Forbidden Substrings In Circular K-Successions

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In this note we define a circular k-succession in a permutation p on $[n]$ as either a pair $p(i)$, $p(i+1)$ if $p(i+1) \equiv p(i)+k \pmod{n}$, or as the pair $p(n)$, $p(1)$ if $p(1) \equiv p(n)+k \pmod{n}$. We count the number of permutations that for fixed k, $k<n$, avoid substrings $j(j+k)$, $1 \leq j \leq n-k$, as well as permutations that avoid substrings $j(j+k) \pmod{n}$ for all j, $1 \leq j \leq n$. We also count circular permutations that avoid such substrings, and show that for substrings $j(j+k) \pmod{n}$, the number of permutations depends on whether n is prime, and more generally, on whether n and k are relatively prime.

Keywords: Circular permutations, circular successions, circular k-successions, k-shifts, k-successions, derangements, forbidden substrings, bijections.

1. Introduction and Previous Results

In [3] we counted the number of permutations according to k-shifts, where we defined $\{d_n^k\}$ as the set of permutations on $[n]$ that for fixed k, $k<n$, avoid substrings $j(j+k)$, $1 \leq j \leq n-k$. The number of such permutations, d_n^k, turned out to be

$$d_n^k = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j!). \tag{1.1}$$

The forbidden substrings in these permutations can be pictured as a diagonal running k places to the right of the main diagonal of an $n \times n$ chessboard (hence the term "k-shifts"). Note that we will also refer to these forbidden substrings that are k spacings apart as "k-successions"; therefore in [3] we counted permutations with no k-successions for $k>1$ (the case $k=1$ was discussed in [2]). Note that there are some references that not only count permutations with no successions but also count permutations with $i=1,2,\ldots,n-1$ successions (the so-called "succession numbers") but they do it only for the case $k=1$, *ie.* for forbidden substrings that are only one spacing apart, $j(j+1)$, so some care should be taken (see [1], for example).

Table 1 in the Appendix provides some d_n^k values. For example, for $n=4$, $k=2$, the forbidden substrings that are 2 spacings apart are $\{13, 24\}$, and

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$d_4^2 = 14$ since there are 14 permutations in S_4 that avoid such substrings or 2-successions.

The sequence $\langle d_n^k \rangle$ is available in OEIS. For example, for $k = 4$, the sequence is now A277563 [4], and for $k = 1$ it is A000255 [5]. Note that for $k = 0$, $d_n^k = Der(n)$, the nth derangement number, which is A000166 in OEIS [9].

In [3] we also counted the number of permutations according to k-shifts (mod n), where we defined $\{D_n^k\}$ as the set of permutations on $[n]$ that for fixed k, $k < n$, avoid substrings $j(j+k)$ for $1 \leq j \leq n-k$, and avoid substrings $j(j+k) \pmod{n}$ for $n - k < j \leq n$. Note that we can summarize in the single definition "avoid substrings $j(j + k) \pmod{n}$ for all j, $1 \leq j \leq n$" if we agree to write n instead of 0 when doing addition (mod n).

The forbidden substrings in this case are easily seen along an $n \times n$ chessboard, where for $j > n - k$, the forbidden positions start again from the first column along a diagonal $n - k$ places below the main diagonal, as in Figure 1 below.

	1	2	3	4
1				$\times$
2	$\times$			
3		$\times$		
4			$\times$	

Figure 1: Forbidden positions in $\{D_4^3\}$.

Figure 1 shows forbidden positions on a 4×4 chessboard that correspond to forbidden substrings of permutations in $\{D_4^3\}$. These forbidden substrings are $\{14; 21, 32, 43\}$. The forbidden substrings below the diagonal are separated by a semicolon; these are the forbidden substrings $j(j+k) \pmod{n}$ for $n-k < j \leq n$. Note that while there are only $n - k$ forbidden substrings in $\{d_n^k\}$, there are n forbidden substrings in $\{D_n^k\}$.

The number of permutations in $\{D_n^k\}$ is given by

$$D_n^k = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n - j)!. \tag{1.2}$$

Furthermore, there are the same number of permutations in $\{D_n^k\}$ as in $\{D_n\}$ whenever n and k are relatively prime ($D_n = D_n^1$; see [3]). Table 4 in the Appendix provides a table for some D_n^k values. For example, for $n = 4$, $k = 3$, we have seen that forbidden substrings in $\{D_4^3\}$ are $\{14; 21, 32, 43\}$ and $D_4^3 = 4$ since there are 4 permutations in S_4 that avoid such substrings or 3-successions. Furthermore, since $(4, 3) = 1$, there is the same number of permutations in $\{D_4^3\}$ as in $\{D_4\}$, which are permutations in S_4 that avoid successions $\{12, 23, 34; 41\}$. The sequence $\langle D_n \rangle$ is A000240 in OEIS [6].

In this note we will discuss permutations in $\{d_n^k\}$ and $\{D_n^k\}$ with no *circular* k-successions, as described in the following definition.

Definition 1.1. We define a *circular k-succession* in a permutation p on $[n]$ as either a pair $p(i)$, $p(i+1)$ if $p(i+1) \equiv p(i)+k \pmod n$, or as the pair $p(n)$, $p(1)$ if $p(1) \equiv p(n)+k \pmod n$. We will denote as $\{d{*}_n^k\}$ the set of permutations in $\{d_n^k\}$ that have no circular k-successions, and similarly, as $\{D{*}_n^k\}$ the set of permutations in $\{D_n^k\}$ that have no circular k-successions. These permutations are in one-line notation.

Hence permutations in $\{d{*}_n^k\}$ avoid substrings $j(j+k)$, $1 \leq j \leq n-k$, and permutations in $\{D{*}_n^k\}$ avoid substrings $j(j+k) \pmod n$ for all j, $1 \leq j \leq n$ (as in $\{d_n^k\}$ and $\{D_n^k\}$, respectively), but in $\{d{*}_n^k\}$ and $\{D{*}_n^k\}$ there is the additional restriction of the circular k-successions.

For example, in $\{D_4^3\}$, we saw above that the forbidden substrings are $\{14; 21, 32, 43\}$, hence the permutation 2134 has a circular 3-succession 21 and the permutation 2413 has a circular 3-succession 32. Hence such permutations won't be allowed in $\{D{*}_4^3\}$ (but 2413 is allowed in $\{D_4^3\}$). Note as mentioned above that the forbidden substrings in $\{D{*}_n^k\}$ are the same as those in $\{D_n^k\}$ and the same is true for $\{d{*}_n^k\}$ and $\{d_n^k\}$, with differences in cardinalities being accounted by circular k-successions, as will be shown below.

Note that in Definition 1.1 we don't really need the definition $\pmod n$ in $\{d{*}_n^k\}$ since forbidden positions in those permutations stay above the main diagonal of an $n \times n$ chessboard.

2. Main Lemmas and Propositions

In [2] we divided permutations according to their starting digit and called these groups *classes*. We showed that permutations in $\{D_n\}$ are equidistributed, meaning that all classes are equinumerous. This is also true for deranged permutations (see [2]).

Now we note that in general $\{d{*}_n^k\}$, which forbids permutations with circular k-successions, will have less permutations than $\{d_n^k\}$. However, if we look at individual classes starting with digits $1, 2, \ldots, n$, we see that there will be classes in $\{d_n^k\}$ and $\{d{*}_n^k\}$ that have the same number of elements, hence they are the same.

Indeed, since forbidden substrings in $\{d_n^k\}$ run strictly over the main diagonal, there will be substrings $j(j+k)$ where $(j+k) \neq i$, $i = 1, 2, \ldots, k$, hence $\{d{*}_n^k\}$ will not have circular k-successions starting with $i = 1, 2, \ldots, k$. Hence classes in $\{d_n^k\}$ and $\{d{*}_n^k\}$ starting with these digits will be the same, and classes starting with $i = k+1, \ldots, n$ will be smaller in $\{d^{*k}_n\}$ than in $\{d^k_n\}$. Hence a total of k classes will be the same in $\{d^{*k}_n\}$ and in $\{d^k_n\}$, and $n-k$ classes will be smaller in $\{d^{*k}_n\}$ than in $\{d^k_n\}$, where $n-k$ is the number of forbidden substrings in $\{d^k_n\}$ and $\{d^{*k}_n\}$.

For example, for $n = 4$, $k = 3$, the forbidden substring in $\{d^3_4\}$ is $\{14\}$, so there won't be permutations in $\{d^{*3}_4\}$ ending in 1 and starting with 4, since they would represent circular 3-successions, which are forbidden. Hence the class of permutations starting with 4 is smaller in $\{d^{*3}_4\}$ than in $\{d^3_4\}$ and the other classes starting with $i = 1, 2, 3$ are equal in both cases. In fact, the cardinalities of the four classes in $\{d^{*3}_4\}$ are $\{4, 4, 4, 4\}$, while the cardinalities of the four classes in $\{d^3_4\}$ are $\{4, 4, 4, 6\}$, the difference being accounted by the permutations 4231 and 4321 which are allowed in $\{d^3_4\}$ but not in $\{d^{*3}_4\}$.

Similarly, for $n = 6$, $k = 2$, forbidden substrings in $\{d^2_6\}$ are $\{13, 24, 35, 46\}$, so there will be forbidden circular 2-successions in $\{d^{*2}_6\}$ starting with 3, 4, 5, 6 but not with 1 and 2. Therefore, the classes in $\{d^2_6\}$ and in $\{d^{*2}_6\}$ starting with 1 and 2 will be the same, and the other classes will be smaller in $\{d^{*2}_6\}$ than in $\{d^2_6\}$. In fact, the cardinalities of the six classes in $\{d^2_6\}$ are $\{53, 53, 64, 64, 64, 64\}$, for a total of 362 permutations, while the cardinality of the classes in $\{d^{*2}_6\}$ is 53 in all six cases, for a total of 318. Hence there are 318 permutations in S_6 that have no circular 2-successions, *ie.* permutations that avoid substrings such as 13 in 213456 or in 345621.

From these examples one might suspect that permutations in $\{d^{*k}_n\}$ are equidistributed (equinumerous), a conjecture which is confirmed in the next section.

2.1 Results for permutations in $\{d^{*k}_n\}$

Proposition 2.1. *For fixed k, $1 \leq k \leq n-1$, if d^{*k}_n denotes the number of permutations on $[n]$ that avoid circular k-successions, then $d^{*k}_n = n d^{k-1}_{n-1}$.*

Proof. Consider a class in $\{d^{*k}_n\}$ where both $\{d^{*k}_n\}$ and $\{d^k_n\}$ have the same number of elements (there is at least one such class for $1 \leq k \leq n-1$), and denote the cardinality of such a class by c. Since a circular permutation is an n-to-1 mapping with respect to a linear one, we have that $d^{*k}_n = n \cdot c$. By inclusion-exclusion, it is straightforward to show that

$$c = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j-1)! \tag{2.1}$$

since the combinatorial term counts the number of ways to get substrings of length j while the term $(n-j-1)!$ counts the $(n-j)!/(n-j)$ circular permutations of forbidden substrings of length j and the remaining elements.

Hence, by Equation (1.1), we see that the right-hand side of Equation (2.1) is just d_{n-1}^{k-1} and the proof follows. □

For example, in $\{d_4^2\}$ we see that the forbidden substrings are $\{13, 24\}$. Recall from [3] that we define a minimal forbidden substring as two consecutive elements ik. We assign to this minimal substring a length equal to one. Recall also that a forbidden substring of length j can be considered as either a single substring of length j or as j overlapping substrings of length 1.

Hence for the term $j = 0$ in Equation 2.1 we see that we can choose no forbidden substrings in $\binom{2}{0}$ ways and we can permute them circularly with the remaining elements 1,2,3,4 in $4!/4$ ways. For the term $j = 1$ we can choose 1 forbidden substring in $\binom{2}{1}$ ways and we can permute it circularly with the remaining elements in $3!/3$ ways (for example, choose the substring 13 and permute the blocks 13, 2, 4 circularly in $3!/3$ ways). For the term $j = 2$ we can choose 2 forbidden substrings in $\binom{2}{2}$ ways and we can permute them circularly with the remaining elements in $2!/2$ ways (that is, permute the blocks 13, 24 circularly in 1! way). Then we have that ${d^*}_n^k = n \cdot c = 4 \cdot 3$, so there are 12 permutations in S_4 that avoid substrings $\{13, 24\}$ in circular 2-successions (for example, permutations such as 1342 and 3421 are not allowed).

Note that even though permutations in $\{d_n^k\}$ are not equidistributed, we gain this property when we forbid circular k-successions in $\{d_n^k\}$, as recorded in the following corollary, which follows from Proposition 2.1, ${d^*}_n^k = n d_{n-1}^{k-1}$.

Corollary 2.2. *The classes in* $\{{d^*}_n^k\}$ *are equidistributed* (*equinumerous*) *and the cardinality of each class is given by* d_{n-1}^{k-1}.

Note that the property of classes being equidistributed is not shared by all types of permutations with forbidden substrings. For example, $\{d_n\}$ and $\{d_n^k\}$ don't have this property, while $\{D_n\}$ and $\{D_n^k\}$ do.

Now we define ${c^*}_n^k$ as the number of circular permutations on $[n]$ (in cycle notation) that avoid substrings $j(j+k)$, $1 \leq j \leq n-k$, *ie.* the same forbidden substrings as in $\{d_n^k\}$ and $\{{d^*}_n^k\}$. We have the following corollary that counts such permutations.

Corollary 2.3. *For fixed* k, $1 \leq k \leq n-1$, *the number of circular permutations* ${c^*}_n^k$ *on* $[n]$ *that avoid substrings* $j(j+k)$, $1 \leq j \leq n-k$, *is given by*

$$ {c^*}_n^k = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j-1)!. \tag{2.2} $$

Proof. This is the number c from the proof of Proposition 2.1. Hence we may write ${d^*}_n^k = n {c^*}_n^k$, where ${c^*}_n^k = d_{n-1}^{k-1}$. □

For example, ${c^*}_5^2 = d_4^1 = 11$, so there are 11 circular permutations in $\{{c^*}_5^2\}$ that avoid substrings $\{13, 24, 35\}$. Furthermore, there are $5 \cdot 11 = 55$ permutations in $\{{d^*}_5^2\}$ that avoid such substrings in circular 2-successions.

Similarly, ${c^*}_4^2 = d_3^1 = 3$, so there are 3 circular permutations that avoid substrings $\{13, 24\}$. These are the permutations $\{(1234), (1423), (1432)\}$, and since $n = 4$, to each of these circular permutations correspond 4 permutations in $\{{d^*}_4^2\}$ with no circular 2-succession (for example the ones for (1234) are $\{1234, 2341, 3412, 4123\}$).

Corollary 2.4. *The number of circular permutations on* $[n]$ *that avoid substrings* $j(j+1)$*,* $1 \leq j \leq n-1$*, (ie.* $k = 1$*) is given by*

$$c_n{}^* = Der(n-1), \tag{2.3}$$

where $Der(n)$ *is the nth derangement number.*

Proof. Since we have that

$$Der(n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)! \tag{2.4}$$

and since Equation 2.2 holds for $k = 1$, the result is immediate (${c^*}_n^1 = {c^*}_n$). □

For example, for $n = 3$, $k = 1$, the number of circular permutations that avoid substrings $\{12, 23\}$ is $Der(2) = 1$ and this circular permutation is (132), which corresponds to the 3 permutations $\{132, 213, 321\}$ in $\{{d^*}_3\}$.

Tables 1 - 3 in the Appendix show some values for d_n^k, ${d^*}_n^k$, and ${c^*}_n^k$.

2.2 Results for permutations in $\{{D^*}_n^k\}$

As in the case of D_n^k , the numbers ${D^*}_n^k$ are more difficult to get. As opposed to permutations in $\{{d^*}_n^k\}$, permutations in $\{{D^*}_n^k\}$ avoid circular k-successions starting with all digits since a total of n substrings are forbidden (as opposed to $n-k$ forbidden substrings in $\{{d^*}_n^k\}$).

We first count the number of permutations in $\{D_n^*\}$, (*ie.* $k = 1$). We have the following proposition.

Proposition 2.5. *The number of permutations in* $\{{D^*}_n\}$ *is given by*

$$D^*{}_n = n \left[\sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j-1)! + (-1)^n \right]. \tag{2.5}$$

Proof. As in the proof of Proposition 2.1, we first obtain the number of circular permutations without forbidden substrings. It is easy to see that the number of such permutations is given by

$$C = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j-1)! + (-1)^n \tag{2.6}$$

since the combinatorial term counts the number of ways to get forbidden substrings while the term $(n-j-1)!$ counts the $(n-j)!/(n-j)$ circular permutations of forbidden substrings and the remaining elements. The result is then obtained by inclusion-exclusion, where for the term $j = n$, we note that the only way to get an n-element substring is by the circular permutation $(12\ldots n)$. □

For example, the forbidden substrings in $\{D^*{}_4\}$ are $\{12, 23, 34; 41\}$. It is easy to count, for instance, that there are $\binom{4}{2}$ forbidden substrings of length $j = 2$ and $2!/2$ circular permutations of these substrings and the remaining elements. For example, a substring of length 2 is given by 123 and we count $2!/2 = 1!$ circular permutation of the blocks 123, 4. Another substring of length 2 (alternatively, two substrings of length 1) is given by 12 34 and we also count $2!/2 = 1!$ circular permutation of these two blocks. Furthermore, the only way to get $j = 4$ forbidden substrings is by the circular permutation (1234), which produces the forbidden substrings $\{12, 23, 34; 41\}$. Hence $C = 1$ by Equation 2.6 and by Equation 2.5 we have that $D^*{}_4 = 4$, which counts the permutations $\{1432, 2143, 3214, 4321\}$ in S_4 that have no circular succession, (*ie.* no circular 1-succession) and avoid substrings $\{12, 23, 34; 41\}$.

If we now move on to permutations without k-successions that avoid substrings $j(j+k) \pmod n$ for $1 \le j \le n$, we see that as in the case of $\{D_n^k\}$, the number of permutations in $\{D^*{}_n^k\}$ depends on whether n is prime, and more generally, on whether n and k are relatively prime. Since the proof follows closely the one for $\{D_n^k\}$, we refer the reader to [3] for the details.

We will only record the main counting result in the proposition below.

Proposition 2.6. *The number of permutations in $\{D^*{}_n^k\}$ with n relative prime to k, $n \ge 3$, $k < n$, is the same as the number of permutations in $\{D^*{}_n\}$.*

The key result in the proof of the proposition is that, if $(n, k) = 1$, there are forbidden substrings of all lengths in $\{D_n^k\}$ (see [3]). But forbidden substrings are the same in $\{D_n^k\}$ and in $\{D^*{}_n^k\}$, so in this case there will also exist forbidden substrings of all lengths in $\{D^*{}_n^k\}$.

The proposition implies the following corollary:

Corollary 2.7. *The number of permutations in $\{D^*{}_p^k\}$, $k = 1, 2, \ldots, p-1$, is the same as the number of permutations in $\{D^*{}_p\}$, p prime, $p \ge 3$.*

Proof. $(p,k)=1$, $k=1,2,\ldots,p-1$. □

Note that Proposition 2.6 is not true if n is not relative prime to k, for example in $\{D^{*2}_{4}\}$. In this case the forbidden substrings are $\{13,24;31,42\}$, and we can't get substrings of lengths 3 or 4. On the other hand, since $(4,3)=1$, we see that forbidden substrings in $\{D^{*3}_{4}\}$ are $\{14;21,32,43\}$, and we can get substrings of all lengths. For example, 2143 is a substring of length 3 (recall that the unit forbidden substring ik has length one, hence the length of forbidden substrings will be one less than the number of elements), and a substring of length 4 (alternatively, 4 substrings of length 1) is given by the circular permutation (1432). We then have that $D^*{}_4=4$ by Equation 2.5 and hence $D^{*3}_{4}=4$ by Proposition 2.6.

As in the case of $\{d^{*k}_{n}\}$, permutations in $\{D^{*k}_{n}\}$ are equidistributed, as established by the following corollary:

Corollary 2.8. *The classes in $\{D^{*k}_{n}\}$ are equidistributed (equinumerous) and the cardinality of each class is given by the term C in Equation 2.6.*

Proof. Equation 2.5 in Proposition 2.5. □

We now define C^{*k}_{n} as the number of circular permutations on $[n]$ (in cycle notation) that avoid substrings $j(j+k) \pmod n$ for $1\le j\le n$, *ie.* the same substrings as in $\{D^{k}_{n}\}$ and $\{D^{*k}_{n}\}$. We have the following corollary that counts such permutations for n and k relatively prime.

Corollary 2.9. *For $(n,k)=1$, the number of circular permutations C^{*k}_{n} on $[n]$ that avoid substrings $j(j+k) \pmod n$ for $1\le j\le n$, is given by*

$$C^{*k}_{n}=\sum_{j=0}^{n-1}(-1)^j\binom{n}{j}(n-j-1)!+(-1)^n. \qquad (2.7)$$

Proof. This is the number C from the proof of Proposition 2.5. Since forbidden substrings in $\{C^{*k}_{n}\}$ are the same as those in $\{D^{*k}_{n}\}$ and since $(n,k)=1$, there are forbidden substrings of all lengths $j=0,1,2,\ldots,n-1,n$, so the sum in Equation 2.7 applies. □

For example, since $(4,3)=1$, we use Equation 2.7 to compute $C^{*3}_{4}=1$, so there is only one circular permutation that avoids substrings $\{14;21,32,43\}$. This circular permutation is (1234), which corresponds to the four permutations $\{1234,2341,3412,4123\}$ in $\{D^{*3}_{4}\}$ with no circular 3-succession. Since $(4,1)=1$, we also have $C^{*1}_{4}=C^*{}_4=1$ by Equation 2.7, and the circular permutation that avoids substrings $\{12,23,34;41\}$ is (1432), which corresponds to the permutations $\{1432,2143,3214,4321\}$ in $\{D^*{}_4\}$, as seen above.

We have that the *gcd* of n and k is very important to determine the number of permutations in $\{C^{*k}_{n}\}$. In fact, if for the same n we have that $(n,k_1)=(n,k_2)$, then $\{C^{*k_1}_n\}$ and $\{C^{*k_2}_n\}$ will have the same number of permutations. The same is true for $\{D^{*k}_n\}$.

Note that usually $C^{*k}_n \neq c^{*k}_n$ except for the case $n = 3$ (and trivially for $n = 2$). Indeed, we have that $C^*{}_3 = c^*{}_3 = 1$ since both numbers count the permutation (132), and $C^{*2}_3 = c^{*2}_3 = 1$ since both numbers count the permutation (123). This in turn implies that $D^*{}_3 = d^*{}_3 = 3$ and $D^{*2}_3 = d^{*2}_3 = 3$.

As a final remark for the case $k = 1$ in circular permutations, recall that $c^{*1}_n = c^*{}_n$, and this counts the number of circular permutations that avoid substrings $j(j+1)$, $1 \leq j \leq n-1$; *ie.* substrings $12, 23, \ldots, (n-1)n$. By Corollary 2.4, $c_n{}^* = Der(n-1)$, so for example, for $n = 4$, we have that $c^*{}_4 = Der(3) = 2$, and the 2 permutations that avoid such substrings are $\{(1324), (1432)\}$.

On the other hand, $C^{*1}_n = C^*{}_n$, and this counts the number of permutations that avoid substrings $j(j+1) \pmod n$ for $1 \leq j \leq n$; *ie.* substrings $\{12, 23, \ldots, (n-1)n, n1\}$. By Corollary 2.9, since $(4, 1) = 1$, we have seen that $C^{*1}_4 = 1$ and the single permutation that avoids such substrings is (1432) (the permutation (1324) from $\{c^*{}_4\}$ is excluded since it has the succession 41, which is forbidden in $\{C^*{}_4\}$).

Note that while in some references such as [1] $C^*{}_n$ is referred to as "the number of circular permutations without a succession", we also consider circular permutations without a succession in $\{c^*{}_n\}$ on the smaller subset of forbidden substrings $\{12, 23, \ldots, (n-1)n\}$. Furthermore, the same reference counts circular permutations with $i = 1, 2, \ldots, n-1$ successions (the so-called "circular succession numbers") but it does it only for forbidden substrings that are one spacing apart, $j(j+k)$, $k = 1$.

For $k > 1$, this note generalizes the enumeration of circular permutations without k-successions (or k-shifts) for both kinds of forbidden substrings (above the diagonal of an $n \times n$ chessboard in $\{c^{*k}_n\}$ and both above and below the diagonal, *ie.* $\pmod n$ in $\{C^{*k}_n\}$). It also enumerates the corresponding permutations in one-line notation in $\{d^{*k}_n\}$ and $\{D^{*k}_n\}$.

For further references, the sequence $\langle C^*{}_n \rangle$ is A000757 in OEIS [7], $\langle D^*{}_n \rangle$ is A167760 [8], $\langle d^*{}_n \rangle$ is A000240 [6], and $\langle c^*{}_n \rangle$ can be looked up in the derangement numbers A000166 [9] due to Corollary 2.4. Note that A000240 not only counts the number of permutations of $[n]$ having no circular succession, $d^*{}_n$, but also the number of permutations D_n on $[n]$ having no substring in $\{12, 23, \ldots, (n-1)n, n1\}$, as well as the number of permutations of $[n]$ having exactly one fixed point (see [2]).

Tables 4 - 6 in the Appendix show some values for D^k_n, D^{*k}_n, and C^{*k}_n.

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APPENDIX

n	Der_n	d_n	d_n^2	d_n^3	d_n^4	d_n^5
1	0					
2	1	1				
3	2	3	4			
4	9	11	14	18		
5	44	53	64	78	96	
6	265	309	362	426	504	600
7	1,854	2,119	2,428	2,790	3,216	3,720
8	14,833	16,687	18,806	21,234	24,024	27,240

Table 1: Some values for d_n^k.

n	${d_n}^*$	${d^*}_n^2$	${d^*}_n^3$	${d^*}_n^4$	${d^*}_n^5$	${d^*}_n^6$
1						
2	0					
3	3	3				
4	8	12	16			
5	45	55	70	90		
6	264	318	384	468	576	
7	1,855	2,163	2,534	2,982	3,528	4,200
8	14,832	16,952	19,424	22,320	25,728	29,760

Table 2: Some values for ${d^*}_n^k$.

n	${c^*}_n$	${c^*}_n^2$	${c^*}_n^3$	${c^*}_n^4$	${c^*}_n^5$	${c^*}_n^6$
1						
2	0					
3	1	1				
4	2	3	4			
5	9	11	14	18		
6	44	53	64	78	96	
7	265	309	362	426	504	600
8	1,854	2,119	2,428	2,790	3,216	3,720

Table 3: Some values for ${c^*}_n^k$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=2$	0					
$n=3$	3	3				
$n=4$	8	8	8			
$n=5$	45	45	45	45		
$n=6$	264	270	240	270	264	
$n=7$	1,855	1,855	1,855	1,855	1,855	1,855

Table 4: Some values for D_n^k.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=2$	0					
$n=3$	3	3				
$n=4$	4	8	4			
$n=5$	40	40	40	40		
$n=6$	216	234	192	234	216	
$n=7$	1,603	1,603	1,603	1,603	1,603	1,603

Table 5: Some values for ${D^*}_n^k$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=2$	0					
$n=3$	1	1				
$n=4$	1	2	1			
$n=5$	8	8	8	8		
$n=6$	36	39	32	39	36	
$n=7$	229	229	229	229	229	229

Table 6: Some values for ${C^*}_n^k$.