

# Determinants of generalized binary band matrices\*

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## Abstract

Under binary matrices we mean matrices whose entries take one of two values. In this paper, explicit formulae for calculating the determinant of some type of binary Toeplitz matrices are obtained. Examples of the application of the determinant of binary Toeplitz matrices for the enumeration of even and odd permutations of different types are given.

## Introduction

Under binary matrices we will mean matrices whose elements can take only two values. Matrices of this kind arise in different mathematical questions. For example, this type includes such popular objects in mathematics and its applications as matrices over the field  $GF(2)$  [1]. The Hadamard's problem of finding the maximal determinant of  $(-1, 1)$ -matrices, i.e. matrices consisting from 1 and  $-1$ , is well known [2]. Further,  $(0, 1)$ -matrices are one of the favorite objects of the enumerative combinatorics [3–6]. Also one can use binary  $(1, x)$ -matrices for enumerative problems [6]. These examples can be extended.

One of the basic notion of the matrix theory is the notion of determinant. There exist effective algorithms for the determinant calculation, for example, the modified method of Gaussian elimination, which run in polynomial time. Nevertheless, in cases of some kinds of matrices it is possible to obtain good explicit formulae for the determinant expression. On the one hand, these formulae allow to draw certain conclusions about properties of matrices. On the other hand, they give even greater gain in the speed of the determinant calculation. Such formulae are known only for very limited class of matrices.

Our paper is motivated to some extent by the recent work [7] in which the explicit formula for the determinant of some binary circulant matrices has been obtained. In the present paper we get explicit formulae for determinants of some kinds of binary Toeplitz matrices. Considered matrices are close in their structure to band matrices, therefore we call them generalized band matrices. For our purposes we use quite elementary methods. Applying the Laplace expansion we obtain recurrent formulae leading to required result.

The paper is organized as follows. In the second section we give and prove the formulae, which allow to efficiently calculate determinants of generalized binary band matrices in an explicit form. In the third section we give a few examples of the application of the determinant of such binary matrices for the enumeration of even and odd permutations of different types.

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*Proof.* Subtracting the penultimate row from the last row and expanding the determinant along the last row, we get that  $f_n = (b - a)f_{n-1}$ . Using the given formula consecutively to  $f_{n-1}$ ,  $f_{n-2}$  and so on, we get (4).  $\square$

Suppose that  $k = n$ , i.e. consider the matrix of order  $n$  of the form:

$$\begin{pmatrix} b & b & b & \dots & b \\ & b & b & \dots & b \\ & & b & \dots & b \\ & & & \ddots & \vdots \\ a & & & & b \end{pmatrix}. \quad (5)$$

By  $g_n$  denote the determinant of such matrix.

**Lemma 1.2.** *One can calculate the determinant of the matrix (5) by the formula:*

$$g_n = (b - a)^{n-1}b. \quad (6)$$

*Proof.* Similarly to the proof of the previous lemma.  $\square$

Consider the matrix (1) of order  $n$  with  $k < n$ . Let  $d_n$  denote the determinant of such matrix.

**Lemma 1.3.** *The following equality holds:*

$$d_n = \begin{cases} (b - a)^k d_{n-k} + (b - a)^{n-1}a, & k < \frac{n}{2}; \\ (b - a)^{n-1}(b + a), & k \geq \frac{n}{2}. \end{cases} \quad (7)$$

*Proof.* Let  $k < \frac{n}{2}$ , i.e. the number of elements  $b$  in the first row of the matrix (1) is less than the number of elements  $a$ . Let us subtract the penultimate row of the matrix from the last row and expand the determinant along the last row. Performing this procedure consecutively  $k$  times, we come to the equality:

$$d_n = (b - a)^k d_{n-k} + (b - a)^k f_{n-k}.$$

Substituting here the formula (4), we get the first row of the equality (7).

Let  $\frac{n}{2} \leq k$ , i.e. the number of elements  $b$  in the first row of the matrix (1) is more or equal to the number of elements  $a$ , but less than  $n$ . Executing the same  $k$  consecutive expansions along the last row as in the first case, we come to the equality:

$$d_n = (b - a)^k (f_{n-k} + g_{n-k}).$$

Substituting here the formulae (4) and (6), we get the second row of (7).  $\square$

*Proof. (of the Theorem 1.1).* Suppose that  $n \equiv p \pmod{k}$ , where  $0 < p \leq k$ , i.e.  $n = km + p$ , where  $m$  is a non-negative integer. Assume that  $k < n$ . Then  $m > 0$  and, using the first row of the formula (7) recurrently  $m - 1$  times, we obtain the equality:

$$d_n = (b - a)^{n-k-p} d_{k+p} + (m - 1)(b - a)^{n-1}a.$$

Since  $\frac{k+p}{2} \leq k$ , then applying the second row of the formula (7) to  $d_{k+p}$  and performing obvious transformations, we get (2).

If  $k = n$ , i.e. the first row of the matrix (1) entirely consists from elements  $b$ , then  $p = n$  and the formula (2) gives us  $d_n = (b - a)^{n-1}b$ , which corresponds to the statement of the Lemma 1.2.  $\square$

**Case 2.** Let  $l > 1$ . In other words, let the  $i$ -th row of the matrix  $A$ ,  $i = 1, \dots, l$ , have the form:

$$\underbrace{[b \ \dots \ b \ a \ \dots \ a]}_n^{k+i-1},$$

and the  $(l + j)$ -th row is obtained from the  $l$ -th one by the removal of  $j$  elements on the right and addition of  $j$  elements  $a$  on the left:

$$A = \begin{pmatrix} b & \dots & b & & & a \\ \vdots & \ddots & & \ddots & & \\ b & & \ddots & & \ddots & \\ & \ddots & & \ddots & & b \\ & & \ddots & & \ddots & \vdots \\ a & & & b & \dots & b \end{pmatrix}. \quad (8)$$

The matrix  $A$  is not symmetric in general, but it is persymmetric (i.e. symmetric with respect of the secondary diagonal) like all Toeplitz matrices.

**Lemma 1.4.** *In the matrix (8) exactly  $\max(k + l - n, 0)$  rows consist entirely from elements  $b$ .*

*Proof.* First note that each row of the matrix  $A$  contains no more than  $k + l - 1$  elements  $b$  by definition. Let  $k + l - n \leq 0$ . Hence  $k + l - 1 < n$ , i.e. there is at least one element  $a$  in each row and there are no rows, consisting entirely from elements  $b$ . Let now  $k + l - n > 0$  or, alternatively,  $k + l - 1 \geq n$ . If  $n = k + i - 1$ ,  $1 \leq i \leq l$ , then the number of rows, that consist entirely from elements  $b$ , will be equal to  $l - i + 1 = k + l - n$ .  $\square$

**Theorem 1.2.** *Let  $1 < l \leq k \leq n$  and  $n \equiv p \pmod{k + l - 1}$ ,  $0 \leq p < k + l - 1$ . Then the determinant of the matrix (8) is equal to:*

$$\det A = \begin{cases} (-1)^{\frac{(k-1)(l-1)n}{k+l-1}} (b-a)^{n-1} \left( b + \frac{n-k-l+1}{k+l-1} a \right), & p = 0, \\ (-1)^{\frac{(k-1)(l-1)(n-1)}{k+l-1}} (b-a)^{n-1} \left( b + \frac{n-1}{k+l-1} a \right), & p = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

*Proof.* By the theorem condition  $n \geq 2$ . Let  $n \equiv p \pmod{k + l - 1}$ ,  $0 \leq p < k + l - 1$ . This is equivalent to  $n = (k + l - 1)s + p$ , where  $s$  is a non-negative integer.

First consider the case when  $s = 0$ , i.e. when  $p = n$  and, respectively,  $n < k + l - 1$ . By lemma 1.4 at least 2 rows of the matrix  $A$  will consist entirely from elements  $b$  in this case, therefore the determinant of the matrix  $A$  will be equal to 0, that corresponds to the formula (9).

Now assume that  $s \geq 1$ . The first row of the matrix  $A$  differs from the second one only by an element in the  $(k + 1)$ -st column. Let us subtract the first row from the second one and expand the determinant along the second row. We will get that  $\det A = (-1)^{k-1} (b - a) \det A'$ , where  $A'$  is a matrix, whose first and second rows differ also only in the  $(k + 1)$ -st column. Performing this procedure  $l - 1$  times, we obtain that



Then, substituting the given formula in (10) and taking into account that  $n = (k + l - 1)s + 1$ , we finally obtain:

$$\begin{aligned} \det A &= (-1)^{(k-1)(l-1)s} (b-a)^{(k+l-1)s} (b+sa) = \\ &= (-1)^{\frac{(k-1)(l-1)(n-1)}{k+l-1}} (b-a)^{n-1} \left( b + \frac{n-1}{k+l-1} a \right). \end{aligned}$$

□

## 2 The application of determinants of binary matrices to the enumeration of permutations

As mentioned in the Introduction, the binary matrices are one of the favorite objects of the enumerative combinatorics. In particular, they are applied for enumeration of permutations with restricted positions. Following [6] let us describe briefly this mechanism.

Let  $A = (a_{ij})$  be a  $(0, 1)$ -matrix of order  $n$ . Each of such matrices defines a class  $\mathcal{B}(A)$  of restricted permutations. Namely a permutation  $p$  belongs to  $\mathcal{B}(A)$  if and only if the inequality  $M_p \leq A$  holds for its incidence matrix  $M_p$ , i.e. each element of the matrix  $M_p$  is not more than the corresponding element of the matrix  $A$ . The matrix  $A$  is called the characteristic matrix of the class  $\mathcal{B}(A)$ . It is not hard to see that the number of permutations in the class  $\mathcal{B}(A)$  is equal to the permanent of the matrix  $A$ :  $|\mathcal{B}(A)| = \text{per } A$ . Denote the number of even and odd permutations from the class  $\mathcal{B}(A)$  by  $E_A$  and  $O_A$ , respectively. It is obvious that  $E_A + O_A = \text{per } A$ . It is easy to see also that  $E_A - O_A = \det A$ . This implies the following formulae for calculating the total number of even and odd permutations from the class  $\mathcal{B}(A)$ :

$$E_A = \frac{\text{per } A + \det A}{2}, \quad O_A = \frac{\text{per } A - \det A}{2}. \quad (11)$$

As an example we calculate the number of even and odd permutations  $\pi \in S_n$  such that  $\pi(i) \neq i, i+1$  for  $i = 1, \dots, n-1$  and  $\pi(n) \neq n$ . The characteristic matrix of this class of permutations is the following  $(0, 1)$ -matrix  $A_n$  of order  $n$ :

$$A_n = \begin{pmatrix} 0 & 0 & & & \\ & 0 & 0 & 1 & \\ & & \ddots & \ddots & \\ & & & & 0 & 0 \\ 1 & & & & & 0 \end{pmatrix}.$$

Such matrices arise in the variation of the famous *ménage problem*, where not a round table, but one side of a rectangular table is considered ([5], ch. 8).

If we denote the permanent of such matrix by  $p_n$ , then the sequence of permanents will satisfy the following recurrence relation:

$$(n-1)p_n = (n^2 - n - 1)p_{n-1} + np_{n-2} + 2(-1)^{n+1}, \quad p_1 = p_2 = 0.$$

One can also calculate these permanents by the following explicit formula:

$$p_n = \sum_{k=0}^n \binom{2n-k}{k} (n-k)! (-1)^k.$$

Let  $d_n$  denote the determinant of the matrix  $A_n$ . Substituting  $b = 0$ ,  $a = 1$ ,  $k = 2$  to the formula (2), we get

$$d_n = (-1)^{n-1} \frac{n-p}{2}, \quad n \equiv p \pmod{2}, \quad 0 < p \leq 2.$$



Let us give an example of the application of this scheme. Let  $\pi$  be a permutation. Recall that a number  $i$  for which  $\pi(i) \geq i$  is called a *weak excedance* of  $\pi$  ([10], p. 40). Let us find the number of even and odd permutations of order  $n$  with exactly  $k$  weak excedances. Consider a binary matrix of order  $n$   $C_n$  whose elements on the main diagonal and above it are equal to  $b$ , and other elements are equal to 1:

$$C_n = \begin{pmatrix} b & b & b & \dots & b \\ & b & b & \dots & b \\ & & b & \dots & b \\ & & & \ddots & \vdots \\ 1 & & & & b \end{pmatrix}.$$

It is well known ([10], p. 39–40) that the permanent of this matrix is equal to the Eulerian polynomial of order  $n$ , and, respectively, the coefficient on  $b^k$  is equal to the Eulerian number  $T(n, k)$  (A008292 in [9]). By the formula (2) we get that  $\det C_n = (b-1)^{n-1}b$ . It follows that the coefficient on  $b^k$  in  $\det C_n$  is equal to:

$$c(n, k) = (-1)^{n-k} \binom{n-1}{k-1}.$$

Hence the number of even and odd permutations of order  $n$  with exactly  $k$  weak excedances is equal to:

$$e_{n,k} = \frac{1}{2} \left( T(n, k) + (-1)^{n-k} \binom{n-1}{k-1} \right), \quad o_{n,k} = \frac{1}{2} \left( T(n, k) - (-1)^{n-k} \binom{n-1}{k-1} \right).$$

Let, for example,  $k = 2$ . Then

$$e_{n,2} = \frac{1}{2} (T(n, 2) + (-1)^n (n-1)), \quad o_{n,2} = \frac{1}{2} (T(n, 2) - (-1)^n (n-1))$$

and we get the following table:

$n$	1	2	3	4	5	6	7	8	9	10	...
$T(n, 2)$	0	1	4	11	26	57	120	247	502	1013	...
$c(n, 2)$	0	1	-2	3	-4	5	-6	7	-8	9	...
$e_{n,2}$	0	1	1	7	11	31	57	127	247	511	...
$o_{n,2}$	0	0	3	4	15	26	63	120	255	502	...

Let us write out all permutations of order 4 with 2 weak excedances: **1423**, **2143**, **2413**, **3124**, **3142**, **3412**, **3421**, **4132**, **4213**, **4312**, **4321** — total 7 even and 4 odd permutations (weak excedances are in bold, even permutations are underlined).

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