Extrema Property of the k-Ranking of Directed Paths and Cycles

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Abstract

A k-ranking of a directed graph G is a labeling of the vertex set of G with k positive integers such that every directed path connecting two vertices with the same label includes a vertex with a larger label in between. The *rank number of* G is defined to be the smallest k such that G has a k-ranking. We find the largest possible directed graph that can be obtained from a directed path or a directed cycle by attaching new edges to the vertices such that the new graphs have the same rank number as the original graphs. The adjacency matrix of the resulting graph is embedded in the Sierpiński triangle.

We present a connection between the number of edges that can be added to paths and the Stirling numbers of the second kind. These results are generalized to create directed graphs which are unions of directed paths and directed cycles that maintain the rank number of a base graph of a directed path or a directed cycle.

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1 Introduction

A vertex coloring of a directed graph is a labeling of its vertices so that no two adjacent vertices receive the same label. In a directed path, edges are oriented in the same direction. A k-ranking of a directed graph is a labeling of the vertex set with k positive integers such that for every directed path connecting two vertices with the same label there is a vertex with a larger label in between. A ranking is *minimal* if the reduction of any label violates the ranking property. The *rank number* $\chi_k(G)$ of a directed graph G is the smallest k such that G has a minimal k-ranking.

It is known that the rank number of a graph G may increase just by adding a new edge, even if the new edge and G share vertices. This raises the question "what is the maximum size of a directed graph that satisfies the property that its rank number is equal to the rank number of its largest directed subpath?" Flórez and Narayan [4, 5] found results related to this question; however, the problem is still open. We believe that studying particular cases will lead to a better understanding of the problem and potential solutions.

In this paper we study the necessary and sufficient conditions for the largest possible directed graph that can be obtained by attaching edges to either a directed path or directed cycle without changing the rank number and the number of vertices. In [4] there is a solution for the undirected case. Here, we analyze cases for which the new directed graph keeps the rank number of the original graph. The maximum number of edges in such graphs is described as well as which edges are present in the graphs.

We build families of directed graphs by adding directed edges (called *admissible*) to a directed path (called the *base*). Those families satisfy the condition that the graphs are maximal graphs with the property that the rank number of each graph equals the rank number of the base directed path. The graphs of the first two families described were constructed recursively by adding all admissible edges (without increasing the number of vertices) to a base directed path. The same idea is extended to directed cycles.

We generalize the concept developed to build the first four families above to other maximal families of graphs preserving the rank number of the base directed path by adding admissible directed paths and directed cycles to a base directed path or directed cycle.

The number of edges and the number of admissible edges of the graphs in the first four families are counted using known numerical sequences. We prove, using the recursive construction, that the maximum number of edges in some of those families of graphs is given by a Stirling number of the second kind.

For those who are interested in computational matters, we provide algorithms, some of which are given in terms of adjacency matrices. We found an interesting connection between one of the adjacency matrices and the Sierpiński triangle. The adjacency matrix of the first graph found in this paper embeds naturally in the Sierpiński sieve triangle.

2 Preliminary Concepts

In this section we review some known concepts and results, introduce new definitions, and give a proof for a lemma.

Let $V := \{v_1, v_2, \ldots, v_\eta\}$ be a set of vertices of a directed graph. An edge (arc) with vertices $\{v_i, v_j\} \subseteq V$, $i < j$, with orientation $v_i \to v_j$ is denoted by $\overrightarrow{e}$ or by $\overrightarrow{v_i v_j}$, and the edge with orientation $v_i \leftarrow v_j$ is denoted by $\overleftarrow{e}$ or by $\overleftarrow{v_i v_j}$. A directed path with vertices V is denoted by $\overrightarrow{P_\eta}$ if its edges are of the form $\overrightarrow{e}$. A directed cycle with vertices V is denoted by $\overrightarrow{C_\eta}$ if its edges are of the form $\overrightarrow{e}$.

Let G be a finite directed graph with vertex set $V(G)$. A *k-ranking* of G is a labeling (or coloring) of $V(G)$ with k positive integers such that every directed path that connects two vertices of the same label (color) contains a vertex of a larger label (color). Thus, a labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, k\}$ is a vertex *k-ranking* of G if for all $u, v \in V(G)$ such that $u \neq v$ and $f(u) = f(v)$, then every directed path connecting u and v contains a vertex w such that $f(w) > f(u)$. Like the chromatic number, the *rank number* of a graph G is defined to be the smallest k such that G has a minimal k-ranking; it is denoted by $\chi_k(G)$.

Let H_1 and H_2 be directed graphs with $V(H_1) \subseteq V(H_2)$ and $E(H_1) \cap E(H_2) = \emptyset$. We say that a directed edge $e \in E(H_1)$ is *admissible* for H_2 if $\chi_r(H_2 \cup \{e\}) = \chi_r(H_2)$, and e is *forbidden* for H_2 if $\chi_r(H_2 \cup \{e\}) > \chi_r(H_2)$.

We distinguish two types of admissible edges. A directed edge e is *admissible of type I* for G if e and the edges in the edge set of G have the same direction. A directed edge e is *admissible of type II* if e and the edges in the edge set of G have opposite direction. Note that admissible edges of type II allow for edges with opposite directions between two vertices.

For example, Figure 1 shows the graph $\overrightarrow{\mathcal{G}}_4 := \overrightarrow{P}_{2^4-1} \cup H(\overrightarrow{\mathcal{G}}_4)$ where $H(\overrightarrow{\mathcal{G}}_4)$ is the graph formed with all admissible edges of type I for $\overrightarrow{P}_{2^4-1}$. In Figure 2 we show the graph $\overleftarrow{\mathcal{G}}_4 := \overrightarrow{P}_{2^4-1} \cup H(\overleftarrow{\mathcal{G}}_4)$ where $H(\overleftarrow{\mathcal{G}}_4)$ is the graph formed with all admissible edges of type II for $\overrightarrow{P}_{2^4-1}$. Since $\overrightarrow{P}_{2^4-1}$ gives rise to both graphs $\overrightarrow{\mathcal{G}}_4$ and $\overleftarrow{\mathcal{G}}_4$, they have the same set of vertices $V = \{v_1, \ldots, v_{15}\}$, where v_1 is leftmost vertex and v_{15} is the rightmost vertex. The numbers on the graphs represent the labelings. That is,

$$\begin{array}{lllll} f(v_1) = 1; & f(v_2) = 2; & f(v_3) = 1; & f(v_4) = 3; & f(v_5) = 1; \\ f(v_6) = 2; & f(v_7) = 1; & f(v_8) = 4; & f(v_9) = 1; & f(v_{10} = 2; \\ f(v_{11}) = 1; & f(v_{12}) = 3; & f(v_{13}) = 1; & f(v_{14}) = 2; & f(v_{15}) = 1\ . \end{array}$$

The largest label in $\overrightarrow{P}_{2^4-1}$ is 4. So, 4 is the rank number of the graphs $\overrightarrow{P}_{2^4-1}$, $\overrightarrow{\mathcal{G}}_4$, and $\overleftarrow{\mathcal{G}}_4$. Thus, $\chi_r(\overrightarrow{\mathcal{G}}_4) = \chi_r(\overleftarrow{\mathcal{G}}_4) = \chi_r(\overrightarrow{P}_{2^4-1}) = 4$.

Bodlaender et al. [1] determined the rank number of a path P_η to be $\lfloor \log_2 \eta \rfloor + 1$. Bruoth and M. Horňák [2] found a similar value, $\lfloor \log_2(\eta - 1) \rfloor + 2$, for C_η. It is easy to see that these two properties are true for both $\chi_r(\overrightarrow{P}_\eta)$ and $\chi_r(\overrightarrow{C}_\eta)$, respectively. A minimal ranking for these two types of graphs can be obtained by labeling the vertices $\{v_i \mid 1 \leq i \leq \eta\}$ with $1, \ldots, \alpha + 1$ where 2^α is the highest power of 2 that divides i. If $\eta = 2^\alpha - 1$, then the minimal ranking of $\chi_r(\overrightarrow{P}_\eta)$ is unique. If $\eta = 2^\alpha$, then the minimal ranking of $\overrightarrow{C}_\eta$ is unique. These two rankings are called the *standard minimal rankings*. The following lemma summarizes these properties.

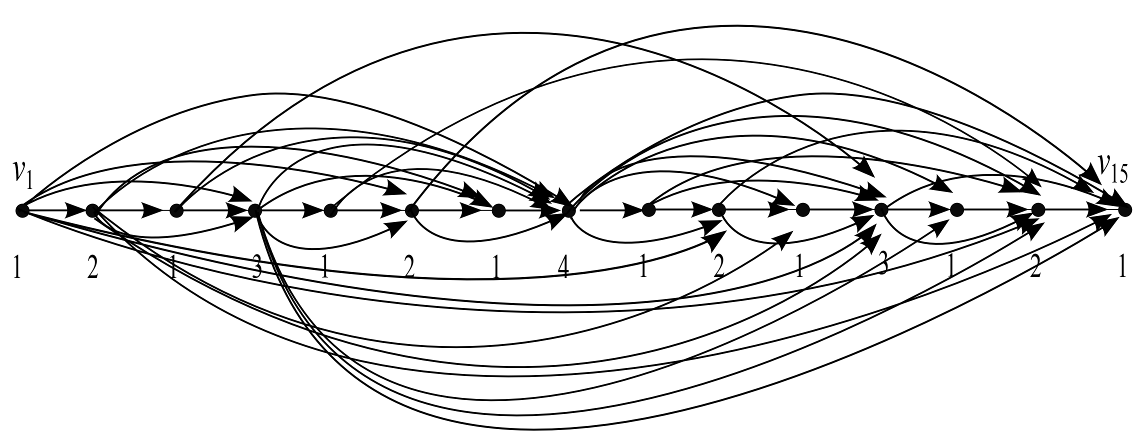

Figure 1: A graph with admissible edges of type I.

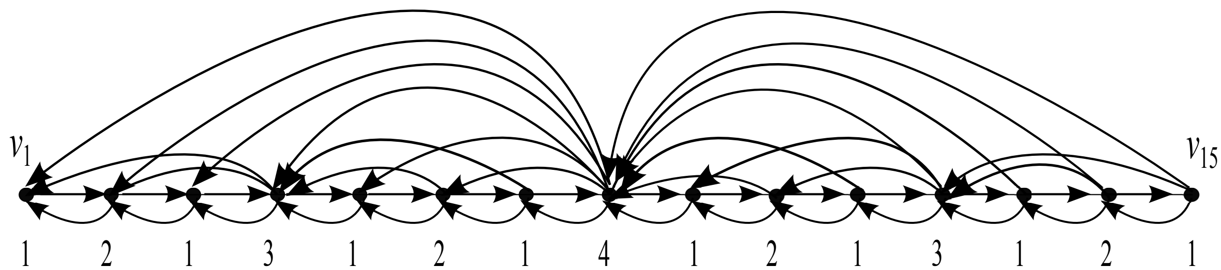

Figure 2: A graph with admissible edges of type II.

Lemma 1 ([1, 2]). *If* Γ *is any of the graphs* $\overrightarrow{P}_{2^k-1}$ *or* $\overrightarrow{C}_{2^k}$*, then* Γ *has a unique minimal ranking and*

$$\chi_k(\Gamma) = \begin{cases} k, & \text{if} \quad \Gamma = \overrightarrow{P}_{2^k-1} \\ k+1, & \text{if} \quad \Gamma = \overrightarrow{C}_{2^k}. \end{cases}$$

We now give some definitions that are going to be used in the results that follow. Let G be a directed graph with f as its k-ranking function. We define $A_j = \{v \in V(G) \mid f(v) \geq j\}$ and use $\mathcal{C}(A_j)$ to denote the set of all components of $G \setminus A_j$ where $0 < j \leq \chi_k(\Gamma)$.

Let $\overrightarrow{\mathcal{G}}_2 := \overrightarrow{P}_3$ with $V(\overrightarrow{\mathcal{G}}_2) = \{v_1, v_2, v_3\}$, and let $\overrightarrow{\mathcal{G}}'_2 := \overrightarrow{P}'_3$ with $V(\overrightarrow{\mathcal{G}}'_2) = \{w_1, w_2, w_3\}$. We define the *direct sum graph of type I* recursively, denoted by $\overrightarrow{\mathcal{G}}_{i+1} = \overrightarrow{\mathcal{G}}_i \oplus \overrightarrow{\mathcal{G}}'_i$, as the graph with vertex set

$$V(\overrightarrow{\mathcal{G}}_{i+1}) = V(\overrightarrow{\mathcal{G}}_i) \cup V(\overrightarrow{\mathcal{G}}'_i) \cup \{v_{2^i}\}$$

where $V(\overrightarrow{\mathcal{G}}_i) = \{v_1, v_2, \ldots, v_{2^i-1}\}$ and $\overrightarrow{\mathcal{G}}'_i$ is the graph obtained from a copy of $\overrightarrow{\mathcal{G}}_i$ by relabeling the vertices of $\overrightarrow{\mathcal{G}}_i$ as follows $w_t := v_t$ for $t = 1, \ldots, 2^i - 1$. That is, $V(\overrightarrow{\mathcal{G}}'_i) = \{w_1, \ldots, w_{2^i-1}\}$. The edge set of $\overrightarrow{\mathcal{G}}_{i+1}$ is

$$E(\overrightarrow{\mathcal{G}}_{i+1}) = E(\overrightarrow{\mathcal{G}}_i) \cup E(\overrightarrow{\mathcal{G}}'_i) \cup E(\overrightarrow{H}_i) \cup \{\overrightarrow{v_{2^i-1}v_{2^i}}, \overrightarrow{v_{2^i}w_1}\},$$

where

$$\overrightarrow{H}_i = \{\overrightarrow{v_j v_{2^i}} | 1 \leq j \leq 2^i - 2\} \cup \{\overrightarrow{v_{2^i} w_j'} | 2 \leq j \leq 2^i - 1\} \cup \{\overrightarrow{v_k w_j'} | \overrightarrow{v_k v_j'} \in E(\overrightarrow{\mathcal{G}}_i)\}. \quad (1)$$

For example, $\overrightarrow{\mathcal{G}}_4$ is depicted in Figure 1.

We use $H(\overrightarrow{\mathcal{G}}_t)$ to denote the graph with the set of vertices equal to the set of vertices of $\overrightarrow{\mathcal{G}}_t$ and the set of edges defined by

$$E(H(\overrightarrow{\mathcal{G}}_t)) := \bigcup_{j=3}^{t} \overrightarrow{H}_j. \quad (2)$$

3 Admissible Edges of Type I

This section proves several properties of the direct sum graph of type I. Assume that all admissible edges and direct sum graphs considered throughout this section are of type I. The path $\overrightarrow{P}_{2^n-1}$ gives rise to the direct sum graph $\overrightarrow{\mathcal{G}}_n$, and they both preserve many properties, including the symmetry described below. We prove that $\overrightarrow{\mathcal{G}}_n$ is the largest graph that shares with $\overrightarrow{P}_{2^n-1}$ the set of vertices, the orientation, the n-rank number, and the same vertex labeling when the labeling is minimum. Note that $\overrightarrow{\mathcal{G}}_n$ is $\overrightarrow{P}_{2^n-1} \cup H(\overrightarrow{\mathcal{G}}_{n-1})$. We prove that $H(\overrightarrow{\mathcal{G}}_{n-1})$ is the set of admissible edges for $\overrightarrow{P}_{2^n-1}$.

Consider the symmetry seen on the graph $\overleftarrow{\mathcal{G}}_4$, (see Figure 2). This symmetry occurs in general in $\overleftarrow{\mathcal{G}}_n$ (see Section 4). However, the symmetry in the graph $\overrightarrow{\mathcal{G}}_4$ is not obvious but does exist (see Figure 1). In Section 5, we give a recursive algorithm to build the adjacency matrix that represents $\overrightarrow{\mathcal{G}}_n$. The adjacency matrix given by Algorithm 1 is symmetric with respect to the antidiagonal (for example, see the matrices in Table 1). This symmetry can be predicted based on how the direct sum graph $\overrightarrow{\mathcal{G}}_n$ is defined.

Recall that the *Stirling numbers of the second kind* $S(n,k)$ count the ways to divide a set of n objects into k nonempty subsets where $n, k \geq 1$. We are interested in the following Stirling numbers, $S(n+1,3) = (1/6)(3^{n+1} - 3(2^{n+1}) + 3)$ and $S(n,2) = 2^{n-1} - 1$. We prove that the number of edges of $\overrightarrow{\mathcal{G}}_n$ and the number of admissible edges for $\overrightarrow{P}_{2^n-1}$ can be described by Stirling numbers of the second kind.

Proposition 2. *If $n \geq 2$ and $\overrightarrow{\mathcal{G}}_n$ is the direct sum graph of type I, then*

1. $\chi_r(\overrightarrow{\mathcal{G}}_n) = \chi_r(\overrightarrow{P}_{2^n-1}) = n$ *and the minimum labeling of* $\overrightarrow{\mathcal{G}}_n$ *is unique,*

2. *the total number of edges in* $\overrightarrow{\mathcal{G}}_n$ *is* $2S(n+1,3) = 3^n - 2^{n+1} + 1$,

3. *an edge* $\overrightarrow{e}$ *is admissible of type I for* $\overrightarrow{P}_{2^n-1}$ *if and and only if* $\overrightarrow{e} \in H(\overrightarrow{\mathcal{G}}_n)$, *where* $H(\overrightarrow{\mathcal{G}}_n)$ *is as in (2), and*

4. *the total number of admissible edges of type I for* $\overrightarrow{P}_{2^n-1}$ *is* $2\left(S(n+1,3) - S(n,2)\right)$.

Proof. **Part 1:** The proof is inductive. Let $T(n)$ be the statement: $\chi_r(\overrightarrow{\mathcal{G}}_n) = \chi_r(\overrightarrow{P}_{2^n-1}) = n$ for $n > 1$ and that $\overrightarrow{\mathcal{G}}_n$ and $\overrightarrow{P}_{2^n-1}$ have the same minimal labeling. For this proof we assume that the labeling of $\overrightarrow{\mathcal{G}}_n$ is minimal.

The proof of $T(2)$ is straightforward from the definition of $\overrightarrow{\mathcal{G}}_2$. We now suppose that $T(n)$ is true for some fixed $n = k$ with $k > 2$. Thus, $\chi_r(\overrightarrow{\mathcal{G}}_k) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$ is true for some fixed $n = k$ with $k > 2$. (We prove $T(k+1)$ is true.)

Consider the graphs $\overrightarrow{P}_{2^k-1}$ and $\overrightarrow{\mathcal{G}}_k$, where $\overrightarrow{\mathcal{G}}_k = \overrightarrow{\mathcal{G}}_{k-1} \oplus \overrightarrow{\mathcal{G}}'_{k-1}$. From the inductive hypothesis and Lemma 1 we know that both graphs have $2^k - 1$ vertices with the same labeling and that this labeling is minimal and unique. From the definition of $\overrightarrow{\mathcal{G}}_{k+1}$ we know that its vertices are $v_1, v_2, \ldots, v_{2^k-1}, v_{2^k}, w_1, w_2, \ldots, w_{2^k-1}$ from left to right. To label $\overrightarrow{\mathcal{G}}_{k+1}$, we define f as follows: the function f keeps the same labels from $\overrightarrow{\mathcal{G}}_k$ for $\{v_1, v_2, \ldots, v_{2^k-1}\}$ and from $\overrightarrow{\mathcal{G}}'_k$ for $\{w_1, w_2, \ldots, w_{2^k-1}\}$ and $f(v_{2^k}) = k+1$ since v_{2^k} needs a new label. The function f is a well defined labeling for $\overrightarrow{\mathcal{G}}_{k+1}$ since $f(v_{2^k}) := k+1$ preserves a good labeling for the edges

$$\{\overrightarrow{v_iv_{2^k}}, \overrightarrow{v_{2^k}w_i} | 1 \le i \le 2^k - 1\}, \qquad \{\overrightarrow{v_iv_j} | \overrightarrow{v_iv_j} \in E(\overrightarrow{\mathcal{G}}_k)\},$$
$$\{\overrightarrow{w_iw_j} | \overrightarrow{w_iw_j} \in E(\overrightarrow{\mathcal{G}}'_k)\}, \quad \text{and} \quad \{\overrightarrow{v_iw_j} | \overrightarrow{v_iv_j} \in E(\overrightarrow{\mathcal{G}}_k)\}.$$

Since one end of each edge in $\{\overrightarrow{v_iv_{2^k}}, \overrightarrow{v_{2^k}w_i} | 1 \le i \le 2^k - 1\}$ is labeled with the highest label, these edges are clearly admissible in $\overrightarrow{\mathcal{G}}_{k+1}$. The edges $\{\overrightarrow{v_iv_j} | \overrightarrow{v_iv_j} \in E(\overrightarrow{\mathcal{G}}_k)\}$ are admissible in $\overrightarrow{\mathcal{G}}_{k+1}$ since they are admissible in $\overrightarrow{\mathcal{G}}_k$. Similarly, the edges $\{\overrightarrow{w_iw_j} | \overrightarrow{w_iw_j} \in E(\overrightarrow{\mathcal{G}}'_k)\}$ are admissible in $\overrightarrow{\mathcal{G}}_{k+1}$. The edges $\{\overrightarrow{v_iw_j} | \overrightarrow{v_iv_j} \in E(\overrightarrow{\mathcal{G}}_k\}$ are admissible in $\overrightarrow{\mathcal{G}}_{k+1}$ since the subgraph of $\overrightarrow{\mathcal{G}}_{k+1}$ induced by the vertices $\{v_1, v_2, \ldots, v_i, w_j, w_{j+1}, \ldots, w_{2^k-1}\}$ has the same labeling as the subgraph of $\overrightarrow{\mathcal{G}}_k$ induced by $\{v_1, v_2, \ldots, v_i, v_j, v_{j+1}, \ldots, v_{2^k-1}\}$ which has a proper rank labeling. Note that f is also a minimal labeling for $\overrightarrow{P}_{2^{k+1}-1}$. This proves $T(k+1)$ is true.

Part 2: Let the total number of edges in $\overrightarrow{\mathcal{G}}_{k+1}$ be denoted by a_{k+1}. From the definition of edges of $\overrightarrow{\mathcal{G}}_{k+1}$ it is easy to see that,

$$\begin{aligned}
a_{k+1} &= |E(\overrightarrow{\mathcal{G}}_k)| + |E(\overrightarrow{\mathcal{G}}'_k)| + |\{\overrightarrow{v_iv_{2^k}} | 1 \le i \le 2^k - 2\}| + |\{\overrightarrow{v_{2^k}w_i} | 2 \le i \le 2^k - 1\}| \\
&\quad + |\{\overrightarrow{v_iw_j} | \overrightarrow{v_iv_j} \in E(\overrightarrow{\mathcal{G}}_k)\}| + |\{\overrightarrow{v_{2^k-1}v_{2^k}}, \overrightarrow{v_{2^k}w_1}\}| \\
&= a_k + a_k + (2^k - 2) + (2^k - 2) + a_k + 2 \\
&= 3a_k + 2(2^k - 1).
\end{aligned}$$

We prove by induction that the number of edges in $\overrightarrow{\mathcal{G}}_n$ is given by $3^n - 2^{n+1} + 1$. Let $T(n)$ be the statement: $a_n = 3^n - 2^{n+1} + 1$ for $n > 1$.

We prove $T(2)$. From definition of $\overrightarrow{\mathcal{G}}_2$, we have $a_2 = 3^2 - 2^3 + 1 = 2$. We now suppose that $T(n)$ is true for some fixed $n = k$ with $k > 2$. Therefore, $a_n = 3^n - 2^{n+1} + 1$ is true for some fixed $n = k$ with $k > 2$. Thus, $a_k = 3^k - 2^{k+1} + 1$ is true. (We prove $T(k+1)$ is true.)

Since $a_{k+1} = 3a_k + 2(2^k - 1)$, we have that

$$\begin{aligned} a_{k+1} &= 3a_k + 2(2^k - 1) \\ &= 3(3^k - 2^{k+1} + 1) + 2(2^k - 1) \\ &= 3^{k+1} - 2^{k+2} + 1, \end{aligned}$$

which shows that $T(k+1)$ holds. Note that $a_n = 3^n - 2^{n+1} + 1$ is twice the Stirling number of second kind. That is, $\overrightarrow{\mathcal{G}}_n$ has $2S(n+1,3) = 3^n - 2^{n+1} + 1$ edges.

Part 3: Suppose that $\overrightarrow{e} \in H(\overrightarrow{\mathcal{G}}_n)$. Then by the definition of $E(H(\overrightarrow{\mathcal{G}}_n))$ in (2) and part 1 of this proposition, $\overrightarrow{e}$ is an admissible edge for $\overrightarrow{P}_{2^n-1}$.

Now suppose that $\overrightarrow{e}$ is an admissible edge for $\overrightarrow{P}_{2^n-1}$. We prove that $\overrightarrow{e} \in H(\overrightarrow{\mathcal{G}}_n)$. Let u and v be the vertices of $\overrightarrow{e}$ with $f(v) < f(u) = j$.

Case 1. Suppose that u and v are in the same component $\mathcal{C}$ of $\overrightarrow{\mathcal{G}}_n \setminus A_{j+1}$. We prove the case in which $\overrightarrow{e} = \overrightarrow{vu}$ and since the proof of the case in which $\overrightarrow{e} = \overrightarrow{uv}$ is similar, we omit it. Recall that $\mathcal{C}$ has $2^j - 1$ vertices and that j is the largest label in $\mathcal{C}$. Since u has the largest label in the component, it corresponds to the vertex in position 2^{j-1}. Then by the definition of $\overrightarrow{e}$, the edge $\overrightarrow{e}$ belongs to $\overrightarrow{H}_j$ as defined in (1). Thus, $\overrightarrow{e} \in H(\overrightarrow{\mathcal{G}}_n)$.

Case 2. Suppose that $u \in \mathcal{C}$ and $v \in \mathcal{C}'$ where $\mathcal{C}$ and $\mathcal{C}'$ are distinct components of $\overrightarrow{\mathcal{G}}_n \setminus A_{j+1}$. Let w be a vertex in $\mathcal{C}'$ with $f(w) = j$, where j is the largest label in $\mathcal{C}'$.

Subcase 1. Suppose that $\overrightarrow{e} = \overrightarrow{uv}$. Note that if $v = w$ or if v is located in $\mathcal{C}'$ in a position to the left of w, then $\overrightarrow{e}$ gives rise to a path connecting u and w which does not contain a larger label in between u and w. That is a contradiction because $\overrightarrow{e}$ is an admissible edge for $\overrightarrow{P}_{2^n-1}$. Thus, v must be located in $\mathcal{C}'$ in a position to the right of w. This implies that $\overrightarrow{e}$ satisfies the condition described in the last set in the definition of $\overrightarrow{H}_j$ in (1).

Subcase 2. Suppose that $\overrightarrow{e} = \overrightarrow{vu}$. Note that if $v = w$ or if v is located in $\mathcal{C}'$ in a position to the right of w, then $\overrightarrow{e}$ gives rise to a path connecting w and u which does not contain a larger label in between u and w. That is a contradiction because $\overrightarrow{e}$ is an admissible edge for $\overrightarrow{P}_{2^n-1}$. Thus, v must be located in $\mathcal{C}'$ in a position to the left of w. This implies that $\overrightarrow{e}$ satisfies the condition described in the last set in the definition of $\overrightarrow{H}_j$ in (1).

Part 4: It is easy to see that $\overrightarrow{P}_{2^n-1}$ has $2^n - 2$ edges, which can be rewritten as $2(2^{n-1} - 1) = 2S(n,2)$. From part 3 of this proposition we know that the set of admissible edges for $\overrightarrow{P}_{2^n-1}$ is $H(\overrightarrow{\mathcal{G}}_n)$. Therefore, the total number of admissible edges is the number of edges in $\overrightarrow{\mathcal{G}}_n$ minus the number of edges in $\overrightarrow{P}_{2^n-1}$ which is $2(S(n+1,3) - S(n,2))$. □

4 Admissible Edges of Type II

This section discusses the admissible edges of type II for $\overrightarrow{P}_{2^n-1}$ using the direct sum graph of type II defined below. Throughout this section, it can be assumed that all admissible edges and direct sum graphs are of type II. We prove that the direct sum graph is a maximum graph such that $\overrightarrow{P}_{2^n-1}$ and $\overleftarrow{\mathcal{G}}_n$ have the same set of vertices, n-rank number, and vertex labeling when the labeling is minimum. The symmetry of $\overleftarrow{\mathcal{G}}_n$ is straightforward, and an example can be seen in Figure 2. Section 5 gives a recursive algorithm for the adjacency matrix for $\overleftarrow{\mathcal{G}}_n$. The symmetry found in $\overleftarrow{\mathcal{G}}_n$ can be clearly seen in the adjacency matrix. It is also anti-diagonal symmetry.

We first discuss several necessary definitions. Let $\overleftarrow{\mathcal{G}}_2 := \overrightarrow{P}_3 \cup E(\overleftarrow{H}_2)$ with vertices $V(\overleftarrow{\mathcal{G}}_2) = \{v_1, v_2, v_3\}$ and edges $E(\overleftarrow{H}_2) = \{\overleftarrow{v_1v_2}, \overleftarrow{v_2v_3}\}$, and let $\overleftarrow{\mathcal{G}}'_2 := \overrightarrow{P}'_3 \cup E(\overleftarrow{H}'_2)$ with vertices $V(\overleftarrow{\mathcal{G}}'_2) = \{w_1, w_2, w_3\}$ and edges $E(\overleftarrow{H}'_2) = \{\overleftarrow{w_1w_2}, \overleftarrow{w_2w_3}\}$. We define the *direct sum graph of type II* recursively, denoted by $\overleftarrow{\mathcal{G}}_{i+1} = \overleftarrow{\mathcal{G}}_i \oplus \overleftarrow{\mathcal{G}}'_i$, as the graph with vertex set

$$V(\overleftarrow{\mathcal{G}}_{i+1}) = V(\overleftarrow{\mathcal{G}}_i) \cup V(\overleftarrow{\mathcal{G}}'_i) \cup \{v_{2^i}\}$$

where $V(\overleftarrow{\mathcal{G}}_i) = \{v_1, v_2, \dots, v_{2^i-1}\}$ and $\overleftarrow{\mathcal{G}}'_i$ is the graph obtained from a copy of $\overleftarrow{\mathcal{G}}_i$ by relabeling the vertices of $\overleftarrow{\mathcal{G}}_i$ as follows $w_t := v_t$ for $t = 1, 2, \dots, 2^i - 1$. That is, $V(\overleftarrow{\mathcal{G}}'_i) = \{w_1, w_2, \dots, w_{2^i-1}\}$. The edge set of $\overleftarrow{\mathcal{G}}_{i+1}$ is

$$E(\overleftarrow{\mathcal{G}}_{i+1}) = E(\overleftarrow{\mathcal{G}}_i) \cup E(\overleftarrow{\mathcal{G}}'_i) \cup E(\overleftarrow{H}_i) \cup \{\overrightarrow{v_{2^i-1}v_{2^i}}, \overrightarrow{v_{2^i}w_1}\},$$

where

$$\overleftarrow{H}_i = \{\overleftarrow{v_jv_{2^i}} | 1 \le j \le 2^i - 1\} \cup \{\overleftarrow{v_{2^i}w_j} | 1 \le j \le 2^i - 1\}. \quad (3)$$

For example, $\overleftarrow{\mathcal{G}}_4$ is depicted in Figure 2.

We use $H(\overleftarrow{\mathcal{G}}_t)$ to denote the graph with the set of vertices equal to the set of vertices of $\overleftarrow{\mathcal{G}}_t$ and the set of edges defined by

$$E(H(\overleftarrow{\mathcal{G}}_t)) := \bigcup_{j=2}^{t} \overleftarrow{H}_j. \quad (4)$$

Note that when the direction is removed from the edges in this construction, the resulting graph is the graph found in [4] for the undirected case. Also note that this construction of $\overleftarrow{\mathcal{G}}_n$ is not unique. That is, there is more than one way to add admissible edges of type II to P_{2^n-1} to create a graph with the maximum number of these admissible edges as possible while maintaining the rank number.

The numerical sequences in Proposition 3 parts (2) and (4) are in Sloane [8] at A058922 and A036799, respectively.

Proposition 3. *If $n \ge 2$ and $\overleftarrow{\mathcal{G}}_n$ is the direct sum graph of type II, then*

1. $\chi_r(\overleftarrow{\mathcal{G}}_n) = \chi_r(\overrightarrow{P}_{2^n-1}) = n$ *and the minimum labeling of* $\overleftarrow{\mathcal{G}}_n$ *is unique,*
2. *the total number of edges in* $\overleftarrow{\mathcal{G}}_n$ *is* $(n-1)2^n$,
3. *an edge* $\overleftarrow{e}$ *is admissible of type II for* P_{2^n-1} *if and and only if* $\overleftarrow{e} \in H(\overleftarrow{\mathcal{G}}_n)$, *where* $H(\overleftarrow{\mathcal{G}}_n)$ *is as in (4), and*
4. *the total number of admissible edges of type II for* $\overrightarrow{P}_{2^n-1}$ *is* $(n-2)2^n + 2$.

Proof. **Part 1:** We prove this part by induction. Let $T(n)$ be the statement: $\chi_r(\overleftarrow{\mathcal{G}}_n) = \chi_r(\overrightarrow{P}_{2^n-1}) = n$ for $n > 1$ and that $\overleftarrow{\mathcal{G}}_n$ and $\overrightarrow{P}_{2^n-1}$ have the same minimal labeling. For this proof suppose that the labeling of $\overleftarrow{\mathcal{G}}_n$ is minimal.

The proof of $T(2)$ is straightforward from the definition of $\overleftarrow{\mathcal{G}}_2$. Suppose that $T(n)$ is true for some fixed $n = k$ with $k > 2$. That is, suppose that $\chi_r(\overleftarrow{\mathcal{G}}_k) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$ is true for some fixed $n = k$ with $k > 2$, and we prove $T(k+1)$ is also true.

Consider the graphs $\overrightarrow{P}_{2^k-1}$ and $\overleftarrow{\mathcal{G}}_k = \overleftarrow{\mathcal{G}}_{k-1} \oplus \overleftarrow{\mathcal{G}}'_{k-1}$. From the inductive hypothesis and Lemma 1 we know that both graphs have $2^k - 1$ vertices with the same labeling and that it is minimal and unique. From the definition of $\overleftarrow{\mathcal{G}}_{k+1}$ we know that its vertices are $v_1, v_2, \ldots, v_{2^k-1}, v_{2^k}, w_1, w_2, \ldots, w_{2^k-1}$ from left to right. To label $\overleftarrow{\mathcal{G}}_{k+1}$, we define f as follows: the function f keeps the same labels from $\overleftarrow{\mathcal{G}}_k$ for $\{v_1, v_2, \ldots, v_{2^k-1}\}$ and from $\overleftarrow{\mathcal{G}}'_k$ for $\{w_1, w_2, \ldots, w_{2^k-1}\}$ and $f(v_{2^k}) = k+1$ since v_{2^k} needs a new label. The function f is a well-defined labeling for $\overleftarrow{\mathcal{G}}_{k+1}$ since $f(v_{2^k}) := k+1$ preserves a proper labeling for the edges

$$\{\overleftarrow{v_i v_{2^k}}, \overleftarrow{v_{2^k} w_i} | 1 \le i \le 2^k - 1\}, \qquad \{\overleftarrow{v_i v_j} | \overleftarrow{v_i v_j} \in E(\overleftarrow{\mathcal{G}}_k)\},$$
$$\{\overleftarrow{w_i w_j} | \overleftarrow{w_i w_j} \in E(\overleftarrow{\mathcal{G}}'_k)\}, \quad \text{and} \quad \{\overrightarrow{v_{2^k-1} v_{2^k}}, \overrightarrow{v_{2^k} w_1}\}.$$

Since one end of each edge in $\{\overleftarrow{v_i v_{2^k}}, \overleftarrow{v_{2^k} w_i} | 1 \le i \le 2^k - 1\}$ and $\{\overrightarrow{v_{2^k-1} v_{2^k}}, \overrightarrow{v_{2^k} w_1}\}$ is labeled with the highest label, it is clear that these edges are admissible for $\overleftarrow{\mathcal{G}}_{k+1}$. The edges $\{\overleftarrow{v_i v_j} | \overleftarrow{v_i v_j} \in E(\overleftarrow{\mathcal{G}}_k)\}$ are admissible in $\overleftarrow{\mathcal{G}}_{k+1}$ since they are admissible in $\overleftarrow{\mathcal{G}}_k$. Similarly, the edges $\{\overleftarrow{w_i w_j} | \overleftarrow{w_i w_j} \in E(\overleftarrow{\mathcal{G}}'_k)\}$ are admissible in $\overleftarrow{\mathcal{G}}_{k+1}$. Note that f is also a minimal labeling for $\overrightarrow{P}_{2^{k+1}-1}$. This proves $T(k+1)$ is true.

Part 2: Let the total number of edges in $\overleftarrow{\mathcal{G}}_{k+1}$ be denoted by b_{k+1}. From the definition of edges of $\overleftarrow{\mathcal{G}}_{k+1}$ it is easy to see that,

$$\begin{aligned} b_{k+1} &= |E(\overleftarrow{\mathcal{G}}_k)| + |E(\overleftarrow{\mathcal{G}}'_k)| + |\{\overleftarrow{v_i v_{2^k}}, \overleftarrow{v_{2^k} w_i} | 1 \le i \le 2^k - 1\}| + |\{\overrightarrow{v_{2^k-1} v_{2^k}}, \overrightarrow{v_{2^k} w_1}\}| \\ &= b_k + b_k + 2(2^k - 1) + 2 \\ &= 2b_k + 2^{k+1}. \end{aligned}$$

We prove by induction that the number of edges in $\overleftarrow{\mathcal{G}}_{k+1}$ is given by $k2^{k+1}$. Let $T(n)$ be the statement: $b_n := (n-1)2^n$ for $n > 1$.

We prove $T(2)$. It is easy to see, from definition of $\overleftarrow{\mathcal{G}}_2$, that $b_2 = (2-1)2^2 = 4$. Suppose that $T(n)$ is true for some fix $n = k$ with $k > 2$. That is, suppose that $b_k = (k-1)2^k$ for some fix $n = k$ with $k > 2$ and we prove $T(k+1)$. Since $b_{k+1} = 2b_k + 2^{k+1}$, we have that

$$\begin{aligned} b_{k+1} &= 2b_k + 2^{k+1} \\ &= 2((k-1)2^k) + 2^{k+1} \\ &= k2^{k+1}. \end{aligned}$$

Thus, $\overleftarrow{\mathcal{G}}_n$ has $(n-1)2^n$ edges.

Part 3: Suppose that $\overleftarrow{e} \in H(\overleftarrow{\mathcal{G}}_n)$. Then by the definition of $E(H(\overleftarrow{\mathcal{G}}_n)$ and part 1 of this proposition, $\overleftarrow{e}$ is an admissible edge for $\overrightarrow{P}_{2^n-1}$.

Now suppose that $\overleftarrow{e}$ is an admissible edge for $\overrightarrow{P}_{2^n-1}$. We prove that $\overleftarrow{e} \in H(\overleftarrow{\mathcal{G}}_n)$ by induction. Let $T(n)$ be the statement: if $\overleftarrow{e}$ is admissible in $\overrightarrow{P}_{2^n-1}$, then $\overleftarrow{e} \in H(\overleftarrow{\mathcal{G}}_n)$ for $n > 1$.

We prove $T(2)$. Let $\overleftarrow{e}$ be admissible in $\overrightarrow{P}_{2^2-1}$. Then either $\overleftarrow{e}$ is in $\overleftarrow{H}_2$ or is $\overleftarrow{v_1v_3}$. The edge $\overleftarrow{v_1v_3}$ leads to a contradiction since $f(v_1) = f(v_3) = 1$ violates a proper labeling. Therefore, if $\overleftarrow{e}$ is admissible in $\overrightarrow{P}_{2^2-1}$, then $\overleftarrow{e} \in \overleftarrow{H}_2$, and thus $\overleftarrow{e} \in H(\overleftarrow{\mathcal{G}}_2)$.

Suppose that $T(n)$ is true for some fixed $n = k$ with $k > 2$. Suppose that $\overleftarrow{e}$ is admissible in $P_{2^{k+1}-1}$ with u and v as endpoints such that $f(v) < f(u) = j$.

Case 1. Suppose that u and v are in the same component $\mathcal{C}$ of $\overleftarrow{\mathcal{G}}_{k+1} \setminus A_{j+1}$. Then u has the largest label in $\mathcal{C}$ and is in position 2^{j-1}. Thus, $\overleftarrow{e} \in \overleftarrow{H}_j$ as defined in (3) and $\overleftarrow{e} \in H(\overleftarrow{\mathcal{G}}_{k+1})$.

Case 2. Suppose that $u \in \mathcal{C}$ and $v \in \mathcal{C}'$ where $\mathcal{C}$ and $\mathcal{C}'$ are distinct components of $\overleftarrow{\mathcal{G}}_{k+1} \setminus A_{j+1}$. Let $w \in \mathcal{C}'$ such that $f(w) = j$. The edge $\overleftarrow{e}$ gives rise to a path connecting u and w which does not contain a larger label in between u and w since each component contains all dual direction edges on the path. Such a path contradicts $\overleftarrow{e}$ being admissible in $\overrightarrow{P}_{2^{k+1}-1}$. Therefore, $\overleftarrow{e}$ is admissible in $\overrightarrow{P}_{2^n-1}$ if and only if $\overleftarrow{e} \in H(\overleftarrow{\mathcal{G}}_n)$.

Part 4: It is easy to see that $\overrightarrow{P}_{2^n-1}$ has $2^n - 2$ edges. From part 3 of this proposition we know that set of admissible edges for $\overrightarrow{P}_{2^n-1}$ is $H(\overleftarrow{\mathcal{G}}_n)$. Therefore, the total number of admissible edges is the number of edges in $\overleftarrow{\mathcal{G}}_n$ minus the number of edges in $\overrightarrow{P}_{2^n-1}$ which is $(n-2)2^n + 2$. □

5 Adjacency Matrices of $\overrightarrow{\mathcal{G}}_n$ and $\overleftarrow{\mathcal{G}}_n$

In this section we give recursive algorithms that highlight the symmetric structure of the graphs $\overrightarrow{\mathcal{G}}_n$ and $\overleftarrow{\mathcal{G}}_n$. The algorithms are based on block-recursive adjacency matrices for direct sum graphs of type I and II. The matrices present symmetry with respect to the antidiagonal rather than the main diagonal (see for example Tables 1 and 2).

In Table 1 we show the matrices A_2 and A_3 that represent direct sum graphs of type I. We observe that A_2 forms three blocks within A_3. Similarly, A_4 will contain three blocks of A_3 and so on. As mentioned previously, this symmetry is not obvious from looking at the corresponding graph, such as in Figure 1. The component symmetry of direct sum graphs of type II is clear from a graph, such as in Figure 2, but the fact that it is antidiagonal symmetry is obvious in the adjacency matrix found in Table 2. It should be clear that matrices for direct sum graphs of type I have the same block-recursive structure as matrices for direct sum graphs of type II, but the contents of the blocks are different.

In Algorithm 1, A_k denotes a $(2^k-1) \times (2^k-1)$ matrix. We use k in this manner because it simplifies our description of the recursion. We denote by 1_k the vector of length $2^k - 1$ where all entries are 1 and the transpose of this vector is denoted by 1_k^T. We denote by 0_k the $2^k \times 2^k$ matrix where all its entries are zero. We divide matrix A_k into blocks with the layout shown in Algorithm 1. With this example in mind, our algorithm for constructing a matrix A_k follows. Note that A_0 and 1_0 are not defined.

We observe, from the recursive definition of the graph $\overrightarrow{\mathcal{G}}_n$, that the adjacency matrix

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \left[\begin{array}{ccc|c|ccc} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Table 1: The adjacency matrices for the direct sum graphs of type I with $2^2 - 1$ and $2^3 - 1$ vertices, respectively.

of $\overrightarrow{\mathcal{G}}_n$ embeds in the Sierpiński sieve triangle. For example, Figure 3 shows A_3 embedded in the corresponding triangle. The entries of A_3 are within the region bounded by the dashed diamond shape in Figure 3 part (a). Thus, the entries on the main diagonal of A_3 correspond to the entries on line nine of the Sierpiński sieve triangle (which is row eight within the diamond). The entries on the super diagonal of A_3 correspond to the entries on line eight of the Sierpiński sieve triangle (which is row seven within the diamond). The entries on the subdiagonal of A_3 correspond to the entries on line 10 of the Sierpiński sieve triangle (which is row nine within the diamond) and so on. Thus, the entry a_{ij} of A_k is given by $\binom{2^k-j+i}{i}$ mod 2. Due to this embedding, $\overrightarrow{\mathcal{G}}_n$ should be called the *directed Sierpiński graph*. In fact, because of this embedding, we can use any number of algorithms to construct A_k and the corresponding graphs. For related graphs, see for example the undirected Pascal graphs defined by Deo and Quinn [3]. The undirected Sierpiński graph (or Hanoi graph) is represented in Figure 3 part (b) (see for example Romik [6]).

One motivation for Deo and Quinn [3] in describing undirected Pascal graphs was to define bidirectional computer network topologies with certain connectivity and cohesion constraints. It should be obvious how the incidence matrix of a type I graph embeds in an undirected Pascal graph–replacing by zero all nonzero entries that are below the main diagonal of the adjacency matrix of an undirected Pascal graph. Modern networks enable roles, communication, protocols, or permissions for which operations on distributed systems are asymmetric. Our directed graphs share some of the properties of Pascal graphs and may be useful in defining asymmetric computer networks with guaranteed properties.

In describing Algorithm 2, we use B, t and u to avoid confusion with Algorithm 1. We use B_t to denote a $(2^t - 1) \times (2^t - 1)$ matrix. As with k above, we use t in this manner because it simplifies our description of the recursion. We denote by 1_t the vector of length $2^t - 1$ where all entries are 1. We denote by 0_t the $(2^t - 1) \times (2^t - 1)$ matrix where all its entries are zero. We denote by J_t a single-entry column vector of length $(2^t - 1)$ where the last element is 1 and all others are 0 and by J'_t a single-entry row vector where the first element is 1 and all others are 0. We divide matrix B_t into blocks with the layout shown in Algorithm 2. For example, in Table 2, we show the matrices B_2 and B_3.

Our algorithm for constructing B_t follows. As with A_k, B_0 and 1_0 are not defined.

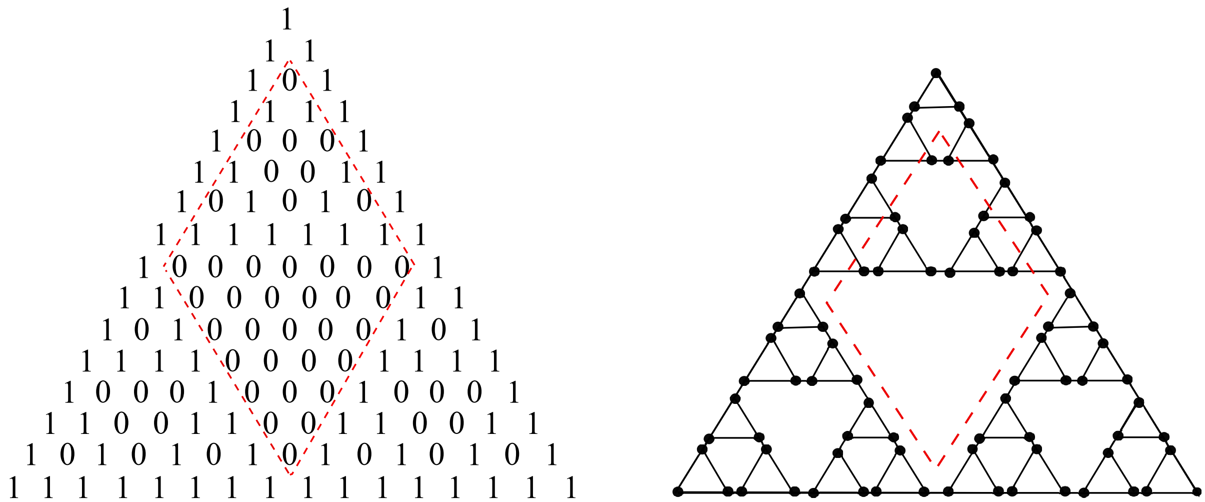

Figure 3: (a) Sierpiński triangle. (b) Sierpiński sieve graph or Hanoi graph.

$$B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Table 2: The adjacency matrices for the direct sum graphs of type II with $2^2 - 1$ and $2^3 - 1$ vertices, respectively.

Algorithm 1 Construct A_k from A_{k-1} recursively.

$1_k = [1]_{(2^k-1)\times 1}$
$0_k = [0]_{2^k \times 2^k}$
$A_1 = [0]$

function BUILDMATRIX(k)
  if $k \equiv 1$ **then**
    $A_k \leftarrow A_1$
  else
    $j \leftarrow k - 1$

$$A_k \leftarrow \begin{bmatrix} A_j & 1_j & A_j \\ & & 1_j^T \\ & 0_j & \\ & & A_j \end{bmatrix}$$

  return A_k

Algorithm 2 Construct B_t from B_{t-1} recursively.

$1_t = \big[1\big]_{(2^t-1)\times 1}$
$0_t = \big[0\big]_{(2^t-1)\times(2^t-1)}$
J_t = single-entry vector $J_{(2^t-1)\times 1}$, where the last element is 1, all others are 0
J'_t = single-entry vector $J_{1\times(2^t-1)}$, where the first element is 1, all others are 0
$B_1 = \big[0\big]$

function BUILDMATRIXII(t)
  if $t \equiv 1$ **then**
    $B_t \leftarrow B_1$
  else
    $u \leftarrow t-1$

$$B_t \leftarrow \begin{bmatrix} B_u & J_u & 0_u \\ 1_u^T & 0 & J'_u \\ 0_u & 1_u & B_u \end{bmatrix}$$

  return B_t

6 Admissible Edges for a Directed Cycle

In this section we use the results from previous sections to find the admissible edges of type I and type II for $\overrightarrow{C}_{2^k}$ and prove similar results. Figure 4 part (a) shows $\overrightarrow{C}_{2^k}$ with the admissible edges of type I. The number of edges in this graph is 65 and the number of admissible edges is 50. Figure 4 part (b) shows $\overrightarrow{C}_{2^k}$ with the admissible edges of type II. The number of edges in this graph is 65 and the number of admissible edges is 49. The number of admissible edges and the number of edges of the new graph are given by known numerical sequences. One of these sequences is the Stirling numbers. In both parts the rank number is equal to the rank number of $\overrightarrow{C}_{2^k}$.

We recall that $V := \{v_1, v_2, \ldots, v_{2^k}\}$ is the set of vertices of $\overrightarrow{C}_{2^k}$, and that $V \setminus \{v_{2^k}\}$ is the set of vertices of $\overrightarrow{P}_{2^k-1}$. We define

$$H_{\overrightarrow{C}} = H(\overrightarrow{\mathcal{G}}_k) \cup \{\overrightarrow{e} \mid \overrightarrow{e} \notin \overrightarrow{C}_{2^k} \text{ with } \overrightarrow{e} = \overrightarrow{v_{2^k}v_i} \text{ where } i \in \{2, \ldots, 2^k-2\}\} \tag{5}$$

where $H(\overrightarrow{\mathcal{G}}_k)$ is the set of admissible edges of type I for $\overrightarrow{P}_{2^k-1}$ (see (2)). We now define

$$H_{\overleftarrow{C}} = H(\overleftarrow{\mathcal{G}}_k) \cup \{\overleftarrow{e} \mid \overleftarrow{e} \notin \overrightarrow{C}_{2^k} \text{ with } \overleftarrow{e} = \overleftarrow{v_{2^k}v_i} \text{ where } i \in \{2, \ldots, 2^k-2\}\} \tag{6}$$

where $H(\overleftarrow{\mathcal{G}}_k)$ is the set of admissible edges of type II for $\overrightarrow{P}_{2^k-1}$ (see (4)).

We use $\overrightarrow{\Omega}_{2^k}$ and $\overleftarrow{\Omega}_{2^k}$ to mean the graphs $\overrightarrow{C}_{2^k} \cup H_{\overrightarrow{C}}$ and $\overrightarrow{C}_{2^k} \cup H_{\overleftarrow{C}}$, respectively. The numerical sequences in Proposition 4 parts (3), (4) and (6) are in Sloane [8] at A001047, A002064 (called Cullen numbers), and A048495, respectively.

Proposition 4. *If $\overrightarrow{\Omega}_{2^k} = \overrightarrow{C}_{2^k} \cup H_{\overrightarrow{C}}$ and $\overleftarrow{\Omega}_{2^k} = \overrightarrow{C}_{2^k} \cup H_{\overleftarrow{C}}$ with $k \geq 2$, then*

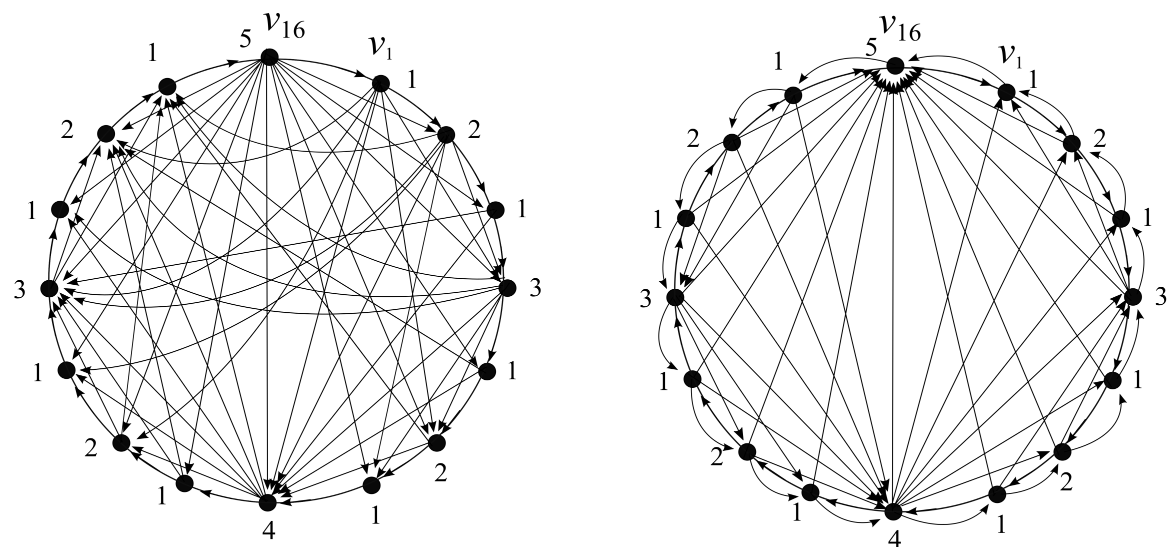

Figure 4: (a) Admissible edges of type I. (b) Admissible edges of type II.

1. *the set* $H_{\overrightarrow{C}}$ *is the set of admissible edges of type I for* $\overrightarrow{C}_{2^k}$ *if and only if*

$$\chi_r(\overrightarrow{\Omega}_{2^k}) = \chi_r(\overrightarrow{C}_{2^k}) = k+1,$$

2. *the set* $H_{\overleftarrow{C}}$ *is the set of admissible edges of type II for* $\overrightarrow{C}_{2^k}$ *if and only if*

$$\chi_r(\overleftarrow{\Omega}_{2^k}) = \chi_r(\overrightarrow{C}_{2^k}) = k+1,$$

3. *the total number of edges in* $\overrightarrow{\Omega}_{2^k}$ *is* $3^k - 2^k$*,*

4. *the total number of edges in* $\overleftarrow{\Omega}_{2^k}$ *is* $k2^k + 1$*,*

5. *the total number of admissible edges of type I for* $\overrightarrow{C}_{2^k}$ *is the Stirling number of the second kind* $2S(k+1, 3) = 3^k - 2^{k+1} + 1$*, and*

6. *the total number of admissible edges of type II for* $\overrightarrow{C}_{2^k}$ *is* $(k-1)2^k + 1$*.*

Proof. We prove parts 1, 3, and 5. The proofs for part 2, 4, and 6 are similar, respectively, and we omit them. We assume all admissible edges are of type I throughout this proof.

Part 1: For the proof of necessity, it is easy to see that if $H_{\overrightarrow{C}}$ is not a set of admissible edges, then this set contains a forbidden edge; therefore the rank number of $\overrightarrow{\Omega}_{2^k}$ is different that the rank number of $\overrightarrow{C}_{2^k}$. That is a contradiction.

We now prove sufficiency. From Lemma 1 we know that $\chi_r(\overrightarrow{C}_{2^k}) = k+1$ and that the minimal ranking is unique. Suppose that $f : V(\overrightarrow{C}_{2^k}) \to \{1, 2, \ldots, k+1\}$ is the ranking function of $\overrightarrow{C}_{2^k}$. So, $f(v_{2^k}) = k+1$.

Let $\overrightarrow{e}_1 := \overrightarrow{v_{2^k-1}v_{2^k}}$ and $\overrightarrow{e}_2 := \overrightarrow{v_{2^k}v_1}$ be two edges of $\overrightarrow{C}_{2^k}$ and let H' be subgraph of $H_{\overrightarrow{C}}$ formed by all edges of $H_{\overrightarrow{C}}$ that have vertices in $V' = V\left(H_{\overrightarrow{C}}\right) \setminus \{v_{2^k}\} = \{v_1, v_2, \ldots, v_{2^k-1}\}$. That is, $H' = H(\overrightarrow{\mathcal{G}}_k)$ and V' is the set of vertices of $\overrightarrow{C}_{2^k} \setminus \{\overrightarrow{e_1}, \overrightarrow{e_2}\}$. Then by Theorem 4, we have $E(H')$ is a set of admissible edges for the graph $\overrightarrow{C}_{2^k} \setminus \{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ if and only if $\chi_r(\overrightarrow{C}_{2^k} \setminus \{e_1, e_2\} \cup H') = k$. The vertices of $\overrightarrow{C}_{2^k} \setminus \{e_1, e_2\} \cup H'$ have the same labels as the vertices V'.

We now prove that $\chi_r(\overrightarrow{C}_{2^k} \cup H_{\overrightarrow{C}}) = k+1$. Let $\overrightarrow{e}$ be an edge in $H_{\overrightarrow{C}} \setminus H'$. That is,

$$\overrightarrow{e} \in \left\{ \overrightarrow{e} \mid \overrightarrow{e} \notin \overrightarrow{C}_{2^k} \text{ with } \overrightarrow{v_{2^k}v_i} \text{ where } i \in \{2, \ldots, 2^k-2\} \right\}.$$

Therefore, the vertices of $\overrightarrow{e}$ are v_{2^k} and v_i for some $2 \le i \le 2^k - 2$. From definition of the labeling function f we know that $f(v_{2^k}) = k+1$ and $f(v_i) < k+1$. We do not create any new paths in $\overrightarrow{\Omega}_{2^k}$ connecting vertices with label $k+1$. This implies that the number of labels does not increase. This proves part 1.

Part 3: We consider the sets of edges W and W' defined as follows:

$$W := \{\overrightarrow{v_{2^k}v_i} | i = 2, \ldots, 2^k-2\} \text{ and } W' := \{\overrightarrow{v_{2^k}v_1}, \overrightarrow{v_{2^k-1}v_{2^k}}\}.$$

The cardinality of W is $2^k - 3$. From Proposition 2 part 1 we know that all admissible edges for $\overrightarrow{P}_{2^k-1}$ are also admissible for $\overrightarrow{C}_{2^k}$. Therefore, the maximum number of edges that can be added to $\overrightarrow{C}_{2^k}$ without changing its rank number is equal to maximum number of edges that can be added to $\overrightarrow{P}_{2^k-1}$ plus all edges in W. Thus, the total number of edges in $\overrightarrow{\Omega}_{2^k}$ is equal to the number of admissible edges for $\overrightarrow{P}_{2^k-1}$ and the number of edges in $\overrightarrow{P}_{2^k-1}$, plus the number of edges in $W \cup W'$. Therefore, the number of edges in $\overrightarrow{\Omega}_{2^k}$ is $3^k - 2^k$.

Part 5: This proof is straightforward by counting the number of edges that are admissible for $\overrightarrow{P}_{2^k-1}$ and adding the number of edges in W.

□

Corollary 5. *The graphs $\overrightarrow{\mathcal{G}}_k$, $\overleftarrow{\mathcal{G}}_k$, $\overrightarrow{\Omega}_{2^k}$, $\overleftarrow{\Omega}_{2^k}$ have unique minimal rankings.*

7 Admissible Graphs for Directed Paths and Cycles

In this section, we explore constructing new graphs by attaching directed paths and directed cycles to the direct sum graphs and the omega graphs built in the previous sections. We give algorithms for labeling the new resulting graphs. The algorithms keep the same rank number as the original graph. Thus, the rank number of the graphs constructed here is either the rank number of $\overrightarrow{P}_{2^k-1}$ or of $\overrightarrow{C}_{2^k}$.

Finding the rank number of a given graph is a hard problem, even for simple graphs. In the previous sections, we took a known graph with known rank number, and we built a new graph that preserves the rank number and as well the set of vertices. In this section, we explore the same idea, but without preserving the set of vertices. Thus, we give some results on how to build new graphs from a base graph such that the new graph is larger than the original in terms of the number of vertices and preserves the rank number of the base graph.

Recall from Section 2 that $\overrightarrow{P}_{2^k-1}$ or $\overrightarrow{C_{2^k}}$ have vertex sets $V = \{v_1, v_2, \ldots, v_{2^k-1}\}$ and $V \cup \{v_{2^k}\}$, respectively. We construct a new graph by attaching a directed path or a directed cycle to the vertex v_i for some i. Let $\{w_1, w_2, \ldots, w_j = v_i\}$ be the edges of a directed path of length j that is attached to $\overrightarrow{P}_{2^k-1}$ or $\overrightarrow{C_{2^k}}$ at the vertex v_i. The path is denoted by $\overrightarrow{P_j}^i$ if its edges are directed as $w_l \to w_{l+1}$, and the path is denoted by $\overleftarrow{P_j}^i$ if its edges are directed as $w_l \leftarrow w_{l+1}$. Notice that the edges of $\overrightarrow{P}_{2^k-1}$ and $\overrightarrow{C_{2^k}}$ are oriented as defined in Section 2.

From Sections 3 and 4 we know that $\overrightarrow{\mathcal{G}}_k := \overrightarrow{P}_{2^k-1} \cup H(\overrightarrow{\mathcal{G}}_k)$ and $\overleftarrow{\mathcal{G}}_k := \overrightarrow{P}_{2^k-1} \cup H(\overleftarrow{\mathcal{G}}_k)$ with vertices V. We also know that $\overrightarrow{\Omega}_{2^k} := \overrightarrow{C}_{2^k} \cup H_{\overrightarrow{C}}$ and $\overleftarrow{\Omega}_{2^k} := \overrightarrow{C}_{2^k} \cup H_{\overleftarrow{C}}$ with vertices $V \cup \{v_{2^k}\}$ where $H_{\overrightarrow{C}}$ is as in (5) and $H_{\overleftarrow{C}}$ is as in (6) .

We say that a directed graph G is *admissible* for a directed graph Γ if $\chi_r(\Gamma \cup G) = \chi_r(\Gamma)$, and G is *forbidden* for Γ if $\chi_r(\Gamma \cup G) > \chi_r(\Gamma)$. As an example of these admissible graphs and Lemma 6, see Figure 5.

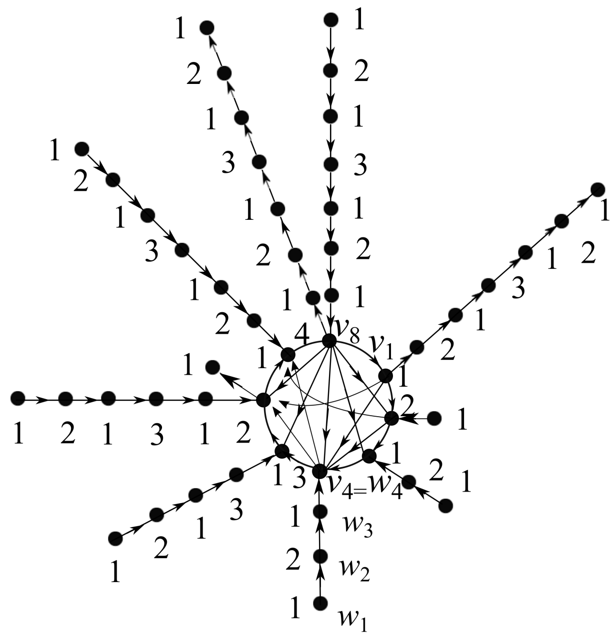

Figure 5: A directed cycle with admissible directed paths.

Lemma 6. *If G_1 is either $\overrightarrow{\mathcal{G}}_k$ or $\overleftarrow{\mathcal{G}}_k$, and G_2 is either $\overrightarrow{\Omega}_{2^k}$ or $\overleftarrow{\Omega}_{2^k}$, then*

1. *the path $\overrightarrow{P_j}^i$ with $1 < j \leq i \leq 2^k - 1$ is an admissible graph for G_1 and*

$$\chi_r(G_1 \cup \overrightarrow{P_j}^i) = \chi_r(\overrightarrow{P}_{2^k-1}) = k,$$

2. *the path $\overleftarrow{P_j}^i$ with $1 \leq j \leq 2^k - i$ is an admissible graph for G_1 and*

$$\chi_r(G_1 \cup \overleftarrow{P_j}^i) = \chi_r(\overrightarrow{P}_{2^k-1}) = k,$$

3. *the path* $\overrightarrow{P}_j^i$ *with* $1 < j \leq i \leq 2^k$ *is an admissible graph for* G_2 *and*

$$\chi_r(G_2 \cup \overrightarrow{P}_j^i) = \chi_r(\overrightarrow{C}_{2^k}) = k+1, \text{ and}$$

4. *the path* $\overleftarrow{P}_j^i$ *with* $1 \leq j \leq 2^k - i$ *is an admissible graph for* G_2 *and*

$$\chi_r(G_2 \cup \overleftarrow{P}_j^i) = \chi_r(\overrightarrow{C}_{2^k}) = k+1.$$

Proof. We prove parts 1 and 2. Parts 3 and 4 are similar, and we omit the proofs. For both parts 1 and 2, we prove the case $G_1 = \overrightarrow{\mathcal{G}}_k$. Since the case $G_1 = \overleftarrow{\mathcal{G}}_k$ is similar, it is omitted. We recall that $V(\overrightarrow{\mathcal{G}}_k) = \{v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_{2^k-1}\}$ is the set of vertices of $\overrightarrow{\mathcal{G}}_k$ and that the edges of $\overrightarrow{\mathcal{G}}_k$ are directed as $v_t \to v_{t+1}$ for $t < 2^k - 1$.

Part 1. Suppose that $\{w_1, \dots, w_j = v_i\}$ is set of vertices of $\overrightarrow{P_j^i}$ for some fixed $1 < j \leq i \leq 2^k - 1$ where $\overrightarrow{P_j^i}$ is the path attached to the vertex $v_i \in G_1$ with edges directed as $w_t \to w_{t+1}$.

Corollary 5 guarantees that G_1 has a unique minimal ranking. Let a minimal ranking function be $f : V(G_1) \to \{1, 2, \dots, k\}$. Since $1 < j \leq i \leq 2^k - 1$, we can label the vertices $\overrightarrow{P_j^i}$ with the labels given by the ranking function f without increasing the rank number of the new graph. Algorithm 3 allows us to do it. (See for example Figure 5.)

Algorithm 3 Labeling vertices of $\overrightarrow{P_j^i}$

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for $t = 0 \to j-1$ **do**
  $f(w_{j-t}) = f(v_{i-t})$;
  $t \leftarrow t+1$

Using Algorithm 3 we obtain

$$f(w_j) = f(v_i), \quad f(w_{j-1}) = f(v_{i-1}), \quad f(w_{j-2}) = f(v_{i-2}), \dots, \text{ and } \quad f(w_1) = f(v_{i-j+1}).$$

Therefore, we have a ranking function $f : V(G_1 \cup \overrightarrow{P}_j^i) \to \{1, 2, \dots, k\}$ that labels all vertices of $G_1 \cup \overrightarrow{P}_j^i$ without increase the rank number. This proves that $\chi_r(G_1 \cup \overrightarrow{P}_j^i) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$.

Part 2. Suppose that $\{w_1, \dots, w_j = v_i\}$ is the set of vertices of $\overleftarrow{P_j^i}$ for some fixed $1 \leq j \leq 2^k - i$ where $\overleftarrow{P_j^i}$ is the path attached to the vertex $v_i \in G_1$ with edges directed as $w_t \leftarrow w_{t+1}$.

Corollary 5 guarantees that G_1 has a unique minimal ranking. Let a minimal ranking function be $f : V(G_1) \to \{1, 2, \dots, k\}$. Since $1 \leq j \leq 2^k - i$, we can label the vertices of $\overleftarrow{P_j^i}$ with the labels given by the ranking function f. This labeling will not increase the rank number of the new graph. Algorithm 4 allows us to do it. (See for example Figure 5.)

Using Algorithm 4 we obtain

$$f(w_j) = f(v_i), \quad f(w_{j-1}) = f(v_{i+1}), \quad f(w_{j-2}) = f(v_{i+2}), \dots, \text{ and } \quad f(w_1) = f(v_{i+j-1}).$$

Therefore, we have a ranking function $f : V(G_1 \cup \overleftarrow{P}_j^i) \to \{1, 2, \dots, k\}$ that labels all vertices of $G_1 \cup \overleftarrow{P}_j^i$ without increasing the rank number. That is, $\chi_r(G_1 \cup \overleftarrow{P}_j^i) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$. ☐

Algorithm 4 Labeling vertices of $\overleftarrow{P_j^i}$

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for $t = 0 \to j-1$ **do**
  $f(w_{j-t}) = f(v_{i+t})$
  $t \leftarrow t+1$

Proposition 7. *If G_1 is either $\overrightarrow{\mathcal{G}}_k$ or $\overleftarrow{\mathcal{G}}_k$, and G_2 is either $\overrightarrow{\Omega}_{2^k}$ or $\overleftarrow{\Omega}_{2^k}$, then*

1. $\chi_r\left(G_1 \cup \bigcup_{1<j\leq i\leq 2^k-1} \overrightarrow{P_j}^i \bigcup_{1\leq j\leq 2^k-i} \overleftarrow{P_j}^i\right) = \chi_r(\overrightarrow{P}_{2^k-1}) = k,$ *and*

2. $\chi_r\left(G_2 \cup \bigcup_{1<j\leq i\leq 2^k} \overrightarrow{P_j}^i \bigcup_{\substack{1<j\leq 2^k-i;\\ i<2^k}} \overleftarrow{P_j}^i\right) = \chi_r(\overrightarrow{C}_{2^k}) = k+1.$

Proof. We prove part 1 for the case $G_1 = \overrightarrow{\mathcal{G}}_k$. The other case where $G_1 = \overleftarrow{\mathcal{G}}_k$ and part 2 are similar and thus omitted. From Lemma 6 part 1 we know that $\chi_r(G_1) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$. Let $f : V(G_1) \to \{1, 2, \ldots, k\}$ be the minimal ranking of $G_1 = \overrightarrow{\mathcal{G}}_k$. Using Algorithms 3 and 4 developed in the proof of Lemma 6, we define a ranking function f' that labels all vertices of all paths of the form of $\overrightarrow{P_j^i}$ or of the form $\overleftarrow{P_j^i}$ attached to G_1. That is, f' is the function defined by Algorithm 3 if the path is of the form $\overrightarrow{P_j^i}$ and f' is the function defined by Algorithm 4 if the path is of the form $\overleftarrow{P_j^i}$. From those algorithms is easy to see that $f'(v) \leq k$.

Let $f^* : V(D) \to \{1, 2, \ldots, k\}$ be the function defined as

$$f^*(v) = \begin{cases} f(v) & \text{if} \quad v \in G_1 \\ f'(v), & \text{if} \quad v \notin G_1, \end{cases}$$

where

$$D := G_1 \cup \bigcup_{1<j\leq i\leq 2^k-1} \overrightarrow{P_j}^i \cup \bigcup_{1\leq j\leq 2^k-i} \overleftarrow{P_j}^i.$$

From the definition of f and f' is easy to see that $f^*(v) \leq k$ for $v \in V(D)$.

We now prove that f^* is a ranking function of D. That is, we want to prove that given any two vertices in D with the same label, every directed path connecting those two vertices has a vertex with larger label. We prove it by contradiction. Suppose that there are two vertices $u, w \in D$ connected by a directed path P with $f^*(w) = f^*(u)$ and for every other vertex $v' \in P$, we have $f^*(v') < f^*(u)$.

Let $V(P) = \{w = w_1, w_2, \ldots, w_j = v_i, v_{i+1}, \ldots, v_l = u_1, u_2, \ldots, u_r = u\}$ be the set of vertices of P, where $\{v_i, v_{i+1}, \ldots, v_l\}$ are vertices in G_1 and $V(P) \setminus \{v_i, v_{i+1}, \ldots, v_l\}$ are vertices in $D \setminus G_1$. Notice that j and r may be equal to one and that i and l may be equal. We suppose that the edges of P are of the form $v_t \to v_{t+1}$, $w_t \to w_{t+1}$, and $u_t \to u_{t+1}$.

From Algorithm 3 in the proof of Lemma 6, we know that $f'(w_{j-s}) = f(v_{i-s})$ for $s = 1, 2, \dots, j-1$, and from Algorithm 4 we know that $f'(u_{j+s}) = f(v_{l+s})$ for $s = 1, 2, \dots r-1$. These imply that there exists a path P' with vertices

$$V(P') = \{v_{i-(j-1)}, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_l, v_{l+1}, \dots, v_{l+r}\}$$

in G_1 satisfying that $f(v_{i-(j-1)}) = f^*(w) = f^*(u) = f(v_{l+r})$ and $f(v') < f^*(w)$ for $v' \in V(P') \setminus \{v_{i-(j-1)}, v_{l+r}\}$. That is contradiction because f is a ranking function of G_1 and $f(v_{i-(j-1)}) = f(v_{l+r})$. This proves that f^* is a ranking function for D. The definition of f^* tells us that $f^*(v) \leq k$ for $v \in V(D)$. Thus, $\chi_r(D) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$. This proves part 1 with $G_1 = \overrightarrow{\mathcal{G}}_k$. □

Recall that $(\alpha_r\alpha_{r-1}\cdots\alpha_1\alpha_0)_2$ with $\alpha_h = 0$ or 1 for $0 \leq h \leq r$ is the binary representation of a positive integer b if $b = \alpha_r 2^r + \alpha_{r-1}2^{r-1} + \cdots + \alpha_1 2^1 + \alpha_0 2^0$. We define h as $h(b) = j$ if α_j is the rightmost nonzero entry of the binary representation of b. Flórez and Narayan [4] proved that if v_m is a vertex of P_{2^k-1} in position m, then $h(m) = f(v_m)$ where f is the ranking function of $\overrightarrow{P}_{2^k-1}$. The same result extends naturally to directed paths.

Let W_1 be a subgraph of a graph W_2. A *vertex of attachment* of W_1 in W_2 is a vertex of W_1 that is incident with some edge of W_2 that is not an edge of W_1 (for this definition see [7] page 11 section I.4).

Let G' be either $\overrightarrow{\mathcal{G}}_k$ or $\overleftarrow{\mathcal{G}}_k$ with vertices $V(G') = \{v_1, v_2, \dots, v_{2^k-1}\}$ and let $G(t)$ be either $\overrightarrow{\Omega}_{2^{h(t)}}$ or $\overleftarrow{\Omega}_{2^{h(t)}}$. We define

$$D(t) := G(t) \cup \bigcup_{1<j\leq i\leq 2^{h(t)}} \overrightarrow{P}_j^i \cup \bigcup_{\substack{1<j\leq 2^{h(t)}-i \\ i<2^{h(t)}}} \overleftarrow{P}_j^i$$

for some $t \in \{2, \dots, 2^k-1\}$.

Let

$$\Gamma := G' \cup \bigcup_{t=2}^{2k-1} D(t).$$

Notice that $D(t)$ has exactly one vertex of attachment in Γ which is given by $v_t \in G'$. As an example of this graph and Proposition 8, see Figure 6.

Let Γ' be the graph formed by G' and the union of a set of graphs N_t for $t \in I$, where I is an index set, such that the graphs G' and N_t intersect exactly in the vertex $v_i \in V(G')$. That is, N_t has exactly one vertex of attachment $v_i \in \Gamma'$. Theorem 9 proves that Γ' generalizes Lemma 6 and Propositions 7 and 8.

Proposition 8. *If for each $t \in \{2, 4, \dots, 2^k-2\}$ the set $\{v_t\} \subset \Gamma$ is the maximum set of vertices of attachment of*

$$G(t) \cup \bigcup_{1<j\leq i\leq 2^{h(t)}} \overrightarrow{P}_j^i \cup \bigcup_{\substack{1<j\leq 2^{h(t)}-i \\ i<2^{h(t)}}} \overleftarrow{P}_j^i$$

in Γ, then $\chi_r(\Gamma) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$.

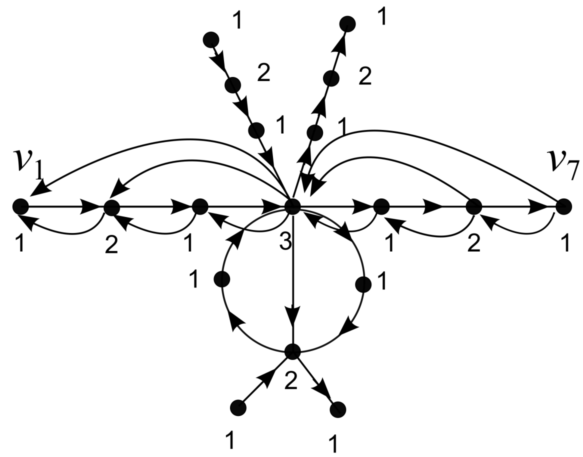

Figure 6: A directed path with admissible directed graphs.

Proof. From Proposition 7, we know that $\chi_r(D(t)) = \chi_r(\overrightarrow{C}_{2^{h(t)}}) \leq h(t)$ where

$$D(t) := G(t) \cup \bigcup_{1<j\leq i\leq 2^{h(t)}} \overrightarrow{P}_j^i \cup \bigcup_{\substack{1<j\leq 2^{h(t)}-i \\ i<2^{h(t)}}} \overleftarrow{P}_j^i$$

for some $t \in \{2, 4, \ldots, 2^k - 2\}$. From the definition of $D(t)$ we can see that $v_t = G' \cap D(t)$. Note that $f(v_t) = h(t)$ and that every other vertex of $D(t)$ has label less than $h(t)$. So, attaching $D(t)$ to G' does not increase the rank number of G'. Since this argument is true for every $t \in \{2, 4, \ldots, 2^k - 2\}$, it proves that $\chi_r(\Gamma) = \chi_r(\overrightarrow{P}_{2^k-1}) = k$. □

Theorem 9. *Let v_i be the vertex of attachment of N_t in Γ' for t in an index set I and some $i \in \{2, \ldots, 2^k - 2\}$. Suppose that $\chi_r(N_t) = h(i)$. If there is a ranking function f_t of N_t such that $f_t(v_i) = h(i)$, then $\chi_r(\Gamma') = \chi_r(\overrightarrow{P}_{2^k-1})$.*

Proof. Since the vertex $v_i = G' \cap N_t$ has label $h(i) = f_t(v_i) = f(v_i)$ where f_t and f are the ranking functions of N_t and G', respectively, every other vertex of N_t has label less than $h(i)$. So, attaching N_t to G' does not increase the rank number of Γ'. Since this argument is true for every $t \in \{2, 4, \ldots, 2^k - 2\}$, it proves that $\chi_r(\Gamma') = \chi_r(\overrightarrow{P}_{2^k-1})$. □

References

[1] H. L. Bodlaender, J. S. Deogun, K. Jansen, T. Kloks, D. Kratsch, H. Müller, and Z. Tuza, Rankings of graphs, *Siam J. Discrete Math.* **11** (1998), 168–181.

[2] E. Bruoth and M. Horňák, Online-ranking numbers for cycles and paths, *Discuss. Math. Graph Theory*, **19** (1999), 175–197.

[3] N. Deo and M. J. Quinn, Pascal graphs and their properties, *Fibonacci Quart.* 21 (1983), no. 3, 203-214.

[4] R. Flórez and D. A. Narayan, Maximizing the number of edges in optimal k-ranking, *AKCE Int. J. Graphs Comb.* **12** no. 1 (2015), 1–8.

[5] R. Flórez and D. A. Narayan, An optimal k-ranking characterization of oriented paths and cycles, *Bull. Inst. Combin. Appl.* **61** (2011), 97–108.

[6] D. Romik, Shortest paths in the tower of Hanoi graph and finite automata, *SIAM J. Discrete Math.* **20** (2006), 610-622.

[7] W. T. Tutte, *Graph Theory*, Cambridge University Press, New York, 2001.

[8] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.