

DIFFERENCE BASES IN CYCLIC GROUPS

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ABSTRACT. A subset B of an Abelian group G is called a *difference basis* of G if each element $g \in G$ can be written as the difference $g = a - b$ of some elements $a, b \in B$. The smallest cardinality $|B|$ of a difference basis $B \subset G$ is called the *difference size* of G and is denoted by $\Delta[G]$. We prove that for every $n \in \mathbb{N}$ the cyclic group C_n of order n has difference size $\frac{1+\sqrt{4|n|-3}}{2} \leq \Delta[C_n] \leq \frac{3}{2}\sqrt{n}$. If $n \geq 9$ (and $n \geq 2 \cdot 10^{15}$), then $\Delta[C_n] \leq \frac{12}{\sqrt{73}}\sqrt{n}$ (and $\Delta[C_n] < \frac{2}{\sqrt{3}}\sqrt{n}$). Also we calculate the difference sizes of all cyclic groups of cardinality ≤ 100 .

1. INTRODUCTION

A subset B of a group G is called a *difference basis* for a subset $A \subset G$ if each element $a \in A$ can be written as $a = xy^{-1}$ for some $x, y \in B$. If the group operation of G is denoted by $+$, then the element xy^{-1} is written as the difference $x - y$ (which justifies the choice of the terminology).

The smallest cardinality of a difference basis for $A \subset G$ is called the *difference size* of A and is denoted by $\Delta[A]$. For example, the set $\{0, 1, 4, 6\}$ is a difference basis for the interval $A = [-6, 6] \cap \mathbb{Z}$ witnessing that $\Delta[A] \leq 4$. In Proposition 2.2(4) we shall prove that the difference size is subadditive in the sense that $\Delta[A \cup B] < \Delta[A] + \Delta[B]$ for any non-empty subsets A, B of a group G .

The definition of a difference basis B for a set A in a group G implies that $|A| \leq |B|^2$ and hence $\Delta[A] \geq \sqrt{|A|}$. The fraction

$$\delta[A] := \frac{\Delta[A]}{\sqrt{|A|}} \geq 1$$

is called the *difference characteristic* of A . The difference characteristic is submultiplicative in the sense that $\delta[G] \leq \delta[H] \cdot \delta[G/H]$ for any normal subgroup H of a finite group G , see [3, 1.1].

In this paper we are interested in evaluating the difference characteristics of finite cyclic groups. In fact, this problem has been studied in the literature (see [5], [10], [15]). In particular, Kozma and Lev [15] proved (using the classification of finite simple groups) that each finite group G has difference characteristic $\delta[G] \leq \frac{4}{\sqrt{3}} \approx 2.3094$. In this paper we shall show that for finite cyclic groups this upper bound can be improved to $\delta[G] \leq \frac{3}{2}$. Moreover, if a finite cyclic group C_n has cardinality $n \geq 9$ (resp. $n \geq 2 \cdot 10^{15}$), then $\delta[C_n] \leq \frac{12}{\sqrt{73}} \approx 1.4045$ (resp. $\Delta[C_n] < \frac{2}{\sqrt{3}} \approx 1.1547$). It is an open problem if $\lim_{n \rightarrow \infty} \delta[C_n] = 1$. However, many subsequences of the sequence $(\delta[C_n])_{n=1}^{\infty}$ indeed converge to 1. In particular, using known results on (relative) difference sets, we shall prove that

$$\lim_{p \rightarrow \infty} \delta[C_{p^2-p}] = \lim_{q \rightarrow \infty} \delta[C_{q^2+q+1}] = \lim_{q \rightarrow \infty} \delta[C_{q^2-q}] = 1$$

where p runs over prime numbers and q runs over prime powers. A number q is called a *prime power* if q is equal to the power p^k of some prime number p .

To derive an upper bound for the difference sizes of arbitrary finite cyclic group, we shall use known information on the difference sizes of order intervals $[-n, n] = \{x \in \mathbb{Z} : |x| \leq n\}$ in the group \mathbb{Z} of integer numbers. Here we exploit the approach first used by Rédei and Rényi [17] and then developed by Leech [14] and Golay [12].

For a model of a cyclic group of order n we take the multiplicative group

$$C_n = \{z \in \mathbb{C} : z^n = 1\}$$

of complex n -th roots of 1.

2. KNOWN RESULTS

In this section we recall some known results on difference bases in finite groups. The following important fact was proved by Kozma and Lev [15] (using the classification of finite simple groups).

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Theorem 2.1 (Kozma, Lev). *Each finite group G has difference characteristic $\delta[G] \leq \frac{4}{\sqrt{3}}$.*

For cyclic groups the upper bound $\frac{4}{\sqrt{3}}$ can be improved to $\frac{3}{2}$, which will be done in Theorem 4.14. For a real number x we put

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\} \text{ and } \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}.$$

The first three statements of the following proposition were proved in [3, 1.1].

Proposition 2.2. *Let G be a finite group. Then*

- (1) $\frac{1+\sqrt{4|G|-3}}{2} \leq \Delta[G] \leq \lceil \frac{|G|+1}{2} \rceil$,
- (2) $\Delta[G] \leq \Delta[H] \cdot \Delta[G/H]$ and $\delta[G] \leq \delta[H] \cdot \delta[G/H]$ for any normal subgroup $H \subset G$;
- (3) $\Delta[G] \leq |H| + |G/H| - 1$ for any subgroup $H \subset G$;
- (4) $\Delta[A \cup B] \leq \Delta[A] + \Delta[B] - 1$ for any non-empty sets $A, B \subset G$.

Proof. Since (1)–(3) are proved in [3, 1.1], we shall explain (4). Given non-empty sets $A, B \subset G$, find difference bases D_A and D_B for the sets A, B of cardinality $|D_A| = \Delta[A]$ and $|D_B| = \Delta[B]$. Taking any point $d \in D_A$ and replacing D_A by its shift $D_A d^{-1}$, we can assume that the unit 1_G of the group G belongs to D_A . By the same reason, we can assume that $1_G \in D_B$. The union $D = D_A \cup D_B$ is a difference basis for $A \cup B$, witnessing that

$$\Delta[A \cup B] \leq |D| \leq |D_A| + |D_B| - 1 = \Delta[A] + \Delta[B] - 1. \quad \square$$

Finite groups G with $\Delta[G] = \lceil \frac{|G|+1}{2} \rceil$ were characterized in [3] as follows.

Theorem 2.3 (Banakh, Gavrylkiv, Nykyforchyn). *For a finite group G*

- (i) $\Delta[G] = \lceil \frac{|G|+1}{2} \rceil > \frac{|G|}{2}$ if and only if G is isomorphic to one of the groups: $C_1, C_2, C_3, C_4, C_2 \times C_2, C_5, D_6, (C_2)^3$;
- (ii) $\Delta[G] = \frac{|G|}{2}$ if and only if G is isomorphic to one of the groups: $C_6, C_8, C_2 \times C_4, D_8, Q_8$.

In this theorem by D_{2n} we denote the dihedral group of cardinality $2n$ and by Q_8 the 8-element group of quaternion units. In [3] the difference sizes $\Delta[G]$ was calculated for all groups G of cardinality $|G| \leq 13$.

TABLE 1. Difference sizes of groups of order ≤ 13

G	C_2	C_3	C_5	C_4	$C_2 \times C_2$	C_6	D_6	C_8	$C_2 \times C_4$	D_8	Q_8	$(C_2)^3$
$\Delta[G]$	2	2	3	3	3	3	4	4	4	4	4	5
G	C_7	C_{11}	C_{13}	C_9	$C_3 \times C_3$	C_{10}	D_{10}	C_{12}	$C_2 \times C_6$	D_{12}	A_4	$C_3 \times C_4$
$\Delta[G]$	3	4	4	4	4	4	4	4	5	5	5	5

Observing that for each cyclic group C_n of cardinality $n \leq 13$ the difference size $\Delta[C_n]$ coincides with the lower bound $\lceil \frac{1+\sqrt{4n-3}}{2} \rceil$ given in Proposition 2.2(1), the authors of [3] posed the following problem.

Problem 2.4 (Banakh, Gavrylkiv, Nykyforchyn). *Is $\Delta[C_n] = \lceil \frac{1+\sqrt{4n-3}}{2} \rceil$ for any finite cyclic groups C_n ?*

Using the results of computer calculations we shall give a negative answer to Problem 2.4. On the other hand, we shall observe that the classical difference sets of Singer [19] witness that $\Delta[C_n] = \frac{1+\sqrt{4n-3}}{2}$ for any number $n = 1 + q + q^2$ where q is a prime power.

3. DIFFERENCE SIZES OF SOME SPECIAL CYCLIC GROUPS

In this section we collect some upper bounds on the difference size $\Delta[C_n]$ of a cyclic group whose order n has some special arithmetic properties, for example, is equal to $q^2 + q + 1$ or $q^2 - 1$ for a prime power q or to $p^2 - p$ for a prime number p . To derive such upper bounds, we shall use known information on (relative) difference sets.

A subset D of a group G is called a *difference set* if each non-idempotent element $g \in G$ can be uniquely written as $g = xy^{-1}$ for some elements $x, y \in D$. This definition implies $|D|^2 - |D| = |G| - 1$ and hence $|D| = \frac{1+\sqrt{4|G|-3}}{2}$. The following fundamental result was proved by Singer [19] in 1938.

Theorem 3.1 (Singer). *For any prime power q the cyclic group C_{q^2+q+1} contains a difference set of cardinality $q + 1$ and hence has difference size $\Delta[C_{q^2+q+1}] = q + 1$.*

Singer's Theorem implies that for any prime power q the cyclic group C_n of cardinality $n = q^2 + q + 1$ has difference size

$$\Delta[C_n] = q + 1 = \frac{1 + \sqrt{4n - 3}}{2}.$$

So, for infinitely many numbers n the lower bound $\frac{1 + \sqrt{4n - 3}}{2}$ for $\Delta[C_n]$ (given in Proposition 2.2(1)) is attained.

The converse result to Singer's Theorem is known in Algebraic Combinatorics as PPC (abbreviated from the Prime Power Conjecture).

Conjecture 3.2 (PPC(n)). *If for a natural number $q < n$ some Abelian group G of order $|G| = q^2 + q + 1$ contains a difference set $D \subset G$, then q is a prime power.*

In [13] PPC(n) is confirmed for all numbers $n < 2\,000\,000$. The Prime Power Conjecture implies the following converse to Theorem 3.1.

Proposition 3.3. *Let G be an Abelian group of difference size $\Delta[G] = \frac{1 + \sqrt{4|G| - 3}}{2}$. If PPC($|G|$) holds, then $q = \Delta[G] - 1$ is a prime power and $|G| = q^2 + q + 1$.*

Proof. Let $q := \Delta[G] - 1$. The equality $1 + q = \frac{1 + \sqrt{4|G| - 3}}{2}$ implies that $|G| = \frac{1}{4}((2q + 1)^2 + 3) = q^2 + q + 1$. Let $D \subset G$ be a difference basis of cardinality $|D| = \Delta[G] = 1 + q$ in G . Consider the surjective map $\xi : D \times D \rightarrow G$, $\xi : (x, y) \mapsto xy^{-1}$. The equality $|D|^2 - |D| = q^2 + q = |G| \setminus \{1_G\}$ implies that for each $g \in G \setminus \{1_G\}$ the preimage $\xi^{-1}(g)$ is a singleton, which means that D is a difference set in G . Now PPC($|G|$) implies that q is a prime power. \square

Singer derived his theorem studying properties of projective planes over finite fields. A corresponding result for affine planes was obtained by Bose [6] and Chowla [7].

Theorem 3.4 (Bose-Chowla). *For any prime power q the set $C_{q^2-1} \setminus C_{q-1}$ has a difference basis of cardinality q in the cyclic group C_{q^2-1} . Consequently, $\Delta[C_{q^2-1} \setminus C_{q-1}] = q$ and*

$$\Delta[C_{q^2-1}] \leq \Delta[C_{q^2-1} \setminus C_{q-1}] + \Delta[C_{q-1}] - 1 = q - 1 + \Delta[C_{q-1}].$$

We shall also need the following theorem essentially proved by Rusza [18].

Theorem 3.5 (Rusza). *For any prime number p the set $C_{p^2-p} \setminus (C_p \cup C_{p-1})$ has a difference basis of cardinality $p - 1$ in the group C_{p^2-p} . Consequently, $\Delta[C_{p^2-p} \setminus (C_p \cup C_{p-1})] = p - 1$ and*

$$\Delta[C_{p^2-p}] \leq p - 3 + \Delta[C_p] + \Delta[C_{p-1}].$$

Proof. Given a prime number p , consider the field $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ of residues modulo p . It is known that the multiplicative group $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$ of this field is isomorphic to the cyclic group C_{p-1} . Since the numbers p and $p - 1$ are relatively prime, the product $\mathbb{Z}_p \times \mathbb{Z}_p^*$ is isomorphic to the cyclic group C_{p^2-p} . So, instead of the group C_{p^2-p} , we can consider the group $\mathbb{Z}_p \times \mathbb{Z}_p^*$ (which is the direct product of the additive and multiplicative groups of the field \mathbb{Z}_p).

We claim that the set $B = \{(x, x) : x \in \mathbb{Z}_p^*\}$ is a difference basis for the set $A = \{(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p^* : x \neq 0 \text{ and } y \neq 1\}$ in the group $\mathbb{Z}_p \times \mathbb{Z}_p^*$. Given any pair $(a, b) \in A$ we need to find two elements $x, y \in \mathbb{Z}_p^*$ such that $(x - y, xy^{-1}) = (a, b)$. Since $b \neq 1$, the element $b - 1$ is invertible in the field \mathbb{Z}_p^* , so we can consider the elements $x = b(b - 1)^{-1}a \neq 0$ and $y = (b - 1)^{-1}a \neq 0$. The pairs (x, x) and (y, y) belong to the set B and their difference $(x - y, xy^{-1})$ in the group $\mathbb{Z}_p \times \mathbb{Z}_p^*$ is equal to (a, b) , witnessing that B is a difference basis for the set A .

Observe that the complement of the set A in the group $\mathbb{Z}_p \times \mathbb{Z}_p^*$ is equal to the union of two subgroups $\{0\} \times \mathbb{Z}_p^*$ and $\mathbb{Z}_p \times \{1\}$. Therefore,

$$\Delta[C_{p^2-p} \setminus (C_p \cup C_{p-1})] = \Delta[A] \leq |B| = p - 1.$$

The upper bound $\Delta[A] \leq p - 1$ combined with the equality

$$\begin{aligned} |A| &= |\mathbb{Z}_p \times \mathbb{Z}_p^*| - |\{0\} \times \mathbb{Z}_p^*| - |\mathbb{Z}_p \times \{1\}| + |\{(0, 1)\}| = \\ &= (p^2 - p) - (p - 1) - p + 1 = p^2 - 3p + 2 = (p - 1)(p - 2) = |B|(|B| - 1) \end{aligned}$$

implies that $\Delta[C_{p^2-p} \setminus (C_p \cup C_{p-1})] = \Delta[A] = p - 1$.

By Proposition 2.2(4),

$$\begin{aligned} \Delta[C_{p^2-p}] &= \Delta[\mathbb{Z}_p \times \mathbb{Z}_p^*] = \Delta[A \cup (\mathbb{Z}_p \times \{1\}) \cup (\{0\} \times \mathbb{Z}_p^*)] \leq \Delta[A] + \Delta[\mathbb{Z}_p \times \{1\}] + \Delta[\{0\} \times \mathbb{Z}_p^*] - 2 \leq \\ &\leq |B| + \Delta[\mathbb{Z}_p] + \Delta[\mathbb{Z}_p^*] - 2 = p - 3 + \Delta[C_p] + \Delta[C_{p-1}]. \end{aligned}$$

\square

4. DIFFERENCE SIZES OF NUMBER INTERVALS

In this section we apply known information on difference sizes of number intervals to evaluating the difference sizes of finite cyclic groups. For integer numbers $a < b$ by $[a, b]$ we shall denote the order-interval $\{x \in \mathbb{Z} : a \leq x \leq b\}$ in the group \mathbb{Z} of integer numbers.

For a natural number $n \in \mathbb{N}$ by $\Delta[n]$ we shall denote the difference size of the interval $[0, n]$. It is equal to the difference size of the intervals $[1, n]$ and $[-n, n]$. Also we put $\bar{\delta}[n] = \frac{\Delta[n]}{\sqrt{n}}$.

For example, the interval $[0, 6]$ has difference size $\Delta[6] = 4$ as witnessed by difference basis $\{0, 1, 4, 6\}$. It is clear that $\Delta[n]$ is a non-decreasing function of the integer parameter n and $\Delta[n](\Delta[n] - 1) \geq n$, which implies that $\Delta[n] > \sqrt{n}$ for all $n \in \mathbb{N}$.

Difference bases for the order intervals $[0, n]$ were studied by Rédei and Rényi [17] who proved that the limit $\lim_{n \rightarrow \infty} \bar{\delta}[n]$ exists and is equal to $\inf_{n \in \mathbb{N}} \bar{\delta}[n]$. Moreover,

$$2.424\dots = 2 + \frac{4}{3\pi} \leq \lim_{n \rightarrow \infty} \bar{\delta}[n]^2 = \inf_{n \in \mathbb{N}} \bar{\delta}[n]^2 \leq \bar{\delta}[6]^2 = \frac{8}{3} = 2.666\dots$$

These lower and upper bounds were improved by Leech [14] and Golay [12] who proved the following theorem.

Theorem 4.1 (Leech-Golay). *For any natural number n we get the lower and upper bounds:*

$$2.434\dots = 2 + \max_{0 < \varphi < \pi} \frac{2 \sin(\varphi)}{\varphi + \pi} \leq \lim_{n \rightarrow \infty} \bar{\delta}[n]^2 = \inf_{n \in \mathbb{N}} \bar{\delta}[n]^2 \leq \bar{\delta}[6166]^2 \leq \frac{128^2}{6166} = 2.6571\dots$$

For small numbers n the difference sizes $\Delta[n]$ of the intervals $[0, n]$ have been calculated by computer. The following table (taken from [14] and [20]) gives the values of $\Delta[n]$ for numbers $n \leq 61$ such that $\Delta[n+1] > \Delta[n]$.

TABLE 2. The values of $\Delta[n]$ for small n .

$n =$	1	3	6	9	13	18	24	29	37	45	51	61
$\Delta[n] =$	2	3	4	5	6	7	8	9	10	11	12	13
$\bar{\delta}[n]^2 \approx$	4	3	2.666	2.777	2.769	2.722	2.666	2.793	2.703	2.688	2.824	2.770

This table shows that for $n \leq 61$ the smallest value $\frac{8}{3}$ of the difference characteristic $\bar{\delta}[n]^2$ is attained for $n \in \{6, 24\}$. Combining the difference basis $\{0, 1, 4, 6\}$ for $[0, 6]$ with the Singer difference sets, Leech [14] and Golay [12] have found four larger numbers n with $\bar{\delta}[n]^2 < \frac{8}{3}$. These (four) numbers are presented in Table 3.

TABLE 3. Some numbers n with $\bar{\delta}[n]^2 < \frac{8}{3}$.

$n =$	4064	4713	5416	6166
$\Delta[n] \leq$	104	112	120	128
$\bar{\delta}^2[n] \leq$	2.6615	2.6616	2.6588	2.6572

Difference sizes of the intervals yield upper bounds on the difference sizes of cyclic groups.

Proposition 4.2. *For any natural number n we get*

$$\Delta[C_n] \leq \Delta[k] \quad \text{and} \quad \bar{\delta}[C_n] \leq \frac{\bar{\delta}[k]}{\sqrt{2}}$$

where $k = \lceil \frac{n-1}{2} \rceil$.

Proof. Given a natural number n , consider the homomorphism $\gamma : \mathbb{Z} \rightarrow C_n$, $\gamma : t \mapsto e^{\frac{2\pi i}{n}t}$. For the number $k = \lceil \frac{n-1}{2} \rceil$, choose a subset $D \subset \mathbb{Z}$ of cardinality $|D| = \Delta[k]$ such that $D - D$ contains the interval $[-k, k]$. Taking into account that $n \leq 2k + 1 = |[-k, k]|$, we conclude that $\gamma(D)\gamma(D)^{-1} = \gamma(D - D) \supset \gamma([-k, k]) = C_n$, which means that the set $B = \gamma(D)$ is a difference basis for the group C_n and hence $\Delta[C_n] \leq |B| \leq |D| = \Delta[k]$.

Observe that $k = \lceil \frac{n-1}{2} \rceil \leq \frac{n}{2}$ and hence

$$\bar{\delta}[C_n] = \frac{\Delta[C_n]}{\sqrt{n}} \leq \frac{\Delta[k]}{\sqrt{k}} \frac{\sqrt{k}}{\sqrt{n}} = \bar{\delta}[k] \cdot \sqrt{\frac{k}{n}} \leq \bar{\delta}[k] \sqrt{\frac{n/2}{n}} = \frac{\bar{\delta}[k]}{\sqrt{2}}.$$

□

Proposition 4.2 and Theorem 4.1 allow us to evaluate lower and upper limits of the difference characteristics of cyclic groups.

Corollary 4.3. *For every natural number n we have the lower and upper bounds:*

$$1 = \inf_{n \in \mathbb{N}} \delta[C_n] = \liminf_{n \rightarrow \infty} \delta[C_n] \leq \limsup_{n \rightarrow \infty} \delta[C_n] \leq \frac{1}{\sqrt{2}} \cdot \inf_{n \in \mathbb{N}} \delta[n] \leq \frac{\delta[6166]}{\sqrt{2}} \leq \frac{64}{\sqrt{3083}} = 1.1526... < \frac{2}{\sqrt{3}} = 1.1547...$$

Corollary 4.3 implies that $\delta[C_n] < \frac{2}{\sqrt{3}}$ for all sufficiently large n . In Theorem 4.12 we shall show that this upper bound holds for all $n \geq 2 \cdot 10^{15}$.

At first we find some upper bounds of the difference sizes of the intervals $[-n, n]$, using the approach first exploited by Rédei and Rényi [17], and then developed by Leech [14] and Golay [12].

To write down these upper bounds, we shall need some information on the numbers $\delta_k[C_m]$, which are defined as follows. For a natural number m and a non-negative number $k < \Delta[C_m]$ let $\delta_k[C_m]$ be the largest integer number $d < m$ for which there exists a set $B \subset [0, m]$ of cardinality $|B| = \Delta[C_m]$ such that $B - B + m\mathbb{Z} = \mathbb{Z}$ and $|B \cap [0, d]| \leq k$. In [12] the numbers $\delta_0[C_m]$ were denoted by c_{\max} and were calculated for all cyclic groups C_m of order $m = 1 + q + q^2$ where $q \leq 32$ is a prime power:

TABLE 4. The numbers $\delta_0[C_m]$ for cyclic groups of order $m = 1 + q + q^2$ for a prime power q .

q	2	3	4	5	7	8	9	11	13	16	17	19	23	25	27	29	31	32
m	7	13	21	31	57	73	91	133	183	273	307	381	553	651	757	871	993	1057
$\delta_0[C_m]$	4	7	10	14	22	28	36	48	56	72	91	98	128	159	172	191	209	198

This table is completed by Table 5 giving the values of $\delta_k[C_m]$ for positive $k < m \leq 307$. These values are found by computer.

TABLE 5. The values of the numbers $\delta_k[C_m]$ for some k and m .

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
m																	
7	4	6	7	-	-	-	-	-	-	-	-	-	-	-	-	-	-
13	7	10	12	13	-	-	-	-	-	-	-	-	-	-	-	-	-
21	10	13	17	20	21	-	-	-	-	-	-	-	-	-	-	-	-
31	14	19	25	28	30	31	-	-	-	-	-	-	-	-	-	-	-
39	18	23	29	35	37	38	39	-	-	-	-	-	-	-	-	-	-
57	22	29	38	43	50	54	56	57	-	-	-	-	-	-	-	-	-
73	28	38	41	52	58	66	70	72	73	-	-	-	-	-	-	-	-
91	36	46	51	62	71	74	84	88	90	91	-	-	-	-	-	-	-
133	48	61	68	79	94	102	114	120	125	130	132	133	-	-	-	-	-
183	56	69	92	98	109	128	139	153	158	168	175	180	182	183	-	-	-
273	72	92	110	130	145	152	175	184	202	218	226	242	258	266	270	272	273
307	91	106	128	152	160	171	199	214	232	245	261	276	284	292	300	304	306

Theorem 4.4. *For any natural number $m \geq 3$ we get the lower bound*

$$\delta_0[C_m] \geq \left\lfloor \frac{m}{\Delta[C_m]} \right\rfloor.$$

If $m = 1 + q + q^2$ for some prime power q , then

$$\delta_0[C_m] \geq \max \left\{ q + 2, \left\lfloor \frac{q^2 + q + 1}{q - \sqrt{q} + 3} \right\rfloor \right\} \geq \max \{ q + 2, q + \sqrt{q} - 3 \}.$$

Proof. Given a natural number $m \geq 3$, fix a difference basis $D \subset C_m$ of cardinality $|D| = \Delta[C_m]$. We lose no generality assuming that D contains the unit 1 of the group C_m . It follows that $|D| < m$. Let $g = e^{\frac{2\pi}{m}i}$ be a generator of the cyclic group C_m . Let $d = \left\lfloor \frac{m}{\Delta[C_m]} \right\rfloor$ and consider the interval $I = \{g^j : 0 \leq j < d\} \subset C_m$, containing d elements of C_m .

Claim 4.5. *There exists $c \in C_m$ such that $cI \cap D = \emptyset$.*

Proof. For a subset $A \subset C_m$ denote by $\chi_A : C_m \rightarrow \{0, 1\}$ its characteristic function (which means that $\chi_A^{-1}(1) = A$). Assume that $cI \cap D \neq \emptyset$ for all $c \in C_m$ and observe that

$$\begin{aligned} m &\leq \sum_{c \in C_m} |cI \cap D| = \sum_{c \in C_m} \sum_{x \in C_m} \chi_{cI}(x) \chi_D(x) = \sum_{x \in C_m} \chi_D(x) \sum_{c \in C_m} \chi_I(c^{-1}x) = \\ &= \sum_{x \in C_m} \chi_D(x) |I| = |I| \cdot |D| = d \cdot \Delta[C_m] \leq m, \end{aligned}$$

which implies that $d \cdot \Delta[C_m] = m$ and $|cI \cap D| = 1$ for all $c \in C_m$. The latter property of D can be used to show that D coincides with the subgroup H generated by g^d , which is not possible as $DD^{-1} = G \neq H$. \square

By Claim 4.5, there exists $c \in C_m$ such that $cI \cap D$ is empty. Then $B = c^{-1}D$ is a difference basis for C_m with $I \cap B = \emptyset$. It follows that the set $A = \{a \in \mathbb{Z} : 0 < a \leq m, g^a \in B\}$ has $A \cap [0, d) = \emptyset$ and witnesses that $\delta_0[C_m] \geq d = \lfloor \frac{m}{\Delta[C_m]} \rfloor$.

Now assume that $m = 1 + q + q^2$ for some prime power q . In this case D is a difference set of cardinality $|D| = 1 + q$. We recall that C_m is the multiplicative subgroup $\{z \in \mathbb{C} : z^m = 1\}$ of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ on the complex plane. Let $I := \{e^{i\phi} : 0 \leq \phi < \pi\} \subset \mathbb{T}$ be the upper half-circle and observe that $\mathbb{T} = I \cup (-I)$.

Claim 4.6. *For some $z \in \mathbb{T}$ the set $I \cap zD$ has cardinality $|I \cap zD| \leq \frac{1+q-\sqrt{q}}{2}$.*

Proof. For a subset $A \subset \mathbb{T}$ by $\chi_A : \mathbb{T} \rightarrow \{0, 1\}$ we denote the characteristic function of the set A in \mathbb{T} , which means that $\chi_A^{-1}(1) = A$. Observe that each element $z \in C_m \cap (iI^{-1})$ has positive real part and the sum $r := \sum_{z \in C_m \cap iI^{-1}} z$ is a positive real number.

Now consider the complex number $\delta := \sum_{z \in C_m} \chi_D(z)z$ and observe that

$$|\delta|^2 = \delta \bar{\delta} = \sum_{x, y \in C_m} \chi_D(x) \chi_D(y) x \bar{y} = \sum_{x \in C_m} \chi_D(x)^2 x x^{-1} + \sum_{x \neq y} \chi_D(x) \chi_D(y) x y^{-1} = |D| + \sum_{z \in C_m \setminus \{1\}} z = |D| - 1 = q.$$

Then $\sum_{z \in D} z = \delta = \sqrt{q} e^{-i\psi}$ for some $\psi \in \mathbb{R}$ and hence the set $D_\psi := D e^{i\psi}$ has $\sum_{z \in D_\psi} z = \sqrt{q}$.

Let $C_\psi = C_m e^{i\psi} \subset \mathbb{T}$. Observe that

$$\begin{aligned} r \cdot \sqrt{q} &= \sum_{x \in C_m} \chi_{iI^{-1}}(x) \cdot \sum_{y \in C_\psi} \chi_{D_\psi}(y) y = \sum_{z \in C_\psi} \sum_{\substack{(x, y) \in C_m \times C_\psi \\ xy = z}} \chi_{iI^{-1}}(x) \cdot \chi_{D_\psi}(y) \cdot z = \\ &= \sum_{z \in C_\psi} \sum_{y \in C_\psi} \chi_{iI^{-1}}(zy^{-1}) \cdot \chi_{D_\psi}(y) \cdot z = \sum_{z \in C_\psi} \sum_{y \in C_\psi} \chi_{zi^{-1}I}(y) \cdot \chi_{D_\psi}(y) \cdot z = \sum_{z \in C_\psi} |i^{-1}zI \cap D e^{i\psi}| \cdot z. \end{aligned}$$

Assuming that $|I \cap zD| > c := \frac{1+q-\sqrt{q}}{2}$ for all $z \in \mathbb{T}$, we conclude that

$$|i^{-1}zI \cap D e^{i\psi}| = |D| - |izI \cap D e^{i\psi}| < (1+q) - c$$

for all $z \in \mathbb{T}$. Taking into account that each complex number $z \in iI$ has negative real part $\Re(z)$,

$$0 < \sum_{z \in C_\psi \cap (-iI)} \Re(z) \leq \sum_{z \in C_m \cap (-iI)} \Re(z) = r$$

and

$$0 = \sum_{z \in C_\psi} \Re(z) = \sum_{z \in C_\psi \setminus iI} \Re(z) + \sum_{z \in C_\psi \cap iI} \Re(z),$$

we conclude that

$$\begin{aligned} r \sqrt{q} &= \sum_{z \in C_\psi} |i^{-1}zI \cap D e^{i\psi}| \cdot z = \sum_{z \in C_\psi \cap (-iI)} |i^{-1}zI \cap D e^{i\psi}| \cdot \Re(z) + \sum_{z \in C_\psi \cap iI} |i^{-1}zI \cap D e^{i\psi}| \cdot \Re(z) < \\ &< (1+q-c) \sum_{z \in C_\psi \cap (-iI)} \Re(z) + c \sum_{z \in C_\psi \cap iI} \Re(z) = (1+q-2c) \sum_{z \in C_\psi \setminus iI} \Re(z) \leq (1+q-2c)r, \end{aligned}$$

which implies that $\sqrt{q} < (1+q-2c)$ and hence $c < \frac{1+q-\sqrt{q}}{2}$. But this contradicts the definition of the constant c . \square

By Claim 4.6, there exists a complex number $c \in \mathbb{T}$ such that $|I \cap cD| \leq \frac{1+q-\sqrt{q}}{2}$. Let $l = |I \cap cD| + 1$ and $J = \{e^{it} : 0 \leq t < \pi/l\} \subset \mathbb{T}$. Observe that the arc I can be covered by $l > |I \cap cD|$ disjoint copies of the arc J . By the Pigeonhole Principle, for some $z \in \mathbb{T}$ the arc zJ is disjoint with the set cD . Then the arc $c^{-1}zJ$ is disjoint with the set D . It is easy to see that the arc $c^{-1}zJ$ contains at least $k := \lfloor \frac{m}{2l} \rfloor$ consecutive points of the group C_m . Therefore, the set $C_m \setminus D$ contains k consecutive points. Replacing D by a suitable shift, we

can assume that those k consecutive points form the set $\{g^j : 0 \leq j < k\}$ where $g = e^{\frac{2\pi}{m}i}$ is the generator of the cyclic group C_m . Then the set $A = \{j \in [0, m] : g^j \in D\}$ is disjoint with the set $[0, k)$ and witnesses that

$$\begin{aligned} \delta_0[C_m] &\geq k = \left\lfloor \frac{m}{2l} \right\rfloor \geq \frac{m}{2l} = \left\lfloor \frac{m}{2(|I \cap cD| + 1)} \right\rfloor = \left\lfloor \frac{m}{1 + q - \sqrt{q} + 2} \right\rfloor = \left\lfloor \frac{q^2 + q + 1}{q - \sqrt{q} + 3} \right\rfloor > \\ &\frac{q^2 + q + 1}{q - \sqrt{q} + 3} - 1 = q + \sqrt{q} - 2 - \frac{4(\sqrt{q} - 1)}{q - \sqrt{q} + 3} > q + \sqrt{q} - 3. \end{aligned}$$

If $q \geq 16$, then

$$\delta_0(C_m) \geq \left\lfloor \frac{q^2 + q + 1}{q - \sqrt{q} + 3} \right\rfloor \geq \left\lfloor \frac{q^2 + q + 1}{q - 1} \right\rfloor = q + 2.$$

If $q < 16$, then the equality $\delta_0[C_m] \geq q + 2$ follows from Table 5. \square

The numbers $\delta_k[C_m]$ are used in the following theorem giving an upper bound for the difference sizes of intervals.

Theorem 4.7. *For any non-negative integer numbers n, m, k with $k < m$ we get the upper bound*

$$\Delta[nm + \delta_k[C_m] - 1] \leq \Delta[n] \cdot \Delta[C_m] + k.$$

Proof. Fix a difference basis $D \subset \mathbb{Z}$ for the interval $[-n, n]$ of cardinality $|D| = \Delta[n]$.

By the definition of the number $\delta_k[C_m]$, there exists a set $A \subset [0, m]$ of cardinality $|A| = \Delta[C_m]$ such that $A - A + m\mathbb{Z} = \mathbb{Z}$ and $|A \cap [0, \delta_k[C_m]]| \leq k$. Find two numbers $\lambda, l \in D$ with $\lambda - l = n$. It is clear that the set

$$B := \{a + md : a \in A, d \in D\} \cup \{a + m(\lambda + 1) : a \in A \cap [0, \delta_k[C_m]]\}$$

has cardinality

$$|B| \leq |D| \cdot |A| + |A \cap [0, \delta_k[C_m]]| \leq \Delta[n] \cdot \Delta[C_m] + k.$$

We claim that the interval $J = \{x \in \mathbb{Z} : |x| < mn + \delta_k[C_m]\}$ is contained in $B - B$. Since the set $B - B$ is symmetric, it suffices to show that each positive number $x \in J$ is contained in $B - B$. Write x as $x = my + z$ for some integer numbers y, z such that $0 \leq y \leq n$ and $0 \leq z < m$. By the choice of A , there are numbers $a, b \in A$ such that $z = a - b + mj$ for some $j \in \mathbb{Z}$. Taking into account that $|mj| = |a - b - z| \leq |a - b| + |z| < m + m = 2m$, we conclude that $|j| \leq 1$ and hence $|y + j| \leq n + 1$.

It follows that $x = my + z = m(y + j) + a - b$. If $|y + j| \leq n$, then we can choose two numbers $u, v \in D$ such that $y + j = u - v$ and conclude that

$$x = m(y + j) + z = m(u - v) + a - b = (mu + a) - (mv + b) \in B - B.$$

So, we assume that $|y + j| = n + 1$. Then $x = m(y + j) + a - b = m(n + 1) + a - b$. Taking into account that $x < mn + \delta_k[C_m]$, we conclude that $a \leq m + a - b = x - mn < \delta_k[C_m]$, and hence $a + m(\lambda + 1) \in B$. Then $x = m(n + 1) + a - b = m(\lambda - l + 1) + a - b = a + m(\lambda + 1) - (b + ml) \in B - B$. Therefore $J \subset B - B$ and $\Delta[nm + \delta_k[C_m] - 1] \leq |B| \leq \Delta[n] \cdot \Delta[C_m] + k$. \square

Corollary 4.8. *Let n be a natural number, q is a prime power and $k = n(1 + q + q^2) + q + 1$. For any natural number $l \leq 2k + 1$ we get the upper bound*

$$\Delta[C_l] \leq \Delta[k] \leq \Delta[n] \cdot (q + 1).$$

Proof. By Theorem 3.1, the cyclic group C_m of order $m = 1 + q + q^2$ has difference size $\Delta[C_m] = q + 1$. By Theorem 4.4, $\delta_0[C_m] \geq q + 2$. By Theorem 4.7,

$$\Delta[C_l] \leq \Delta[k] \leq \Delta[nm + \delta_0[C_m] - 1] \leq \Delta[n] \cdot \Delta[C_m] = \Delta[n] \cdot (q + 1).$$

\square

Applying Corollary 4.8 with $n = 6$ we derive another corollary.

Corollary 4.9. *For any prime power q and a natural number l with $l \leq 15 + 14q + 12q^2$ we get the upper bound $\Delta[C_l] \leq 4(q + 1)$.*

For a real number x by $q(x)$ we denote the smallest prime power, which is larger or equal than x . It is easy to see that for real numbers $x \geq 12$ and $y \geq 0$ the inequality $x \leq 15 + 14y + 12y^2$ is equivalent to $y \geq \frac{-7 + \sqrt{12x - 131}}{12}$. This observation, combined with Corollary 4.9 yields the following upper bound for $\Delta[C_n]$.

Corollary 4.10. *Each finite cyclic group C_n of order $n \geq 11$ has difference size $\Delta[C_n] \leq 4 + 4q\left(\frac{-7 + \sqrt{12n - 131}}{12}\right)$.*

For a real number x by $p(x)$ we denote the smallest prime number greater or equal to x . It is clear that $x \leq q(x) \leq p(x)$. By [2], $p(x) = x + O(x^{21/40})$. The (still unproven) Andrica's Conjecture [1] says that $p(x) < x + 2\sqrt{x} + 1$ for all $x \geq 1$. This conjecture was confirmed by Imran Ghory [11] for all $x \leq 1.3 \times 10^{16}$.

Corollary 4.11. *Each finite cyclic group C_n of cardinality $11 \leq n \leq 2 \cdot 10^{33}$ has difference size*

$$\Delta[C_n] < \frac{1}{3}\sqrt{12n-131} + \frac{4}{\sqrt{3}}\sqrt[4]{12n-131} + \frac{17}{3}.$$

If the Andrica Conjecture is true, then this upper bound holds for all numbers n .

Proof. Given a number $n \leq 2 \cdot 10^{33}$, consider the real number $x := \frac{-7 + \sqrt{12n-131}}{12}$. The inequality $n \leq 2 \cdot 10^{33}$ implies that $x \leq 1.3 \cdot 10^{16}$, so we can apply the result of Ghory [11], and conclude that $p(x) < x + 2\sqrt{x} + 1$. By Corollary 4.10,

$$\begin{aligned} \Delta[C_n] &\leq 4(p(x) + 1) < 4x + 8\sqrt{x} + 8 = \frac{-7 + \sqrt{12n-131}}{3} + 8\sqrt{\frac{-7 + \sqrt{12n-131}}{12}} + 8 < \\ &< \frac{1}{3}\sqrt{12n-131} + \frac{4}{\sqrt{3}}\sqrt[4]{12n-131} + \frac{17}{3}. \end{aligned}$$

If the Andrica's Conjecture is true, then the same argument works for all n . \square

Applying Corollary 4.10 with $n = 6166$ and the upper bound $q(x) \leq p(x) = x + O(x^{21/40})$ from [2], we get a more refined upper bound for $\Delta[C_m]$.

Theorem 4.12. *For any prime power q and a natural number $n \leq 12335 + 12334q + 12332q^2$ we get the upper bound $\Delta[C_n] \leq 128(q + 1)$. Consequently, for any $n \geq 926$ we get the upper bound*

$$\Delta[C_n] \leq 128 + 128 \cdot q \left(\frac{-6167 + \sqrt{12332n - 114083331}}{12332} \right) = \frac{64}{\sqrt{3084}}\sqrt{n} + O(n^{21/80}).$$

If $n \geq 2 \cdot 10^{15}$, then $\Delta[C_n] < \frac{2}{\sqrt{3}}\sqrt{n}$.

Proof. By Theorem 3.1, the cyclic group of order $m = 1 + q + q^2$ has difference size $\Delta[C_m] = q + 1$. By Theorem 4.4, $\delta_0[C_m] \geq q + 2$. Since $n \leq 12335 + 12334q + 12332q^2 = 2(6166(1 + q + q^2) + q + 1) + 1$, we can apply Theorem 4.7 and obtain the upper bound

$$\Delta[C_n] \leq \Delta[6166m + \delta_0[C_m] + 1] \leq \Delta[6166] \cdot \Delta[C_m] = 128(q + 1).$$

If $n \geq 926$, then the real number $x := \frac{-6167 + \sqrt{12332n - 114083331}}{12332}$ is well-defined and

$$n = 12335 + 12334x + 12332x^2 \leq 12335 + 12334q(x) + 12332q(x)^2.$$

By [2], $q(x) \leq p(x) = x + O(x^{21/40})$. Then

$$\Delta[C_n] \leq 128(q(x) + 1) = 128x + O(x^{21/40}) = \frac{128}{\sqrt{12332}}\sqrt{n} + O(n^{21/80}) = \frac{64}{\sqrt{3083}}\sqrt{n} + O(n^{21/80}).$$

Now assume that $n \geq 2 \cdot 10^{15}$. In this case $x = \frac{-6167 + \sqrt{12332n - 114083331}}{12332} > 396738$. By [8, 6.8], $p(x) \leq x(1 + \frac{1}{25 \ln^2 x})$. Then $\Delta[C_n] \leq 128(p(x) + 1)$ and we see that the inequality $\Delta[C_n] < \frac{2}{\sqrt{3}}\sqrt{n}$, follows from the inequality

$$128 + 32 \cdot \frac{-6167 + \sqrt{12332n - 114083331}}{3083} \left(1 + \frac{1}{25 \ln^2 \left(\frac{-6167 + \sqrt{12332n - 114083331}}{12332} \right)} \right) < \frac{2}{\sqrt{3}}\sqrt{n},$$

holding for all $n \geq 2 \cdot 10^{15}$. \square

Now we evaluate the difference sizes of intervals $[0, n]$ and cyclic groups C_n for relatively small n .

Applying Theorem 4.7 to the known values of the difference sizes $\Delta[3] = 3$, $\Delta[4] = 6$, $\Delta[8] = 24$, known values of difference sizes of cyclic groups given in Table 7, and known values of the numbers $\delta_k[C_m]$ given in Tables 4 and 5, we obtain the upper bounds for the difference sizes $\Delta[n]$ of intervals of length $n \leq 6166$, given in Table 6.

TABLE 6. The values of $\Delta[n]$ for relatively small n .

$n =$	72	84	87	105	109	135	138	142	145	200	204	208	211	251	256	262	268
$\Delta[n] \leq$	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$n =$	363	465	581	684	845	1153	1389	1709	1932	2383	3445	4064	4713	5416	6166		
$\Delta[n] \leq$	32	36	40	45	48	56	64	68	72	80	96	104	112	120	128		

Now we can prove an upper bound for $\bar{\delta}[C_n]$, holding for small n .

Theorem 4.13. *The upper bound $\bar{\delta}[C_n] \leq \frac{12}{\sqrt{73}} = 1.40449\dots$ holds for all $n \geq 9$. Moreover, if $n \neq 292$, then $\bar{\delta}[C_n] \leq \frac{24}{\sqrt{293}} = 1.40209\dots$*

Proof. For $n \in [9, 100]$ the inequality $\bar{\delta}[C_n] \leq \frac{24}{\sqrt{293}}$ can be verified using known values of $\Delta[C_n]$, see Table 7.

For $n \in [101, 11981]$ the upper bound $\bar{\delta}[C_n] \leq \frac{12}{\sqrt{73}}$ follows from the upper bound $\Delta[C_n] \leq \Delta[\lceil \frac{n-1}{2} \rceil]$ given in Proposition 4.2 and the upper bounds for the difference sizes $\Delta[n]$ given in Table 6. For example, let us prove the upper bound $\bar{\delta}[C_n] \leq \frac{12}{\sqrt{73}}$ for the number $n = 292$.

By Proposition 4.2, $\Delta[C_{292}] \leq \Delta[146]$. Looking at Table 2, we see that $\Delta[146] \leq \Delta[200] \leq 24$. Consequently, $\bar{\delta}[C_{292}] = \frac{\Delta[C_{292}]}{\sqrt{292}} \leq \frac{24}{\sqrt{4 \cdot 73}} = \frac{12}{\sqrt{73}}$. For all other numbers $n \in [101, 11981]$ by the same method we get the inequality $\bar{\delta}[C_n] \leq \frac{24}{\sqrt{293}}$.

For $n \geq 11982$ we shall apply the known fact (see the sequence <https://oeis.org/A166968> on the On-line Encyclopedia of Integer Sequences) saying that for every $k \geq 32$ the interval $[k, \frac{7}{6}k]$ contains a prime number. Consider the real number $x = \frac{-7 + \sqrt{12n-131}}{12} > 31$ and let q be the smallest prime power which is greater or equal to x . It follows that

$$q \leq \frac{7}{6} [x] < \frac{7}{6} (x + 1) = \frac{7}{6} \cdot \frac{5 + \sqrt{12n - 131}}{12}.$$

The inequality $q \geq x$ implies $n \leq 15 + 14q + 12q^2$. By Corollary 4.10, $\Delta[C_n] \leq 4(q+1) \leq \frac{7}{6} \cdot \frac{5 + \sqrt{12n-131}}{3} + 4$. To prove that $\bar{\delta}[C_n] \leq \frac{24}{\sqrt{293}}$, it remains to check that $\frac{7}{6} \cdot \frac{5 + \sqrt{12n-131}}{3} + 4 \leq \frac{24}{\sqrt{293}} \sqrt{n}$. The elementary calculations show that this inequality holds for all $n \geq 11436$. \square

Theorem 4.13 and known values of $\bar{\delta}[C_n]$ for $n < 9$ allow us to find the largest value of the difference characteristics $\bar{\delta}[C_n]$.

Corollary 4.14. $\max_{n \in \mathbb{N}} \bar{\delta}[C_n] = \bar{\delta}[C_4] = \frac{3}{2}$.

Also we can establish some upper bounds for the difference sizes of cyclic groups whose cardinality has some special arithmetic properties.

Corollary 4.15. *For any prime power $q \neq 3$ the cyclic group C_{q^2-1} has difference size*

$$\Delta[C_{q^2-1}] \leq q - 1 + \frac{12}{\sqrt{73}} \sqrt{q-1} < q - 1 + \sqrt{2q-2}.$$

Proof. For $q \in \{2, 4, 5, 7, 8, 9\}$ the upper bound

$$\Delta[C_{q^2-1}] \leq q - 1 + \frac{12}{\sqrt{73}} \sqrt{q-1}$$

can be verified using known values of the difference sizes $\Delta[C_{q^2-1}]$ given in Table 7.

So, assume that $q > 9$ and hence $q \geq 11$. In this case Theorem 4.13 guarantees that $\Delta[C_{q-1}] \leq \frac{12}{\sqrt{73}} \sqrt{q-1}$. Applying Theorem 3.4, we conclude that the group C_{q^2-1} has difference size

$$\Delta[C_{q^2-1}] \leq q - 1 + \Delta[C_{q-1}] \leq q - 1 + \frac{12}{\sqrt{73}} \sqrt{q-1}.$$

\square

By analogy we can derive the following upper bound for $\Delta[C_{p^2-p}]$ from Theorem 3.5.

Corollary 4.16. *For any prime number p the cyclic group C_{p^2-p} has difference size*

$$\Delta[C_{p^2-p}] \leq p - 3 + \frac{12}{\sqrt{73}} (\sqrt{p} + \sqrt{p-1}).$$

In Table 7 we present the results of computer calculation of the difference sizes of cyclic groups of order ≤ 100 . In this table

$$lb[C_n] := \left\lceil \frac{1 + \sqrt{4n-3}}{2} \right\rceil$$

is the lower bound given in Proposition 2.2(1) and

$$ub[C_n] := \begin{cases} q + 1 & \text{if } n = q^2 + q + 1 \text{ for some prime power } q, \\ q - 1 + \Delta[C_{q-1}] & \text{if } n = q^2 - 1 \text{ for some prime power } q, \\ p - 3 + \Delta[C_p] + \Delta[C_{p-1}] & \text{if } n = p^2 - p \text{ for some prime number } p, \end{cases}$$

is the upper bound for $\Delta[C_n]$ given in Theorems 3.1, 3.4 and 3.5. With the boldface font we denote the numbers $n \in \{7, 13, 21, 31, 57, 73, 91\}$, equal to $1 + q + q^2$ for a prime power q . For such numbers we know the exact value $\Delta[C_n] = lb[C_n] = 1 + q$.

TABLE 7. Difference sizes of cyclic groups C_n for $n \leq 100$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$lb[C_n]$	1	2	2	3	3	3	3	4	4	4	4	4	4	5	5	5	5	5	5	5
$\Delta[C_n]$	1	2	2	3	3	3	3	4	4	4	4	4	4	5	5	5	5	5	5	6
$ub[C_n]$		2	2			4	3	4					4		5					8
$\Delta[\lceil \frac{n-1}{2} \rceil]$	1	2	2	3	3	3	3	4	4	4	4	4	4	5	5	5	5	5	5	6
$ C_n $	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$lb[C_n]$	5	6	6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7
$\Delta[C_n]$	5	6	6	6	6	6	6	6	7	7	6	7	7	7	7	7	7	8	7	8
$ub[C_n]$	5			7							6									
$\Delta[\lceil \frac{n-1}{2} \rceil]$	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7	8	8	8
$ C_n $	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$lb[C_n]$	7	7	7	8	8	8	8	8	8	8	8	8	8	8	8	8	8	9	9	9
$\Delta[C_n]$	8	8	8	8	8	8	8	8	8	8	8	9	9	9	9	9	8	9	9	9
$ub[C_n]$		10						9									8			
$\Delta[\lceil \frac{n-1}{2} \rceil]$	8	8	8	8	8	8	8	8	8	9	9	9	9	9	9	9	9	9	9	10
$ C_n $	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
$lb[C_n]$	9	9	9	9	9	9	9	9	9	9	9	9	9	10	10	10	10	10	10	10
$\Delta[C_n]$	9	9	9	9	9	10	10	10	10	10	10	10	9	10	10	10	10	10	10	11
$ub[C_n]$			10										9							12
$\Delta[\lceil \frac{n-1}{2} \rceil]$	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	11	11	11	11	11
$ C_n $	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
$lb[C_n]$	10	10	10	10	10	10	10	10	10	10	10	11	11	11	11	11	11	11	11	11
$\Delta[C_n]$	11	11	11	11	11	11	11	11	11	11	10	11	12	12	12	12	12	12	12	12
$ub[C_n]$											10									
$\Delta[\lceil \frac{n-1}{2} \rceil]$	11	11	11	11	11	11	11	11	11	11	11	11	12	12	13	13	13	13	13	13

Remark 4.17. For $n = 20$ we get the strict inequality $\Delta[C_n] > lb[C_n]$, which answers Problem 2.4 in negative.

The results of computer calculations suggest the following questions.

Question 4.18. Is $|\Delta[C_{n+1}] - \Delta[C_n]| \leq 1$ for every $n \in \mathbb{N}$?

Question 4.19. Is $\sup_{n \in \mathbb{N}} (\Delta[C_n] - lb[C_n]) < \infty$?

The following problem seems to be the most intriguing (see <http://mathoverflow.net/questions/262317>).

Problem 4.20. Is $\Delta[C_n] = (1 + o(1)) \cdot \sqrt{n}$ as $n \rightarrow \infty$? Equivalently, is $\lim_{n \rightarrow \infty} \bar{\delta}[C_n] = 1$?

Theorem 3.1, Corollaries 4.15, 4.16, and Proposition 2.2(2) allows us to produce many subsequences of the sequence $(\bar{\delta}[C_n])_{n=1}^\infty$ tending to the unit. In particular,

$$\lim_{q \rightarrow \infty} \bar{\delta}[C_{q^2+q+1}] = \lim_{q \rightarrow \infty} \bar{\delta}[C_{q^2-1}] = \lim_{p \rightarrow \infty} \bar{\delta}[C_{p^2-p}] = 1,$$

where q runs over prime powers and p runs over prime numbers to infinity. On the other hand, we do not know the answers to the following problems (which are weaker versions of Problem 4.20).

Problem 4.21. Is $\lim_{p \rightarrow \infty} \bar{\delta}[C_p] = 1$?

Problem 4.22. Let p be a prime number. Is $\lim_{k \rightarrow \infty} \bar{\delta}[C_{p^k}] = 1$?

Problem 4.23. Is $\limsup_{n \rightarrow \infty} \bar{\delta}[C_n] = \limsup_{p \rightarrow \infty} \bar{\delta}[C_p]$?

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