \left(a_1 \widehat{\underset{\mathcal{L}_{q_0} \Delta}{q_1 \ q_2}} a_2 \right) \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} a_3 &= \omega_{q_0}(\omega_{q_1}(a_1) \odot \omega_{q_2}(a_2)) \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} a_3 \\ &= \omega_{p_0}(\omega_{p_1}(\omega_{q_0}(\omega_{q_1}(a_1) \odot \omega_{q_2}(a_2))) \odot \omega_{p_2}(a_3)) \\ &= \omega_{p_0}(\omega_{r_1}(\omega_{r_0}(\omega_{q_1}(a_1) \odot \omega_{q_2}(a_2))) \odot \omega_{p_2}(a_3)) \\ &= \left(a_1 \widehat{\underset{\mathcal{L}_{r_0} \Delta}{q_1 \ q_2}} a_2 \right) \widehat{\underset{\mathcal{L}_{p_0} \Delta}{r_1 \ r_2}} a_3, \end{aligned} \quad (4.4.11)$$

so that (4.4.2a) holds.

(b) When $p_1 \star q_0 = r_2 \star r_0 = \mathbf{1}_{\mathcal{M}}$, since by (4.4.7), $\omega_{p_1} \circ \omega_{q_0} = \omega_{r_2} \circ \omega_{r_0} = \text{Id}_{\mathcal{A}}$ and since \odot is associative,

$$\begin{aligned} \left(a_1 \widehat{\underset{\mathcal{L}_{q_0} \Delta}{q_1 \ q_2}} a_2 \right) \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} a_3 &= \omega_{q_0}(\omega_{q_1}(a_1) \odot \omega_{q_2}(a_2)) \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} a_3 \\ &= \omega_{p_0}(\omega_{p_1}(\omega_{q_0}(\omega_{q_1}(a_1) \odot \omega_{q_2}(a_2))) \odot \omega_{p_2}(a_3)) \\ &= \omega_{p_0}(\omega_{q_1}(a_1) \odot \omega_{q_2}(a_2) \odot \omega_{p_2}(a_3)) \\ &= \omega_{p_0}(\omega_{q_1}(a_1) \odot (\omega_{q_2}(a_2) \odot \omega_{p_2}(a_3))) \\ &= \omega_{p_0}(\omega_{q_1}(a_1) \odot \omega_{r_2}(\omega_{r_0}(\omega_{q_2}(a_2) \odot \omega_{p_2}(a_3)))) \\ &= a_1 \widehat{\underset{\mathcal{L}_{p_0} \Delta}{q_1 \ r_2}} \left(a_2 \widehat{\underset{\mathcal{L}_{r_0} \Delta}{q_2 \ p_2}} a_3 \right), \end{aligned} \quad (4.4.12)$$

so that (4.4.2b) holds.

(c) When $p_2 \star q_0 = r_2 \star r_0 \neq \mathbf{1}_{\mathcal{M}}$, since by (4.4.8), $\omega_{p_2} \circ \omega_{q_0} = \omega_{r_2} \circ \omega_{r_0}$,

$$\begin{aligned} a_1 \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} \left(a_2 \widehat{\underset{\mathcal{L}_{q_0} \Delta}{q_1 \ q_2}} a_3 \right) &= a_1 \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ p_2}} \omega_{q_0}(\omega_{q_1}(a_2) \odot \omega_{q_2}(a_3)) \\ &= \omega_{p_0}(\omega_{p_1}(a_1) \odot \omega_{p_2}(\omega_{q_0}(\omega_{q_1}(a_2) \odot \omega_{q_2}(a_3)))) \\ &= \omega_{p_0}(\omega_{p_1}(a_1) \odot \omega_{r_2}(\omega_{r_0}(\omega_{q_1}(a_2) \odot \omega_{q_2}(a_3)))) \\ &= a_1 \widehat{\underset{\mathcal{L}_{p_0} \Delta}{p_1 \ r_2}} \left(a_2 \widehat{\underset{\mathcal{L}_{r_0} \Delta}{q_1 \ q_2}} a_3 \right), \end{aligned} \quad (4.4.13)$$

so that (4.4.2c) holds.

Therefore, \mathcal{A} is an NC \mathcal{M} -algebra. \square

By Theorem 4.4.1, \mathcal{A} has the structure of an NC \mathcal{M} -algebra. Hence, there is a left action \cdot of the operad NC \mathcal{M} on the tensor algebra of \mathcal{A} of the form

$$\cdot : \text{NC}\mathcal{M}(n) \otimes \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}, \quad n \geq 1, \quad (4.4.14)$$

whose definition comes from the ones of the operations (4.4.9) and Relation (1.2.19). We describe here an algorithm to compute the action of any element of NC \mathcal{M} of arity n on tensors $a_1 \otimes \cdots \otimes a_n$ of $\mathcal{A}^{\otimes n}$. First, if \mathfrak{b} is an \mathcal{M} -bubble of arity n ,

$$\mathfrak{b} \cdot (a_1 \otimes \cdots \otimes a_n) = \omega_{\mathfrak{b}_0} \left(\prod_{i \in [n]} \omega_{\mathfrak{b}_i} (a_i) \right), \quad (4.4.15)$$

where the product of (4.4.15) denotes the iterated version of the associative product \odot of \mathcal{A} . When \mathfrak{p} is a noncrossing \mathcal{M} -clique of arity n , \mathfrak{p} acts recursively on $a_1 \otimes \cdots \otimes a_n$ as follows. One has

$$\mathfrak{p} \cdot a_1 = a_1 \quad (4.4.16)$$

when $\mathfrak{p} = \circ - \circ$, and

$$\mathfrak{p} \cdot (a_1 \otimes \cdots \otimes a_n) = \mathfrak{b} \cdot \left((\mathfrak{r}_1 \cdot (a_1 \otimes \cdots \otimes a_{|\mathfrak{r}_1|})) \otimes \cdots \otimes (\mathfrak{r}_k \cdot (a_{|\mathfrak{r}_1|+\cdots+|\mathfrak{r}_{k-1}|+1} \otimes \cdots \otimes a_n)) \right), \quad (4.4.17)$$

where, by setting \mathfrak{t} as the bubble tree $\text{bt}(\mathfrak{p})$ of \mathfrak{p} (see Section 4.1.2), \mathfrak{b} and $\mathfrak{r}_1, \dots, \mathfrak{r}_k$ are the unique \mathcal{M} -bubble and noncrossing \mathcal{M} -cliques such that $\mathfrak{t} = \mathfrak{c}(\mathfrak{b}) \circ [\text{bt}(\mathfrak{r}_1), \dots, \text{bt}(\mathfrak{r}_k)]$.

Here are few examples of the construction provided by Theorem 4.4.1.

Noncommutative polynomials and selected concatenation: Let us consider the unitary magma \mathbb{S}_ℓ of all subsets of $[\ell]$ with the union as product. Let $A := \{a_j : j \in [\ell]\}$ be an alphabet of noncommutative letters. We define on the associative algebra $\mathbb{K}\langle A \rangle$ of polynomials on A the linear maps

$$\omega_S : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle, \quad S \in \mathbb{S}_\ell, \quad (4.4.18)$$

as follows. For any $u \in A^*$ and $S \in \mathbb{S}_\ell$, we set

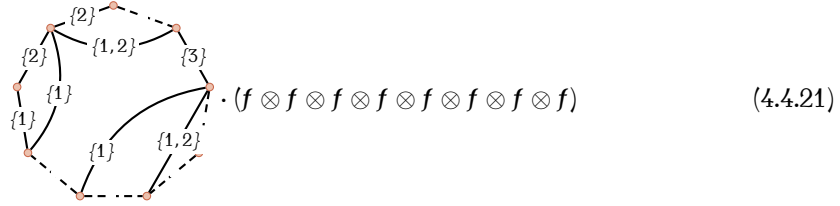
$$\omega_S(u) := \begin{cases} u & \text{if } |u|_{a_j} \geq 1 \text{ for all } j \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (4.4.19)$$

Since, for all $u \in A^*$, $\omega_\emptyset(u) = u$ and $(\omega_S \circ \omega_{S'}) (u) = \omega_{S \cup S'}(u)$, and \emptyset is the unit of \mathbb{S}_ℓ , we obtain from Theorem 4.4.1 that the operations (4.4.9) endow $\mathbb{K}\langle A \rangle$ with an NC \mathbb{S}_ℓ -algebra structure. For instance, when $\ell := 3$, one has

$$(a_1 + a_1 a_3 + a_2 a_2) \underset{\langle \{2,3\} \rangle}{\overset{\{1\} \{2\}}{\wedge}} (1 + a_3 + a_2 a_1) = a_1 a_3 a_2 a_1, \quad (4.4.20a)$$

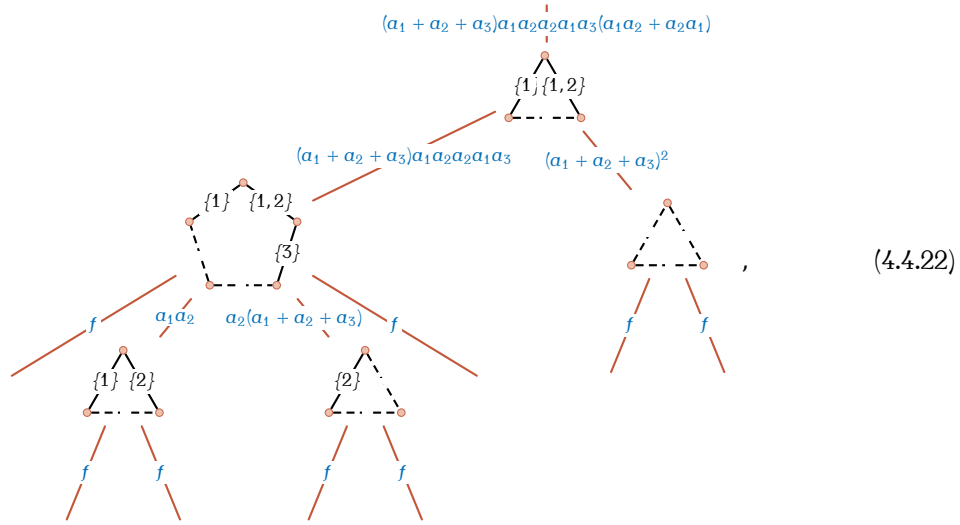
$$(a_1 + a_1 a_3 + a_2 a_2) \underset{\langle \{1,3\} \rangle}{\overset{\{1\} \emptyset}{\wedge}} (1 + a_3 + a_2 a_1) = 2 a_1 a_3 + a_1 a_3 a_3 + a_1 a_3 a_2 a_1. \quad (4.4.20b)$$

Besides, to compute the action



$$\cdot (f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f) \quad (4.4.21)$$

where $f := a_1 + a_2 + a_3$, we use the above algorithm and (4.4.15) and (4.4.17). By presenting the computation on the bubble tree of the noncrossing \mathbb{S}_3 -clique of (4.4.21), we obtain



$$(a_1 + a_2 + a_3)a_1a_2a_2a_1a_3(a_1a_2 + a_2a_1) \quad (4.4.22)$$

so that (4.4.21) is equal to the polynomial $(a_1 + a_2 + a_3)a_1a_2a_2a_1a_3(a_1a_2 + a_2a_1)$.

Noncommutative polynomials and constant term product: Consider here the unitary magma \mathbb{D}_0 . Let $A := \{a_1, a_2, \dots\}$ be an infinite alphabet of noncommutative letters. We define on the associative algebra $\mathbb{K}\langle A \rangle$ of polynomials on A the linear maps

$$\omega_{\mathbb{1}}, \omega_0 : \mathbb{K}\langle A \rangle \rightarrow \mathbb{K}\langle A \rangle, \quad (4.4.23)$$

as follows. For any $u \in A^*$, we set $\omega_{\mathbb{1}}(u) := u$, and

$$\omega_0(u) := \begin{cases} 1 & \text{if } u = \epsilon, \\ 0 & \text{otherwise.} \end{cases} \quad (4.4.24)$$

In other terms, $\omega_0(f)$ is the constant term, denoted by $f(0)$, of the polynomial $f \in \mathbb{K}\langle A \rangle$. Since $\omega_{\mathbb{1}}$ is the identity map on $\mathbb{K}\langle A \rangle$ and, for all $u \in A^*$,

$$(\omega_0 \circ \omega_0)(f) = (f(0))(0) = f(0) = \omega_0(f), \quad (4.4.25)$$

we obtain from Theorem 4.4.1 that the operations (4.4.9) endow $\mathbb{K}\langle A \rangle$ with a NCD_0 -algebra structure. For instance, for all polynomials f_1 and f_2 of $\mathbb{K}\langle A \rangle$, we have

$$f_1 \hat{\triangleleft}_{\mathbb{1}}^{\mathbb{1}} f_2 = f_1 f_2, \quad (4.4.26a) \quad f_1 \overset{0}{\triangleleft}_{\mathbb{1}}^{\mathbb{1}} f_2 = f_1(0) f_2, \quad (4.4.26c)$$

$$f_1 \overset{\mathbb{1}}{\triangleleft}_0^{\mathbb{1}} f_2 = (f_1 f_2)(0) = f_1(0) f_2(0), \quad (4.4.26b) \quad f_1 \hat{\triangleleft}_{\mathbb{1}}^0 f_2 = f_1(f_2(0)). \quad (4.4.26d)$$

From (4.4.26c) and (4.4.26d), when $f_1(0) = 1 = f_2(0)$,

$$f_1 \left(\underset{\triangleleft}{\overset{\circ}{\mathbb{1}}} + \underset{\triangleleft}{\overset{\circ}{\mathbb{1}}} \right) f_2 = f_1(0) f_2 + f_1(f_2(0)) = f_1 + f_2. \quad (4.4.27)$$

4.4.4. From monoids. If \mathcal{M} is a monoid, with binary associative operation \star and unit $\mathbb{1}_{\mathcal{M}}$, we denote by $\mathbb{K}\langle \mathcal{M}^* \rangle$ the space of all noncommutative polynomials on \mathcal{M} , seen as an alphabet, with coefficients in \mathbb{K} . This space can be endowed with an NC \mathcal{M} -algebra structure as follows.

For any $x \in \mathcal{M}$ and any word $w \in \mathcal{M}^*$, let

$$x * w := (x \star w_1) \dots (x \star w_{|w|}). \quad (4.4.28)$$

This operation $*$ is linearly extended on the right on $\mathbb{K}\langle \mathcal{M}^* \rangle$.

Proposition 4.4.2. *Let \mathcal{M} be a finite monoid. The vector space $\mathbb{K}\langle \mathcal{M}^* \rangle$ endowed with the binary linear operations*

$$\underset{\triangleleft}{\overset{\mathfrak{p}_1 \ \mathfrak{p}_2}{\mathfrak{p}_0}} : \mathbb{K}\langle \mathcal{M}^* \rangle \otimes \mathbb{K}\langle \mathcal{M}^* \rangle \rightarrow \mathbb{K}\langle \mathcal{M}^* \rangle, \quad \mathfrak{p} \in \mathcal{T}_{\mathcal{M}}, \quad (4.4.29)$$

defined for each \mathcal{M} -triangle \mathfrak{p} and any $f_1, f_2 \in \mathbb{K}\langle \mathcal{M}^* \rangle$ by

$$f_1 \underset{\triangleleft}{\overset{\mathfrak{p}_1 \ \mathfrak{p}_2}{\mathfrak{p}_0}} f_2 := \mathfrak{p}_0 * ((\mathfrak{p}_1 * f_1) (\mathfrak{p}_2 * f_2)), \quad (4.4.30)$$

is an NC \mathcal{M} -algebra.

Proof. This follows from Theorem 4.4.1 as a particular case of the general construction it provides. Indeed, $\mathbb{K}\langle \mathcal{M}^* \rangle$ is an associative algebra for the concatenation product of words. Moreover, by defining maps $\omega_x : \mathbb{K}\langle \mathcal{M}^* \rangle \rightarrow \mathbb{K}\langle \mathcal{M}^* \rangle$, $x \in \mathcal{M}$, linearly by $\omega_x(u) := x * u$ for any word $u \in \mathcal{M}^*$, we obtain, since \mathcal{M} is a monoid, that this family of maps satisfies (4.4.7) and (4.4.8). Now, since the definition (4.4.30) is the specialization of the definition (4.4.10) in this particular case, the statement of the proposition follows. \square

Here are few examples of the construction provided by Proposition 4.4.2.

Words and double shifted concatenation: Consider the monoid \mathbb{N}_{ℓ} for an $\ell \geq 1$. By Proposition 4.4.2, the operations (4.4.29) endow $\mathbb{K}\langle \mathbb{N}_{\ell}^* \rangle$ with a structure of an NC \mathbb{N}_{ℓ} -algebra. For instance, in $\mathbb{K}\langle \mathbb{N}_2^* \rangle$, one has

$$0211 \underset{\triangleleft}{\overset{\circ}{\mathbb{1}}} 312 = 3100023. \quad (4.4.31)$$

Words and erasing concatenation: Consider the monoid \mathbb{D}_{ℓ} for an $\ell \geq 0$. By Proposition 4.4.2, the operations (4.4.29) endow $\mathbb{K}\langle \mathbb{D}_{\ell}^* \rangle$ with a structure of an NC \mathbb{D}_{ℓ} -algebra. For instance, for all words u and v of \mathbb{D}_{ℓ}^* , we have

$$u \underset{\triangleleft}{\overset{\mathbb{1}}{\mathbb{1}}} v = uv, \quad (4.4.32a) \quad u \underset{\triangleleft}{\overset{\mathbb{1}}{\mathbb{1}}} v = 0^{|u|+|v|}, \quad (4.4.32c)$$

$$u \underset{\triangleleft}{\overset{\mathbb{1}}{d_i}} v = (uv)_{d_i}, \quad (4.4.32b) \quad u \underset{\triangleleft}{\overset{d_i \ d_j}{\mathbb{1}}} v = u_{d_i} v_{d_j}, \quad (4.4.32d)$$

where, for any word w of \mathbb{D}_{ℓ}^* and any element d_j of \mathbb{D}_{ℓ} , $j \in [\ell]$, w_{d_j} is the word obtained by replacing each occurrence of $\mathbb{1}$ by d_j and each occurrence of d_i , $i \in [\ell]$, by 0 in w .

4.5. **Koszul dual.** Since by Theorem 4.2.8, the operad $\text{NC}\mathcal{M}$ is binary and quadratic, this operad admits a Koszul dual $\text{NC}\mathcal{M}^!$. We end the study of $\text{NC}\mathcal{M}$ by collecting the main properties of $\text{NC}\mathcal{M}^!$.

4.5.1. *Presentation.* Let $\mathfrak{R}_{\text{NC}\mathcal{M}}^!$ be the subspace of $\text{Free}(\text{Vect}(\mathcal{T}_{\mathcal{M}}))(3)$ generated by the elements

$$\sum_{\substack{p_1, q_0 \in \mathcal{M} \\ p_1 \star q_0 = \delta}} c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{q}_0 \end{array} \right), \quad p_0, p_2, q_1, q_2 \in \mathcal{M}, \delta \in \bar{\mathcal{M}}, \quad (4.5.1a)$$

$$\sum_{\substack{p_1, q_0 \in \mathcal{M} \\ p_1 \star q_0 = \mathbf{1}_{\mathcal{M}}}} c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{q}_0 \end{array} \right) - c \left(\begin{array}{c} \text{q}_1 \quad \text{p}_1 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_2 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{q}_0 \end{array} \right), \quad p_0, p_2, q_1, q_2 \in \mathcal{M}, \quad (4.5.1b)$$

$$\sum_{\substack{p_2, q_0 \in \mathcal{M} \\ p_2 \star q_0 = \delta}} c \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{q}_0 \end{array} \right), \quad p_0, p_1, q_1, q_2 \in \mathcal{M}, \delta \in \bar{\mathcal{M}}, \quad (4.5.1c)$$

where p and q are \mathcal{M} -triangles.

Proposition 4.5.1. *Let \mathcal{M} be a finite unitary magma. Then, the Koszul dual $\text{NC}\mathcal{M}^!$ of $\text{NC}\mathcal{M}$ admits the presentation $(\mathcal{T}_{\mathcal{M}}, \mathfrak{R}_{\text{NC}\mathcal{M}}^!)$.*

Proof. Let

$$f := \sum_{t \in T_3} \lambda_t t \quad (4.5.2)$$

be a generic element of $\mathfrak{R}_{\text{NC}\mathcal{M}}^!$, where T_3 is the set of all syntax trees on $\mathcal{T}_{\mathcal{M}}$ or arity 3 and the λ_t are coefficients of \mathbb{K} . By definition of Koszul duality of operads, $\langle r, f \rangle = 0$ for all $r \in \mathfrak{R}_{\text{NC}\mathcal{M}}$, where $\langle -, - \rangle$ is the scalar product defined in (1.2.16). Then, since $\mathfrak{R}_{\text{NC}\mathcal{M}}^!$ is the subspace of $\text{Free}(\text{Vect}(\mathcal{T}_{\mathcal{M}}))(3)$ generated by (4.2.1a), (4.2.1b), and (4.2.1c), one has

$$\lambda \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{q}_0 \end{array} \right) - \lambda \left(\begin{array}{c} \text{r}_1 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{r}_0 \end{array} \right) = 0, \quad p_1 \star q_0 = r_1 \star r_0 \neq \mathbf{1}_{\mathcal{M}}, \quad (4.5.3a)$$

$$\lambda \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_1 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{q}_0 \end{array} \right) + \lambda \left(\begin{array}{c} \text{q}_1 \quad \text{r}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_2 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{r}_0 \end{array} \right) = 0, \quad p_1 \star q_0 = r_2 \star r_0 = \mathbf{1}_{\mathcal{M}}, \quad (4.5.3b)$$

$$\lambda \left(\begin{array}{c} \text{p}_1 \quad \text{p}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{q}_0 \end{array} \right) - \lambda \left(\begin{array}{c} \text{p}_1 \quad \text{r}_2 \\ \swarrow \quad \searrow \\ \text{p}_0 \end{array} \right) \circ_2 c \left(\begin{array}{c} \text{q}_1 \quad \text{q}_2 \\ \swarrow \quad \searrow \\ \text{r}_0 \end{array} \right) = 0, \quad p_2 \star q_0 = r_2 \star r_0 \neq \mathbf{1}_{\mathcal{M}}, \quad (4.5.3c)$$

Proposition 4.5.2. *Let \mathcal{M} be a finite unitary magma. Then, the dimension of the space $\mathfrak{R}_{\text{NC}\mathcal{M}}^!$ satisfies*

$$\dim \mathfrak{R}_{\text{NC}\mathcal{M}}^! = 2m^5 - m^4, \quad (4.5.9)$$

where $m := \#\mathcal{M}$.

Proof. To compute the dimension of the space of relations $\mathfrak{R}_{\text{NC}\mathcal{M}}^!$ of $\text{NC}\mathcal{M}^!$, we consider the presentation of $\text{NC}\mathcal{M}^!$ provided by Proposition 4.5.1. Consider the space \mathfrak{R}_1 generated by the family consisting in the elements (4.5.1a). Since this family is linearly independent and each of its element is totally specified by a tuple $(p_0, p_2, q_1, q_2, \delta) \in \mathcal{M}^4 \times \bar{\mathcal{M}}$, we obtain

$$\dim \mathfrak{R}_1 = m^4(m-1). \quad (4.5.10)$$

For the same reason, the dimension of the space \mathfrak{R}_3 generated by the elements (4.5.1c) satisfies $\dim \mathfrak{R}_3 = \dim \mathfrak{R}_1$. Now, let \mathfrak{R}_2 be the space generated by the elements (4.5.1b). Since this family is linearly independent and each of its elements is totally specified by a tuple $(p_0, p_2, q_1, q_2) \in \mathcal{M}^4$, we obtain

$$\dim \mathfrak{R}_2 = m^4. \quad (4.5.11)$$

Therefore, since

$$\mathfrak{R}_{\text{NC}\mathcal{M}}^! = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \oplus \mathfrak{R}_3, \quad (4.5.12)$$

we obtain the stated formula (4.5.9) by summing the dimensions of \mathfrak{R}_1 , \mathfrak{R}_2 , and \mathfrak{R}_3 . \square

Observe that, by Propositions 4.2.2 and 4.5.2, we have

$$\begin{aligned} \dim \mathfrak{R}_{\text{NC}\mathcal{M}} + \dim \mathfrak{R}_{\text{NC}\mathcal{M}}^! &= 2m^6 - 2m^5 + m^4 + 2m^5 - m^4 \\ &= 2m^6 \\ &= \dim \text{Free}(\text{Vect}(\mathcal{T}_{\mathcal{M}}))(3), \end{aligned} \quad (4.5.13)$$

as expected by Koszul duality, where $m := \#\mathcal{M}$.

4.5.2. Dimensions.

Proposition 4.5.3. *Let \mathcal{M} be a finite unitary magma. The Hilbert series $\mathcal{H}_{\text{NC}\mathcal{M}^!}(t)$ of $\text{NC}\mathcal{M}^!$ satisfies*

$$t + (m-1)t^2 + (2m^2t - 3mt + 2t - 1)\mathcal{H}_{\text{NC}\mathcal{M}^!}(t) + (m^3 - 2m^2 + 2m - 1)\mathcal{H}_{\text{NC}\mathcal{M}^!}(t)^2 = 0, \quad (4.5.14)$$

where $m := \#\mathcal{M}$.

Proof. Let $G(t)$ be the generating series such that $G(-t)$ satisfies (4.5.14). Therefore, $G(t)$ satisfies

$$-t + (m-1)t^2 + (-2m^2t + 3mt - 2t - 1)G(t) + (m^3 - 2m^2 + 2m - 1)G(t)^2 = 0, \quad (4.5.15)$$

and, by solving (4.5.15) as a quadratic equation where t is the unknown, we obtain

$$t = \frac{1 + (2m^2 - 3m + 2)G(t) - \sqrt{1 + 2(2m^2 - m)G(t) + m^2G(t)^2}}{2(m-1)}. \quad (4.5.16)$$

Moreover, by Proposition 4.1.6 and (4.1.14), by setting $F(t) := \mathcal{H}_{\text{NC}\mathcal{M}}(-t)$, we have

$$F(G(t)) = \frac{1 + (2m^2 - 3m + 2)G(t) - \sqrt{1 + 2(2m^2 - m)G(t) + m^2G(t)^2}}{2(m - 1)} = t, \quad (4.5.17)$$

showing that $F(t)$ and $G(t)$ are the inverses for each other for series composition.

Now, since by Theorem 4.2.9, $\text{NC}\mathcal{M}$ is a Koszul operad, the Hilbert series of $\text{NC}\mathcal{M}$ and $\text{NC}\mathcal{M}^\dagger$ satisfy (1.2.17). Therefore, (4.5.17) implies that the Hilbert series of $\text{NC}\mathcal{M}^\dagger$ is the series $\mathcal{H}_{\text{NC}\mathcal{M}^\dagger}(t)$, satisfying the stated relation (4.5.14). \square

We deduce from Proposition 4.5.3 that the Hilbert series of $\text{NC}\mathcal{M}^\dagger$ satisfies

$$\mathcal{H}_{\text{NC}\mathcal{M}^\dagger}(t) = \frac{1 - (2m^2 - 3m + 2)t - \sqrt{1 - 2(2m^3 - 2m^2 + m)t + m^2t^2}}{2(m^3 - 2m^2 + 2m - 1)}, \quad (4.5.18)$$

where $m := \#\mathcal{M} \neq 1$.

Proposition 4.5.4. *Let \mathcal{M} be a finite unitary magma. For all $n \geq 2$,*

$$\dim \text{NC}\mathcal{M}^\dagger(n) = \sum_{0 \leq k \leq n-2} m^{n+1}(m(m-1) + 1)^k (m(m-1))^{n-k-2} \text{nar}(n, k). \quad (4.5.19)$$

Proof. The proof consists in enumerating dual \mathcal{M} -cliques, introduced in upcoming Section 4.5.3. Indeed, by Proposition 4.5.5, $\dim \text{NC}\mathcal{M}^\dagger(n)$ is equal to the number of dual \mathcal{M} -cliques of arity n . The expression for $\dim \text{NC}\mathcal{M}^\dagger(n)$ claimed by (4.5.19) can be proved by using similar arguments as the ones intervening in the proof of Proposition 4.1.7 for the expression (4.1.15) of $\dim \text{NC}\mathcal{M}(n)$. \square

We can use Proposition 4.5.4 to compute the first dimensions of $\text{NC}\mathcal{M}^\dagger$. For instance, depending on $m := \#\mathcal{M}$, we have the following sequences of dimensions:

$$1, 1, 1, 1, 1, 1, 1, 1, \quad m = 1, \quad (4.5.20a)$$

$$1, 8, 80, 992, 13760, 204416, 3180800, 51176960, \quad m = 2, \quad (4.5.20b)$$

$$1, 27, 1053, 51273, 2795715, 163318599, 9994719033, 632496651597, \quad m = 3, \quad (4.5.20c)$$

$$1, 64, 6400, 799744, 111923200, 16782082048, 2636161024000, 428208345579520, \quad m = 4. \quad (4.5.20d)$$

The second one is Sequence A234596 of [Sl0]. The last two sequences are not listed in [Sl0] at this time. It is worthwhile to observe that the dimensions of $\text{NC}\mathcal{M}^\dagger$ when $\#\mathcal{M} = 2$ are the ones of the operad BNC of bicolored noncrossing configurations (see Section 5.2).

4.5.3. *Basis.* To describe a basis of $\text{NC}\mathcal{M}^1$, we introduce the following sort of \mathcal{M} -decorated cliques. A *dual \mathcal{M} -clique* is an \mathcal{M}^2 -clique such that its base and its edges are labeled by pairs $(a, a) \in \mathcal{M}^2$, and all solid diagonals are labeled by pairs $(a, b) \in \mathcal{M}^2$ with $a \neq b$. Observe that a non-solid diagonal of a dual \mathcal{M} -clique is labeled by $(1_{\mathcal{M}}, 1_{\mathcal{M}})$. All definitions about \mathcal{M} -cliques of Section 2.1 remain valid for dual \mathcal{M} -cliques. For example,



is a noncrossing dual \mathbb{N}_3 -clique.

Proposition 4.5.5. *Let \mathcal{M} be a finite unitary magma. The underlying graded vector space of $\text{NC}\mathcal{M}^1$ is the linear span of all noncrossing dual \mathcal{M} -cliques.*

Proof. The statement of the proposition is equivalent to the fact that the generating series of noncrossing dual \mathcal{M} -cliques is the Hilbert series $\mathcal{H}_{\text{NC}\mathcal{M}^1}(t)$ of $\text{NC}\mathcal{M}^1$. From the definition of dual \mathcal{M} -cliques, we obtain that the set of the dual \mathcal{M} -cliques of arity n , $n \geq 1$, is in bijection with the set of the \mathcal{M}^2 -Schröder trees of arity n having the outgoing edges from the root and the edges connecting internal nodes with leaves labeled by pairs $(a, a) \in \mathcal{M}^2$, and the edges connecting two internal nodes labeled by pairs $(a, b) \in \mathcal{M}^2$ with $a \neq b$. The map bt defined in Section 4.1.2 (see also Section 4.1.3) realizes such a bijection. Let $T(t)$ be the generating series of these \mathcal{M}^2 -Schröder trees, and let $S(t)$ be the generating series of the \mathcal{M}^2 -Schröder trees of arities greater than 1 and such that the outgoing edges from the roots and the edges connecting two internal nodes are labeled by pairs $(a, b) \in \mathcal{M}^2$ with $a \neq b$, and the edges connecting internal nodes with leaves are labeled by pairs $(a, a) \in \mathcal{M}^2$. From the description of these trees, one has

$$S(t) = m(m-1) \frac{(mt + S(t))^2}{1 - mt - S}, \tag{4.5.22}$$

where $m := \#\mathcal{M}$. Moreover, when $m \neq 1$, $T(t)$ satisfies

$$T(t) = t + \frac{S(t)}{m-1}, \tag{4.5.23}$$

and we obtain that $T(t)$ admits (4.5.18) as solution. Then, by Proposition 4.5.3, when $m \neq 1$, this implies the statement of the proposition. When $m = 1$, it follows from Proposition 4.5.1 that $\text{NC}\mathcal{M}^1$ is isomorphic to the associative operad As . Hence, in this case, $\dim \text{NC}\mathcal{M}^1(n) = 1$ for all $n \geq 1$. Since there is exactly one dual \mathcal{M} -clique of arity n for any $n \geq 1$, the statement of the proposition is satisfied. \square

Proposition 4.5.5 gives a combinatorial description of the elements of $\text{NC}\mathcal{M}^1$. Nevertheless, we do not know for the time being a partial composition on the linear span of these elements providing a realization of $\text{NC}\mathcal{M}^1$.

5. CONCRETE CONSTRUCTIONS

The clique construction provides alternative definitions of known operads. We explore here the cases of the operad NCP of based noncrossing trees, the operad \mathcal{FF}_4 of formal fractions, the operad BNC of bicolored noncrossing configurations and, the operads MT and DMT of multi-fildes and double multi-fildes.

5.1. Rational functions and related operads. We use here the (noncrossing) clique construction to interpret few operads related to the operad RatFct of rational functions (see Section 2.2.8).

5.1.1. Dendriform and based noncrossing tree operads. The *operad of based noncrossing trees* NCP is an operad introduced in [Cha07]. This operad is generated by two binary elements $<$ and $>$ subjected to exactly one quadratic nontrivial relation. The algebras over NCP are *L-algebras* and have been studied in [Ler11]. We do not describe NCP in details here because this is not essential for the sequel. We just explain how to construct NCP through the clique construction and interpret a known link between NCP and the dendriform operad through the rational functions associated with \mathbb{Z} -cliques (see Section 2.2.8).

Let \mathcal{O}_{NCP} be the suboperad of CZ generated by

$$\left\{ \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right\}. \quad (5.1.1)$$

By using Proposition 4.3.2, we find that the Hilbert series $\mathcal{H}_{\mathcal{O}_{\text{NCP}}}(t)$ of \mathcal{O}_{NCP} satisfies

$$t - \mathcal{H}_{\mathcal{O}_{\text{NCP}}}(t) + 2\mathcal{H}_{\mathcal{O}_{\text{NCP}}}(t)^2 - \mathcal{H}_{\mathcal{O}_{\text{NCP}}}(t)^3 = 0. \quad (5.1.2)$$

The first dimensions of \mathcal{O} are

$$1, 2, 7, 30, 143, 728, 3876, 21318, \quad (5.1.3)$$

and form Sequence A006013 of [Slo]. Moreover, one can see that

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \circ_1 \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \circ_2 \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}, \quad (5.1.4)$$

is the only nontrivial relation of degree 2 between the generators of \mathcal{O}_{NCP} .

Proposition 5.1.1. *The operad \mathcal{O}_{NCP} is isomorphic to the operad NCP.*

Proof. Let $\phi : \mathcal{O}_{\text{NCP}}(2) \rightarrow \text{NCP}(2)$ be the linear map satisfying

$$\phi \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = <, \quad (5.1.5a)$$

$$\phi \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = >, \quad (5.1.5b)$$

where $<$ and $>$ are the two binary generators of NCP. In [Cha07], a presentation of NCP is described wherein its generators satisfy one nontrivial relation of degree 2. This relation can be obtained by replacing each \mathbb{Z} -clique appearing in (5.1.4) by its image by ϕ . For this reason, ϕ uniquely extends into an operad morphism. Moreover, because the image of ϕ contains all the generators of NCP, this morphism is surjective. Finally, the Hilbert series

$$\begin{array}{c} \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \circ_1 \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} = \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \circ_2 \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array}, \quad (5.1.12f) \end{array} \quad \begin{array}{c} \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \circ_1 \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} = \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \circ_2 \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array}, \quad (5.1.12h) \end{array}$$

$$\begin{array}{c} \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ 1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \circ_1 \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ 1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} = \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ 1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \circ_2 \begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ 1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array}, \quad (5.1.12g) \end{array}$$

of all nontrivial relations of degree 2 between the generators of $\mathcal{O}_{\mathcal{FF}_4}$.

Proposition 5.1.2. *The operad $\mathcal{O}_{\mathcal{FF}_4}$ is isomorphic to the operad \mathcal{FF}_4 .*

Proof. Let $\phi : \mathcal{O}_{\mathcal{FF}_4}(2) \rightarrow \mathcal{FF}_4(2)$ be the linear map satisfying

$$\phi \left(\begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad 1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \right) = \frac{1}{\{1\}\{1,2\}}, \quad (5.1.13a) \quad \phi \left(\begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ 1 \quad 1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \right) = \frac{1}{\{1,2\}}, \quad (5.1.13c)$$

$$\phi \left(\begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ 1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \right) = \frac{1}{\{2\}\{1,2\}}, \quad (5.1.13b) \quad \phi \left(\begin{array}{c} \wedge \\ \swarrow \quad \searrow \\ -1 \quad -1 \\ \swarrow \quad \searrow \\ \swarrow \quad \searrow \\ -1 \quad -1 \end{array} \right) = \frac{1}{\{1\}\{2\}}. \quad (5.1.13d)$$

In [CHN16], a presentation of \mathcal{FF}_4 is described wherein its generators satisfy eight nontrivial relations of degree 2. These relations can be obtained by replacing each \mathbb{Z} -clique appearing in (5.1.12a)–(5.1.12h) by its image by ϕ . For this reason, ϕ uniquely extends into an operad morphism. Moreover, because the image of ϕ contains all the generators of \mathcal{FF}_4 , this morphism is surjective. Finally, again by [CHN16], the Hilbert series of \mathcal{FF}_4 satisfies (5.1.10), so that $\mathcal{O}_{\mathcal{FF}_4}$ and \mathcal{FF}_4 have the same dimensions. Therefore, ϕ is an operad isomorphism. \square

Proposition 5.1.2 shows hence that the operad \mathcal{FF}_4 can be built through the construction C. Observe also that, as a consequence of Proposition 5.1.2, all suboperads of \mathcal{FF}_4 defined in [CHN16] that are generated by a subset of (5.1.8) can be constructed by the clique construction.

5.2. Operad of bicolored noncrossing configurations. The *operad of bicolored noncrossing configurations* BNC is an operad defined in [CG14]. For any $n \geq 2$, $\text{BNC}(n)$ is the linear span of all *bicolored noncrossing configurations*, where such objects are regular polygons \mathfrak{c} with $n + 1$ edges and such that any arc of \mathfrak{c} is blue, red, or uncolored, no blue or red arc crosses another blue or red arc, and all red arcs are diagonals. These objects can be seen as particular cliques, so that all definitions of Section 2.1.1 remain valid here. For instance,



(5.2.1)

is a bicolored noncrossing configuration of arity 9 (blue arcs are drawn as continuous segments and red arcs, as dashed ones). Moreover, $\text{BNC}(1)$ is the linear span of the singleton containing the only polygon of arity 1 with its only arc is uncolored. The partial composition of BNC is defined, in a geometric way, as follows. For any bicolored noncrossing configurations \mathfrak{c} and \mathfrak{d} of respective arities n and m , and $i \in [n]$, the bicolored noncrossing configuration $\mathfrak{c} \circ_i \mathfrak{d}$ is obtained by gluing the base of \mathfrak{d} onto the i th edge of \mathfrak{c} , and then,

- (a) if the base of ∂ and the i th edge of c are both uncoloured, the arc $(i, i + m)$ of $c \circ_i \partial$ becomes red;
- (b) if the base of ∂ and the i th edge of c are both blue, the arc $(i, i + m)$ of $c \circ_i \partial$ becomes blue;
- (c) otherwise, the base of ∂ and the i th necessarily have different colors; in this case, the arc $(i, i + m)$ of $c \circ_i \partial$ is uncolored.

For example,

(5.2.2a)

(5.2.2b)

(5.2.2c)

Let us now consider the unitary magma $\mathcal{M}_{\text{BNC}} := \{\mathbb{1}, a, b\}$ wherein operation \star is defined by the Cayley table

\star	$\mathbb{1}$	a	b
$\mathbb{1}$	$\mathbb{1}$	a	b
a	a	a	$\mathbb{1}$
b	b	$\mathbb{1}$	b

(5.2.3)

In other words, \mathcal{M}_{BNC} is the unitary magma wherein a and b are idempotent, and $a \star b = \mathbb{1} = b \star a$. Observe that \mathcal{M}_{BNC} is a commutative unitary magma, but, since

$$(b \star a) \star a = \mathbb{1} \star a = a \neq b = b \star \mathbb{1} = b \star (a \star a), \tag{5.2.4}$$

the operation \star is not associative.

Let $\phi : \text{BNC} \rightarrow \text{NC}\mathcal{M}_{\text{BNC}}$ be the linear map defined in the following way. For any bicolored noncrossing configuration c , $\phi(c)$ is the noncrossing \mathcal{M}_{BNC} -clique of $\text{NC}\mathcal{M}_{\text{BNC}}$ obtained by replacing all blue arcs of c by arcs labeled by a , all red diagonals of c by diagonals labeled by b , all uncolored edges and bases of c by edges labeled by b , and all uncolored diagonals of c by diagonals labeled by $\mathbb{1}$. For instance,

(5.2.5)

Proposition 5.2.1. *The linear span of $\circ\text{--}\circ$ together with all noncrossing \mathcal{M}_{BNC} -cliques without edges nor bases labeled by $\mathbb{1}$ forms a suboperad of $\text{NC}\mathcal{M}_{\text{BNC}}$ isomorphic to BNC . Moreover, ϕ is an isomorphism between these two operads.*

Proof. Let us denote by \mathcal{O}_{BNC} the subspace of $\text{NC}\mathcal{M}_{\text{BNC}}$ described in the statement of the proposition. First of all, it follows from the definition of the partial composition of $\text{NC}\mathcal{M}_{\text{BNC}}$ that \mathcal{O}_{BNC} is closed under the partial composition operation. Hence, and since \mathcal{O}_{BNC} contains the unit of $\text{NC}\mathcal{M}_{\text{BNC}}$, \mathcal{O}_{BNC} is an operad. Second, observe that the image of ϕ is the underlying space of \mathcal{O}_{BNC} and, from the definition of the partial composition of BNC, one can check that ϕ is an operad morphism. Finally, since ϕ is a bijection from BNC to \mathcal{O}_{BNC} , the statement of the proposition follows. \square

Proposition 5.2.1 shows hence that the operad BNC can be built through the noncrossing clique construction. Moreover, observe that in [CG14], an automorphism of BNC called *complementary* is considered. The complementary of a bicolored noncrossing configuration is an involution acting by modifying the colors of some arcs of its arcs. Under our setting, this automorphism translates simply as the map $C\theta : \mathcal{O}_{\text{BNC}} \rightarrow \mathcal{O}_{\text{BNC}}$ where \mathcal{O}_{BNC} is the operad isomorphic to BNC described in the statement of Proposition 5.2.1 and $\theta : \mathcal{M}_{\text{BNC}} \rightarrow \mathcal{M}_{\text{BNC}}$ is the unitary magma automorphism of \mathcal{M}_{BNC} satisfying $\theta(1) = 1$, $\theta(a) = b$, and $\theta(b) = a$.

Besides, it is shown in [CG14] that the set of all bicolored noncrossing configurations of arity 2 is a minimal generating set of BNC. Thus, by Proposition 5.2.1, the set

$$\left\{ \begin{array}{c} \text{a} \quad \text{a} \\ \diagdown \quad \diagup \\ \text{a} \quad \text{a} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{a} \end{array}, \begin{array}{c} \text{a} \quad \text{b} \\ \diagdown \quad \diagup \\ \text{a} \quad \text{b} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{b} \end{array}, \begin{array}{c} \text{b} \quad \text{a} \\ \diagdown \quad \diagup \\ \text{b} \quad \text{a} \\ \diagup \quad \diagdown \\ \text{b} \quad \text{a} \end{array}, \begin{array}{c} \text{b} \quad \text{b} \\ \diagdown \quad \diagup \\ \text{b} \quad \text{b} \\ \diagup \quad \diagdown \\ \text{b} \quad \text{b} \end{array}, \begin{array}{c} \text{a} \quad \text{a} \\ \diagdown \quad \diagup \\ \text{b} \quad \text{b} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{a} \end{array}, \begin{array}{c} \text{a} \quad \text{b} \\ \diagdown \quad \diagup \\ \text{b} \quad \text{a} \\ \diagup \quad \diagdown \\ \text{a} \quad \text{b} \end{array}, \begin{array}{c} \text{b} \quad \text{a} \\ \diagdown \quad \diagup \\ \text{a} \quad \text{b} \\ \diagup \quad \diagdown \\ \text{b} \quad \text{a} \end{array}, \begin{array}{c} \text{b} \quad \text{b} \\ \diagdown \quad \diagup \\ \text{a} \quad \text{a} \\ \diagup \quad \diagdown \\ \text{b} \quad \text{b} \end{array} \right\} \quad (5.2.6)$$

is a minimal generating set of the suboperad \mathcal{O}_{BNC} of $\text{NC}\mathcal{M}_{\text{BNC}}$ isomorphic to BNC. As a consequence, all the suboperads of BNC defined in [CG14] which are generated by a subset of the set of the generators of BNC can be constructed by the noncrossing clique construction. This includes, among others, the magmatic operad, the free operad on two binary generators, the operad of noncrossing plants [Cha07], the dipterous operad [LR03, Zin12], and the 2-associative operad [LR06, Zin12].

5.3. Operads from language theory. We provide constructions of two operads coming from formal language theory by using the clique construction.

5.3.1. Multi-tildes. Multi-tildes are operators introduced in [CCM11] in the context of formal language theory as a convenient way to express regular languages. A *multi-tilde* is a pair (n, \mathfrak{s}) where n is a positive integer and \mathfrak{s} is a subset of $\{(x, y) \in [n]^2 : x \leq y\}$. The *arity* of the multi-tilde (n, \mathfrak{s}) is n . As shown in [LMN13], the linear span of all multi-tildes admits a very natural structure of an operad. This operad, denoted by MT, is defined as follows. For any $n \geq 1$, $\text{MT}(n)$ is the linear span of all multi-tildes of arity n and the partial composition $(n, \mathfrak{s}) \circ_i (m, \mathfrak{t})$, $i \in [n]$, of two multi-tildes (n, \mathfrak{s}) and (m, \mathfrak{t}) is defined linearly by

$$(n, \mathfrak{s}) \circ_i (m, \mathfrak{t}) := (n + m - 1, \{\text{sh}_{i,m}(x, y) : (x, y) \in \mathfrak{s}\} \cup \{\text{sh}_{0,i}(x, y) : (x, y) \in \mathfrak{t}\}), \quad (5.3.1)$$

where

$$\text{sh}_{i,p}(x, y) := \begin{cases} (x, y) & \text{if } y \leq i - 1, \\ (x, y + p - 1) & \text{if } x \leq i \leq y, \\ (x + p - 1, y + p - 1) & \text{otherwise.} \end{cases} \quad (5.3.2)$$

For instance, one has

$$(5, \{(1, 5), (2, 4), (4, 5)\}) \circ_4 (6, \{(2, 2), (4, 6)\}) = (10, \{(1, 10), (2, 9), (4, 10), (5, 5), (7, 9)\}), \quad (5.3.3a)$$

$$(5, \{(1, 5), (2, 4), (4, 5)\}) \circ_5 (6, \{(2, 2), (4, 6)\}) = (10, \{(1, 10), (2, 4), (4, 10), (6, 6), (8, 10)\}). \quad (5.3.3b)$$

Observe that the multi-tilde $(1, \emptyset)$ is the unit of MT.

Let $\phi : \text{MT} \rightarrow \mathbb{C}\mathbb{D}_0$ be the map linearly defined as follows. For any multi-tilde (n, \mathfrak{s}) different from $(1, \{(1, 1)\})$, $\phi((n, \mathfrak{s}))$ is the \mathbb{D}_0 -clique of arity n defined, for any $1 \leq x < y \leq n + 1$, by

$$\phi((n, \mathfrak{s}))(x, y) := \begin{cases} 0 & \text{if } (x, y - 1) \in \mathfrak{s}, \\ \mathbb{1} & \text{otherwise.} \end{cases} \quad (5.3.4)$$

For instance,

$$\phi((5, \{(1, 5), (2, 4), (4, 5)\})) = \text{Diagram} \quad (5.3.5)$$

Proposition 5.3.1. *The operad $\mathbb{C}\mathbb{D}_0$ is isomorphic to the suboperad of MT consisting in the linear span of all multi-tildes except the nontrivial multi-tilde $(1, \{(1, 1)\})$ of arity 1. Moreover, ϕ is an isomorphism between these two operads.*

Proof. A direct consequence of the definition (5.3.4) of ϕ is that this map is an isomorphism of vector spaces. Moreover, it follows from the definitions of the partial compositions of MT and $\mathbb{C}\mathbb{D}_0$ that ϕ is an operad morphism. \square

By Proposition 5.3.1, one can interpret the partial compositions (5.3.3a) and (5.3.3b) of multi-tildes as partial compositions of \mathbb{D}_0 -cliques. This give respectively

$$\text{Diagram} \circ_4 \text{Diagram} = \text{Diagram} \quad (5.3.6a)$$

$$\text{Diagram} \circ_5 \text{Diagram} = \text{Diagram} \quad (5.3.6b)$$

5.3.2. *Double multi-tildes.* Double multi-tildes are natural generalizations of multi-tildes, introduced in [GLMN16]. A *double multi-tilde* is a triple $(n, \mathfrak{s}, \mathfrak{t})$ where (n, \mathfrak{t}) and (n, \mathfrak{s}) are both multi-tildes of the same arity n . The *arity* of the double multi-tilde $(n, \mathfrak{s}, \mathfrak{t})$ is n . As shown in [GLMN16], the linear span of all double multi-tildes admits a structure of an operad. This operad, denoted by DMT, is defined as follows. For any $n \geq 1$, $\text{DMT}(n)$ is the linear span of all double multi-tildes of arity n and the partial composition $(n, \mathfrak{s}, \mathfrak{t}) \circ_i (m, \mathfrak{u}, \mathfrak{v})$, $i \in [n]$, of two double multi-tildes $(n, \mathfrak{s}, \mathfrak{t})$ and $(m, \mathfrak{u}, \mathfrak{v})$ is defined linearly by

$$(n, \mathfrak{s}, \mathfrak{t}) \circ_i (m, \mathfrak{u}, \mathfrak{v}) := (n, \mathfrak{s} \circ_i \mathfrak{u}, \mathfrak{t} \circ_i \mathfrak{v}), \quad (5.3.7)$$

where the two partial compositions \circ_i of the right member of (5.3.7) are the ones of MT. We can observe that DMT is isomorphic to the Hadamard product $\text{MT} * \text{MT}$. For instance, one has

$$(3, \{(2, 2)\}, \{(1, 2), (1, 3)\}) \circ_2 (2, \{(1, 1)\}, \{(1, 2)\}) = (4, \{(2, 2), (2, 3)\}, \{(1, 3), (1, 4), (2, 3)\}). \quad (5.3.8)$$

The unit of DMT is $(1, \emptyset, \emptyset)$.

Consider now the operad $\mathbb{C}\mathbb{D}_0^2$ and let $\phi : \text{DMT} \rightarrow \mathbb{C}\mathbb{D}_0^2$ be the map linearly defined as follows. The image by ϕ of $(1, \emptyset, \emptyset)$ is the unit of $\mathbb{C}\mathbb{D}_0^2$ and, for any double multi-tilde $(n, \mathfrak{s}, \mathfrak{t})$ of arity $n \geq 2$, $\phi((n, \mathfrak{s}, \mathfrak{t}))$ is the \mathbb{D}_0^2 -clique of arity n defined, for any $1 \leq x < y \leq n + 1$, by

$$\phi((n, \mathfrak{s}, \mathfrak{t}))(x, y) := \begin{cases} (0, \mathbb{1}) & \text{if } (x, y - 1) \in \mathfrak{s} \text{ and } (x, y - 1) \notin \mathfrak{t}, \\ (\mathbb{1}, 0) & \text{if } (x, y - 1) \notin \mathfrak{s} \text{ and } (x, y - 1) \in \mathfrak{t}, \\ (0, 0) & \text{if } (x, y - 1) \in \mathfrak{s} \text{ and } (x, y - 1) \in \mathfrak{t}, \\ (\mathbb{1}, \mathbb{1}) & \text{otherwise.} \end{cases} \quad (5.3.9)$$

For instance,

$$\phi((4, \{(2, 2), (2, 3)\}, \{(1, 3), (1, 4), (2, 3)\})) = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \quad (5.3.10)$$

Proposition 5.3.2. *The operad $\mathbb{C}\mathbb{D}_0^2$ is isomorphic to the suboperad of DMT consisting in the linear span of all double multi-tildes except the three nontrivial double multi-tildes of arity 1. Moreover, ϕ is an isomorphism between these two operads.*

Proof. There are two ways to prove the first assertion of the statement of the proposition. On the one hand, this property follows from Proposition 2.1.2 and Proposition 5.3.1. On the other hand, the whole statement of the proposition is a direct consequence of the definition (5.3.9) of ϕ , showing that ϕ is an isomorphism of vector spaces, and, from the definitions of the partial compositions of DMT and $\mathbb{C}\mathbb{D}_0^2$ showing that ϕ is an operad morphism. \square

By Proposition 5.3.2, one can interpret the partial composition (5.3.8) of double multi-tildes as a partial composition of \mathbb{D}_0^2 -cliques. This gives

$$\begin{array}{c} \text{---} (0, 1) \text{---} \\ | \quad \diagup \\ | \quad (1, 0) \text{---} \\ | \quad \diagdown \\ \text{---} (1, 0) \text{---} \end{array} \circ_2 \begin{array}{c} \text{---} (0, 1) \text{---} \\ \diagdown \quad \diagup \\ \text{---} (1, 0) \text{---} \end{array} = \begin{array}{c} \text{---} (0, 1) \text{---} \\ | \quad \diagup \\ \text{---} (0, 0) \text{---} \\ | \quad \diagdown \\ \text{---} (1, 0) \text{---} \\ | \quad \diagdown \\ \text{---} (1, 0) \text{---} \end{array} \quad (5.3.11)$$

CONCLUSION AND PERSPECTIVES

This work presents and study the clique construction \mathcal{C} , producing operads from unitary magmas. We have seen that \mathcal{C} has many both algebraic and combinatorial properties. Among its most notable ones, $\mathcal{C}\mathcal{M}$ admits several quotients involving combinatorial families of decorated cliques, admits a binary and quadratic suboperad $\text{NC}\mathcal{M}$ which is a Koszul, and contains a lot of already studied and classic operads. Besides, in the course of this work, whose text is already long enough, we have put aside a bunch of questions. Let us address these here.

When \mathcal{M} is a \mathbb{Z} -graded unitary magma, a link between $\mathcal{C}\mathcal{M}$ and the operad of rational functions RatFct [Lod10] has been developed in Section 2.2.8 by means of a morphism F_θ between these two operads. We have observed that F_θ is not injective (see (2.2.31a) and (2.2.31b)). A description of the kernel of F_θ , even when \mathcal{M} is the unitary magma \mathbb{Z} , seems not easy to obtain. Trying to obtain this description is a first perspective of this work.

Here is a second perspective. In Section 3, we have defined and briefly studied some quotients and suboperads of $\mathcal{C}\mathcal{M}$. In particular, we have considered the quotient $\text{Deg}_1\mathcal{M}$ of $\mathcal{C}\mathcal{M}$, involving \mathcal{M} -cliques of degrees at most 1. As mentioned, $\text{Deg}_1\mathbb{D}_0$ is an operad defined on the linear span of involutions (except the nontrivial involution of \mathfrak{S}_2). A complete study of this operad seems worthwhile, including a description of a minimal generating set, a presentation by generators and relations, a description of its partial composition on the H-basis and on the K-basis, and a realization of this operad in terms of standard Young tableaux.

The last question we develop here concerns the Koszul dual $\text{NC}\mathcal{M}^!$ of $\text{NC}\mathcal{M}$. Section 4.5 contains results about this operad, like a description of its presentation and a formula for its dimensions. We have also established the fact that, as graded vector spaces, $\text{NC}\mathcal{M}^!$ is isomorphic to the linear span of all noncrossing dual \mathcal{M} -cliques. To obtain a realization of $\text{NC}\mathcal{M}^!$, it is now enough to endow this last space with an adequate partial composition. This is the last perspective we address here.

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