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ENUMERATION OF CARLITZ MULTIPERMUTATIONS

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Abstract. A multipermutation with k copies each of $1 \dots n$ is Carlitz if neighbours are different. We enumerate these objects for $k = 2, 3, 4$ and derive recurrences. In particular, we prove and improve a conjectured recurrence for $k = 3$, stated in OEIS, the Online Encyclopedia of Integer Sequences.

1. Introduction

Leonard Carlitz [1] enumerated compositions with adjacent parts being different. We will count multipermutations of $1^k, 2^k, \dots, n^k$ with the same condition.

Definition 1.1. *A multipermutation is* Carlitz *if adjacent elements are different.*

For $k = 1$, these are just the $n!$ ordinary permutations, but for $k > 1$ there are few results. OEIS has entries A114938 for $k = 2$, where an expression and a three-term recurrence is given, and A193638 for $k = 3$, but with no formula and only a conjectured recurrence.

Let $A_k(n)$ be the set of Carlitz multipermutations of $1^k, 2^k, \dots, n^k$ and let $a_k(n) = |A_k(n)|$. The simplest examples are

$$A_2(2) = \{1212, 2121\}, \quad a_2(2) = 2$$

$$A_2(3) = \{121323, 123123, 123132, 123213, 123231, \dots\}, \quad a_2(3) = 30$$

Table 1. Number of Carlitz mutipermutations

$a_k(n)$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$k = 1$	1	1	2	6	24	120	720
$k = 2$	1	0	2	30	864	39 480	2 631 600
$k = 3$	1	0	2	174	41 304	19 606 320	16 438 575 600

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The numbers grow very fast. An upper bound is of course $(kn)!/(k!)^n$, the number of all multipermutations.

We see that $A_2(2)$ has two elements, but only one *pattern*, $xyxy$. If we identify elements with the same pattern, we get a smaller set $A'_k(n)$. Every pattern may be realized in $n!$ ways as a multipermutation, so $a'_k(n) = a_k(n)/n!$ as the examples show.

$A'_2(2) = \{1212\}, \quad a'_2(2) = 1$

$A'_2(3) = \{121323, 123123, 123132, 123213, 123231\}, \quad a'_2(3) = 5$

As representative we choose the *ordered multipermutation*, where the elements appear in order. For any pattern, such as $zyzxyxyxz$, the order condition determines what numeral each letter represents, in this case 121323231.

Sometimes, it seems more natural to work with $a'_k(n)$, sometimes $a_k(n)$ is more convenient. OEIS has entries A278990 for $a'_2(n)$, with formula and a three-term recurrence, and A190826 for $a'_3(n)$ with no formula and an only conjectured recurrence.

TABLE 2. Number of ordered Carlitz mutipermutations

$a'_k(n)$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$k=1$	1	1	1	1	1	1	1
$k=2$	1	0	1	5	36	329	3 655
$k=3$	1	0	1	29	1 721	163 386	22 831 355

2. Inclusion-exclusion formulas

Computing $a_2(n)$ by inclusion-exclusion is Example 2.2.3 in [3]. We show the method for $a_2(3) = 30$.

To begin with, there are $6!/2^3 = 90$ multipermutations of 1 1 2 2 3 3. We subtract all containing the subpattern 11, i.e. multipermutations of the five symbols 11 2 2 3 3. These are $5!/2^2$. The same goes for 22 and 33 so we subtract $\binom{3}{1}5!/2^2 = 90$. Patterns with both 11 and 22 were subtracted twice, so we add $4!/2^1$ for every such pair, totalling $\binom{3}{2}4!/2^1 = 36$. Finally, patterns with all three 11, 22, 33 must be subtracted, that is $\binom{3}{3}3!/2^0 = 6$.

The general formula looks like this.

Proposition 2.1.

$$a_2(n) = \sum_{s+t=n} \left[(-1)^t \binom{n}{s} \frac{(2s+t)!}{(2!)^s} \right]$$

The sum is to be taken over nonnegative s, t that add upp to n. Here s counts symbols that are separate, like $..x..x..$, and t counts sybols that appear together, like $..xx..$, so there are $2s + t$ blocks to permute and s indistinguishable pairs.

The case $k = 3$ is trickier as we now have three subpatterns to consider. If s of the symbols appear separated, like $.x.x.x.$, t of the symbols appear two-plus-one, like $.xx.x.$, and u of the symbols appear united, like $.xxx.$, inclusion-exclusion will produce a surprisingly simple formula. A more thorough treatment is given in Martin's thesis [7].

Theorem 2.2.

$$a_3(n) = \sum_{s+t+u=n} \left[(-1)^t \binom{n}{s, t, u} \frac{(3s + 2t + u)!}{(3!)^s} \right]$$

Proof. A direct application of inclusion-exclusion would be possible if we knew how many multipermutations contain 11, how many contain 11 and 22 etc. The $t = 2, u = 0$ counts permutations of blocks, some of length 1 and some of length 2, for example 11 and 22. This will produce all desired multipermutations, but some of them will be counted twice, for 111 is the same sequence as 111. So we must subtract permutations where the ones are united, and this explains the term $t = 1, u = 1$. But now again we must add permutations with both 111 and 222 and this explains the term $t = 0, u = 2$. □

Let us try to compute $a_3(3) = 174$ with the formula.

$$1\cdot\frac{9!}{6^3} - 3\cdot\frac{8!}{6^2} + 3\cdot\frac{7!}{6^2} + 3\cdot\frac{7!}{6} - 6\cdot\frac{6!}{6} + 3\cdot\frac{5!}{6} - 1\cdot\frac{6!}{1} + 3\cdot\frac{5!}{1} - 3\cdot\frac{4!}{1} + 1\cdot\frac{3!}{1} = 174$$

It is easy to write down similar formulas for $k \geq 4$. We just give $k = 4$ as an example. The proof has no new twists, so we omit it. Just note that v and w count $xx..xx$ resp. $xxxx$.

Theorem 2.3.

$$a_4(n) = \sum_{s+t+u+v+w=n} \left[(-1)^{t+w} \binom{n}{s, t, u, v, w} \frac{(4s + 3t + 2u + 2v + w)!}{(4!)^s (2!)^{v+t}} \right]$$

We were able to give each term a combinatorial interpretation but the formulas are not new. Ira Gessel [2] used rook polynomials to derive more general expressions than these and Jair Taylor [4] proved the same formulas directly from the generating function. Their elegant version of Th.2.3 is

$$a_4(n) = \Phi((\frac{t^3}{6} - t^2 + t)^n) \ ,$$

where $\Phi(t^n) = n!$, so after expansion each power of t is replaced with a factorial.

3. Recurrences

For many purposes, recurrences are superior to the explicit formulas of last section. We will show how to get recurrences for $a'_k(n)$, the number of ordered Carlitz multipermutations. Recall that $a'_k(n) = a_k(n)/n!$.

The OEIS [5] gives conjectural three-term recurrences for $a_2(n)$ and $a'_2(n)$, a four-term recurrence for $a_3(n)$ and a five-term recurrence for $a'_3(n)$. All these conjectures will be proved below.

Theorem 3.1. *The sequence p_n, recursively defined by*

$$p_{n+1} = (2n+1)p_n + p_{n-1},\ p_0 = 1,\ p_1 = 0,$$

counts ordered Carlitz words of $1^2, \ldots, n^2$.

Proof. As $p_n = a'_2(n)$, $p_2 = 1$ counts the word `1212` and $p_3 = 5$ counts the words `010212,012012,012102,012120,012021`, using symbols `012`. The first four words are of the type `0..`$\hat{0}$`.`, that is the zero may be removed without violating the Carlitz property, but the fifth word is of the type `0..x0x..`
Now, we count words in $0^2, 1^2, \ldots, n^2$. according to type.

`0..`$\hat{0}$`.` is counted by $2np_n$ (insert $\hat{0}$ anywhere).

`0..x0x` for $x > 1$ is counted by p_n (transform `1..1` $\mapsto$ `0..x0x`).

`0101..` is counted by p_{n-1} (prefix `0101`). □

In our example, `1212` $\mapsto$ `02x0x2`, which is the same pattern as `012021`.

Theorem 3.2. *The sequences p_n, q_n, recursively defined by*

$$\begin{aligned} 2p_{n+1} &= (3n+3)q_n - 2(3n+1)p_n + 2p_{n-1}, & p_0 = 1,\ p_1 = 0, \\ q_n &= (3n+2)p_n + 2q_{n-1}, & q_0 = 0, \end{aligned}$$

count ordered Carlitz words of $1^3, \ldots, n^3$ resp. of $0^2, 1^3, \ldots, n^3$.

Proof. $p_2 = 1$ counts the word `121212` and $q_2 = 8$ counts the words `01`$\hat{0}$`21212,...,0121212`$\hat{0}$`,01202121,01212021`. The first six words of the type `0..`$\hat{0}$`.` are counted by $3np_n$, the last two `0..x0x..x.` and `0..x..x0x.` with $x > 1$ by $2p_n$. Finally, `0101..1.` and `01..101.` are counted by $2q_{n-1}$. This proves the recurrence for q_n.

We now count p_{n+1} by cases according to type of `0`. As there are two noninitial zeros, the cases will sum to $2p_{n+1}$.
`0..0..`$\hat{0}$`.` is counted by $(3n-1)q_n$ (insert $\hat{0}$ in empty slot).

`0..0..x..x0x.` for $x > 1$ is counted by $2(q_n - p_n - q_{n-1})$, for our transformation `1..0..1..1.` $\mapsto$ `0..0..x..x0x.` does not work for `101..1.` (counted by p_n) or for `10..1..1.` (counted by q_{n-1}) .

`0101..1..0` and the equinumerous `01..101..0` split into subcases depending on the position of the other zero.

`010101..` is counted by p_{n-1}.

`01010..1.`, `0101..01.` and `0101..10.` are counted by $3q_{n-1}$.

`0101..0..1.` and `0101..1..0.` are counted by $2p_n$.

Collecting terms and replacing $2q_{n-1}$ with $q_n - (3n+2)p_n$ we get the recurrence for p_{n+1}. □

Corollary 3.3. *The recursively defined sequence*

$$p_{n+1} = \lambda p_n + \mu p_{n-1} + \nu p_{n-2},\ p_0 = 1,\ p_1 = 0,\ p_2 = 1$$

where $\lambda = (9n^2+9n+8)/2\ + 2/n$, $\mu = (6n+3) - 4/n$, $\nu = -2 - 2/n$ *counts ordered Carlitz words of* $1^3, \dots, n^3$

Proof. Lowering indices in Th.3.2 we get

$$2p_n = 3nq_{n-1} - 2(3n-2)p_{n-1} + 2p_{n-2}$$

Adding $-2 - \frac{2}{n}$ times this to the $2p_{n+1}$-recurrence and then using the q_n-recurrence , we get the desired four-term recurrence. □

The five-term recurrence in OEIS entry A190826 was found by Richard J. Mathar using an ansatz with twenty unknown coefficients [6]. It is of course easily derived by adding two versions of our four-term recurrence, one of them with lowered indices.

The four-term recurrence in OEIS entry A193638 was found by Alois P. Heintz. It is now a corollary obtained by multiplication with $(n+1)!$. Recurrences for $a'_k(n)$ with $k > 3$ may be derived in exactly the same way. We state the result for $k = 4$ here and leave the details to the reader.

Theorem 3.4. *The sequences* p_n, q_n, r_n, *recursively defined by*

$$\begin{aligned} 3p_{n+1} &= (4n+1)q_n + 3(10q_{n-1} - r_n + 4r_{n-1} + (6n+7)p_n + p_{n-1}) \\ 2q_n &= (4n+6)r_n + 6r_{n-1} - (16n+6)p_n \\ r_n &= (4n+3)p_n + 3q_{n-1},\ p_0 = 1,\ p_1 = 0,\ q_0 = 0,\ r_0 = 0 \end{aligned}$$

count ordered Carlitz words of $1^4, \dots, n^4$ *resp. of* $0^3, 1^4, \dots, n^4$, *and of* $0^2, 1^4, \dots, n^4$.

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