Semi-Baxter and strong-Baxter: two relatives of the Baxter sequence

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Abstract

In this paper, we enumerate two families of pattern-avoiding permutations: those avoiding the vincular pattern $2\underline{41}3$, which we call semi-Baxter permutations, and those avoiding the vincular patterns $2\underline{41}3$, $3\underline{14}2$ and $3\underline{41}2$, which we call strong-Baxter permutations. We call semi-Baxter numbers and strong-Baxter numbers the associated enumeration sequences. We prove in addition that the semi-Baxter numbers enumerate plane permutations (avoiding $2\underline{14}3$). The problem of counting these permutations was open and has given rise to several conjectures, which we also prove in this paper.

For each family (that of semi-Baxter – or equivalently, plane – and that of strong-Baxter permutations), we describe a generating tree, which translates into a functional equation for the generating function. For semi-Baxter permutations, it is solved using (a variant of) the kernel method, giving an expression for the generating function. From it, we obtain closed formulas for the semi-Baxter numbers, a recurrence that they satisfy, as well as their asymptotic behavior. For strong-Baxter permutations, we show that their generating function is (a slight modification of) that of a family of walks in the quarter plane, which is known to be non D-finite.

Along the way, we define families of labeled Dyck paths enumerated by the semi-Baxter, strong-Baxter, and also by the Baxter and factorial numbers. To our knowledge, the family of Baxter paths that we define is the first family where objects are single lattice paths which is enumerated by the well-known sequence of Baxter numbers.

1 Introduction

The main purpose of this article is the study of two enumeration sequences, which we call the *semi-Baxter sequence* and the *strong-Baxter sequence*. They enumerate, among other objects, families of pattern-avoiding permutations closely related to the well-known family of Baxter permutations, and to the slightly less popular one of twisted Baxter permutations, which are both counted by the sequence of Baxter numbers [20, sequence A001181].

Recall that a permutation $\pi = \pi_1\pi_2\dots\pi_n$ contains the vincular[1] pattern $2\underline{41}3$ if there exists a subsequence $\pi_i\pi_j\pi_{j+1}\pi_k$ of π (with $i < j < k-1$), called an *occurrence* of the pattern, that satisfies $\pi_{j+1} < \pi_i < \pi_k < \pi_j$. Containment and occurrences of the patterns $3\underline{14}2$, $3\underline{41}2$, $2\underline{14}3$ and $\underline{14}23$ are defined similarly. A permutation not containing a pattern avoids it. Baxter permutations [8, among many others] are those that avoid both $2\underline{41}3$ and $3\underline{14}2$, while twisted Baxter permutations [13, and references therein] are the ones avoiding $2\underline{41}3$ and $3\underline{41}2$. We denote by $Av(P)$ the family of permutations avoiding all patterns in P.

The two sequences that will be our main focus are first the one enumerating permutations avoiding $2\underline{41}3$, called *semi-Baxter permutations*, and second the one enumerating permutations avoiding all three patterns $2\underline{41}3$, $3\underline{14}2$ and $3\underline{41}2$, called *strong-Baxter permutations*. Remark

[1] Throughout the article, we adopt the convention of denoting by the symbol $\underline{\ \ }$ the elements that are required to be adjacent in an occurrence of a vincular pattern, rather than using the historical notation with dashes wherever elements are not required to be consecutive. For instance, our pattern $2\underline{41}3$ is sometimes written $2-41-3$ in the literature.

that a permutation avoiding the (classical) pattern 231 necessarily avoids $2\,\underline{41}\,3$, $3\,\underline{14}\,2$ and $3\,\underline{41}\,2$, and recall that $Av(231)$ is enumerated by the sequence of Catalan numbers. The definitions in terms of pattern-avoidance and the enumeration results given above can be summarized as shown in Figure 1.

$$
\begin{array}{ccccccccc}
\text{Catalan} & \leq & \text{strong-Baxter} & \leq & \text{Baxter} & \leq & \text{semi-Baxter} & \leq & \text{factorial} \\[1em]
 & & Av(2\,\underline{41}\,3, & \subsetneq & Av(2\,\underline{41}\,3, 3\,\underline{14}\,2) & \subsetneq & & & \text{all} \\
Av(231) & \subseteq & 3\,\underline{14}\,2, & & & & Av(2\,\underline{41}\,3) & \subseteq & \text{permutations} \\
 & & 3\,\underline{41}\,2) & \subsetneq & Av(2\,\underline{41}\,3, 3\,\underline{41}\,2) & \subsetneq & & &
\end{array}
$$

Figure 1: Sequences from Catalan to factorial numbers, with nested families of pattern-avoiding permutations that they enumerate.

Our first and main contribution is the study of the two sequences of semi-Baxter and strong-Baxter numbers.

We deal with the semi-Baxter sequence (enumerating semi-Baxter permutations) in Section 3. It has been proved in [19] (as a special case of a general statement) that this sequence also enumerates *plane permutations*, defined by the avoidance of $2\,\underline{14}\,3$. This sequence is referenced as A117106 in [20]. We first give a more specific proof that plane permutations and semi-Baxter permutations are equinumerous, by providing a common generating tree (or succession rule) with two labels for these two families. Basics and references about generating trees can be found in Subsection 2.1.

We solve completely the problem of enumerating semi-Baxter permutations (or equivalently, plane permutations), pushing further the techniques that were used to enumerate Baxter permutations in [8]. Namely, we start from the functional equation associated with our succession rule for semi-Baxter permutations, and we solve it using variants of the kernel method [8, 18]. This results in an expression for the (D-finite) generating function for semi-Baxter permutations. From it, we obtain several formulas for the semi-Baxter numbers: first, a complicated closed formula; second, a simple recursive formula; and third, three simple closed formulas that were conjectured by D. Bevan [5].

The problem of enumerating plane permutations was posed by M. Bousquet-Mélou and S. Butler in [10]. Some conjectures related to this enumeration problem were later proposed, in particular by D. Bevan [4, 5] and M. Martinez and C. Savage [21]. Not only do we solve the problem of enumerating plane permutations (or equivalently, semi-Baxter permutations) completely, but we also prove these conjectures. We mention that it has been conjectured in [3] by A. Baxter and M. Shattuck that permutations avoiding $\underline{14}\,23$ are also enumerated by the same sequence, but we have not been able to prove it.

In Section 5, we focus on the study of strong-Baxter permutations and of the strong-Baxter sequence. Again, we provide a generating tree for strong-Baxter permutations, and translate the corresponding succession rule into a functional equation for their generating function. However, we do not solve the equation using the kernel method. Instead, from the functional equation, we prove that the generating function for strong-Baxter permutations is a very close relative of the one for a family of walks in the quarter plane studied in [6]. As a consequence, the generating function for strong-Baxter permutations is not D-finite. Families of permutations with non D-finite generating functions are quite rare in the literature on pattern-avoiding permutations (although mostly studied for classical patterns, instead of vincular ones): this makes the example of strong-Baxter permutations particularly interesting.

The second contribution of our work is to define families of lattice paths that are enumerated by the sequences of Figure 1.

This part of our work is motivated by the combination of two observations. First, among the families of combinatorial objects that are enumerated by Catalan numbers, the one of Dyck paths is certainly one of the most well-known. Second, although the Baxter numbers are a natural generalization of the Catalan sequence (see for example the discussion in [12]), we are not aware of any family of lattice paths enumerated by the Baxter sequence. Of course, there are the triples of non-intersecting lattice paths [7], but to our knowledge no combinatorial family where the objects are *single* paths is known to be enumerated by the Baxter numbers. Mimicking the second row of the table in Figure 1, we aim at defining nested families of lattice paths enumerated by the sequences of the first row of this figure. This results in particular in a definition of a family of lattice paths enumerated by the Baxter numbers, which we believe was missing so far in the literature on Baxter numbers.

The definitions of the families of factorial, semi-Baxter, Baxter and strong-Baxter paths appear along the paper, in Subsections 2.2, 3.5, 4.3 and 5.3. Each is a family of labeled Dyck paths, where some of the up steps (called the *free up steps*) carry integer labels in a prescribed range. That these families are nested is clear from the constraints imposed on the labels of the free up steps, which are always more restrictive. To prove that the family of Baxter paths is enumerated by the Baxter sequence, we exhibit a generating tree for Baxter paths that corresponds to the succession rule Ω_{Bax} associated with this sequence. We proceed similarly for semi-Baxter and strong-Baxter paths. For the case of factorial paths, we follow the same idea, but an additional (elementary) argument is needed to establish that the obtained succession rule corresponds to factorial numbers.

For each family, we believe that giving first the definition of the paths, and then proving that they grow according to the corresponding succession rule, as we will do, is the clearest way to go. However, with this presentation, the way we have been able to find appropriate definitions of these families of paths remains hidden. We believe it deserves a few words in this introduction. The idea is to observe that the succession rules for the sequences of Figure 1, considered from right to left, each are a specialization of the previous one. What we mean exactly by specialization is explained in further details along the paper, following the same idea as in [12]. From the growth of, say, semi-Baxter paths along Ω_{semi}, and the restriction of Ω_{semi} to Ω_{Bax}, we then examine how the restriction on the labels in the succession rules can be interpreted as restrictions on the objects produced in the generating tree (in our case, with stronger constraints on the labels of the free up steps).

With the approach, we have been able to describe restricted families of factorial paths, all containing Dyck paths, associated with rules Ω_{semi}, Ω_{Bax} and Ω_{strong}. Note however that, in the case of permutations, we had two restricted families enumerated by the Baxter numbers: that of Baxter permutations and that of twisted Baxter ones. The succession rule Ω_{Bax} is classically associated with Baxter permutations, while there is another succession rule (Ω_{TBax}) corresponding also to Baxter numbers and associated rather with twisted Baxter permutations. Since both Ω_{Bax} and Ω_{TBax} are restrictions of Ω_{semi} (while being at the same time generalizations of Ω_{strong}), the same approach as described above could be applied to define another family of paths, which we could call twisted Baxter paths, enumerated by the Baxter numbers, and which grow according to Ω_{TBax}. We have explored this route, but it did not result in any nice conditions on the labels of the free up steps that would be worthy of being reported in this article.

The article is next organized as follows. Section 2 recalls basics and easy facts about the Catalan and factorial sequences and objects. Sections 3, 4 and 5 then focus on the sequences of semi-Baxter numbers, Baxter numbers, and strong Baxter numbers, respectively, and on the associated objects.

2 Catalan and factorial sequences, succession rules, and lattice paths

With the classical example of Dyck paths, we start by reviewing some basics about generating trees and succession rules. These are then adapted to define a family of labeled Dyck paths counted by the sequence of factorial numbers, which we call *factorial paths*. In the following sections, for each sequence considered, we will define a subset of factorial paths that the sequence enumerates.

2.1 Dyck paths and the Catalan succession rule

A *Dyck path* of size n is a path in the plane, using up steps $U = (1, 1)$ and down steps $D = (1, -1)$, starting at the point of coordinate $(0, 0)$ and ending in $(2n, 0)$, while always staying above the x-axis. Of course, a Dyck path can be represented by a word on the alphabet $\{U, D\}$ (although not all such words are acceptable). Dyck path are enumerated by the sequence of Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ [20, A000108]. Later in our work, we will consider families of *labeled Dyck paths*, where some of the up steps carry integer labels.

Generating trees and succession rules will also be important for our work. We give a brief general presentation below. Details can be found for instance in [1, 2, 8, 25]. We also review the classical generating tree for Dyck paths and the classical succession rule for Catalan numbers, since we will build on them later in this work.

Consider any combinatorial class $\mathcal{C}$, that is to say any set of discrete objects equipped with a notion of size, such that there is a finite number of objects of size n for any integer n. Assume also that $\mathcal{C}$ contains exactly one object of size 1. A *generating tree* for $\mathcal{C}$ is an infinite rooted tree, whose vertices are the objects of $\mathcal{C}$, each appearing exactly once in the tree, and such that objects of size n are at distance n from the root (with the convention that the root is at distance 1 from itself). The children of some object $c \in \mathcal{C}$ are obtained by adding an *atom* (*i.e.* a piece of object that makes its size increase by 1) to c. Of course, since every object should appear only once in the tree, not all additions are possible. We should ensure the unique appearance property by considering only additions that follow some restricted rules. We will call the *growth* of $\mathcal{C}$ the process of adding atoms following these prescribed rules.

Such a growth for Dyck paths is described in [2]. The atoms are *peaks*, that is to say UD factors. To ensure that all Dyck paths appear exactly once in the generating tree, peaks are inserted in the *last descent* of the path, that is to say, in the longest suffix containing only the letter D. More precisely, the children of a Dyck path $w{\cdot}UD^k$ are $w{\cdot}U\boldsymbol{UD}D^k, w{\cdot}UD\boldsymbol{UD}D^{k-1}, \ldots, w{\cdot}UD^{k-1}\boldsymbol{UD}D$ and $w \cdot UD^k\boldsymbol{UD}$. The first few levels of the generating tree for Dyck paths are shown in Figure 2 (left).

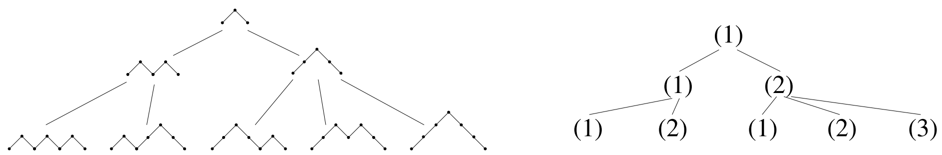

Figure 2: Two ways of looking at the generating tree for Dyck paths: with objects (left) and with labels from the succession rule Ω_{Cat} (right).

Of importance for enumeration purposes is the general shape of a generating tree, not the specific objects labeling its nodes. From now on, when we write generating tree, we intend this shape of the tree, without the objects labeling the nodes. A *succession rule* is a compact way of representing such a generating tree for a combinatorial class $\mathcal{C}$ without referring to its objects, but identifying them with *labels*. Therefore, a succession rule is made of one starting label corresponding to the label of the root and of productions encoding the way labels spread in the generating tree. As we explain in [12], the sequence enumerating the class $\mathcal{C}$ can be recovered from the succession rule itself, without reference to the specifics of the objects in $\mathcal{C}$: indeed, the nth term of the sequence is the total number of labels (counted with repetition) that are produced from the root by $n - 1$ applications of the productions, or equivalently, the number of nodes at distance n from the root in the generating tree.

From the growth of Dyck paths described above, we classically obtain the following succession rule associated with Catalan numbers (corresponding to the tree shown in Figure 2, right):

$$\Omega_{Cat} = \begin{cases} (1) \\ (k) \rightsquigarrow (1), (2), \ldots, (k), (k+1). \end{cases}$$

The intended meaning of the label (k) is the number of D steps in the last descent of a path.

The class of Dyck paths is just one of many families of discrete objects counted by the Catalan numbers. Among these, of particular interest for us is the class $Av(231)$ of permutations avoiding the pattern 231. Indeed, in this article, we will be interested in the enumeration of several families of restricted permutations that contain $Av(231)$ (as well as its symmetries $Av(132)$, $Av(213)$ and $Av(312)$).

2.2 Factorial paths and associated succession rule

The *factorial numbers* [20, A000142] are well-known for counting permutations of size n. As indicated above, our work enumerates families of restricted permutations containing $Av(231)$. For each enumeration sequence so obtained, we will define a family of labeled Dyck paths counted by the same sequence. Therefore, we start by introducing *factorial paths*, a family of labeled Dyck paths in bijective correspondence with the set of all permutations: all our paths afterward will be restricted factorial paths.

Definition 1. *A* free up step *in a Dyck path is any up step which does not immediately follow a down step,* i.e., *any step U which does not appear in a DU factor. If U is a free up step, we usually write it $\overline{U}$.*

Definition 2. *A* factorial path *of size n is a Dyck path of size n in which every free up step $\overline{U}$ is labeled with an integer in $[1, d+1]$, where d is the total number of D steps occurring before $\overline{U}$ in the path.*

Figure 3 shows two examples: one factorial path and one labeling of the same underlying Dyck path which does not satisfy the condition in Definition 2.

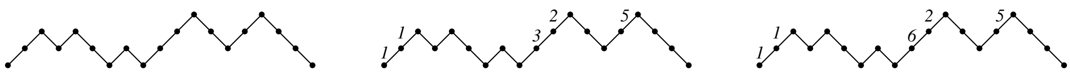

Figure 3: A Dyck path (left), a labeling of its free up steps which makes it a factorial path (middle), and another labeling which does not give a factorial path, label 6 being too large (right).

The proof of Proposition 3 below is a good warm-up exercise before we turn to similar but more complicated arguments in the next sections.

Proposition 3. *Factorial paths grow according to the succession rule*

$$\Omega_{fact} = \begin{cases} (1,1) \\ (k,n) \rightsquigarrow (1,n+1), (2,n+1), \ldots, (k,n+1), (k+1,n+1)^{n-k+1}. \end{cases}$$

As a consequence, they are enumerated by the factorial numbers.

Proof. As in the classical case of Dyck paths, we make factorial paths grow by insertion of a peak in the last descent. This growth can be captured by a succession rule as follows. To any factorial path, we assign the label (k, n) where k is the number of D steps in the last descent, and n is the size of the path.

Consider a factorial path f whose underlying unlabeled Dyck path is $d = w \cdot UD^k$. The Dyck path obtained adding a peak at the beginning of the last descent, $w \cdot U\boldsymbol{UD}D^k$, is the only child of

d such that the added up step is free. In the children of f, this step $\overline{U}$ may have any label between 1 and $n-k+1$, since $n-k$ is the number of D steps before $\overline{U}$ (in either f or d). This gives the $(k+1, n+1)^{n-k+1}$ part of the production, which replaces the single $(k+1)$ in the case of Ω_{Cat}.

Next, in all other children $w \cdot UD\boldsymbol{UDD}D^{k-1}, \ldots, w \cdot UD^{k-1}\boldsymbol{UDD}, w \cdot UD^k\boldsymbol{UD}$ of d, the added U step is not free, so it receives no label. This corresponds to the $(1, n+1), (2, n+1), \ldots, (k, n+1)$ part of the production, similarly to the first k productions of Ω_{Cat}.

The proof that the number of factorial paths of size n is $n!$ follows then easily by induction. Indeed, there is of course 1 factorial path of size 1, and from the production, we see that each factorial path of size n has in total $k + (n-k+1) = n+1$ children. □

Remark 4. From Proposition 3, it follows that there exists an (implicit) bijection between factorial paths of size n and permutations of size n. It is however not hard to provide such a bijection explicitly.

To see this, let us first introduce the *skew representation* of a factorial path, illustrated in Figure 4. Introducing the new west step $W = (-1, 1)$, it is the path obtained replacing any free up step $\overline{U}$ labeled by i by $W^{i-1}UD^{i-1}$, and removing pairs of steps D and W that overlap (or equivalently, removing recursively all DW factors). It is not hard to see that through this construction, factorial paths of size n are in one-to-one correspondence with *underdiagonal paths* of size n, that is to say paths starting at $(0,0)$, ending at $(2n, 0)$, staying above the x axis and below the line $y = x$, using U, D, and W steps, and such that no steps D and W overlap. Of course, underdiagonal paths of size n contain n up steps. Moreover, denoting e_i the distance from the ith up steps to the line $y = x$, it holds that $e_i \leq i-1$. Sequences $(e_1, \ldots, e_n)$ such that $e_i \leq i-1$ for all i are known as *inversion sequences*. The above construction from underdiagonal paths to inversion sequences is actually easily invertible, providing a bijection between them. Finally, there is a well-known one-to-one correspondence between inversion sequences and permutations, yielding the desired bijection. The interested reader may find more about inversion sequences and this bijection in [21].

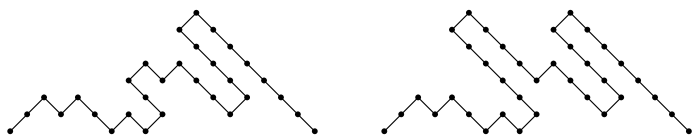

Figure 4: The skew representations of the middle and right paths of Figure 3. Only the first one encodes a factorial path, and is therefore an underdiagonal path. It corresponds to the inversion sequence $(0, 0, 1, 3, 4, 2, 3, 6, 2)$, and hence to the permutation 498316527.

3 Semi-Baxter numbers

3.1 Definition, context, and summary of our results

Definition 5. *A* semi-Baxter permutation *is a permutation that avoids the pattern* $2\,\underline{41}\,3$.

Definition 6. *The sequence of* semi-Baxter numbers*, (SB_n), is defined by taking SB_n to be the number of semi-Baxter permutations of size n.*

The name "semi-Baxter" has been chosen because $2\,\underline{41}\,3$ is one of the two patterns (namely, $2\,\underline{41}\,3$ and $3\,\underline{14}\,2$) whose avoidance defines the family of so-called Baxter permutations [17, 14], enumerated by the Baxter numbers [20, sequence A001181]. (Remark that up to symmetry, we could have defined semi-Baxter permutations by the avoidance of $3\,\underline{14}\,2$, obtaining the same sequence.) Note that $2\,\underline{41}\,3$ is also one of the two patterns (namely, $2\,\underline{41}\,3$ and $3\,\underline{41}\,2$) whose avoidance defines the family of so-called twisted Baxter permutations [24, 26], also enumerated by the Baxter numbers.

The first few terms of the sequence of semi-Baxter numbers are

$$1, 2, 6, 23, 104, 530, 2958, 17734, \ldots$$

The family of semi-Baxter permutations already appears in the literature, at least on a few occasions. Indeed, it is an easy exercise to see that the avoidance of $2\,\underline{4\,1}\,3$ is equivalent to that of the barred pattern $25\bar{3}14$, which has been studied by L. Pudwell in [23]. (The definition of barred patterns, which is not essential to our work, can be found in [23].) In this work, by means of enumeration schemes L. Pudwell suggests that the enumerative sequences of semi-Baxter permutations and *plane permutations* (see Definition 8 p.8) coincide. This conjecture has later been proved as a special case of a general statement in [19, Corollary 1.9(b)]. In Corollary 10 we give an alternative and self-contained proof that plane permutations and semi-Baxter permutations are indeed equinumerous. The sequence enumerating plane permutations has already been registered on the OEIS [20] as sequence A117106, which is then our sequence (SB_n).

The enumeration of plane permutations has received a fair amount of attention in the literature. The problem first arose as an open problem in [10]. It was studied with a quite experimental perspective, as one case of many, through enumeration schemes by L. Pudwell in [23]. Then, D. Bevan computed the first 37 terms of their enumerative sequence [4], by iterating a functional equation provided in [4, Theorem 13.1]. Although [4] gives a functional equation for the generating function of semi-Baxter numbers, there is no formula (closed or recursive) for SB_n. There is however a conjectured explicit formula related to Apéry numbers, which, in addition, gives information about their asymptotic behavior. Another recursive formula for SB_n has been conjectured by M. Martinez and C. Savage in [21], in relation with *inversion sequences* avoiding some patterns (definition and precise statement are provided in Subsection 3.4). Finally, closed formulas for SB_n have been conjectured by D. Bevan in [5].

Our results about semi-Baxter numbers are the following. Most importantly, we solve the problem of enumerating semi-Baxter permutations, as well as plane permutations. We provide a common succession rule that governs their growth, presented in Subsection 3.2. By means of standard tools we translate the succession rule into a functional equation whose solution is the generating function of semi-Baxter numbers. Then Subsection 3.3 gives a closed expression for the generating function of semi-Baxter numbers, together with closed, recursive and asymptotic formulas for SB_n. The results of this subsection are proved in Subsections 3.6 and 3.7 following the same method as in [11]: the functional equation is solved using the obstinate kernel method, a first closed formula for SB_n is obtained by the Lagrange inversion, the recursive formula follows from it applying the method of *creative telescoping* [22], which can then be applied again to prove that the explicit formulas for SB_n conjectured in [5] are correct, and finally careful estimates of binomial coefficients occurring in these formulas give the asymptotic behavior of SB_n. In Subsection 3.4 we additionally prove the formulas for SB_n conjectured in [4] (in relation with Apéry numbers) and in [21] (in relation with inversion sequences). Finally, in Subsection 3.5, we define a family of labeled Dyck paths, which we call *semi-Baxter paths*, and we show that they are enumerated by (SB_n).

3.2 Succession rule for semi-Baxter permutations and plane permutations

Throughout the paper we will build permutations of increasing sizes by performing "local expansions" on the right of any permutation π. More precisely, when inserting $a \in \{1, \ldots, n+1\}$ on the right of any π of size n, we obtain the permutation $\pi' = \pi'_1 \ldots \pi'_n \pi'_{n+1}$ where $\pi'_{n+1} = a$, $\pi'_i = \pi_i$ if $\pi_i < a$ and $\pi'_i = \pi_i + 1$ if $\pi_i \geq a$. We use the notation $\pi \cdot a$ to denote π'. For instance, $1\,4\,2\,3 \cdot 3 = 1\,5\,2\,4\,3$. This is easily understood on the diagrams representing permutations (which consist of points in the Cartesian plane at coordinates (i, π_i)): a local expansion corresponds to adding a new point on the right of the diagram, which lies vertically between two existing points (or below the lowest, or above the highest), and finally normalizing the picture obtained – see Figure 5.

Proposition 7. *Semi-Baxter permutations can be generated by the following succession rule:*

$$\Omega_{semi} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\dots,(h,k+1) \\ \qquad\qquad (h+k,1),\dots,(h+1,k). \end{cases}$$

Proof. First, observe that removing the last element of a permutation avoiding $2\underline{41}3$, we obtain a permutation that still avoids $2\underline{41}3$. So, a generating tree for semi-Baxter permutations can be obtained with local expansions on the right.

For π a semi-Baxter permutation of size n, the *active sites* are by definition the points a (or equivalently the values a) such that $\pi \cdot a$ is also semi-Baxter, *i.e.*, avoids $2\underline{41}3$. The other points a are called non-active sites. An occurrence of $2\underline{31}$ in π is a subsequence $\pi_j\pi_i\pi_{i+1}$ (with $j < i$) such that $\pi_{i+1} < \pi_j < \pi_i$. Obviously, the non-active sites a of π are characterized by the fact that $a \in (\pi_j, \pi_i]$ for some occurrence $\pi_j\pi_i\pi_{i+1}$ of $2\underline{31}$. We call a *non-empty descent* of π a pair $\pi_i\pi_{i+1}$ such that there exists π_j that makes $\pi_j\pi_i\pi_{i+1}$ an occurrence of $2\underline{31}$. Note that in the case where $\pi_{n-1}\pi_n$ is a non-empty descent, choosing $\pi_j = \pi_n + 1$ always gives an occurrence of $2\underline{31}$, and it is the smallest possible value of π_j for which $\pi_j\pi_{n-1}\pi_n$ is an occurrence of $2\underline{31}$.

To each semi-Baxter permutation π of size n, we assign a label (h,k), where h (resp. k) is the number of the active sites of π smaller than or equal to (resp. greater than) π_n. Remark that $h, k \geq 1$, since π_n and $\pi_n + 1$ are always active sites. Moreover, the label of the permutation $\pi = 1$ is $(1,1)$, which is the root in Ω_{semi}.

Consider a semi-Baxter permutation π of size n and label (h,k). Proving Proposition 7 amounts to showing that permutations $\pi \cdot a$ have labels $(1,k+1),\dots,(h,k+1),(h+k,1),\dots,(h+1,k)$ when a runs over all active sites of π. Figure 5, which shows an example of semi-Baxter permutation π with label $(2,2)$ and all the corresponding $\pi \cdot a$ with their labels, should help understanding the case analysis that follows. Let a be an active site of π.

Assume first that $a > \pi_n$ (this happens exactly k times), so that $\pi \cdot a$ ends with an ascent. The occurrences of $2\underline{31}$ in $\pi \cdot a$ are the same as in π. Consequently, the active sites are not modified, except that the active site a of π is now split into two actives sites of $\pi \cdot a$: one immediately below a and one immediately above. It follows that $\pi \cdot a$ has label $(h+k+1-i, i)$, if a is the i-th active site from the top. Since i ranges from 1 to k, this gives the second row of the production of Ω_{semi}.

Assume next that $a = \pi_n$. Then, $\pi \cdot a$ ends with a descent, but an empty one. Similar to the above case, we therefore get one more active site in $\pi \cdot a$ than in π, and $\pi \cdot a$ has label $(h, k+1)$, the last label in the first row of the production of Ω_{semi}.

Finally, assume that $a < \pi_n$ (this happens exactly $h-1$ times). Now, $\pi \cdot a$ ends with a non-empty descent, which is $(\pi_n + 1)a$. It follows from the discussion at the beginning of this proof that all sites of $\pi \cdot a$ in $(a+1, \pi_n+1]$ become non-active, while all others remain active if they were so in π (again, with a replaced by two active sites surrounding it, one below it and one above). If a is the i-th active site from the bottom, it follows that $\pi \cdot a$ has label $(i, k+1)$, hence giving all missing labels in the first row of the production of Ω_{semi}. □

Definition 8. *A* plane permutation *is a permutation that avoids the vincular pattern* $2\underline{14}3$ *(or equivalently, the barred pattern* $21\bar{3}54$*).*

Proposition 9. *Plane permutations can be generated by* Ω_{semi}.

Proof. The proof of this statement follows applying the same steps as in the proof of Proposition 7. First, observe that removing the last element of a permutation avoiding $2\underline{14}3$, we obtain a permutation that still avoids $2\underline{14}3$. So, a generating tree for plane permutations can be obtained with local expansions on the right.

For π a plane permutation of size n, the active sites are by definition the values a such that $\pi \cdot a$ is also plane, *i.e.*, avoids $2\underline{14}3$. The other points a are called non-active sites. An occurrence

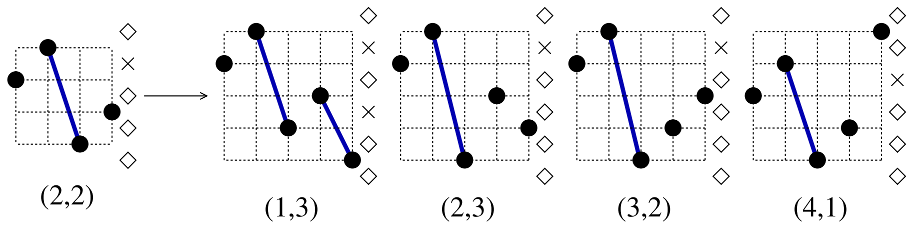

Figure 5: The growth of semi-Baxter permutations. Active sites are marked with $\Diamond$, non-active sites by $\times$, and non-empty descents are represented with bold blue lines.

of $2\underline{13}$ in π is a subsequence $\pi_j\pi_i\pi_{i+1}$ (with $j < i$) such that $\pi_i < \pi_j < \pi_{i+1}$. Note that the non-active sites a of π are characterized by the fact that $a \in (\pi_j, \pi_{i+1}]$ for some occurrence $\pi_j\pi_i\pi_{i+1}$ of $2\underline{13}$. We call a *non-empty ascent* of π a pair $\pi_i\pi_{i+1}$ such that there exists π_j that makes $\pi_j\pi_i\pi_{i+1}$ an occurrence of $2\underline{13}$. As in the proof of Proposition 7, if $\pi_{n-1}\pi_n$ is a non-empty ascent, $\pi_j = \pi_{n-1}+1$ is the smallest value of π_j such that $\pi_j\pi_{n-1}\pi_n$ is an occurrence of $2\underline{13}$.

Now, to each plane permutation π of size n, we assign a label (h, k), where h (resp. k) is the number of the active sites of π greater than (resp. smaller than or equal to) π_n. Remark that $h, k \geq 1$, since 1 and $\pi_n + 1$ are always active sites. Moreover, the label of the permutation $\pi = 1$ is $(1, 1)$, which is the root in Ω_{semi}. The proof is concluded by showing that the permutations $\pi \cdot a$ have labels $(1, k+1), \ldots, (h, k+1), (h+k, 1), \ldots, (h+1, k)$, when a runs over all active sites of π.

If $a \leq \pi_n$, $\pi \cdot a$ ends with a descent, and it follows as in the proof of Proposition 7 that the active sites of $\pi \cdot a$ are the same as those of π (with a split into two sites). This gives the second row of the production of Ω_{semi} (the label $(h+k+1-i, i)$ for $1 \leq i \leq k$ corresponding to a being the i-th active site from the bottom).

If $a = \pi_n + 1$, $\pi \cdot a$ ends with an empty ascent, and hence has label $(h, k+1)$ again as in the proof of Proposition 7.

Finally, if $a > \pi_n + 1$ (which happens $h - 1$ times), $\pi \cdot a$ ends with a non-empty ascent. The discussion at the beginning of the proof implies that all sites of $\pi \cdot a$ in $(\pi_n + 1, a]$ are deactivated while all others remain active. If a is the i-th active site from the top, it follows that $\pi \cdot a$ has label $(i, k+1)$, hence giving all missing labels in the first row of the production of Ω_{semi}. $\square$

Because the two families of semi-Baxter and of plane permutations grow according to the same succession rule, we obtain the following.

Corollary 10. *Semi-Baxter permutations and plane permutations are equinumerous. In other words, SB_n is also the number of plane permutations of size n.*

Note that the two generating trees for semi-Baxter and for plane permutations which are encoded by Ω_{semi} are of course isomorphic: this provides a size-preserving bijection between these two families. It is however not defined directly on the object themselves, but only referring to the generating tree structure.

3.3 Properties of the semi-Baxter sequence and of its generating function

For $h, k \geq 1$, let $S_{h,k}(x) \equiv S_{h,k}$ denote the size generating function for semi-Baxter permutations having label (h, k). The rule Ω_{semi} translates into a functional equation for the generating function $S(x; y, z) \equiv S(y, z) = \sum_{h,k\geq 1} S_{h,k} y^h z^k$.

Proposition 11. *The generating function $S(y, z)$ satisfies the following functional equation:*

$$S(y,z) = xyz + \frac{xyz}{1-y}\left(S(1,z) - S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z) - S(y,y)\right). \tag{1}$$

Proof. Starting from the growth of semi-Baxter permutations according to Ω_{semi} we write:

$$\begin{aligned}
S(y,z) &= xyz + x\sum_{h,k\geq 1} S_{h,k}\left((y+y^2+\cdots+y^h)z^{k+1} + (y^{h+k}z + y^{h+k-1}z^2+\cdots+y^{h+1}z^k)\right)\\
&= xyz + x\sum_{h,k\geq 1} S_{h,k}\left(\frac{1-y^h}{1-y}y\,z^{k+1} + \frac{1-\left(\frac{y}{z}\right)^k}{1-\frac{y}{z}}y^{h+1}z^k\right)\\
&= xyz + \frac{xyz}{1-y}\left(S(1,z)-S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z)-S(y,y)\right). \qquad \square
\end{aligned}$$

From Proposition 11, a lot of information can be derived about the generating function $S(1,1)$ of semi-Baxter numbers, and about these numbers themselves. The results we obtain are stated below, but the proofs are postponed to Subsections 3.6 and 3.7. A Maple worksheet recording the computations in these proofs is available from the authors' webpage[2].

First, using the "obstinate kernel method" (used for instance in [8] to enumerate Baxter permutations), we can give a more practical expression for S. As usual, we let $\bar{a}$ denote $1/a$.

Theorem 12. *Let $W(x;a)\equiv W$ be the unique formal power series in x such that*

$$W = x\bar{a}(1+a)(W+1+a)(W+a).$$

The series solution $S(y,z)$ of eq. (1) satisfies $S(1+a,1+a)=\Omega_{\geq}[F(a,W)]$, where $\Omega_{\geq}[F(a,W)]$ stands for the formal power series in x obtained by considering only those terms in the series expansion that have non-negative powers of a, and the function $F(a,W)$ is defined by

$$\begin{aligned}
F(a,W) = \quad & (1+a)^2\,x + \left(\bar{a}^5+\bar{a}^4+2+2a\right)\,x\,W\\
&+\left(-\bar{a}^5-\bar{a}^4+\bar{a}^3-\bar{a}^2-\bar{a}+1\right)\,x\,W^2 - \left(\bar{a}^4-\bar{a}^2\right)\,x\,W^3.
\end{aligned} \tag{2}$$

Note that in Theorem 12, W and $F(a,W)$ are algebraic series in x whose coefficients are Laurent polynomials in a. It follows, as in [8, page 6], that $S(1+a,1+a)=\Omega_{\geq}[F(a,W)]$ is D-finite, and hence also $S(1,1)$.

Using the Lagrange Inversion, we can derive from Theorem 12 an explicit but complicated expression for the coefficients of $S(1,1)$, which is reported in Corollary 23 in Subsection 3.7. Surprisingly this complicated expression hides a very simple recurrence that was conjectured in [4].

Proposition 13. *The numbers SB_n are recursively characterized by $SB_0=0$, $SB_1=1$ and for $n\geq 2$*

$$SB_n = \frac{11n^2+11n-6}{(n+4)(n+3)}SB_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)}SB_{n-2}. \tag{3}$$

From the recurrence of Proposition 13, we can in turn prove closed formulas for semi-Baxter numbers, which have been conjectured in [5]. These are much simpler than the one given in Corollary 23 by the Lagrange inversion, and also very much alike the summation formula for Baxter numbers (which we recall in Subsection 4.1).

Theorem 14. *For any $n\geq 2$, the number SB_n of semi-Baxter permutations of size n satisfies*

$$\begin{aligned}
SB_n &= \frac{24}{(n-1)n^2(n+1)(n+2)}\sum_{j=0}^{n}\binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1}\\
&= \frac{24}{(n-1)n^2(n+1)(n+2)}\sum_{j=0}^{n}\binom{n}{j+2}\binom{n+1}{j}\binom{n+j+2}{j+3}\\
&= \frac{24}{(n-1)n^2(n+1)(n+2)}\sum_{j=0}^{n}\binom{n+1}{j+3}\binom{n+2}{j+1}\binom{n+j+3}{j}.
\end{aligned}$$

[2] for instance at `http://user.math.uzh.ch/bouvel/publications/Semi-Baxter.mw`

There is actually a fourth formula that has been conjectured in [5], namely

$$SB_n = \frac{24}{(n-1)n(n+1)^2(n+2)} \sum_{j=0}^{n} \binom{n+1}{j}\binom{n+1}{j+3}\binom{n+j+2}{j+2}.$$

Taking the multiplicative factors inside the sums, it is easy to see (for instance going back to the definition of binomial coefficients as quotients of factorials) that it is term by term equal to the second formula of Theorem 14.

From the first (or any) of the formulas above, we can derive the dominant asymptotics of SB_n.

Corollary 15. *Let $\varphi = \frac{1}{2}(\sqrt{5}-1)$. It holds that*

$$SB_n \sim A\,\frac{\mu^n}{n^6},$$

where $A = \frac{12}{\pi}\,5^{-1/4}\varphi^{-15/2} \approx 94.34$ and $\mu = \varphi^{-5} = (11+5\sqrt{5})/2$.

3.4 Proofs of two conjectures about semi-Baxter numbers

As indicated in Subsection 3.1, in addition to the formulas reported in Theorem 14 above, two conjectural formulas for SB_n have been proposed in the literature, in different contexts.

The first one is attributed to M. Van Hoeij and reported by D. Bevan in [4]. The conjecture is an explicit formula for semi-Baxter numbers that involves Apéry numbers $a_n = \sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}$ (sequence A005258 on [20]). We will prove the validity of this conjecture using the recursive formula for semi-Baxter numbers (Proposition 13) and the following recurrence satisfied by Apéry numbers:

$$a_{n+1} = \frac{11n^2+11n+3}{(n+1)^2}\,a_n + \frac{n^2}{(n+1)^2}\,a_{n-1}. \tag{4}$$

Proposition 16 ([4], Conjecture 13.2)**.** *For $n \geq 2$,*

$$SB_n = \frac{24((5n^3-5n+6)a_{n+1} - (5n^2+15n+18)a_n)}{5(n-1)n^2(n+2)^2(n+3)^2(n+4)}$$

Proof. For the sake of brevity we write $A(n) = 5n^3-5n+6$ and $B(n) = 5n^2+15n+18$ so that the statement becomes

$$SB_n = \frac{24(A(n)\,a_{n+1} - B(n)\,a_n)}{5(n-1)n^2(n+2)^2(n+3)^2(n+4)}. \tag{5}$$

The validity of eq. (5) is proved by induction on n: for $n = 2, 3$, it holds $SB_2 = \frac{A(2)a_3 - B(2)a_2}{2000} = \frac{36\cdot147 - 68\cdot19}{2000} = 2$ and $SB_3 = \frac{A(3)a_4 - B(3)a_3}{23625} = \frac{126\cdot1251 - 108\cdot147}{23625} = 6$.

Then, suppose that eq. (5) is valid for $n-1$ and $n-2$. In order to prove it for n, consider the recursive formula of eq. (3) and substitute in it SB_{n-1} and SB_{n-2} by using eq. (5). Now, after some work of manipulation and by using eq. (4) we can write SB_n as in eq. (5). $\square$

Remark 17. With Corollary 23, Theorem 14 and Proposition 16, we get five expressions for the nth semi-Baxter number as a sum over j. Note that although the sums are equal, the corresponding summands in each sum are not (this is readily checked for $n = 8$ and $j = 5$ for instance). Therefore, Corollary 23, Theorem 14 and Proposition 16 give five essentially different ways of expressing the semi-Baxter numbers.

The second conjecture appears in the work of M. Martinez and C. Savage on pattern-avoiding inversion sequences [21].

Recall that an *inversion sequence* of size n is an integer sequence $(e_1, e_2, \dots, e_n)$ satisfying $0 \le e_i < i$ for all $i \in \{1, 2, \dots, n\}$. In [21] the authors introduce the notion of pattern avoidance in inversion sequences in quite general terms.

Of interest to us here is only the set $\mathbf{I}_n(210, 100)$ of inversion sequences avoiding the patterns 210 and 100, which can be characterized as follows. A *weak left-to-right maximum* of an inversion sequence $(e_1, e_2, \dots, e_n)$ is an entry e_i satisfying $e_i \ge e_j$ for all $j \le i$. Every inversion sequence e can be decomposed in e^{top}, which is the (weakly increasing) sequence of weak left-to-right maxima of e, and e^{bottom}, which is the (possibly empty) sequence of the remaining entries of e.

Proposition 18 ([21], Observation 10)**.** *An inversion sequence e avoids 210 and 100 if and only if e^{top} is weakly increasing and e^{bottom} is strictly increasing.*

The enumeration of inversion sequences avoiding 210 and 100 is solved in [21], with a summation formula as reported in Proposition 19 below. Let $\text{top}(e) = \max(e^{top})$ and $\text{bottom}(e) = \max(e^{bottom})$. If e^{bottom} is empty, the convention is to take $\text{bottom}(e) = -1$.

Proposition 19 ([21], Theorem 32)**.** *Let $Q_{n,a,b}$ be the number of $e \in \mathbf{I}_n(210, 100)$ with $top(e) = a$ and $bottom(e) = b$. Then $Q_{n,a,b} = \sum_{i=-1}^{b-1} Q_{n-1,a,i} + \sum_{j=b+1}^{a} Q_{n-1,j,b}$, with initial conditions $Q_{n,a,b} = 0$, if $n \le a$, and $Q_{n,a,-1} = \frac{n-a}{n}\binom{n-1+a}{a}$. Hence,*

$$|\mathbf{I}_n(210, 100)| = \sum_{a=0}^{n-1} \sum_{b=-1}^{a-1} Q_{n,a,b} = \frac{1}{n+1}\binom{2n}{n} + \sum_{a=0}^{n-1} \sum_{b=0}^{a-1} Q_{n,a,b}. \qquad (6)$$

We prove the following conjecture of [21].

Theorem 20. *There are as many inversion sequences of size n avoiding 210 and 100 as plane permutations of size n. In other words $|\mathbf{I}_n(210, 100)| = SB_n$.*

Proof. We prove the statement by showing that $\cup_n \mathbf{I}_n(210, 100)$ grows according to Ω_{semi}. Given an inversion sequence $e \in \mathbf{I}_n(210, 100)$, we make it grow by adding a rightmost entry.

Let a be the $\text{top}(e)$ and b be the $\text{bottom}(e)$. From Proposition 18, it follows that $f = (e, p) = (e_1, \dots, e_n, p)$ is an inversion sequence of size $n+1$ avoiding 210 and 100 if and only if $n \ge p > b$. Moreover, if $p \ge a$, then $f^{top} = (e^{top}, p)$ and $f^{bottom} = e^{bottom}$; and if $b < p < a$, then $f^{top} = e^{top}$ and $f^{bottom} = (e^{bottom}, p)$.

Now, we assign to any $e \in \mathbf{I}_n(210, 100)$ the label (h, k), where $h = a - b$ and $k = n - a$. The sequence $e = (0)$ has label $(1, 1)$, since $a = \text{top}(e) = 0$ and $b = \text{bottom}(e) = -1$. Let e be an inversion sequence of $\mathbf{I}_n(210, 100)$ with label (h, k). The labels of the inversion sequences of $\mathbf{I}_{n+1}(210, 100)$ produced adding a rightmost entry p to e are

- $(h+k, 1), (h+k-1, 2), \dots, (h+1, k)$ when $p = n, n-1, \dots, a+1$,
- $(h, k+1)$ when $p = a$,
- $(1, k+1), \dots, (h-1, k+1)$ when $p = a-1, \dots, b+1$,

which concludes the proof that $\cup_n \mathbf{I}_n(210, 100)$ grows according to Ω_{semi}. □

3.5 Semi-Baxter paths

To our knowledge, only families of restricted permutations are known to be counted by the semi-Baxter numbers. Here, we provide a new occurrence of these numbers in terms of labeled Dyck paths, as a restricted family of factorial paths.

Definition 21. *A* semi-Baxter path *is a factorial path where the labels of the free up steps satisfy the following additional constraint: for every pair of free up steps (U', U'') with U' occurring before U'' and no free up step between them, the label of U'' is in the range $[1, h]$, where $h \ge 1$ is the sum of the label of U' with the number of D steps between U' and U''.*

A semi-Baxter path can be obtained from a Dyck path by properly labeling its free up steps as shown in Figure 6 (left). The factorial path in Figure 6 (right) is not a semi-Baxter path, because the last free up step is labeled by 6, which is here a value outside of the range of Definition 21. Observe that unlike in factorial paths, the sequence of labels corresponding to consecutive free up steps in semi-Baxter paths has to be non-increasing.

Figure 6: A semi-Baxter path (left) and a factorial path which is not a semi-Baxter path (right).

Let $\mathcal{SB}_n$ denote the set of semi-Baxter paths of size n. With Proposition 22 below, we prove that $\mathcal{SB} = \cup_n \mathcal{SB}_n$ is enumerated by sequence of semi-Baxter numbers.

Proposition 22. *Semi-Baxter paths grow according to the rule Ω_{semi}.*

Proof. Similarly to Dyck paths and factorial paths, we make semi-Baxter paths grow by insertion of a peak in the last descent, as shown in Figure 7. To any $s \in \mathcal{SB}_n$, denoting e the label of its rightmost free up step $\overline{U}$ (which always exists, since the first step of the path is always a free up step), we assign the label (h, k), where h is equal to e plus the number of down steps between $\overline{U}$ and the rightmost up step of s (they might coincide) and k is the number of steps of the last descent of s. With this labeling, we shall see that the growth of semi-Baxter paths can be encoded by Ω_{semi}.

The unique semi-Baxter path in $\mathcal{SB}_1$ receives the label $(1, 1)$. From any $s \in \mathcal{SB}_n$ of label (h, k), we perform two kinds of insertions, which we shall see correspond to all productions of (h, k) in Ω_{semi}.

a) We add a peak at the beginning of the last descent of s. This means that the added U step follows another up step and hence is free, while the number of down steps in the last descent increases by one. Moreover, U receives a label, which can be any value in the range $[1, h]$. If i is the label assigned to U, for $1 \leq i \leq h$, then the path produced has label $(i, k + 1)$.

b) We add a peak immediately after any down step of the last descent of s. In this case the added step U is not free, and hence carry no label. Denoting $s = w \cdot UD^k$ (with this U possibly labeled), the children of s so produced are $w \cdot UD^j\boldsymbol{UD}D^{k-j}$ for $1 \leq j \leq k$, so they have labels $(h + j, k + 1 - j)$. ☐

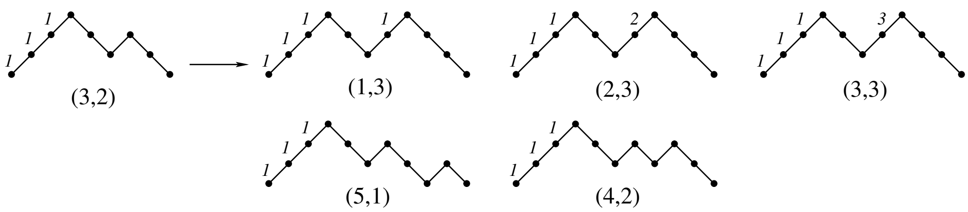

Figure 7: The growth of a semi-Baxter path of label $(3, 2)$.

3.6 Generating function of semi-Baxter numbers: proof

Recall that $S(y, z)$ denotes the multivariate generating function of semi-Baxter permutations. In Theorem 12, we have given an expression for $S(1 + a, 1 + a)$, which we now prove.

Proof of Theorem 12. The linear functional equation of eq. (1) has two catalytic variables, y and z and its solution $S(y,z)$ is not symmetric in y and z. To solve eq. (1) it is convenient to set $y = 1+a$ and collect all the terms having $S(1+a,z)$ in them, obtaining the *kernel form* of the equation:

$$K(a,z)S(1+a,z) = xz(1+a) + \frac{xz(1+a)}{a}S(1,z) - \frac{xz(1+a)}{z-1-a}S(1+a,1+a), \tag{7}$$

where the kernel function is $K(a,z) = 1 - \frac{xz(1+a)}{a} - \frac{xz(1+a)}{z-1-a}$. For brevity, we refer to the right-hand side of eq. (7) as $R(x,a,z,S(1,z),S(1+a,1+a))$.

The kernel function is quadratic in z. Denoting $Z_+(a)$ and $Z_-(a)$ the solutions of $K(a,z)=0$ with respect to z, and $Q = \sqrt{a^2 - 2ax - 6a^2x + x^2 + 2ax^2 + a^2x^2 - 4a^3x}$, we have

$$\begin{aligned} Z_+(a) &= \frac{1}{2}\frac{a+x+ax-Q}{x(1+a)} = (1+a) + (1+a)^2x + \frac{(1+a)^3(1+2a)}{a}x^2 + O(x^3), \\ Z_-(a) &= \frac{1}{2}\frac{a+x+ax+Q}{x(1+a)} = \frac{a}{(1+a)x} - a - (1+a)^2x - \frac{(1+a)^3(1+2a)}{a}x^2 + O(x^3). \end{aligned}$$

The kernel root Z_- is not a well-defined power series in x, whereas the other kernel root Z_+ is a power series in x whose coefficients are Laurent polynomials in a. So, setting $z = Z_+$, the function $S(1+a,z)$ is a convergent power series in x and the right-hand side of eq. (7) is equal to zero, *i.e.* $R(x,a,Z_+,S(1,Z_+),S(1+a,1+a)) = 0$.

At this point we follow the usual kernel method approach (see for instance [8]) and attempt to eliminate the term $S(1,Z_+)$ by exploiting transformations that leave the kernel, $K(a,z)$, unchanged. Examining the kernel shows that the transformations

$$\Phi : (a,z) \to \left(\frac{z-1-a}{1+a}, z\right) \quad \text{and} \quad \Psi : (a,z) \to \left(a, \frac{z+za-1-a}{z-1-a}\right)$$

leave the kernel unchanged and generate a group of order 10.

Among all the elements of this group we consider the following pairs $(f_1(a,z), f_2(a,z))$:

$$[a,z] \underset{\Phi}{\longleftrightarrow} \left[\frac{z-1-a}{1+a}, z\right] \underset{\Psi}{\longleftrightarrow} \left[\frac{z-1-a}{1+a}, \frac{z-1}{a}\right] \underset{\Phi}{\longleftrightarrow} \left[\frac{z-1-a}{az}, \frac{z-1}{a}\right] \underset{\Psi}{\longleftrightarrow} \left[\frac{z-1-a}{az}, \frac{1+a}{a}\right].$$

These have been chosen since, for each of them, $f_1(a,Z_+)$ and $f_2(a,Z_+)$ are well-defined power series in x with Laurent polynomial coefficients in a. Moreover, they share the property that $S(1+f_1(a,Z_+), f_2(a,Z_+))$ are convergent power series in x. It follows that, substituting each of these pairs for (a,z) in eq. (7), we obtain a system of five equations, whose left-hand sides are all 0, and with six overlapping unknowns:

$$\begin{cases} 0 = R(x,a,Z_+,S(1,Z_+),S(1+a,1+a)) \\ 0 = R\left(x, \frac{Z_+-1-a}{1+a}, Z_+, S(1,Z_+), S(1+\frac{Z_+-1-a}{1+a}, 1+\frac{Z_+-1-a}{1+a})\right) \\ 0 = R\left(x, \frac{Z_+-1}{a}, \frac{Z_+-1-a}{1+a}, S(1,\frac{Z_+-1}{a}), S(1+\frac{Z_+-1-a}{1+a}, 1+\frac{Z_+-1-a}{1+a})\right) \\ 0 = R\left(x, \frac{Z_+-1-a}{aZ_+}, \frac{Z_+-1}{a}, S(1,\frac{Z_+-1}{a}), S(1+\frac{Z_+-1-a}{aZ_+}, 1+\frac{Z_+-1-a}{aZ_+})\right) \\ 0 = R\left(x, \frac{Z_+-1-a}{aZ_+}, \frac{1+a}{a}, S(1,\frac{1+a}{a}), S(1+\frac{Z_+-1-a}{aZ_+}, 1+\frac{Z_+-1-a}{aZ_+})\right). \end{cases}$$

Eliminating all unknowns except $S(1+a,1+a)$ and $S(1,1+\bar{a})$, this system reduces (after some work) to the following equation, where $P(a,z) = \frac{(-z+1+a)}{z(z-1)a^4}(-za^4 + z^2a^4 - za^3 + z^2a^3 - z^3a^2 -$

$2a^2 + z^2a^2 + za^2 - 4a + 5az - 3az^2 + z^3a + 3z - z^2 - 2)$:

$$S(1+a, 1+a) - \frac{(1+a)^2 x}{a^4} S(1, 1+\bar{a}) + P(a, Z_+) = 0. \tag{8}$$

The form of eq. (8) allows us to separate its terms according to the power of a:

- $S(1+a, 1+a)$ is a power series in x with polynomial coefficient in a whose lowest power of a is 0,
- $S(1, 1+\bar{a})$ is a power series in x with polynomial coefficient in $\bar{a}$ whose highest power of a is 0; consequently, and since $\frac{(1+a)^2 x}{a^4} = x(a^{-4} + 2a^{-3} + a^{-2})$, we obtain that $\frac{(1+a)^2 x}{a^4} S(1, 1+\bar{a})$ is a power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is -2.

Hence when we expand the series $-P(a, Z_+)$ as a power series in x, the non-negative powers of a in the coefficients must be equal to those of $S(1+a, 1+a)$, while the negative powers of a come from $\frac{(1+a)^2 x}{a^4} S(1, 1+\bar{a})$.

Then, in order to have a better expression for $P(a, z)$, we perform a further substitution setting $z = w+1+a$. More precisely, let $W \equiv W(x; a)$ be the power series in x defined by $W = Z_+ - (1+a)$. We have the following expression for $F(a, W) := -P(a, Z_+)$:

$$\begin{aligned} F(a, W) = -P(a, Z_+) = -P(a, W+1+a) = \quad & (1+a)^2 x + \left(\tfrac{1}{a^5} + \tfrac{1}{a^4} + 2 + 2a\right) x W \\ & + \left(-\tfrac{1}{a^5} - \tfrac{1}{a^4} + \tfrac{1}{a^3} - \tfrac{1}{a^2} - \tfrac{1}{a} + 1\right) x W^2 \\ & - \left(\tfrac{1}{a^4} - \tfrac{1}{a^2}\right) x W^3. \end{aligned}$$

Moreover, since $K(a, W+1+a) = 0$, the function W is recursively defined by

$$W = x\bar{a}(1+a)(W+1+a)(W+a), \tag{9}$$

as claimed. □

3.7 Formulas for semi-Baxter numbers: proofs

From the expression of $S(1+a, 1+a)$ obtained above, the Lagrange inversion allows us to derive an explicit expression for the semi-Baxter numbers, as shown below in Corollary 23. This first formula for SB_n is complicated, but we show how to obtain from it the simple recurrence of Proposition 13. Next, this recurrence allow us to prove the conjectured simpler formulas for SB_n stated in Theorem 14. Finally, from one of these nice formulas, we prove the asymptotic estimate of SB_n stated in Corollary 15.

Corollary 23. *The number SB_n of semi-Baxter permutations of size n satisfies:*

$$\begin{aligned} \text{for all } n \geq 1,\ SB_{n+1} = \quad & \tfrac{1}{n} \textstyle\sum_{j=0}^{n} \binom{n}{j} \Big[2 \binom{n+1}{j+2} \binom{n+j+2}{n+2} + \binom{n}{j+1} \binom{n+j+2}{n-3} + 3 \binom{n}{j+4} \binom{n+j+4}{n+1} \\ & + 2 \binom{n}{j+2} \binom{n+j+4}{n} \left(2 - \tfrac{n+j+5}{n+1} - \tfrac{n}{j+5}\right) + \tfrac{2n}{j+3} \binom{n}{j+2} \binom{n+j+2}{n} \Big]. \end{aligned}$$

Proof. The nth semi-Baxter number, SB_n, is the coefficient of x^n in $S(1,1)$, which we denote as usual $[x^n]S(1,1)$. Notice that this number is also the coefficient $[a^0 x^n]S(1+a, 1+a)$, and so by Theorem 12 it is the coefficient of $a^0 x^n$ in $F(a, W) = -P(a, W+1+a)$, namely

$$\begin{aligned} SB_n = \quad [a^0 x^{n-1}] \quad & \Big((1+a)^2 + \left(\tfrac{1}{a^5} + \tfrac{1}{a^4} + 2 + 2a\right) W + \left(-\tfrac{1}{a^5} - \tfrac{1}{a^4} + \tfrac{1}{a^3} - \tfrac{1}{a^2} - \tfrac{1}{a} + 1\right) W^2 \\ & + \left(\tfrac{1}{a^4} - \tfrac{1}{a^2}\right) W^3 \Big). \end{aligned}$$

This expression can be evaluated from $[a^s x^k]W^i$, for $i = 1, 2, 3$. Precisely,

$$\begin{aligned} SB_n = \quad & [a^5 x^{n-1}]W + [a^4 x^{n-1}]W + 2[a^0 x^{n-1}]W + 2[a^{-1} x^{n-1}]W - [a^5 x^{n-1}]W^2 - [a^4 x^{n-1}]W^2 \\ & + [a^3 x^{n-1}]W^2 - [a^2 x^{n-1}]W^2 - [a^1 x^{n-1}]W^2 + \left[a^0 x^{n-1}\right] W^2 + [a^4 x^{n-1}]W^3 - [a^2 x^{n-1}]W^3. \end{aligned}$$

The Lagrange inversion and eq. (9) then prove that

$$[a^s x^k]W^i = \frac{i}{k}\sum_{j=0}^{k-i}\binom{k}{j}\binom{k}{j+i}\binom{k+j+i}{j+s}, \text{ for } i=1,2,3.$$

We can then substitute this into the above expression for SB_n and so, for $n \geq 2$, express SB_n as $SB_n = \sum_{j=0}^{n-1} F_{SB}(n,j)$, where

$$\begin{aligned} F_{SB}(n,j) \quad = \quad & \tfrac{1}{n-1}\tbinom{n-1}{j}\Big\{\tbinom{n-1}{j+1}\Big[\tbinom{n+j+1}{j+5}+2\tbinom{n+j+1}{j}\Big]+2\tbinom{n-1}{j+2}\Big[-\tbinom{n+j+2}{j+5}+\tbinom{n+j+1}{j+3} \\ & -\tbinom{n+j+2}{j+2}+\tbinom{n+j+1}{j}\Big] + 3\tbinom{n-1}{j+3}\Big[\tbinom{n+j+2}{j+4}-\tbinom{n+j+2}{j+2}\Big]\Big\}. \end{aligned} \tag{10}$$

Then, manipulating the products in each term by means of binomial coefficient identities we obtain the announced explicit formula for the semi-Baxter coefficients:

$$\begin{aligned} \text{for all } n \geq 1,\ SB_{n+1} = \quad & \tfrac{1}{n}\sum_{j=0}^{n}\tbinom{n}{j}\Big[2\tbinom{n+1}{j+2}\tbinom{n+j+2}{n+2}+\tbinom{n}{j+1}\tbinom{n+j+2}{n-3}+3\tbinom{n}{j+4}\tbinom{n+j+4}{n+1} \\ & +2\tbinom{n}{j+2}\tbinom{n+j+4}{n}\big(2-\tfrac{n+j+5}{n+1}-\tfrac{n}{j+5}\big)+\tfrac{2n}{j+3}\tbinom{n}{j+2}\tbinom{n+j+2}{n}\Big]. \end{aligned}$$ □

Proof of Proposition 13. From Corollary 23, we can write $SB_n = \sum_{j=0}^{n-1} F_{SB}(n,j)$, where the summand $F_{SB}(n,j)$ given by eq. (10) is hypergeometric, and we prove the announced recurrence using *creative telescoping* [22]. The Maple package ZEILBERGER implements this approach: using $F_{SB}(n,j)$ as input, it yields

$$\begin{aligned} (n+5)(n+6)\cdot F_{SB}(n+2,j) - (11n^2+55n+60)\cdot F_{SB}(n+1,j) - n(n-1)\cdot F_{SB}(n,j) \\ = G_{SB}(n,j+1) - G_{SB}(n,j), \end{aligned} \tag{11}$$

where $G_{SB}(n,j)$ is known as the certificate. It has the additional property that $G_{SB}(n,j)/F_{SB}(n,j)$ is a rational function of n and j. The expression $G_{SB}(n,j)$ is quite cumbersome and we do not report it here — it can be readily reconstructed using ZEILBERGER as done in the Maple worksheet associated with our paper.

To complete the proof of the recurrence it is sufficient to sum both sides of eq. (11) over j, j ranging from 0 to $n+1$. Since the coefficients on the left-hand side of eq. (11) are independent of j, summing it over j gives

$$\begin{aligned} & (n+5)(n+6)\cdot SB_{n+2} - (11n^2+55n+60)\cdot SB_{n+1} - n(n-1)\cdot SB_n \\ & \quad - (11n^2+55n+60)\cdot F_{SB}(n+1,n+1) - n(n-1)\cdot(F_{SB}(n,n)+F_{SB}(n,n+1)) \end{aligned} \tag{12}$$

Summing the right-hand side over j gives a telescoping series, and simplifies as $G_{SB}(n,n+2) - G_{SB}(n,0)$. From the explicit expression of $F_{SB}(n,j)$ and $G_{SB}(n,j)$, it is elementary to check that

$$F_{SB}(n+1,n+1) = F_{SB}(n,n) = F_{SB}(n,n+1) = G_{SB}(n,n+2) = G_{SB}(n,0) = 0.$$

Summing eq. (11) therefore gives

$$(n+5)(n+6)\cdot SB_{n+2} - (11n^2+55n+60)\cdot SB_{n+1} - n(n-1)\cdot SB_n = 0.$$

Shifting $n \mapsto n-2$ and rearranging finally gives the recurrence of Proposition 13. □

Proof of Theorem 14. For each of the summation formulas given in Theorem 14, we apply the method of creative telescoping, as in the proof of Proposition 13. In all three cases, this produces a recurrence satisfied by these numbers, and every time we find exactly the recurrence given in Proposition 13. Checking that the initial terms of the sequences coincide completes the proof. □

Proof of Corollary 15. From Theorem 14, and letting

$$A(n;j) \equiv A(j) = \frac{24}{(n-1)n^2(n+1)(n+2)} \binom{n}{j+2} \binom{n+2}{j} \binom{n+j+2}{j+1},$$

we have that for all $n \geq 2$, $SB_n = \sum_{j=0}^{n} A(j)$. From this expression, the proof of Corollary 15 follows the same strategy as in [11, Section 2.6]. We first show that the summands $A(j)$ form a unimodal sequence, and we identify the value j_0 where $A(j)$ is maximal. Second, we find an estimate of $A(j)$ when j is close to j_0 (in an interval of width $\mathcal{O}(n^{1/2+\varepsilon})$). We next split the sum $\sum_{j=0}^{n} A(j)$ into two parts: the terms for j outside of this interval, and those for j inside. The third step is to prove that the first part is negligible with respect to the second part. And the fourth step is to estimate the second part of the sum, using the estimate of $A(j)$ when j is close to j_0.

This is achieved in a series of four lemmas below. Combining Lemmas 27 and 28, it then follows immediately that, for any $\varepsilon \in (0, 1/6)$, we have

$$SB_n = A\mu^n n^{-6} \left(1 + O\left(n^{3\varepsilon - 1/2}\right)\right),$$

where $\varphi = \frac{1}{2}(\sqrt{5} - 1)$, $A = \frac{12}{\pi} 5^{-1/4} \varphi^{-15/2} \approx 94.34$ and $\mu = \varphi^{-5} = (11 + 5\sqrt{5})/2$. And this completes the proof of Corollary 15. □

Remark 24. The computations presented in the lemmas below are actually simpler than those in [11, Section 2.6], because we are interested in the dominant asymptotics only. But following [11] more closely and keeping higher order terms in our expansions, one could establish further subdominant terms in the asymptotic expansion of SB_n.

Lemma 25. *For large enough n, the numbers $A(j)$ form a unimodal sequence. Moreover, the maximum of this sequence occurs at $j = \varphi n + O(1)$, for $\varphi = \frac{1}{2}(\sqrt{5} - 1)$.*

Proof. Let R be the ratio $R \equiv R(j) = \frac{A(j)}{A(j+1)} = \frac{(j+1)(j+2)(j+3)}{(n+2-j)(n-j-2)(n+j+3)}$. Rewrite it as $R = F \cdot G \cdot H$, where $F = \frac{(j+1)}{(n-j-2)}$, $G = \frac{(j+2)}{(n-j+2)}$ and $H = \frac{j+3}{n+j+3}$. Computing the derivatives in j of F, G and H, we check that each of them is an increasing function of j, thus R is increasing with j. Moreover, for n large enough, it holds that $R(0) \leq 1$ and $R(n-3) \geq 1$. Therefore, $A(j)$ is a unimodal (and actually log-concave) sequence and there exists $j_0 \in [0, n-3]$ such that $R(j_0) = 1$.

To find this value j_0 where $A(j)$ reaches its maximum, we simply set $R = 1$ and solve it for j. The equation $R = 1$ is quadratic in j and we choose the only solution lying in the range $[0, n-3]$. Expanding it in n gives $j_0 = \varphi n - \frac{3}{2} + \frac{3}{10}\sqrt{5} + O(n^{-1})$, with $\varphi = \frac{1}{2}(\sqrt{5} - 1)$. □

Lemma 26. *Let $\varepsilon \in (0, 1/6)$ and $j = \varphi n + r$, with $r = s\sqrt{n}$ and $|s| \leq n^{\varepsilon}$. Then:*

$$A(j) = \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3 + 1/2)\cdot s^2} \left(1 + O\left(n^{3\varepsilon - 1/2}\right)\right).$$

Moreover, this estimate is uniform in j.

Proof. We start with an estimate of each of the binomial coefficients occurring in $A(j)$, obtained using expansions of the Gamma function.

First, recall the Stirling expansion $\Gamma(n+1) = n^n \cdot \sqrt{2\pi n} \cdot e^{-n} \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$. Hence, with $j = \varphi n + r$ and $r = s\sqrt{n}$, we also have

$$\begin{aligned} \Gamma(j+3) &= (j+2)^{j+2} \cdot \sqrt{2\pi} \cdot \sqrt{\varphi n} \left(1 + O(n^{\varepsilon - 1/2})\right) \cdot e^{-(j+2)} \cdot \left(1 + O\left(\frac{1}{j+2}\right)\right) \\ &= (j+2)^{j+2} \cdot \sqrt{2\pi\varphi n} \cdot e^{-(j+2)} \cdot \left(1 + O(n^{\varepsilon - 1/2})\right), \end{aligned}$$

and similarly $\Gamma(n-j-1) = (n-j-2)^{n-j-2} \cdot \sqrt{2\pi(1-\varphi)n} \cdot e^{-(n-j-2)} \cdot \left(1 + O(n^{\varepsilon - 1/2})\right)$.

Note that the above expansions for $\Gamma(j+3)$ and $\Gamma(n-j-1)$ are both uniform in j. This will also be the case for the other expansions obtained later in this proof, but we will not remark on it every time.

We further expand

$$\begin{aligned}
\log \tfrac{n^n}{(j+2)^{j+2}(n-j-2)^{n-j-2}} &= n \log \tfrac{n}{n-j-2} - (j+2) \log \tfrac{j+2}{n-j-2} \\
&= n \log \left(\tfrac{1}{1-\varphi} \cdot \tfrac{1}{1-\frac{s}{(1-\varphi)\sqrt{n}} - \frac{2}{(1-\varphi)n}} \right) - (j+2) \log \left(\tfrac{\varphi}{1-\varphi} \cdot \tfrac{1+\frac{s}{\varphi\sqrt{n}}+\frac{2}{\varphi n}}{1-\frac{s}{(1-\varphi)\sqrt{n}} - \frac{2}{(1-\varphi)n}} \right) \\
&= n \left(\log \tfrac{1}{1-\varphi} + \tfrac{s}{(1-\varphi)\sqrt{n}} + \tfrac{s^2}{2(1-\varphi)^2 n} + \tfrac{2}{(1-\varphi)n} + O(n^{3\varepsilon-3/2}) \right) \\
&\quad - (j+2) \left(\log \tfrac{\varphi}{1-\varphi} + \tfrac{s}{\varphi(1-\varphi)\sqrt{n}} + \tfrac{s^2(2\varphi-1)}{2\varphi^2(1-\varphi)^2 n} + \tfrac{2}{(1-\varphi)n} + O(n^{3\varepsilon-3/2}) \right) \\
&= n \log \tfrac{1}{1-\varphi} + \tfrac{s\sqrt{n}}{(1-\varphi)} + \tfrac{s^2}{2(1-\varphi)^2} + \tfrac{2}{(1-\varphi)} + O(n^{3\varepsilon-1/2}) \\
&\quad - (j+2) \log \tfrac{\varphi}{1-\varphi} - \tfrac{s\sqrt{n}}{(1-\varphi)} - \tfrac{s^2}{2\varphi(1-\varphi)^2} - \tfrac{2}{(1-\varphi)} + O(n^{3\varepsilon-1/2}) \\
&= n \log \tfrac{1}{1-\varphi} - (j+2) \log \tfrac{\varphi}{1-\varphi} + \tfrac{s^2}{2(1-\varphi)^2} \left(1 - \tfrac{1}{\varphi} \right) + O(n^{3\varepsilon-1/2}).
\end{aligned}$$

So, exponentiating, we obtain

$$\frac{n^n}{(j+2)^{j+2}(n-j-2)^{n-j-2}} = \left(\frac{1}{1-\varphi}\right)^n \left(\frac{1-\varphi}{\varphi}\right)^{j+2} \cdot e^{-s^2/2\varphi(1-\varphi)} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

This gives our estimate for the binomial coefficient $\binom{n}{j+2} = \frac{\Gamma(n+1)}{\Gamma(j+3)\Gamma(n-j-1)}$ occurring in $A(j)$:

$$\binom{n}{j+2} = \frac{1}{\sqrt{2\pi\varphi(1-\varphi)n}} \cdot \left(\frac{1}{1-\varphi}\right)^n \left(\frac{1-\varphi}{\varphi}\right)^{j+2} \cdot e^{-s^2/2\varphi(1-\varphi)} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

We need to compute similar estimates for the other two binomial coefficients occurring in $A(j)$, namely $\binom{n+2}{j}$ and $\binom{n+j+2}{j+1}$. Skipping the details, we obtain

$$\begin{aligned}
\binom{n+2}{j} &= \frac{1}{\sqrt{2\pi\varphi(1-\varphi)n}} \cdot \left(\frac{1}{1-\varphi}\right)^{n+2} \left(\frac{1-\varphi}{\varphi}\right)^{j} \cdot e^{-s^2/2\varphi(1-\varphi)} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right) \\
\text{and } \binom{n+j+2}{j+1} &= \sqrt{\frac{1+\varphi}{2\pi\varphi n}} \cdot (1+\varphi)^{n+j+2} \varphi^{-(j+1)} \cdot e^{-s^2/2} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).
\end{aligned}$$

Consequently, since $\varphi^2 = 1-\varphi$ and $1+\varphi = 1/\varphi$, we obtain

$$\binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1} = \frac{1}{\sqrt{8\pi^3}} \cdot \varphi^{-9} \cdot \varphi^{-5n} \cdot n^{-3/2} \cdot e^{-(1/\varphi^3+1/2)\cdot s^2} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

This finally gives us our claimed estimate for $A(j) = \frac{24}{(n-1)n^2(n+1)(n+2)}\binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1}$:

$$A(j) = \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3+1/2)\cdot s^2} \cdot \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right). \qquad \square$$

Lemma 27. *Let $\varepsilon \in (0, 1/6)$. Then for all $m \geq 0$,*

$$\sum_{|j-\varphi n|>n^{1/2+\varepsilon}} A(j) = o(\varphi^{-5n} n^{-m}).$$

Proof. Let $j_+ = \lfloor \varphi n + n^{1/2+\varepsilon} \rfloor$ and $j_- = \lceil \varphi n - n^{1/2+\varepsilon} \rceil$. By Lemma 25 (unimodality), we have

$$\sum_{|j-\varphi n|>n^{1/2+\varepsilon}} A(j) \leq (\varphi n - n^{1/2+\varepsilon}) \cdot A(j_-) + (n - \varphi n - n^{1/2+\varepsilon}) \cdot A(j_+).$$

Moreover, by Lemma 26, it holds that

$$A(j_\pm) = \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3+1/2)\cdot n^{2\varepsilon}} \cdot \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

It follows that for every $m \geq 0$,

$$(\varphi n - n^{1/2+\varepsilon})A(j_-) = o(\varphi^{-5n}n^{-m}) \text{ and } (n - \varphi n - n^{1/2+\varepsilon})A(j_+) = o(\varphi^{-5n}n^{-m}). \qquad \square$$

Lemma 28. *Let* $\varepsilon \in (0, 1/6)$*. Then,*

$$\sum_{|j-\varphi n|\leq n^{1/2+\varepsilon}} A(j) = A\mu^n n^{-6}\left(1 + O\left(n^{3\varepsilon-1/2}\right)\right),$$

where $A = \frac{12}{\pi}\, 5^{-1/4}\varphi^{-15/2} \approx 94.34$ *and* $\mu = \varphi^{-5} = (11 + 5\sqrt{5})/2$.

Proof. The estimate of Lemma 26 being uniform in j, we can write

$$\begin{aligned}\sum_{|j-\varphi n|\leq n^{1/2+\varepsilon}} A(j) &= \sum_{|j-\varphi n|\leq n^{1/2+\varepsilon}} \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3+1/2)\cdot(j-\varphi n)^2/n} \cdot \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right) \\ &= \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right) \sum_{|j-\varphi n|\leq n^{1/2+\varepsilon}} e^{-(1/\varphi^3+1/2)\cdot(j-\varphi n)^2/n}.\end{aligned}$$

Using the Euler-Maclaurin formula, we rewrite this sum as an integral as follows:

$$\begin{aligned}\sum_{|j-\varphi n|\leq n^{1/2+\varepsilon}} e^{-(1/\varphi^3+1/2)\cdot(j-\varphi n)^2/n} &= \sqrt{n} \cdot \int_{-\infty}^{\infty} e^{-(1/\varphi^3+1/2)s^2} \mathrm{d}s + o(n^{-m}) \\ &= \sqrt{n} \cdot \sqrt{\frac{2\pi\varphi^3}{\sqrt{5}}} + o(n^{-m}).\end{aligned}$$

In this formula, m is any positive integer (since the error term in the Euler-Maclaurin formula is smaller than any polynomial in n). Note also that the leading $\sqrt{n}$ comes from changing integration variables from j to s (with $j = \varphi n + s\sqrt{n}$). The estimate given in Lemma 28 finally follows by elementary computations. $\square$

4 Baxter numbers

This section starts with an overview of some known results about Baxter numbers. We believe it helps understanding the relations, similarities and differences between this well-known sequence and the two main sequences studied in our work (semi-Baxter numbers in Section 3 and strong-Baxter numbers in Section 5). Next, studying two families of restricted semi-Baxter permutations enumerated by Baxter numbers, we show that Ω_{semi} generalizes two known succession rules for Baxter numbers. Finally, we introduce *Baxter paths*, a family of restricted semi-Baxter paths enumerated by Baxter numbers as well. To our knowledge, this is the first definition of a combinatorial family enumerated by Baxter numbers where the objects are single paths (although triples of non-intersecting lattice paths are known to be enumerated by Baxter numbers, see [7] for instance).

4.1 Baxter numbers and restricted permutations

Baxter permutations (see [17] among others) are usually defined as permutations avoiding the two vincular patterns $2\,\underline{41}\,3$ and $3\,\underline{14}\,2$. Denoting B_n the number of Baxter permutations of size n, the sequence (B_n) is known as the sequence of *Baxter numbers*. It is identified as sequence A001181 in [20] and its first terms are $1, 2, 6, 22, 92, 422, 2074, 10754, \ldots$. Since [14], an explicit formula for B_n has been known:

$$\text{for all } n \geq 1,\ B_n = \frac{2}{n(n+1)^2} \sum_{j=1}^{n} \binom{n+1}{j-1} \binom{n+1}{j} \binom{n+1}{j+1}.$$

In [8], M. Bousquet-Mélou investigates further properties of Baxter numbers. The above formula can also be found in [8, Theorem 1]. Moreover, using the succession rule reviewed in Proposition 29 below, [8] characterizes the generating function of Baxter numbers as the solution of a bivariate functional equation. It is then solved with the obstinate kernel method, implying that the generating function for Baxter numbers is D-finite [8, Theorem 4]. Although technical details differ, it is the same approach than the one we used in Section 3. In the light of our recurrence for semi-Baxter numbers (see Proposition 13), it is also interesting to note that Baxter numbers satisfy a similar recurrence, attributed to R. Stanley and reported by M. Somos in [20], namely

$$B_0 = 0, \quad B_1 = 1, \quad \text{and for } n \geq 2, B_n = \frac{7n^2 + 7n - 2}{(n+3)(n+2)} B_{n-1} + \frac{8(n-2)(n-1)}{(n+3)(n+2)} B_{n-2}.$$

In addition to Baxter permutations, several combinatorial families are enumerated by Baxter numbers. See for instance [15] which collects some of them and provides links between them. We will be specifically interested in a second family of restricted permutations which is also enumerated by Baxter numbers, namely the *twisted Baxter permutations*, defined by the avoidance of $2\,\underline{41}\,3$ and $3\,\underline{41}\,2$ [24, 26].

4.2 Succession rules for Baxter and twisted Baxter permutations

It is clear from their definition in terms of pattern-avoidance that the families of Baxter and twisted Baxter permutations are subsets of the family of semi-Baxter permutations. Therefore, the growth of semi-Baxter permutations provided in Subsection 3.2 can be restricted to each of these families, producing a succession rule for Baxter numbers. In the following, we present these two restrictions, which happen to be (variants of) well-known succession rules for Baxter numbers. This reinforces our conviction that the generalization of Baxter numbers to semi-Baxter numbers is natural.

Let us first consider Baxter permutations. To that effect, recall that a LTR (left-to-right) maximum of a permutation π is an element π_i such that $\pi_i > \pi_j$ for all $j < i$. Similarly, a RTL maximum (resp. RTL minimum) of π is an element π_i such that $\pi_i > \pi_j$ (resp. $\pi_i < \pi_j$) for all $j > i$. Following [8, Section 2.1] we can make Baxter permutations grow by adding new maximal elements to them, which may be inserted either immediately before a LTR maximum or immediately after a RTL maximum. Giving to any Baxter permutation the label (h, k) where h (resp. k) is the number of its RTL (resp. LTR) maxima, this gives the most classical succession rule for Baxter numbers.

Proposition 29 ([8], Lemma 2)**.** *Baxter permutations grow according to*

$$\Omega_{Bax} = \left\{ \begin{array}{l} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\ldots,(h,k+1) \\ \qquad\qquad\; (h+1,1),\ldots,(h+1,k). \end{array} \right.$$

But note that Baxter permutations are invariant under the 8 symmetries of the square. Consequently, up to a 90° rotation, inserting a new maximum element in a Baxter permutation can be easily regarded as inserting a new element on the right of a Baxter permutation (as we did for semi-Baxter permutations). Those are then inserted immediately below a RTL minimum or immediately above a RTL maximum. Note that, in a semi-Baxter permutation, these are always active sites, so the generating tree associated with Ω_{Bax} is a subtree of the generating tree associated with Ω_{semi}. Through the rotation, the interpretation of the label (h, k) of a Baxter permutation is modified as follows: h (resp. k) is the number of its RTL minima (resp. RTL maxima), that is to say of active sites below (resp. above) the last element of the permutation. As expected, this coincides with the interpretation of labels in the growth of semi-Baxter permutations according to Ω_{semi}.

Turning to twisted Baxter permutations, specializing the growth of semi-Baxter permutations, we obtain the following.

Proposition 30. *Twisted Baxter permutations grow according to*

$$\Omega_{TBax} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k), \ldots, (h-1,k), (h,k+1) \\ \qquad\qquad (h+k,1), \ldots, (h+1,k). \end{cases}$$

Proof. As in the proof of Proposition 7, we let twisted Baxter permutations grow by performing local expansions on the right, as illustrated in Figure 8. (This is possible since removing the last element in a twisted Baxter permutation produces a twisted Baxter permutation.)

Let π be a twisted Baxter permutation of size n. By definition an active site s of π is an element a such that $\pi \cdot a$ avoids the two forbidden patterns. Then, we assign to π a label (h,k), where h (resp. k) is the number of active sites smaller than or equal to (resp. greater than) π_n. As in the proof of Proposition 7, the permutation 1 has label $(1,1)$ and now we describe the labels of the permutations $\pi \cdot a$ when a runs over all the active sites of π.

If $a < \pi_n$, then $\pi \cdot a$ ends with a non-empty descent and, as in the proof of Proposition 7, all sites of π in the range $(a+1, \pi_n + 1]$ become non-active in $\pi \cdot a$ (due to the avoidance of $2\,\underline{41}\,3$). Moreover, due to the avoidance of $3\,\underline{41}\,2$, the site immediately above a in $\pi \cdot a$ also becomes non-active. All other active sites of π remain active in $\pi \cdot a$, hence giving the labels (i,k), for $1 \le i < h$, in the productions of Ω_{TBax} ((i,k) corresponds to the case where a is the ith active site from the bottom).

If $a = \pi_n$, no sites of π become non-active, giving the label $(h, k+1)$.

If $a > \pi_n$, then $\pi \cdot a$ ends with an ascent and no site of π become non-active. Hence, we obtain the missing labels in the production of Ω_{TBax}: $(h+k+1-i, i)$, for $1 \le i \le k$ (label $(h+k+1-i, i)$ corresponds to a being the ith active site from the top). $\square$

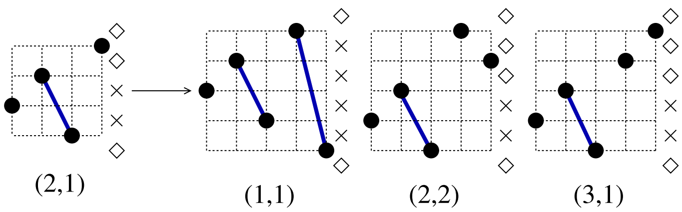

Figure 8: The growth of twisted Baxter permutations (with notation as in Figure 5).

We remark that although Ω_{TBax} is not precisely the succession rule presented in [13] for twisted Baxter permutations, it is an obvious variant of it: indeed, starting from the rule of [13], it is enough to replace every label (q,r) by $(r+1, q-1)$ to recover Ω_{TBax}.

It follows immediately from the proof of Proposition 30 that Ω_{TBax} is a specialization of Ω_{semi}. With Ω_{Bax}, we therefore obtain two such specializations. In addition, we can observe that the productions of Ω_{TBax} on second line are the same as in Ω_{semi}, whereas the productions on the first line of Ω_{Bax} are the same as Ω_{semi}. This means that the restrictions imposed by these two specializations are "independent". We will combine them in Section 5, obtaining a succession rule which consists of the first line of Ω_{TBax} and the second line of Ω_{Bax}.

4.3 Baxter paths

Recall that semi-Baxter paths defined in Subsection 3.5 grow according to Ω_{semi}. Since Ω_{Bax} is a restriction of Ω_{semi}, this allows to define a family of labeled Dyck paths (our *Baxter paths*) which grow according to Ω_{Bax}, and therefore are enumerated by the Baxter numbers.

Note that this is in theory also possible considering the restriction Ω_{TBax} of Ω_{semi}; however, we have not been able to translate this restriction into a nice characterization on the allowed labels on the free up steps, unlike in the case of Ω_{Bax} which we now present.

Definition 31. *A* Baxter path *is a factorial path where the labels of the free up steps satisfy the following additional constraint: for every pair of free up steps (U', U'') with U' occurring before U'' and no free up step between them, the label of U'' is in the range $[1, h]$, where $h \geq 1$ is the sum of the label of U' with the number of DU factors between U' and U''.*

Comparing with Definition 21, it is obvious that the set of Baxter paths is included in that of semi-Baxter paths. Examples of a Baxter path and of a non-Baxter path are shown in Figure 9.

Figure 9: A Baxter path (left) and a semi-Baxter path which is not a Baxter path (right).

Proposition 32. *Baxter paths grow according to the rule Ω_{Bax}.*

Proof. As usual, we make Baxter paths grow by insertion of a peak in the last descent, as shown in Figure 10. To any Baxter path b, denoting e the label of its rightmost free up step $\overline{U}$, we assign the label (h, k), where h is equal to e plus the number of DU factors occurring after $\overline{U}$ and k is the number of steps of the last descent of b. As shown below, with this labeling, the growth of Baxter paths is encoded by Ω_{Bax}.

The unique path of size 1 (which is UD with the U labeled with 1) has label $(1, 1)$. Then, from any Baxter path b with label (h, k) we perform insertions. These are of one of the two following types.

a) We add a peak at the beginning of the last descent of b. The U step added is free, and therefore receives a label, which is by definition in the range $[1, h]$. Meanwhile, the number of steps in the last descent increases by 1, giving all labels in the first line of the production of Ω_{Bax}.

b) We add a peak immediately after any down step of the last descent of b. The added U step following a down step, it carries no label. Therefore, if $b = w \cdot UD^k$ (with this U possibly labeled), the children of b are $w \cdot UD^j \boldsymbol{UD} D^{k-j}$ for $1 \leq j \leq k$, so they have labels $(h + 1, k - j + 1)$, giving the second line of the production of Ω_{Bax}. $\square$

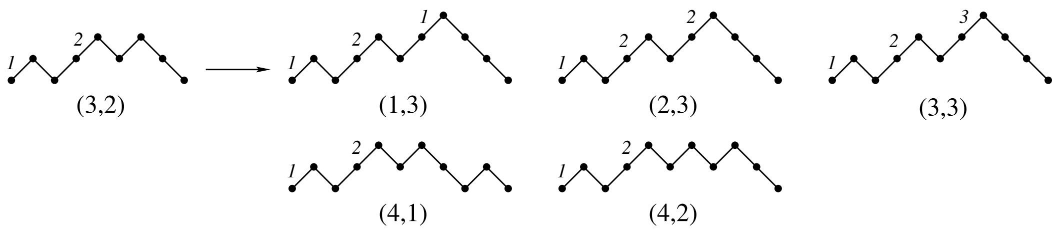

Figure 10: The growth of a Baxter path of label $(3, 2)$.

5 Strong-Baxter numbers

While Section 3 was studying a sequence larger than the Baxter numbers (with a family of permutations containing both the Baxter and twisted Baxter permutations), we now turn to a sequence smaller than the Baxter numbers (associated with a family of permutations included in both families of Baxter and twisted Baxter permutations). We present a succession rule for this sequence, properties of its generating function, and a family of restricted factorial paths that it enumerates.

5.1 Strong-Baxter numbers, strong-Baxter permutations, and their succession rule

Definition 33. *A* strong-Baxter permutation *is a permutation that avoids all three vincular patterns* $2\,\underline{41}\,3$*,* $3\,\underline{14}\,2$ *and* $3\,\underline{41}\,2$*.*

Definition 34. *The sequence of* strong-Baxter numbers *is the sequence that enumerates strong-Baxter permutations.*

We have added the sequence enumerating strong-Baxter permutations to the OEIS, where it is now registered as [20, A281784]. It starts with:

$$1, 2, 6, 21, 82, 346, 1547, \ldots$$

The pattern-avoidance definition makes it clear that the family of strong-Baxter permutations is the intersection of the two families of Baxter and twisted Baxter permutations. In that sense, such permutations "satisfy two Baxter conditions", hence the name strong-Baxter.

A succession rule for strong-Baxter numbers is given by the following proposition.

Proposition 35. *Strong-Baxter permutations grow according to the following succession rule:*

$$\Omega_{strong} = \left\{ \begin{array}{l} (1,1) \\ (h,k) \rightsquigarrow (1,k), \ldots, (h-1,k), (h,k+1) \\ \qquad\qquad (h+1,1), \ldots, (h+1,k). \end{array} \right.$$

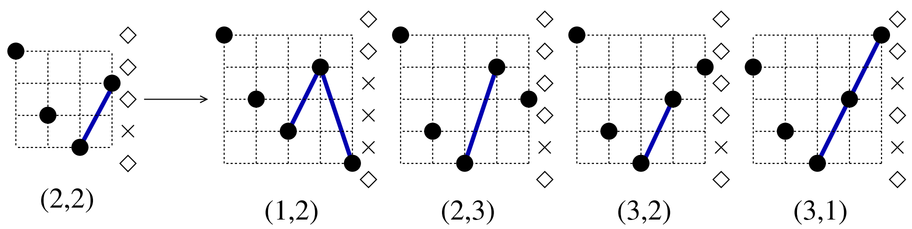

Figure 11: The growth of strong-Baxter permutations (with notation as in Figure 5).

Proof. As in the proof of Propositions 7 and 30, we build a generating tree for strong-Baxter permutations performing local expansions on the right, as illustrated in Figure 11. Note that this is possible since removing the last point from any strong-Baxter permutation gives a strong-Baxter permutation.

Let π be a strong-Baxter permutation of size n. By definition, the active sites of π are the a's such that $\pi \cdot a$ is a strong-Baxter permutations. The label given to π is (h,k), where h (resp. k) is the number of active sites that are smaller than or equal to (resp. greater than) π_n. As in the proof of Proposition 7, the permutation 1 has label $(1,1)$, and we now need to describe, for π of label (h,k), the labels of the permutations $\pi \cdot a$ when a runs over all active sites of π. So, let a be such an active site.

If $a < \pi_n$, then $\pi \cdot a$ ends with a non-empty descent. As in the proof of Proposition 7, all sites of $\pi \cdot a$ in $(a+1, \pi_n+1]$ become non-active (due to the avoidance of $2\underline{41}3$). Moreover, due to the avoidance of $3\underline{41}2$, the site immediately above a in $\pi \cdot a$ also become non-active. All other active sites of π remain active in $\pi \cdot a$, hence giving the labels (i,k) for $1 \le i < h$ in the production of Ω_{strong} (again, i is such that a is the i-th active site from the bottom).

If $a = \pi_n$, no site of π becomes non-active, giving the label $(h, k+1)$ in the production of Ω_{strong}.

Finally, if $a > \pi_n$, then $\pi \cdot a$ ends with an ascent. Because of the avoidance of $3\underline{14}2$, we need to consider the occurrences of $2\underline{13}$ in π to identify which active sites of π become non-active in $\pi \cdot a$. It follows from a discussion similar to that in the proof of Proposition 7 that all sites of $\pi \cdot a$ in $[\pi_n+1, a)$ become non-active. Hence, we obtain the missing labels in the production of Ω_{strong}: $(h+1, i)$ for $1 \le i \le k$ (where i indicates that a is the i-th active site from the top). □

In the same sense that both Ω_{Bax} and Ω_{TBax} specialize Ω_{semi}, it is easy to see that the succession rule Ω_{strong} is a specialization of the rule Ω_{Bax} (for Baxter permutations) as well as of the rule Ω_{TBax} (for twisted Baxter permutations). In this case, the rule Ω_{strong} associated with the intersection of these two families is simply obtained by taking, for each object produced, the minimum label among the two labels given by Ω_{Bax} and Ω_{TBax}. This appears clearly in the following representation:

$$\begin{array}{lllcccccccc}
\Omega_{semi}: & (h,k) & \to & (1,k+1) & \dots & (h-1,k+1) & (h,k+1) & (h+k,1) & \dots & (h+1,k) \\
\Omega_{Bax}: & (h,k) & \to & (1,k+1) & \dots & (h-1,k+1) & (h,k+1) & (h+1,1) & \dots & (h+1,k) \\
\Omega_{TBax}: & (h,k) & \to & (1,k) & \dots & (h-1,k) & (h,k+1) & (h+k,1) & \dots & (h+1,k) \\
\Omega_{strong}: & (h,k) & \to & (1,k) & \dots & (h-1,k) & (h,k+1) & (h+1,1) & \dots & (h+1,k).
\end{array}$$

This is easily explained. Note first that in all four cases h (resp. k) records the number of active sites below (resp. above) the rightmost element of a permutation. Then, it is enough to remark that among the active sites of a semi-Baxter permutation (avoiding $2\underline{41}3$), the avoidance of $3\underline{41}2$ deactivates only sites above the rightmost element of the permutation, while the avoidance of $3\underline{14}2$ deactivates only sites below it.

5.2 Generating function of strong-Baxter numbers

Let $I_{h,k}(x) \equiv I_{h,k}$ denote the generating function for strong-Baxter permutations having label (h,k), with $h,k \ge 1$, and let $I(x;y,z) \equiv I(y,z) = \sum_{h,k\ge1} I_{h,k} y^h z^k$. (The notation I stands for Intersection, of the families of Baxter and twisted Baxter permutations.)

Proposition 36. *The generating function $I(y,z)$ satisfies the following functional equation:*

$$I(y,z) = xyz + \frac{x}{1-y}(y\,I(1,z) - I(y,z)) + xz\,I(y,z) + \frac{xyz}{1-z}(I(y,1) - I(y,z)). \tag{13}$$

Proof. From the growth of strong-Baxter permutations according to Ω_{strong} we write:

$$\begin{aligned}
I(y,z) &= xyz + x\sum_{h,k\ge1} I_{h,k}\left((y+y^2+\dots+y^{h-1})z^k + y^h z^{k+1} + y^{h+1}(z+z^2+\dots+z^k)\right) \\
&= xyz + x\sum_{h,k\ge1} I_{h,k}\left(\frac{1-y^{h-1}}{1-y} y\, z^k + y^h z^{k+1} + \frac{1-z^k}{1-z} y^{h+1}\, z\right) \\
&= xyz + \frac{x}{1-y}\left(y\,I(1,z) - I(y,z)\right) + xz\,I(y,z) + \frac{xyz}{1-z}\left(I(y,1) - I(y,z)\right).
\end{aligned}$$ □

In order to study the nature of the generating function $I(1,1)$ for strong-Baxter numbers, we look at the kernel of eq. (13), which is

$$K(y,z) = 1 + x\left(\frac{1}{1-y} - z + \frac{yz}{1-z}\right). \tag{14}$$

We perform the substitutions $y = 1 + a$ and $z = 1 + b$ so that eq. (14) is rewritten as

$$K(1+a, 1+b) = 1 - xQ(a, b) \quad \text{where} \quad Q(a, b) = \frac{1}{a} + \frac{1}{b} + \frac{a}{b} + a + 2 + b. \tag{15}$$

The kernel $K(1 + a, 1 + b)$ is not symmetric in a and b. As in the proof of Theorem 12 (see Subsection 3.6), we look for the birational transformations Φ and Ψ in a and b that leave the kernel unchanged, which are:

$$\Phi : (a, b) \to \left(a, \frac{1+a}{b}\right), \quad \text{and} \quad \Psi : (a, b) \to \left(-\frac{b}{a(1+b)}, b\right).$$

One observes, using Maple for example, that the group generated by these two transformations is not of small order. We actually suspect that it is of infinite order, preventing us from using the obstinate kernel method to solve eq. (13).

Nevertheless, after the substitution $y = 1 + a$ and $z = 1 + b$, the kernel we obtain in eq. (15) resembles kernels of functional equations associated with the enumeration of families of walks in the (positive) quarter plane [9].

Proposition 37. *Let $W(t; a, b)$ be the generating function for walks confined in the quarter plane and using $\{(-1, 0), (0, -1), (1, -1), (1, 0), (0, 1)\}$ as step set, where t counts the number of steps and a (resp. b) records the x-coordinate (resp. y-coordinate) of the ending point. The function $W(t; a, b)$ satisfies the following functional equation:*

$$W(t; a, b) = 1 + t\left(\frac{1}{a} + \frac{1}{b} + \frac{a}{b} + a + b\right) W(t; a, b) - \frac{t}{a} W(t; 0, b) - t\frac{(1+a)}{b} W(t; a, 0). \tag{16}$$

Not only can we take inspiration from the literature on walks in the quarter plane for our problem of solving eq. (13). Actually, modifying the step set, we can even arrange that $K(1 + a, 1 + b)$ is exactly the kernel arising in the functional equation for enumerating a family of walks.

Lemma 38. *Let $W_2(t; a, b)$ be the generating function for walks confined in the quarter plane and using $\{(-1, 0), (0, -1), (1, -1), (1, 0), (0, 1), (0, 0), (0, 0)\}$ as step (multi-)set, where t counts the number of steps and a (resp. b) records the x-coordinate (resp. y-coordinate) of the ending point. The difference with the step set of Proposition 37 is that we have added two copies of the trivial step $(0, 0)$, which are distinguished (they can be considered as counterclockwise and clockwise loops for instance).*

The generating functions $W(t; a, b)$ and $W_2(t; a, b)$ are related by

$$W_2(x; a, b) = W\left(\frac{x}{1-2x}; a, b\right) \frac{1}{1-2x} \tag{17}$$

Moreover, writing $J(x; a, b) := I(x; 1 + a, 1 + b)$ the generating function for strong-Baxter permutations, it holds that

$$J(x; a, b) = (1 + a)(1 + b)\, x\, W_2(x; a, b). \tag{18}$$

Proof. First, walks counted by W_2 can be described from walks counted by W as follows: a W_2-walk is a (possibly empty) sequence of trivial steps, followed by a W-walk where, after each step, we insert a (possibly empty) sequence of trivial steps. This simple combinatorial argument shows that $W_2(x; a, b) = W(\frac{x}{1-2x}; a, b)\frac{1}{1-2x}$.

Next, consider the kernel form of eq. (13) after substituting $y = 1 + a$ and $z = 1 + b$, which is

$$(1 - xQ(a, b))J(x; a, b) = x(1 + a)(1 + b) - x\,\frac{1+a}{a}\,J(x; 0, b) - x\,\frac{(1+a)(1+b)}{b}\,J(x; a, 0). \tag{19}$$

Compare it to the kernel form of eq. (16):

$$(1 - t(Q(a, b) - 2))W(t; a, b) = 1 - \frac{t}{a}\,W(t; 0, b) - t\,\frac{(1+a)}{b}\,W(t; a, 0). \tag{20}$$

Substituting t with $\frac{x}{1-2x}$ in eq. (20), and multiplying this equation by $(1 + a)(1 + b)x$, we see that $(1 + a)(1 + b)\, x\, W_2(x; a, b)$ satisfies eq. (19), proving our claim. □

With results of [6], this easily gives the following theorem.

Theorem 39. *The generating function $I(1,1)$ of strong-Baxter numbers is not D-finite. The same holds for the refined generating function $I(a+1,b+1)$.*

Proof. With the notation of Lemma 38, our goal is to prove that $J(x;a,b)$ and $J(x;0,0)$ are not D-finite. Recall from eq. (18) that $J(x;a,b) = (1+a)(1+b)\,x\,W_2(x;a,b)$, so $J(x;0,0)$ and $W_2(x;0,0)$ (resp. $J(x;a,b)$ and $W_2(x;a,b)$) coincide up to a factor x (resp. $(1+a)(1+b)\,x$). Therefore, proving that $W_2(x;0,0)$ and $W_2(x;a,b)$ are non D-finite is enough.

It is proved in [6] that neither $W(t;a,b)$ nor $W(t;0,0)$ are D-finite. Consequently, since $\frac{1}{1-2x}$ and $\frac{x}{1-2x}$ are algebraic series, it follows from eq. (17) both $W_2(x;a,b)$ and $W_2(x;0,0)$ are not D-finite. □

Moreover, some information on the asymptotic behavior of the number of strong-Baxter permutations can be derived starting from the connection to walks confined in the quarter plane. In [6] the following proposition is presented.

Proposition 40 (Denisov and Wachtel)**.** *Let $\mathfrak{S} \subseteq \{0,\pm1\}^2$ be a step set which is not confined to a half-plane. Let e_n denote the number of $\mathfrak{S}$-excursions of length n confined to the quarter plane $\mathbb{N}^2$ and using only steps in $\mathfrak{S}$. Then, there exist constants K, ρ, and α which depend only on $\mathfrak{S}$, such that:*

- *if the walk is aperiodic, $e_n \sim K\,\rho^n\,n^\alpha$,*
- *if the walk is periodic (then of period 2), $e_{2n} \sim K\,\rho^{2n}\,(2n)^\alpha$, $\;e_{2n+1} = 0$.*

From [6], the growth constant ρ_W associated with $W(t;0,0)$ is an algebraic number whose approximate value is 4.729031538. We show below that the growth constant of strong-Baxter numbers is closely related to ρ_W.

Corollary 41. *The growth constant for the strong-Baxter numbers is $\rho_W + 2 \approx 6.729031538$.*

Proof. From Lemma 38, $I(x;1,1) = x\,W_2(x;0,0) = x\,W(\frac{x}{1-2x};0,0)\,\frac{1}{1-2x}$. And from the discussion above, $\frac{1}{\rho_W}$ is the radius of convergence of $W(t;0,0)$. The radius of convergence of $g(x) = \frac{x}{1-2x}$ is $\frac{1}{2}$, and $\lim_{\substack{x\to 1/2\\ x<1/2}} g(x) = +\infty > \frac{1}{\rho_W}$. So, the composition $W(g(x);0,0)$ is supercritical (see [16, p. 411]), and the radius of convergence of $W(\frac{x}{1-2x};0,0)$ is $g^{-1}\left(\frac{1}{\rho_W}\right) = \frac{1}{\rho_W+2}$. Since $\frac{1}{\rho_W+2}$ is smaller than the radius of convergence $\frac{1}{2}$ of $\frac{1}{1-2x}$, $\frac{1}{\rho_W+2}$ is also the radius of convergence of $x\,W(\frac{x}{1-2x};0,0)\,\frac{1}{1-2x} = I(x;1,1)$, proving our claim. □

5.3 Strong-Baxter paths

Like we did for the semi-Baxter and Baxter sequences, we can provide a restriction on factorial paths which yields the strong-Baxter sequence.

Definition 42. *A* strong-Baxter path *is a factorial path where labels of free up steps satisfy the following additional constraint: for every pair of free up steps (U', U'') with U' occurring before U'' and no free up step between them, the label of U'' is in the range $[1,k]$, where $k \geq 1$ is the sum of the label of U' with the number of UDU factors between U' (included) and U''.*

It follows immediately from their definition that the family of Baxter paths (and, hence that of semi-Baxter paths) contains strong-Baxter paths as a subfamily (see also examples in Figure 12).

As for the previously considered families of restricted factorial paths, our goal is to prove the following.

Proposition 43. *Strong-Baxter paths grow according to the rule Ω_{strong}.*

Figure 12: A strong-Baxter path (left) and a Baxter path which is not a strong-Baxter path (right).

As before, we will make paths grow by insertion of a peak in the last descent. There is however a subtlety in the way this growth is encoded in the labels (h, k). First, remark that Ω_{Bax} is completely symmetric in h and k. Therefore, interchanging our interpretation of the labels (that it to say, taking h to be the number of steps of the last descent and k to be the label of the rightmost free up step plus the number of DU factors after it), the same growth for Baxter paths would also be encoded by Ω_{Bax}. This "interchanged" interpretation of labels is the appropriate one to prove Proposition 43.

Proof of Proposition 43. As announced, we make strong-Baxter paths grow by insertion of a peak in the last descent, as shown in Figure 13. To any strong-Baxter path s, denoting e the label of its rightmost free up step $\overline{U}$, we assign the label (h, k), where h is the number of steps of the last descent of s and k is equal to e plus the number of UDU factors occurring after $\overline{U}$ (included).

The unique strong-Baxter path of size 1, UD where U is labeled with 1, has label $(1, 1)$.

Let s be a strong-Baxter path of label (h, k). The insertions of a peak in the last descent of s produce the following strong-Baxter paths, whose labels correspond to the production of Ω_{strong}.

a) We add a peak at the beginning of the last descent of s. The added U step is free, and receives a label which is any label in the range $[1, k]$. Denoting i be the label assigned to U, the produced path has label $(h + 1, i)$.

b) We add a peak immediately after any down step of the last descent of s. The added U step following a down step, it carries no label. Therefore, if $s = w \cdot UD^h$ (with this U possibly labeled), the children of s are $w \cdot UD^j \boldsymbol{UD}D^{h-j}$ for $1 \leq j \leq h$. When $j = 1$, one UDU factor is created after the rightmost free up step of s, and the obtained path has label $(h, k + 1)$. Otherwise, no such factor is created, and the obtained paths have labels $(h - j + 1, k)$ for $1 < j \leq h$. □

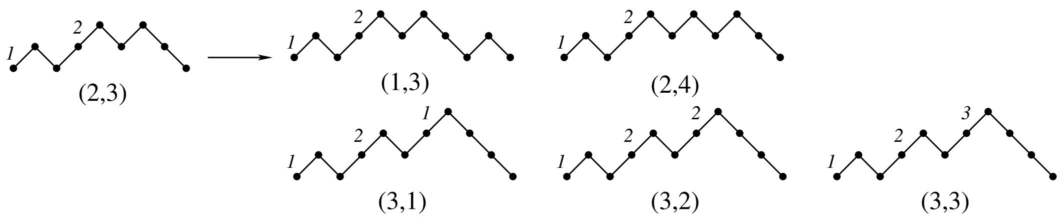

Figure 13: The growth of a strong-Baxter path of label $(2, 3)$.

Acknowledgements

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We also thank Christian Krattenthaler for independently suggesting this method, and pointing to the reference [11].

Finally, we thank Andrew Baxter for clarifying the status of the conjecture about $\underline{14}23$-avoiding permutations, which brought reference [19] to our attention.

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