arXiv:1702.04485v1 [math.GR] 15 Feb 2017

COMBINATORIAL RESULTS FOR CERTAIN SEMIGROUPS OF ORDER-DECREASING PARTIAL ISOMETRIES OF A FINITE CHAIN

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Abstract

Let $\mathcal{I}_n$ be the symmetric inverse semigroup on $X_n = \{1, 2, \ldots, n\}$ and let $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ be its subsemigroups of order-decreasing partial isometries and of order-preserving and order-decreasing partial isometries of X_n, respectively. In this paper we investigate the cardinalities of some equivalences on $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ which lead naturally to obtaining the order of the semigroups.[1] [2]

MSC2010: 20M18, 20M20, 05A10, 05A15.

1 Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$ and $\mathcal{I}_n$ be the partial one-to-one transformation semigroup on X_n under composition of mappings. Then $\mathcal{I}_n$ is an *inverse* semigroup (that is, for all $\alpha \in \mathcal{I}_n$ there exists a unique $\alpha' \in \mathcal{I}_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$). The importance of $\mathcal{I}_n$ (more commonly known as the *symmetric inverse semigroup or monoid*) to inverse semigroup theory may be likened to that of the symmetric group $\mathcal{S}_n$ to group theory. Every finite inverse semigroup S is embeddable in $\mathcal{I}_n$, the analogue of Cayley's theorem for finite groups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of $\mathcal{I}_n$, see for example [3, 5, 6, 10, 14, 19, 20].

A transformation $\alpha \in \mathcal{I}_n$ is said to be *order-preserving (order-reversing)* if $(\forall x, y \in \operatorname{Dom}\alpha)\ x \leq y \implies x\alpha \leq y\alpha\ (x\alpha \geq y\alpha)$ and, an *isometry (or distance-preserving)* if $(\forall x, y \in \operatorname{Dom}\alpha)\ | x - y | = | x\alpha - y\alpha |$. We shall denote by $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, the semigroups of partial isometries and of order-preserving partial isometries of an $n-$chain, respectively. Eventhough semigroups of partial isometries on more restrictive but richer mathematical structures have been studied by Wallen [21], and Bracci and Picasso [4] the study of the corresponding semigroups on chains was only initiated recently by Al-Kharousi et al. [1, 2]. A little while later, Kehinde et al. [13] studied $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$, the order-decreasing analogues of $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, respectively.

Analogous to Al-Kharousi et al. [2], this paper investigates the combinatorial properties of $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$, thereby complementing the results in Kehinde

[1] *Key Words*: partial one-one transformation, partial isometries, height, right (left) waist, right (left) shoulder and fix of a transformation, idempotents and nilpotents.

[2] Financial support from Sultan Qaboos University Internal Grant: IG/SCI/DOMS/13/06 is gratefully acknowledged.

et al. [13] which dealt mainly with the algebraic and rank properties of these semigroups. In this section we introduce basic definitions and terminology as well as quote some elementary results from Section 1 of Al-Kharousi et al. [1] and Kehinde et al. [13] that will be needed in this paper. In Section 2 we obtain the cardinalities of two equivalences defined on $\mathcal{ODDP}_n$ and $\mathcal{DDP}_n$. These equivalences lead to formulae for the orders of $\mathcal{ODDP}_n$ and $\mathcal{DDP}_n$ as well as new triangles of numbers that were as a result of this work recently recorded in [18].

For standard concepts in semigroup and symmetric inverse semigroup theory, see for example [12, 16]. In particular $E(S)$ denotes the set of idempotents of S. Let

$$\mathcal{DDP}_n = \{\alpha \in \mathcal{DP}_n : (\forall\ x \in \operatorname{Dom}\alpha)\ x\alpha \leq x\}. \tag{1}$$

be the subsemigroup of $\mathcal{I}_n$ consisting of all order-decreasing partial isometries of X_n. Also let

$$\mathcal{ODDP}_n = \{\alpha \in \mathcal{DDP}_n : (\forall\ x, y \in \operatorname{Dom}\alpha)\ x \leq y \Longrightarrow x\alpha \leq y\alpha\} \tag{2}$$

be the subsemigroup of $\mathcal{DDP}_n$ consisting of all order-preserving and order-decreasing partial isometries of X_n. Then we have the following result.

Lemma 1.1 *$\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ are subsemigroups of $\mathcal{I}_n$.*

Remark 1.2 *$\mathcal{DDP}_n = \mathcal{DP}_n \cap \mathcal{I}_n^-$ and $\mathcal{ODDP}_n = \mathcal{ODP}_n \cap \mathcal{I}_n^-$, where $\mathcal{I}_n^-$ is the semigroup of partial one-to-one order-decreasing transformations of X_n [19].*

Next, let α be an arbitrary element in $\mathcal{I}_n$. The *height* or *rank* of α is $h(\alpha) = |\operatorname{Im}\alpha|$, the *right [left] waist* of α is $w^+(\alpha) = max(\operatorname{Im}\alpha)$ $[w^-(\alpha) = min(\operatorname{Im}\alpha)]$, the *right [left] shoulder* of α is $\varpi^+(\alpha) = max(\operatorname{Dom}\alpha)$ $[\varpi(\alpha) = min(\operatorname{Dom}\alpha)]$, and *fix* of α is denoted by $f(\alpha)$, and defined by $f(\alpha) = |F(\alpha)|$, where

$$F(\alpha) = \{x \in X_n : x\alpha = x\}.$$

Next we quote some parts of [1, Lemma 1.2] that will be needed as well as state some additional observations that will help us understand more the cycle structure of order-decreasing partial isometries.

Lemma 1.3 *Let $\alpha \in \mathcal{DP}_n$. Then we have the following:*

(a) The map α is either order-preserving or order-reversing. Equivalently, α is either a translation or a reflection.

(b) If $f(\alpha) = p > 1$ then $f(\alpha) = h(\alpha)$. Equivalently, if $f(\alpha) > 1$ then α is a partial identity.

(c) If α is order-preserving and $f(\alpha) \geq 1$ then α is a partial identity.

(d) If α is order-preserving then it is either strictly order-decreasing ($x\alpha < x$ for all x in $\operatorname{Dom}\alpha$) or strictly order-increasing ($x\alpha > x$ for all x in $\operatorname{Dom}\alpha$) or a partial identity.

(e) If $F(\alpha) = \{i\}$ (for $1 \leq i \leq n$) then for all $x \in \operatorname{Dom}\alpha$ we have that $x + x\alpha = 2i$.

(f) If α is order-decreasing and $i \in F(\alpha)$ ($1 \leq i \leq n$) then for all $x \in \operatorname{Dom}\alpha$ such that $x < i$ we have $x\alpha = x$.

(g) If α is order-decreasing and $F(\alpha) = \{i\}$ then $\operatorname{Dom}\alpha \subseteq \{i, i+1, \ldots, n\}$.

2 Combinatorial results

For a nice survey article concerning combinatorial problems in the symmetric inverse semigroup and some of its subsemigroups we refer the reader to Umar [20].
As in Umar [20], for natural numbers $n \geq p \geq m \geq 0$ and $n \geq i \geq 0$ we define

$$F(n; p_i) = \mid \{\alpha \in S : h(\alpha) = \mid \operatorname{Im}\alpha \mid = i\} \mid, \tag{3}$$

$$F(n; m_i) = \mid \{\alpha \in S : f(\alpha) = i\} \mid \tag{4}$$

where S is any subsemigroup of $\mathcal{I}_n$. From [2, Proposition 2.4] we have

Theorem 2.1 *Let $S = \mathcal{ODP}_n$. Then $F(n; p) = \frac{(2n-p+1)}{p+1}\binom{n}{p}$, where $n \geq p \geq 1$.*

We now have

Proposition 2.2 *Let $S = \mathcal{ODDP}_n$. Then $F(n; p) = \binom{n+1}{p+1}$, where $n \geq p \geq 1$.*

Proof. By virtue of Lemma 1.3[d] and Theorem 2.1 we see that

$$\begin{aligned} F(n;p) &= \frac{1}{2}\left[\frac{2n-p+1}{p+1}\binom{n}{p} - \binom{n}{p}\right] + \binom{n}{p} \\ &= \frac{1}{2}\left[\frac{2(n-p)}{p+1}\binom{n}{p}\right] + \binom{n}{p} \\ &= \frac{n-p}{p+1}\binom{n}{p} + \binom{n}{p} = \binom{n}{p+1} + \binom{n}{p} = \binom{n+1}{p+1}. \end{aligned}$$

□

The proof of the next lemma is routine using Proposition 2.2

Lemma 2.3 *Let $S = \mathcal{ODDP}_n$. Then $F(n;p) = F(n-1;p-1) + F(n-1;p)$, for all $n \geq p \geq 2$.*

Theorem 2.4 $\mid \mathcal{ODDP}_n \mid = 2^{n+1} - (n+1)$.

Proof. It is enough to observe that $\mid \mathcal{ODDP}_n \mid = \sum_{p=0}^{n} F(n;p)$.

Lemma 2.5 *Let $S = \mathcal{ODDP}_n$. Then $F(n;m) = \binom{n}{m}$, for all $n \geq m \geq 1$.*

Proof. It follows directly from Lemma 1.3[b,c] and the fact that all idempotents are necessarily order-decreasing. □

Proposition 2.6 *Let $U_n = \{\alpha \in \mathcal{ODDP}_n : f(\alpha) = 0\}$. Then $\mid U_n \mid =$ $\mid \mathcal{ODDP}_{n-1} \mid$.*

Proof. The proof is similar to that of [19, Theorem 4.3]. □

Remark 2.7 *The triangles of numbers $F(n;p)$ and $F(n;m)$, have as a result of this work appeared in Sloane [18] as [A184049] and [A184050], respectively.*

Now we turn our attention to counting order-reversing partial isometries. First recall from [13, Section3.2(c)] that order-decreasing and order-reversing partial isometries exist only for heights less than or equal to $n/2$. We now have

Lemma 2.8 *Let $S = \mathcal{DDP}_n^*$ be the set of order-reversing partial isometries of X_n. Then $F(n;p_0) = 1$ and $F(n;p_1) = \binom{n+1}{2}$, for all $n \geq 1$.*

Proof. These follow from the simple observation that

$$\{\alpha \in \mathcal{ODDP}_n : h(\alpha) = 0 \text{ or } 1\} = \{\alpha \in \mathcal{DDP}_n^* : h(\alpha) = 0 \text{ or } 1\}$$

and Proposition 2.2. □

Lemma 2.9 *Let $\alpha \in \mathcal{DDP}_n^*$. Then for all $p \geq 1$ we have $F(2p+1, p+1) = 1$ and $F(2p, p) = 3$.*

Proof. **(i)** By Lemma 1.3[f,g] we see that for $i \in \{0, 1, \ldots, p\}$, $\binom{p+1+i}{p+1-i}$ is the unique order-reversing isometry of height $p+1$; and **(ii)** for $i \in \{0, 1, \ldots, p-1\}$, $\binom{p+i}{p-i}$, $\binom{p+1+i}{p-i}$ and $\binom{p+1+i}{p+1-i}$ are the only order-reversing isometries of height p. □

The following technical lemma will be useful later.

Lemma 2.10 *Let $\alpha \in \mathcal{DDP}_n^*$. Suppose $\varpi^+(\alpha) - r \in \operatorname{Dom}\alpha$ and $\varpi^+(\alpha) - s \notin \operatorname{Dom}\alpha$ for all $1 \leq s < r$. Then $\varpi(\alpha) > r$.*

Proof. By order-reversing we see that $(\varpi^+(\alpha))\alpha = w^-(\alpha)$ and $(\varpi(\alpha))\alpha = w^+(\alpha)$. Thus $\varpi^+(\alpha) - r \geq \varpi(\alpha) \implies \varpi^+(\alpha) - \varpi(\alpha) \geq r$. So by isometry we have $w^+(\alpha) - w^-(\alpha) = \varpi^+(\alpha) - \varpi(\alpha) \geq r \implies w^+(\alpha) \geq w^-(\alpha) + r \implies w^+(\alpha) > r \implies \varpi(\alpha) > r$, as required. □

Lemma 2.11 *Let $S = \mathcal{DDP}_n^*$. Then $F(n;p) = F(n-2;p-1) + F(n-2;p)$, for all $n \geq p \geq 2$.*

Proof. Let $\alpha \in \mathcal{DDP}_n^*$ and $h(\alpha) = p$. Define $A = \{\alpha \in \mathcal{DDP}_{n-2}^* : h(\alpha) = p\}$ and $B = \{\alpha \in \mathcal{DDP}_{n-2}^* : h(\alpha) = p-1\}$. Clearly, $A \cap B = \emptyset$. Define a map $\theta : \{\alpha \in \mathcal{DDP}_n^* : h(\alpha) = p\} \to A \cup B$ by $(\alpha)\theta = \alpha'$ where
(i) $x\alpha' = x\alpha\,(x \in \operatorname{Dom}\alpha)$, if $\alpha \in A$. It is clear that α' is an order-decreasing isometry and $h(\alpha) = p$;
(ii) if $\{n-1, n\} \subseteq \operatorname{Dom}\alpha\}$ and $\alpha \in B$, let $\operatorname{Dom}\alpha' = \{x-1 : x \in \operatorname{Dom}\alpha$ and $x < n\}$ and $(x-1)\alpha' = x\alpha - 1 \leq x-1$ and so α' is order-decreasing and $h(\alpha) = p-1$;
(iii) if $\{n-2, n-1\} \subseteq \operatorname{Dom}\alpha\}$ and $\alpha \in B$, let $\operatorname{Dom}\alpha' = \{x-1 : x \in \operatorname{Dom}\alpha$ and $x < n-1\}$ and $(x-1)\alpha' = x\alpha - 1 \leq x-1$ and so α' is order-decreasing and $h(\alpha) = p-1$;
(iv) otherwise, if $\alpha \in B$, let $\operatorname{Dom}\alpha' = \{x - r : x \in \operatorname{Dom}\alpha$ and $x < \varpi^+(\alpha)\}$, where r is such that $\varpi^+(\alpha) - r \in \operatorname{Dom}\alpha$ and $\varpi^+(\alpha) - s \notin \operatorname{Dom}\alpha$ for all $1 \leq s < r$. Define $(x-r)\alpha' = x\alpha - r \leq x - r$ and so α' is order-decreasing and Lemma 2.10 ensures that $h(\alpha) = p-1$.

Moreover, in (ii) and (iii), we have $\mid (x-1)\alpha' - (y-1)\alpha' \mid = \mid (x\alpha - 1) - (y\alpha - 1) \mid = \mid x\alpha - y\alpha \mid = \mid x - y \mid = \mid (x-1) - (y-1) \mid$, and in (iv), we have $\mid (x-r)\alpha' - (y-r)\alpha' \mid = \mid (x\alpha - r) - (y\alpha - r) \mid = \mid x\alpha - y\alpha \mid = \mid x - y \mid =$
$\mid (x-r) - (y-r) \mid$. Hence α' is an isometry.

Also observe that in (ii), we have $\varpi^+(\alpha') = n-2$; in (iii) we have $\varpi^+(\alpha') = n-3$; and in (iv) we have $\varpi^+(\alpha') < n-3$. These observations coupled with the definitions of α' ensures that θ is a bijection.
To show that θ is onto it is enough to note that we can in a symmetric manner define θ^{-1} from $A \cup B \to \{\alpha \in \mathcal{DDP}_n^* : h(\alpha) = p\}$. This establishes the statement of the lemma. □

The next lemma which can be proved by induction, is necessary.

Lemma 2.12 *Let $S = \mathcal{DDP}_n^*$. Then we have the following:*

$$\sum_{i \geq 0} \binom{n-1-2i}{2} = \begin{cases} \frac{(n+1)(n-1)(2n-3)}{24}, & \text{if } n \text{ is odd;} \\ \frac{n(n-2)(2n+1)}{24}, & \text{if } n \text{ is even.} \end{cases}$$

Lemma 2.13 *Let $S = \mathcal{DDP}_n^*$. Then we have the following:*

$$F(n;p_2) = \begin{cases} \frac{(n+1)(n-1)(2n-3)}{24}, & \text{if } n \text{ is odd;} \\ \frac{n(n-2)(2n+1)}{24}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. By applying Lemmas 2.8 and 2.11 sucessively we get

$$\begin{aligned} F(n;p_2) &= F(n-2;p_1)+F(n-2;p_2) = F(n-2;p_2)+\binom{n-1}{2} \\ &= F(n-4;p_2)+\binom{n-3}{2}+\binom{n-1}{2} \\ &= F(n-6;p_2)+\binom{n-5}{2}+\binom{n-3}{2}+\binom{n-1}{2}. \end{aligned}$$

By iteration the result follows from Lemma 2.12 and the facts that $F(2;p_2)=0$ and $F(3;p_2)=1=\binom{2}{2}$. □

Proposition 2.14 *Let $S=\mathcal{DDP}_n^*$. Then for all $\lfloor (n+1)/2\rfloor \ge p \ge 1$, we have* $F(n;p)=\begin{cases} \frac{(n+1)(n-1)(n-3)\cdots(n-2p+3)(2n-3p+3)}{2^p(p+1)!}, & \text{if } n \text{ is odd;} \\ \frac{n(n-2)(n-4)\cdots(n-2p+2)(2n-p+3)}{2^p(p+1)!}, & \text{if } n \text{ is even.} \end{cases}$.

Proof. (By Induction).

Basis Step: $F(n;p_1)=\binom{n+1}{1+1}=\binom{n+1}{2}$ is true by Lemma 2.8 and the observation made in its proof, while the formula for $F(n;p_2)$ is true by Lemma 2.13.

Inductive Step: Suppose $F(m;p)$ is true for all $\lfloor (n+1)/2\rfloor > m \ge p \ge 1$.

Case 1. If m is odd, consider (using the induction hypothesis)

$$\begin{aligned} F(m+2;p) &= F(m;p)+F(m;p-1) \\ &= \frac{(m+1)(m-1)(m-3)\cdots(m-2p+3)(2m-3p+3)}{2^p(p+1)!} \\ &+ \frac{(m+1)(m-1)(m-3)\cdots(m-2p+5)(2m-3p+6)}{2^{p-1}p!} \\ &= \frac{(m+3)(m+1)(m-1)\cdots(m-2p+5)(2m-3p+7)}{2^p(p+1)!}, \end{aligned}$$

which is the formula for $F(m+2;p)$ when m is odd.

Case 2. If m is even, consider (using the induction hypothesis)

$$\begin{aligned} F(m+2;p) &= F(m;p)+F(m;p-1) \\ &= \frac{m(m-2)(m-4)\cdots(m-2p+2)(2m-p+3)}{2^p(p+1)!} \\ &+ \frac{m(m-2)(m-4)\cdots(m-2p+4)(2m-p+4)}{2^{p-1}p!} \\ &= \frac{(m+2)m(m-2)\cdots(m-2p+4)(2m-p+7)}{2^p(p+1)!}, \end{aligned}$$

which is the formula for $F(m+2;p)$ when m is even. $\square$

Proposition 2.15 *Let $S = \mathcal{DDP}_n^*$ and let $b_n = \sum_{p \geq 0} F(n;p)$. Then for $n \geq 0$, we have*

1. $b_{2n+1} = 5 \cdot 2^{n+1} - 4n - 8$;

2. $b_{2n} = 7 \cdot 2^n - 4n - 6$.

Proof. Apply induction and use the fact that $\mid \mathcal{DDP}_n^* \mid = \sum_{p=0}^{n} F(n;p)$.

Proposition 2.16 *Let $S = \mathcal{DDP}_n$. Then*

(1) *if n is odd and $(n+1)/2 \geq p \geq 2$*
$F(n;p) = \frac{(n+1)(n-1)(n-3)\cdots(n-2p+3)(2n-3p+3)}{2^p(p+1)!} + \binom{n+1}{p+1}$;

(2) *if n is even and $n/2 \geq p \geq 2$*
$F(n;p) = \frac{n(n-2)(n-4)\cdots(n-2p+2)(2n-p+3)}{2^p(p+1)!} + \binom{n+1}{p+1}$;

(3) *if $\lfloor (n+1)/2 \rfloor < p$, $F(n;p) = \binom{n+1}{p+1}$.*

Proof. It follows from Propositions 2.2 & 2.14 and Lemmas 1.3[c] & 2.8. $\square$

Combining Theorem 2.4, Lemmas 1.3[a,c] & 2.9, Proposition 2.15 and the observation made in the proof of Lemma 2.8 we get the order of $\mathcal{DDP}_n$ which we record as a theorem below.

Theorem 2.17 *Let $\mathcal{DDP}_n$. Then for all $n \geq 0$ we have*

(1) $\mid \mathcal{DDP}_{2n+1} \mid = 2^{2n+2} + 5 \cdot 2^{n+1} - (2n^2 + 9n + 12)$;

(2) $\mid \mathcal{DDP}_{2n} \mid = 2^{2n+1} + 7 \cdot 2^n - (2n^2 + 7n + 8)$.

Lemma 2.18 *Let $S = \mathcal{DDP}_n$. Then $F(n;m) = \binom{n}{m}$, for all $n \geq m \geq 2$.*

Proof. It follows directly from [13, Lemma 3.18] and the fact that all idempotents are necessarily order-decreasing. $\square$

Proposition 2.19 *Let $S = \mathcal{DDP}_n$. Then $F(2n;m_1) = 2^{n+1} - 2$ and $F(2n-1;m_1) = 3 \cdot 2^{n-1} - 2$, for all $n \geq 1$.*

Proof. Let $F(\alpha) = \{i\}$. Then by Lemma 1.3[e], for any $x \in \mathrm{Dom}\,\alpha$ we have $x + x\alpha = 2i$. Thus, by Lemma 1.3[g], there $2i - 2$ possible elements for $\mathrm{Dom}\,\alpha$: $(x, x\alpha) \in \{(i,i), (i+1, i-1), (i+2, i-2), \ldots, (2i-1, 1)\}$. However, (excluding (i,i)) we see that there are $\sum_{j=0} \binom{i-1}{j} = 2^{i-1}$, possible partial isometries with $F(\alpha) = \{i\}$, where $2i - 1 \leq n \iff i \leq (n+1)/2$. Moreover, by symmetry we see that $F(\alpha) = \{i\}$ and $F(\alpha) = \{n - i + 1\}$ give rise to equal number of decreasing partial isometries. Note that if n is odd (even) the equation $i = n - i + 1$ has one (no) solution. Hence, if $n = 2a - 1$ we have

$$2\sum_{i=1}^{a-1} 2^{i-1} + 2^{a-1} = 2(2^{a-1} - 1) + 2^{a-1} = 3.2^{a-1} - 2$$

decreasing partial isometries with exactly one fixed point; if $n = 2a$ we have

$$2\sum_{i=1}^{a} 2^{i-1} = 2(2^a - 1) = 2^{a+1} - 2$$

decreasing partial isometries with exactly one fixed point. □

Theorem 2.20 *Let $\mathcal{DDP}_n$. Then*

$$a_n = \mid \mathcal{DDP}_n \mid = 3a_{n-1} - 2a_{n-2} - 2^{\lfloor \frac{n}{2} \rfloor} + n + 1,$$

with $a_0 = 1$ and $a_{-1} = 0$.

Proof. It follows from Propositions 2.6 & 2.19, Lemma 2.18 and the fact that $\mid \mathcal{DDP}_n \mid = \sum_{m=0}^{n} F(n; m)$. □

Remark 2.21 *The triangle of numbers $F(n;m)$ and sequence $\mid \mathcal{DDP}_n \mid$ have as a result of this work appeared in Sloane [18] as [A184051] and [A184052], respectively. However, the triangles of numbers $F(n;p)$ for $\mathcal{DDP}_n$ and ${\mathcal{DDP}^*}_n$ and the sequence $\mid \mathcal{DDP}^*_n \mid$ are as at the time of submitting this paper not in Sloane [18]. For some computed values of $F(n;p)$, see Tables 3.1 and 3.2.*

$n \backslash p$	0	1	2	3	4	5	6	7	$\sum F(n;p) = \mid \mathcal{DDP}^*_n \mid$
0	1								1
1	1	1							2
2	1	3	0						4
3	1	6	1	0					8
4	1	10	3	0	0				14
5	1	15	7	1	0	0			24
6	1	21	13	3	0	0	0		38
7	1	28	22	8	1	0	0	0	60

Table 3.1

$n\backslash p$	0	1	2	3	4	5	6	7	$\sum F(n;p) = \mid \mathcal{DDP}_n \mid$
0	1								1
1	1	1							2
2	1	3	1						5
3	1	6	5	1					13
4	1	10	13	5	1				30
5	1	15	27	16	6	1			66
6	1	21	48	38	21	7	1		137
7	1	28	78	78	57	28	8	1	279

Table 3.2

3 Number of $\mathcal{D}^*$-classes

For the definitions of the Green's relations ($\mathcal{L}, \mathcal{R}$ and $\mathcal{D}$) and their starred analogues ($\mathcal{L}^*, \mathcal{R}^*$ and $\mathcal{D}^*$), we refer the reader to Howie [12] and Fountain [8], (respectively) or Ganyushkin and Mazorchuk [9].

First, notice that from [1, Lemma 2.1] we deduce that number of $\mathcal{L}$-classes in $K(n,p) = \{\alpha \in \mathcal{DP}_n : h(\alpha) = p\}$ (as well as the number of $\mathcal{R}$-classes there) is $\binom{n}{p}$. To describe the $\mathcal{D}$-classes in $\mathcal{DP}_n$ and $\mathcal{ODP}_n$, first we recall (from [1]) that the *gap* and *reverse gap of the image set of* α *(with* $h(\alpha) = p$*) are ordered* $(p-1)$*-tuples* defined as follows:

$$g(\operatorname{Im}\alpha) = (\mid a_2\alpha - a_1\alpha \mid, \mid a_3\alpha - a_2\alpha \mid, \ldots, \mid a_p\alpha - a_{p-1}\alpha \mid)$$

and

$$g^R(\operatorname{Im}\alpha) = (\mid a_p\alpha - a_{p-1}\alpha \mid), \ldots, \mid a_3\alpha - a_2\alpha \mid, \mid a_2\alpha - a_1\alpha \mid),$$

where $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix}$ with $1 \leq a_1 < a_2 < \cdots < a_p \leq n$. Further, let $d_i = \mid a_{i+1}\alpha - a_i\alpha \mid$ for $i = 1, 2, \ldots, p-1$. Then

$$g(\operatorname{Im}\alpha) = (d_1, d_2, \ldots, d_{p-1}) \text{ and } g^R(\operatorname{Im}\alpha) = (d_{p-1}, d_{p-2}, \ldots, d_1).$$

For example, if

$$\alpha = \begin{pmatrix} 1 & 2 & 4 & 7 & 8 \\ 3 & 4 & 6 & 9 & 10 \end{pmatrix}, \beta = \begin{pmatrix} 2 & 4 & 7 & 8 \\ 10 & 8 & 5 & 4 \end{pmatrix} \in \mathcal{DP}_{10}$$

then $g(\operatorname{Im}\alpha) = (1,2,3,1)$, $g(\operatorname{Im}\beta) = (2,3,1)$, $g^R(\operatorname{Im}\alpha) = (1,3,2,1)$ and $g^R(\operatorname{Im}\beta) = (1,3,2)$. Next, let $d(n,p)$ be the number of distinct ordered p-tuples: $(d_1, d_2, \ldots, d_p)$ with $\sum_{i=1}^{p} d_i = n$. This is clearly the number of *compositions* of n into p parts. Thus, we have

Lemma 3.1 *[17, p.151]* $d(n,p) = \binom{n-1}{p-1}$.

We shall henceforth use the following well-known binomial identity when needed:

$$\sum_{m=p}^{n} \binom{m}{p} = \binom{n+1}{p+1}.$$

We take this opportunity to state and prove a result which was omitted in [2].

Theorem 3.2 *Let* $S = \mathcal{ODP}_n$. *Then*

(1) *the number of* $\mathcal{D}$*-classes in* $K(n,p)$ $(p \geq 1)$ *is* $\binom{n-1}{p-1}$*;*

(2) *the number of* $\mathcal{D}$*-classes in* S *is* $1 + 2^{n-1}$.

Proof.

(1) It follows from [1, Theorem 2.5]: $(\alpha, \beta) \in \mathcal{D}$ if and only if $g(\operatorname{Im}\alpha) = g(\operatorname{Im}\beta)$; [1, Lemma 3.3]: $p - 1 \leq \sum_{i=1}^{p-1} d_i \leq n - 1$; Lemma 3.1; and so the number of $\mathcal{D}$-classes is $\sum_{i=p-1}^{n-1} d(i, p-1) = \sum_{i=p-1}^{n-1} \binom{i-1}{p-2} = \binom{n-1}{p-1}$.

(2) The number of $\mathcal{D}$-classes in S is $1 + \sum_{p=1}^{n} \binom{n-1}{p-1} = 1 + 2^{n-1}$.

□

The following results from [13] will be needed:

Lemma 3.3 *[13, Lemma 2.3] Let* $\alpha, \beta \in \mathcal{DDP}_n$ *or* $\mathcal{ODDP}_n$. *Then*

(1) $\alpha \leq_{\mathcal{R}^*} \beta$ *if and only if* $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$*;*

(2) $\alpha \leq_{\mathcal{L}^*} \beta$ *if and only if* $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$*;*

(3) $\alpha \leq_{\mathcal{H}^*} \beta$ *if and only if* $\operatorname{Dom}\alpha \subseteq \operatorname{Dom}\beta$ *and* $\operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$.

From [13, (3)], for $\alpha, \beta \in \mathcal{DDP}_n$, we have $(\alpha, \beta) \in \mathcal{D}^*$ if and only if

$$g(\operatorname{Im}\alpha) = \begin{cases} g(\operatorname{Im}\beta); \text{ or} \\ g^R(\operatorname{Im}\beta), \text{ if } p \leq a_p - a_1 \leq (n-1)/2. \end{cases} \tag{5}$$

Similarly, from [13, (4)], for $\alpha, \beta \in \mathcal{ODDP}_n$, we have

$$(\alpha, \beta) \in \mathcal{D}^* \text{ if and only if } g(\operatorname{Im}\alpha) = g(\operatorname{Im}\beta). \tag{6}$$

Now a corollary of Theorem 3.2 follows:

Corollary 3.4 *Let* $S = \mathcal{ODDP}_n$. *Then*

(1) the number of $\mathcal{D}^*$-classes in $K(n,p)$ $(p \geq 1)$ is $\binom{n-1}{p-1}$;

(2) the number of $\mathcal{D}^*$-classes in S is $1 + 2^{n-1}$.

Observe that for all $\alpha \in \mathcal{DP}_n$ with $h(\alpha) = p$,

$$a_p - a_1 = \sum_{i=1}^{p-1}(a_{i+1} - a_i) = \sum_{i=1}^{p-1} d_i, \tag{7}$$

where $g(\mathrm{Dom}\,\alpha) = (d_1, d_2, \ldots, d_{p-1})$. Moreover, an ordered p-tuple: $(d_1, d_2, \ldots, d_p)$ is said to be *symmetric* if

$$(d_1, d_2, \ldots, d_p) = (d_1, d_2, \ldots, d_p)^R = (d_p, d_{p-1}, \ldots, d_1).$$

Now, let $d_s(n,p)$ be the number of distinct symmetric ordered p-tuples: $(d_1, d_2, \ldots, d_p)$ with $\sum_{i=1}^{p} d_i = n$. Then we have

Lemma 3.5 *[2, Lemma 3.5]* $d_s(n;p) = \begin{cases} 0, & \text{if } n \text{ is odd and } p \text{ is even;} \\ \binom{\lfloor \frac{n-1}{2} \rfloor}{\lfloor \frac{p-1}{2} \rfloor}, & \text{otherwise.} \end{cases}$

Now by virtue of (5) and [1, Theorem 2.5], it is not difficult to see that the number of $\mathcal{D}^*$-classes in $\mathcal{DDP}_n$ is the same as the number of $\mathcal{D}$-classes in $\mathcal{ODP}_n$ less those pairs that are merged into single $\mathcal{D}^*$-classes in $\mathcal{DDP}_n$. Thus, we have

Lemma 3.6 *Let $g(m,p)$ be the number of $\mathcal{D}$-classes in $\mathcal{ODP}_n$ (consisting of maps of height p and $\sum d_i = m$) that are merged into single $\mathcal{D}^*$-classes in $\mathcal{DDP}_n$. Then $m \leq (n-1)/2$, and*

$$g(m,p) = \begin{cases} \frac{1}{2}\binom{m-1}{p-2}, & \text{if } n \text{ is odd and } p \text{ is odd;} \\ \frac{1}{2}\left[\binom{m-1}{p-2} - \binom{\lfloor \frac{m-1}{2} \rfloor}{\lfloor \frac{p-2}{2} \rfloor}\right], & \text{otherwise.} \end{cases}$$

Proof. The result follows from (5), Lemmas 3.1 & 3.5 and the observation that

$$g(n,p) = \frac{d(n-1,p-1) - d_s(n-1,p-1)}{2}.$$

□

Now have the main result of this section.

Theorem 3.7 *Let $B(n,p)$ be the number of $\mathcal{D}$-classes in $\mathcal{ODP}_n$ (consisting of maps of height p) that are merged into single $\mathcal{D}^*$-classes in $\mathcal{DDP}_n$. Then for $n \geq p \geq 1$, we have*

$$B(n,p) = \begin{cases} \frac{1}{2}\left[\binom{\lfloor \frac{n-1}{2} \rfloor}{p-1} - \binom{\lfloor \frac{n-1}{4} \rfloor}{\frac{p-1}{2}}\right], & \text{if } p \text{ is odd;} \\ \frac{1}{2}\left[\binom{\lfloor \frac{n-1}{2} \rfloor}{p-1} - 2\binom{\lfloor \frac{n-1}{4} \rfloor}{\frac{p}{2}}\right], & \text{if } n \equiv 1, 2 \,(\mathrm{mod}\, 4), \text{ \& } p \text{ is even;} \\ \frac{1}{2}\left[\binom{\lfloor \frac{n-1}{2} \rfloor}{p-1} - 2\binom{\lfloor \frac{n-3}{4} \rfloor}{\frac{p}{2}} - \binom{\lfloor \frac{n-3}{4} \rfloor}{\frac{p-2}{2}}\right], & \text{if } n \equiv -1, 0 \,(\mathrm{mod}\, 4), \text{ \& } p \text{ is even.} \end{cases}$$

Proof. The result follows from (5), (7) and Lemma 3.6. To see this, let $n \equiv 0\,(mod\,4)$ and p be even. Then $n = 4k$ for some integer k, and

$$\begin{aligned}
B(n,p) &= \sum_{m=p}^{\lfloor\frac{n-1}{2}\rfloor} g(m,p) = \sum_{m=p}^{2k-1} g(m,p) \\
&= g(p,p) + g(p+2,p) + \cdots + g(2k-2,p) \\
&+ g(p+1,p) + g(p+3,p) + \cdots + g(2k-1,p) \\
&= \frac{1}{2}\left[\binom{p-1}{p-2} - \binom{\frac{p-2}{2}}{\frac{p-2}{2}} + \binom{p+1}{p-2} - \binom{\frac{p}{2}}{\frac{p-2}{2}} + \cdots + \binom{2k-3}{p-2} - \binom{k-2}{\frac{p-2}{2}}\right] \\
&+ \frac{1}{2}\left[\binom{p}{p-2} - \binom{\frac{p}{2}}{\frac{p-2}{2}} + \binom{p+2}{p-2} - \binom{\frac{p+2}{2}}{\frac{p-2}{2}} + \cdots + \binom{2k-2}{p-2} - \binom{k-1}{\frac{p-2}{2}}\right] \\
&= \frac{1}{2}\left[\binom{2k-1}{p-1} - 2\binom{k-1}{\frac{p}{2}} - \binom{k-1}{\frac{p-2}{2}}\right] \\
&= \frac{1}{2}\left[\binom{\frac{n-2}{2}}{p-1} - 2\binom{\frac{n-4}{4}}{\frac{p}{2}} - \binom{\frac{n-4}{4}}{\frac{p-2}{2}}\right].
\end{aligned}$$

All the other cases are handled similarly. □

Now have the main result of this section.

Corollary 3.8 *The number of $\mathcal{D}^*$-classes in $\mathcal{DDP}_n$ (consisting of maps of height $p \geq 1$) is $\binom{n-1}{p-1} - B(n,p)$.*

Proof. The result follows from Theorem 3.7 and the remarks preceding Lemma 3.6. □

Corollary 3.9 *The number of $\mathcal{D}^*$-classes in $\mathcal{DDP}_n$ denoted by d_n is*

$$d_n = \begin{cases} 2^{n-1} - 2^{\lfloor\frac{n-3}{2}\rfloor} + \cdot 2^{\lfloor\frac{n+1}{4}\rfloor}, & \text{if } n \equiv -1, 0\,(mod\,4); \\ 2^{n-1} - 2^{\lfloor\frac{n-3}{2}\rfloor} + 3\cdot 2^{\lfloor\frac{n-3}{4}\rfloor}, & \text{if } n \equiv 1, 2\,(mod\,4). \end{cases}$$

Proof. The result follows from Theorem 3.7 and Corollary 3.8. To see this, let $n \equiv 1, 2\,(mod\,4)$. Then $n = 4k+1, 4k+2$ for some integer k, and

$$\begin{aligned}
d_n &= 1 + \sum_{p=1}^{n}\binom{n-1}{p-1} - \sum_{p=1}^{\lfloor\frac{n-1}{2}\rfloor} B(n,p) = 1 + 2^{n-1} - \sum_{p=1}^{2k} B(n,p) \\
&= 1 + 2^{n-1} - [B(n,1) + B(n,3) + \cdots + B(n,2k-1)] \\
&- [B(n,2) + B(n,4) + \cdots + B(n,2k)] \\
&= 1 + 2^{n-1} - \frac{1}{2}\left[\binom{2k}{0} - \binom{k}{0} + \binom{2k}{2} - \binom{k}{1} + \cdots + \binom{2k}{2k-2} - \binom{k}{k-1}\right]
\end{aligned}$$

$$
\begin{aligned}
- & \frac{1}{2}\left[\binom{2k}{1} - 2\binom{k}{1} + \binom{2k}{3} - 2\binom{k}{2} + \cdots + \binom{2k}{2k-1} - 2\binom{k}{k}\right] \\
= & 1 + 2^{n-1} - \frac{1}{2}\left[(2^{2k} - 1) - (3 \cdot 2^k + 1) + 2\right] \\
= & 2^{n-1} - 2^{\lfloor \frac{n-3}{2} \rfloor} + 3 \cdot 2^{\lfloor \frac{n-3}{4} \rfloor}.
\end{aligned}
$$

The case $n \equiv -1, 0\, (mod\, 4)$ is handled similarly. □

Acknowledgements. The second named author would like to thank Bowen University, Iwo and Sultan Qaboos University for their financial support and hospitality, respectively.

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