

Tree tribes and lower bounds for switching lemmas

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Abstract

We show tight upper and lower bounds for switching lemmas obtained by the action of random p -restrictions on boolean functions that can be expressed as decision trees in which every vertex is at a distance of at most t from some leaf, also called t -clipped decision trees. More specifically, we show the following:

1. If a boolean function f can be expressed as a t -clipped decision tree, then under the action of a random p -restriction ρ , the probability that the smallest depth decision tree for $f|_\rho$ has depth greater than d is upper bounded by $(4p2^t)^d$.
2. For every t , there exists a function g_t that can be expressed as a t -clipped decision tree, such that under the action of a random p -restriction ρ , the probability that the smallest depth decision tree for $g_t|_\rho$ has depth greater than d is *lower* bounded by $(c_0 p 2^t)^d$, for $0 \leq p \leq c_p 2^{-t}$ and $0 \leq d \leq c_d \frac{\log n}{2^t \log t}$, where c_0, c_p, c_d are universal constants.

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Contents

1	Introduction	3
2	Preliminaries	6
2.1	Decision trees and random restrictions	6
2.2	Polynomials and linear operators	7
2.3	Basic equalities and inequalities	8
3	Upper bound	9
3.1	Recurrence for γ	10
3.2	Upper bound on γ	12
3.3	Observations from the upper bound	14
4	Tree tribes	16
4.1	Tree tribes	16
4.2	Properties of tree tribes	16
5	Lower bound	21
5.1	The case for $d = 1$	21
5.1.1	Recurrence relations	23
5.1.2	The constant coefficients, $[1]P_0(r)$ and $[1]P_1(r)$	25
5.1.3	The coefficients of p , $[p]P_0(r)$ and $[p]P_1(r)$	26
5.1.4	Bounding higher powers of p in $P_0(r) + P_1(r)$	30
5.1.5	Lower bound	36
5.2	Inductive step	36
5.2.1	The issue with taking $d = 0$ as the base case	36
5.2.2	Lower bound for $d \geq 2$	37
A	Proofs from Section 5	42

1 Introduction

One useful and powerful idea to separate a boolean function g from some set \mathcal{F} of boolean functions over $\{0,1\}^n$ is to use *restrictions*. By showing that restricted to some subset of $\{0,1\}^n$, the functions in \mathcal{F} become *simple*, but the function g does not become simple, it can be concluded that $g \notin \mathcal{F}$. The extent to which functions in \mathcal{F} become simple is captured by switching lemmas, which informally try to answer the following question: Given a family \mathcal{F} of boolean functions characterized by some parameter t , and a family \mathcal{S} of distributions over subsets of $\{0,1\}^n$ characterized by some parameter p , how complex does a function $f \in \mathcal{F}$ remain after it is restricted to a subset S chosen according to some distribution $D \in \mathcal{S}$? Defining the measure of complexity of a function and the sets \mathcal{F} and \mathcal{S} gives a switching lemma of a particular type.

Switching lemmas originated with the works of [FSS84] and [Ajt83], and were proved in their strongest form for DNFs by Hastad [Has86], which answered the question stated above as follows: Let \mathcal{F} be the set of DNFs (or CNFs) of width t over n boolean variables. Let \mathcal{S} contain exactly one distribution over subsets of $\{0,1\}^n$, also called a random p -restriction, which chooses the subset S as $S = S_1 \oplus \dots \oplus S_n$ where \oplus is the direct sum, by independently choosing each S_i as the set $\{0,1\}$ with probability p , the set $\{0\}$ with probability $\frac{1-p}{2}$, and the set $\{1\}$ with probability $\frac{1-p}{2}$. The measure of complexity of a function is the depth of the smallest depth decision tree that decides it. With these instantiations, it was shown in [Has86] that the probability that the decision tree for a t -DNF has depth greater than d after it is restricted to a random p -restriction is upper bounded by $(5pt)^d$.

The original proof of this result in [Has86] used conditioning on values of the variables under the applied restriction, and later [Raz93] gave an alternate combinatorial proof of the same fact. An excellent survey and explanations of many of these results is in [Bea94]. Apart from giving an alternate proof that parity is not in AC_0 , switching lemmas and their variants for different families of functions and restrictions have found a large number of applications, to obtain lower bounds on circuit size and depth and oracle separations of complexity classes [Sip83, Ajt83, FSS84, Yao85, Has86, Lyn86, Cai89, Ajt89, Bea90, RST15, Has16], and limitations on bounded depth proof systems [Ajt90, BIK⁺92, BP93, BPU92, KPW95, PBI93, IS01, PRST16], amongst others.

Our results

A natural generalization of t -DNFs are functions that can be expressed as t -clipped decision trees, first introduced in [PRST16]. A t -clipped decision tree is a decision tree in which every vertex is at a distance of at most t from some leaf. As observed in [PRST16], every t -DNF can be expressed as a t -clipped decision tree.¹ Casting t -DNFs as t -clipped decision trees, [PRST16] proved a strong switching lemma for t -clipped decision trees restricted to randomly chosen affine subspaces of $\{0,1\}^n$. Describing the results in [PRST16] requires considerable setup, and we will do that in the next section.

Our first result is an improved upper bound on the action of random p -restrictions on t -clipped decision trees. We write $\text{DT}_{\text{depth}}(g)$ for the depth of the least depth decision tree for the function g , and for simplicity, we write $f|_{\rho}$ to mean that f is restricted to a subset of $\{0,1\}^n$ chosen according to ρ .

¹Note however that the other way round is not true, specifically because for boolean variables $a, b, x, a \cdot x \vee \bar{x} \cdot b \not\equiv a \cdot x \vee b$ for $a = 0, x = 1, b = 1$.

Theorem 1. For any boolean function f that has a t -clipped decision tree, for a random p -restriction ρ ,

$$\Pr_{\rho}[DT_{depth}(f|_{\rho}) \geq d] \leq (4p2^t)^d.$$

For the case of random p -restrictions, Theorem 1 is an improvement by a factor of $(10t)^d$ over the bounds in [PRST16]. Our proof of Theorem 1 is recursive, and uses conditioning on variables similar to that in [Has86]. We would like to remark that although switching lemmas in general are proved in two essentially equivalent ways - one that uses conditioning on variables and another that uses combinatorial arguments - the method of conditioning is seen to give slightly stronger bounds, as seen by the slightly better constant in [Has86] over [Raz93], and an improvement similar to 1 in [Has16] over [RST15]. Theorem 1 is another case in point for random p -restrictions over the result in [PRST16].

A natural question is whether the bound in Theorem 1 is tight, or can further be improved to a bound of $O(pt)^d$ similar to that for t -DNFs. However, we show that it is asymptotically tight up to constant factors.

Theorem 2. For every $t \geq 1$, there is a function g_t expressible as a t -clipped decision tree, such that for a random p -restriction ρ , for $0 \leq p \leq c_p 2^{-t}$ and $0 \leq d \leq c_d \left(\frac{\log n}{2^t \log t}\right)$,

$$\Pr_{\rho}[DT_{depth}(g_t|_{\rho}) \geq d] \geq (c_0 p 2^t)^d,$$

where c_p , c_d and c_0 are universal constants.

The functions g_t that we create for Theorem 2 are explicit constructions, and we call them tree tribes, or more specifically, clipped xor tree tribes. It follows from Theorem 2 that these functions, expressible at t -clipped decision trees, are more resilient to random p -restrictions than t -DNFs. Further, the fourier coefficients of these functions have combinatorial properties that might be of independent interest, and we think that these functions might serve as counterexamples or “extreme points” for other problems.

The proof of Theorem 2 is recursive, and proceeds by analyzing the coefficients of polynomials that arise in the analysis of tree tribes. Note that in Theorem 2, since we want to lower bound the probability of the event $DT_{depth}(g_t|_{\rho}) \geq d$, we require that the decision tree with the least depth (and thus every decision tree) for $g_t|_{\rho}$ must have depth greater than d with sufficient probability. To achieve this, our proof proceeds as follows: If T is the decision tree for g_t , we lower bound the probability of finding “paths with a split” in $T|_{\rho}$. A “path with a split” is a subtree of $T|_{\rho}$, which consists of a path of distinct variables y_1, \dots, y_d (where y_1 is closest to the root in $T|_{\rho}$), such that y_d is connected to two leaves with different values (more specifically, y_d has a path to a leaf labelled 0 and a leaf labelled 1). Any decision tree for such a subtree of $T|_{\rho}$ must have depth at least d (at least if the variables y_1, \dots, y_d do not appear elsewhere in the tree), by exactly the same argument that any decision tree for the OR function on d variables must have depth at least d . However, we need a sufficient number of vertices in the tree before we get sufficient probability mass for the event of finding such a path with a split, and this is the reason we get an upper bound on the depth d for which Theorem 2 holds.

The application of Theorem 2 is to bounded depth Frege proof systems, which we discuss next.

Implications for proof complexity

Proof complexity, originating with [CR79], tries to answer whether NP is same as coNP, starting with the observation that if an unsatisfiable 3-SAT formula has a short proof of unsatisfiability in *any* propositional proof system, then $NP = coNP$. However, since we expect NP to be different from coNP, one way to attack the problem is to show that in many specific proof systems, some unsatisfiable formula requires long proofs.

A natural proof system to consider is the Frege proof system, in which each line of the proof is an AC_0 formula over boolean variables, with the connectives $\neg, \vee, \wedge, \implies$, has modus ponens as the only rule of inference, and has a small set of simple axioms. In a series of works, it was shown that any constant depth Frege proof for the Pigeonhole Principle (PHP) requires superpolynomial size [Ajt88], any polynomial size Frege proof for PHP requires $\Omega(\log \log n)$ depth [BPU92, KPW95, PBI93], and similar bounds hold for Tseitin formulas (contradictions) over the complete graph [UF+96]. Alternately, an upper bound was shown in [Bus87], by which both PHP and Tseitin formulas over any graph have polynomial size, $O(\log n)$ depth Frege proofs.

This gap between the power of $O(\log \log n)$ and $O(\log n)$ depth Frege proofs was long open, until the recent work [PRST16], in which it was shown that there is a Tseitin formula over a 3-regular expander graph, such that any $O(\sqrt{\log n})$ depth Frege proof for it must have super polynomial size. A crucial idea in [PRST16] is the use of random projections, which has recently been used to obtain powerful correlation bounds within constant depth circuits in the breakthrough work [RST15], and further improved in [Has16].

At the heart of the result in [PRST16] is a delicately constructed switching lemma, such that when applied to Tseitin formulas over 3-regular expanders, the resulting graph remains a 3-regular expander. An exemplary exposition of the switching lemma used in [PRST16] is given in [Ros16], and we present the description given there.

Let the universe $U = \{0, 1\}^n$ where we consider the natural equivalence between elements of U and subsets of $[n]$. For $A \in U$ and $B \subseteq U$, we say that the set B shatters A if for every $A' \subseteq A$, there exists $B' \in B$ such that $B' \cap A = A'$. Let B be some affine subspace of U . A decision tree T is B -independent, if B shatters the set of variables on every root to leaf path in T . Let $\mathcal{F}_{t,B}$ be the set of all functions for which there is a B -independent t -clipped decision tree. We say that an arbitrary distribution V over the random linear subspaces of B is p -bounded, if for every $J \in U$, the probability that a subspace chosen according to V shatters J is upper bounded by $p^{|J|}$. Let $S_{p,B}$ be the set of all distributions $W = V + u$, where V is a p -bounded distribution and u is chosen uniformly from B . We write $\rho \leftarrow W$ to denote an affine subspace chosen according to W . Given this setup, the following theorem is shown in [PRST16, Ros16].

Theorem 3. [PRST16] *For any arbitrary affine subspace B , for every $f \in \mathcal{F}_{t,B}$ and $W \in S_{p,B}$,*

$$\Pr_{\rho \leftarrow W}[DT_{depth}(f|_{\rho}) \geq d] \leq (40pt2^t)^d.$$

The proof of Theorem 3 is via a beautiful combinatorial argument, that applies restrictions to *all* the variables in a t -clipped decision tree directly, and notably, does not use recursion. The reader is referred to [PRST16, Ros16] for the proof. Casting t -DNFs as t -clipped decision trees and using Theorem 3, it was shown in [PRST16] that any depth d Frege proof for a carefully constructed Tseitin contradiction τ over 3-regular expanders must have size at least $\exp(\Omega(\frac{\log n}{d})^2)$. Improving the factor $(40pt2^t)^d$ to $O(pt)^d$ in Theorem 3 would imply exponential lower bounds on the size of

Frege proofs for τ ; more specifically, it would imply that any depth d Frege proof for τ over 3-regular expanders must have size at least $\exp(\Omega(n^{1/d}))$, matching the optimal lower bounds known for boolean circuits [Has86].

However, note that in the parameters for Theorem 3, if we let $B = U = \{0,1\}^n$, it shatters every set $A \in U$. Further, we can choose $Y \in \mathcal{S}_{p,B}$ to simply be a random p -restriction - this is equivalent to a distribution $W = V + u$, where the distribution V chooses a random subspace of U by independently choosing the i 'th standard basis vector with probability p , and u is a uniformly random bit string in U . With these instantiations, the following is an immediate corollary of Theorem 2.

Corollary 4. *For every t , there is an affine subspace B of $\{0,1\}^n$, and $g_t \in \mathcal{F}_{t,B}$ and $Y \in \mathcal{S}_{p,B}$, such that for $0 \leq p \leq c_p 2^{-t}$ and $0 \leq d \leq c_d \left(\frac{\log n}{2^t \log t}\right)$,*

$$\Pr_{\rho \leftarrow Y}[DT_{depth}(g_t|\rho) \geq d] \geq (c_0 p 2^t)^d,$$

where c_p, c_d and c_0 are universal constants.

Note that Theorem 4 shows that the bounds in Theorem 3 of [PRST16] are almost tight. As a consequence, it shows that for small values of d , up till $\sim 2^{-t} \log n$, there are functions that have t -clipped decision trees, but for which exponential lower bounds on the size of Frege proofs are not possible via improvements to the switching lemma stated in Theorem 3. Note however that it leaves open the possibility of obtaining such exponential bounds for t -DNFs if they are treated *directly* and not cast as t -clipped decision trees.

We start by proving Theorem 1 in Section 3, since observations from the improved upper bound proof, stated in Subsection 3.3, will lead to the definition of tree tribes in Section 4, which will further help in proving Theorem 2 in Section 5.

2 Preliminaries

2.1 Decision trees and random restrictions

We will consider only boolean functions over the hypercube, $f : \{0,1\}^n \rightarrow \{0,1\}$. When we consider the function f in the fourier basis, we assume that it is a boolean function over $\{1,-1\}^n$. The fourier expansion of $f : \{1,-1\}^n \rightarrow \{1,-1\}$ is given by $f(x) = \sum_{S \subseteq [n]} \hat{f}_S \chi_S(x)$, where $\chi_S(x) = \prod_{i \in S} x_i$ and for every $S, \hat{f}_S \in \mathbb{R}$. The bias of f is defined as

$$\text{bias}(f) = \left| \Pr_x[f(x) = 0] - \frac{1}{2} \right|,$$

and the correlation between boolean functions f and g is given by

$$\text{Corr}(f, g) = \Pr_x[f(x) = g(x)].$$

The influence of a variable x_i is defined as the probability, over choosing a random bit string on variables different from x_i , that flipping the value of x_i flips the value of the function.

Definition 5. (*Decision trees*) Given a set of variables $X = \{x_1, \dots, x_n\}$, a *decision tree* T is a rooted binary tree $T = (V, E, X, \sigma_V, \sigma_E)$, where the functions $\sigma_V : V \rightarrow X$ and $\sigma_E : E \rightarrow \{0, 1\}$ respectively label the vertices with variables in X and edges with 0 or 1, and the leaves are labelled with either 0 or 1. Further, in every root to leaf path, any variable of X appears *at most once*. Given a boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ over the variables X , we say that T is a decision tree for f or that f is computed/evaluated by T , if, for every root to leaf path $\pi = \{x_{i_1}, e_{i_1}, \dots, x_{i_l}, e_{i_l}, b\}$ where $x_{i_k} \in X$, $e_{i_k} \in \{0, 1\}$ and where the value of the leaf is $b \in \{0, 1\}$, the function f evaluates to b on the subcube in which the variables $(x_{i_1}, \dots, x_{i_l})$ are assigned the value $(e_{i_1}, \dots, e_{i_l})$. Given a decision tree T , we use the symbols $f(T)$ to denote the function computed by the tree on variables X .

Definition 6. (DT_{depth}) For any root to leaf path $\pi = (x_{i_1}, e_{i_1}, \dots, x_{i_l}, e_{i_l}, b)$, we say that the *length* of the path is the number of edges in it, i.e. $|\pi| = l$. The *depth* of a decision tree T , denoted by $\text{depth}(T)$ is the length of the *longest* root to leaf path in T . For any boolean function f , we denote $DT_{\text{depth}}(f)$ to be the minimum depth amongst all decision trees T for f . We also say that the variables *queried* along the path π were $(x_{i_1}, \dots, x_{i_l})$, and the values *assigned* or *received* or the values to which the variables *evaluated* were $(e_{i_1}, \dots, e_{i_l})$.

Definition 7. (*Clipped decision trees*) A decision tree T is t -clipped, if every vertex of T is at a distance of at most t from some leaf.

Definition 8. (*Random restrictions*) Given a set of variables $X = \{x_1, \dots, x_n\}$, a restriction ρ is a string in $\{0, 1, *\}^n$, i.e., $\rho : X \rightarrow \{0, 1, *\}$, where $\rho(x_i) = *$ means that the variable is left unset, i.e., $\rho(x_i) = x_i$. A random restriction is a distribution over $\{0, 1, *\}^n$. A restriction ρ is said to be a random p -restriction, if ρ is a distribution that choses a subset $S \subseteq X$ of variables with probability $|S|^p$ and assigns $*$ to them, and uniformly assigns the value 0 or 1 to the remaining variables. Equivalently, ρ independently assigns each variable the value $*$ with probability p and 0 or 1 with probability $q = \frac{1-p}{2}$.

We will restrict the symbols p and q to that specific meaning throughout, even when we treat them as formal variables.

Remark 9. Note that a random p -restrictions ρ has product structure due to the independence between various variables, i.e. $\rho = \rho_1 \rho_2$, where ρ_1 and ρ_2 are random p -restrictions over the variable sets $\{x_1, \dots, x_m\}$ and $\{x_{m+1}, \dots, x_n\}$ respectively.

Given a boolean function f on n variables, we write $f|_\rho$ for the boolean function obtained by restricting f according to a subset of $\{0, 1\}^n$ chosen according to ρ , and say that f is restricted to ρ or f is hit by ρ . We will also use the notation $T|_\rho$, which would mean $f(T)|_\rho$. We will mainly study the probability of the event that the DT_{depth} of $f|_\rho$ is greater than d when ρ is a random p -restriction, i.e.,

$$\Pr_\rho[DT_{\text{depth}}(f|_\rho) \geq d].$$

2.2 Polynomials and linear operators

A univariate polynomial Q in the variable p will be an infinite dimensional vector over \mathbb{R} , in the vector space $\mathbb{R}[p]$, with a finite number of non-zero coefficients. We will denote Q as $Q = \sum_{i \geq 0} c_i p^i$. Using the standard notation for generating functions, for every $i \geq 0$, we define the operator $[p^i]$ as

$$[p^i]Q = c_i.$$

Further, we define the operator $[\uparrow p^i]$ as

$$[\uparrow p^i]Q = \sum_{j \geq i} ([p^j]Q) p^{j-i} = \sum_{j \geq i} c_j p^{j-i}.$$

Definition 10. (*Absolute maximizer*) Given a closed set $\mathcal{D} \subseteq [0, 1]$ and some polynomial Q in $\mathbb{R}[p]$, we define the absolute maximizers G_i of $[\uparrow p^i]Q$ as

$$G_i(Q) = \max_{p \in \mathcal{D}} |[\uparrow p^i]Q|.$$

The following claim is immediate from the definitions.

Lemma 11. *For any two univariate polynomials $Q, R \in \mathbb{R}[p]$, the following hold:*

1. $[p^i](Q + R) = [p^i]Q + [p^i]R$
2. $[\uparrow p^i](Q + R) = [\uparrow p^i]Q + [\uparrow p^i]R$
3. $[p^i](QR) = \sum_{j=0}^i [p^j]Q [p^{i-j}]R$
4. $[p^i]p^j Q = 0$ if $j > i$
5. $[p^i]p^j Q = [p^{i-j}]Q$ if $j \leq i$
6. $G_i(Q \pm R) \leq G_i(Q) + G_i(R)$.

2.3 Basic equalities and inequalities

We mention some basic equalities and inequalities that we will repeatedly use:

Lemma 12. *For any integer $n \geq 1$, the following hold:*

1.
$$\sum_{i=0}^n p^i = \frac{1 - p^{n+1}}{1 - p}$$
2.
$$\sum_{k=1}^n \frac{k}{2^k} = 2 - \frac{n+2}{2^n} \leq 2$$
3.
$$\sum_{k=2}^n \binom{k}{2} \frac{1}{2^k} = 2 - \frac{n^2 + 3n + 4}{2^{n+1}} \leq 2$$
4.
$$\sum_{i=0}^n \binom{n}{i} q^{n-i} p^i (i+1) = pn(q+p)^{n-1} + (q+p)^n$$
5. For $p \in [0, 1]$,
$$\frac{1}{p} (1 - (1-p)^t) \leq t$$

6. For $p \in [0, 1]$,

$$\frac{1}{p^2} ((1-p)^t - 1 + tp) \leq \binom{t}{2}$$

Proof. (1), (2) and (3) follow by direct calculations. For (4), write $x(q+x)^n = \sum_{i=0}^n \binom{n}{i} q^{n-i} x^{i+1}$, differentiate both sides with respect to x , and set $x = p$. We show (6) and the proof of (5) is similar. Use induction on t to show that the function $\frac{1}{p^2} ((1-p)^t - 1 + tp)$ is a non-increasing function of p . It is trivially true for $t = 1$, let it be true for $t - 1$. Rewriting,

$$\begin{aligned} \frac{1}{p^2} ((1-p)^t - 1 + tp) &= \frac{1}{p^2} \left((1-p) \left((1-p)^{t-1} - 1 + p(t-1) \right) + (1-p) - p(1-p)(t-1) - 1 + tp \right) \\ &= (1-p) \frac{1}{p^2} \left((1-p)^{t-1} - 1 + p(t-1) \right) + t - 1 \end{aligned}$$

which is a non-increasing function of p , since $(1-p)$ is a decreasing function of p in $[0, 1]$ and by the induction hypothesis, $\frac{1}{p^2} ((1-p)^{t-1} - 1 + p(t-1))$ is a non-increasing function of p . Thus, the function is maximized at $p = 0$, setting which proves the claim. \square

3 Upper bound

We start by showing an improved upper bound on the probability that a function represented by a clipped decision tree has depth greater than d after it is hit with a random p -restriction. Our proof will be recursive and use conditioning similar to that in [Has86].

Let T be a t -clipped decision tree and $f = f(T)$ be the corresponding boolean function. We assume that T has n variables, but note that since the final bounds in Theorem 1 are independent of n , it is safe to keep the intuition that T is virtually an infinite tree. Let T be hit by a random p -restriction ρ . We want to show that

$$\Pr_{\rho}[\text{DT}_{\text{depth}}(f|_{\rho}) \geq d] \leq (4p2^t)^d.$$

We will use induction on d to prove the claim. To set up the induction, we will need the following definitions.

Definition 13. A decision tree T is (t_0, t) -clipped for $t_0 \leq t$, if the root has distance at most t_0 to some leaf, and every other vertex has distance at most t to some leaf.

Definition 14. For integers t_0, t, n, d , define the probabilities $\gamma_{d,n}(t_0, t)$ as follows:

$$\gamma_{d,n}(t_0, t) = \max_{\substack{(t_0, t)\text{-clipped trees } T \text{ that} \\ \text{decide any function on } n \text{ variables}}} \Pr_{\rho}[\text{DT}_{\text{depth}}(T|_{\rho}) \geq d]$$

The probabilities γ have been specifically defined to make the parameter n irrelevant, as long as any recursive inequality for γ only reduces n . It is simple to see that γ is monotone in n .

Lemma 15. If $n' \leq n$, then $\gamma_{d,n'}(t_0, t) \leq \gamma_{d,n}(t_0, t)$.

Proof. Assume $n = n' + 1$, and the claim follows. Let T' be a (t_0, t) clipped tree on n' vertices, which achieves the maximum for $\gamma_{d,n'}(t_0, t)$ for some d . Let x be the last variable queried in T' along some root to leaf path, such that the subtrees rooted at x are both leaves. Note that such a variable always exists in a finite tree. When x is queried and it evaluates to 1, let the value of the leaf be $a \in \{0, 1\}$. Let y be a new variable different from all the variables appearing in T' . Let T be a new tree created as follows: T is same as T' , except that when the variable x is queried, if it evaluates to 1, we query the variable y , and the value of both the leaves at y is a . Note that essentially, y will never be queried by any decision tree. T is also (t_0, t) -clipped, and since $f(T') = f(T)$ for any $y \in \{0, 1, *\}$, they have the same minimum depth under the action of any random restriction, and the claim follows. \square

3.1 Recurrence for γ

We proceed by writing a recurrence for $\gamma_{d,n}(t_0, t)$.

Let T be the (t_0, t) -clipped decision tree for which $\gamma_{d,n}(t_0, t)$ has maximum value. Without loss of generality, let x_1 be the root variable queried in T . Let $(x_1, e_1, \dots, x_{t_0}, e_{t_0}, a)$ be the path from the root to the leaf at distance t_0 which evaluates to $a \in \{0, 1\}$. Since any variable is assigned 0 or 1 with equal probability, without loss of generality, let $e_1 = 0$. Let the subtree out of the 0-edge at x_1 be T_0 and the subtree out of the 1-edge be T_1 .

Under the action of a random p -restriction ρ , if x_1 is assigned 0 by ρ , note that we get a $(t_0 - 1, t)$ clipped decision tree T_0 on n_0 variables where $n_0 < n$. By our definition of decision trees 5, since x_1 appears as the root of T , it cannot appear again as a variable in T_0 , and thus T_0 is indeed a $(t_0 - 1, t)$ -clipped tree. If x_1 is assigned 1 by ρ , similarly, we get a (t, t) clipped decision tree T_1 on n_1 variables where $n_1 < n$.

Let the event $E_{T,d}$ be defined as follows:

$$E_{T,d} \equiv \text{DT}_{\text{depth}}(T|_{\rho}) \geq d.$$

Lemma 16. *In case x_1 is assigned $*$ by a random p -restriction ρ ,*

$$E_{T,d} \subseteq E_{T_0,d-1} \cup E_{T_1,d-1}.$$

Proof. We show the contrapositive. Let ρ' be the restriction on variables different from x_1 . As stated before, by definition 5, T_0 and T_1 do not contain x_1 , and thus $T_0|_{\rho} = T_0|_{\rho'}$ and $T_1|_{\rho} = T_1|_{\rho'}$. Let V_0 and V_1 be decision trees such that $\text{depth}(V_0) = \text{DT}_{\text{depth}}(T_0|_{\rho})$ and $\text{depth}(V_1) = \text{DT}_{\text{depth}}(T_1|_{\rho})$ (V_0 and V_1 are random variables). Thus, if $\text{DT}_{\text{depth}}(T_0|_{\rho}) < d - 1$, i.e. the smallest depth of the decision tree computing $T_0|_{\rho}$ is strictly less than $d - 1$, and similarly $\text{DT}_{\text{depth}}(T_1|_{\rho}) < d - 1$, then the decision tree which has x_1 as the root, and V_0 as the left subtree and V_1 as the right subtree would correctly evaluate the function $T|_{\rho}$ and have depth strictly less than d , which would mean that $\text{DT}_{\text{depth}}(T|_{\rho}) < d$, and the event $E_{T,d}$ cannot happen. \square

Let $\rho = \rho_{x_1}\rho'$ where ρ' is a random restriction on variables different from x_1 . Thus, we can

write the following,

$$\begin{aligned}
\Pr_{\rho}[E_{T,d}] &= \Pr_{\rho}[E_{T,d}|\rho(x_1) = 0] \Pr_{\rho}[\rho(x_1) = 0] + \Pr_{\rho}[E_{T,d}|\rho(x_1) = 1] \Pr_{\rho}[\rho(x_1) = 1] \\
&\quad + \Pr_{\rho}[E_{T,d}|\rho(x_1) = *] \Pr_{\rho}[\rho(x_1) = *] \\
&\leq \Pr_{\rho'}[E_{T_0,d}]q + \Pr_{\rho'}[E_{T_1,d}]q + \Pr_{\rho'}[E_{T_0,d-1} \cup E_{T_1,d-1}]p \\
&\leq \Pr_{\rho'}[E_{T_0,d}]q + \Pr_{\rho'}[E_{T_1,d}]q + \Pr_{\rho'}[E_{T_0,d-1}]p + \Pr_{\rho'}[E_{T_1,d-1}]p
\end{aligned} \tag{1}$$

where the second line used Lemma 16, the fact that the subtrees at x_1 do not contain the variable x_1 and that ρ has product structure since each of the variables are assigned values independently, and the last line used the union bound. Further, note that since $\Pr_{\rho}[E_{T,d}] \leq \gamma_{d,n}(t_0, t)$ if T is (t_0, t) -clipped and has n variables, we can rewrite the inequality 1 in terms of the $\gamma_{d,n}(t_0, t)$, and using Lemma 15, we get,

$$\begin{aligned}
\gamma_{d,n}(t_0, t) &\leq q\gamma_{d,n_0}(t_0 - 1, t) + q\gamma_{d,n_1}(t, t) + p\gamma_{d-1,n_0}(t_0 - 1, t) + p\gamma_{d-1,n_1}(t, t) \\
&\leq q\gamma_{d,n}(t_0 - 1, t) + q\gamma_{d,n}(t, t) + p\gamma_{d-1,n}(t_0 - 1, t) + p\gamma_{d-1,n}(t, t).
\end{aligned}$$

The parameters n and t can be made implicit, and we can rewrite the recurrence succinctly as

$$\gamma_d(t_0) \leq q\gamma_d(t_0 - 1) + q\gamma_d(t) + p\gamma_{d-1}(t_0 - 1) + p\gamma_{d-1}(t), \tag{2}$$

and for every integer d , we set

$$\gamma_d(0) = 0. \tag{3}$$

Notice that at this point, we can use induction for $\gamma_{d-1}(t)$ but not the other terms. We now show the following recursive claim for $\gamma_d(t_0)$.

Lemma 17. *After m iterations, the recursion is,*

$$\gamma_d(t_0) \leq \sum_{i=0}^m \binom{m}{i} q^{m-i} p^i \gamma_{d-i}(t_0 - m) + \sum_{i=1}^m q^i \gamma_d(t) + \sum_{j=1}^m p^j \gamma_{d-j}(t) \left(\sum_{i=0}^{m-j} \binom{j+i}{i} q^i \right). \tag{4}$$

Proof. The base case for $m = 1$ is given by equation 2. Let the recurrence be true for m . Thus,

$$\gamma_d(t_0) \leq \sum_{i=0}^m \binom{m}{i} q^{m-i} p^i \gamma_{d-i}(t_0 - m) + \sum_{i=1}^m q^i \gamma_d(t) + \sum_{j=1}^m p^j \gamma_{d-j}(t) \left(\sum_{i=0}^{m-j} \binom{j+i}{i} q^i \right).$$

Let $t'_0 = t_0 - m - 1$. Using the recursion in equation 2 for $\gamma_{d-i}(t_0 - m)$, we can write

$$\begin{aligned}
\gamma_d(t_0) &\leq \sum_{i=0}^m \binom{m}{i} q^{m-i} p^i (q\gamma_{d-i}(t'_0) + q\gamma_{d-i}(t) + p\gamma_{d-i-1}(t'_0) + p\gamma_{d-i-1}(t)) \\
&\quad + \sum_{i=1}^m q^i \gamma_d(t) + \sum_{j=1}^m p^j \gamma_{d-j}(t) \left(\sum_{i=0}^{m-j} \binom{j+i}{i} q^i \right).
\end{aligned}$$

We can sum the terms based on whether the parameter in γ is t'_0 or t . Summing the first and the third terms above, we can rewrite them as

$$\begin{aligned}
A_1 &= \sum_{i=0}^m \binom{m}{i} q^{m-i} p^i (q\gamma_{d-i}(t'_0) + p\gamma_{d-i-1}(t'_0)) \\
&= q^{m+1}\gamma_d(t'_0) + p^{m+1}\gamma_{d-m-1}(t'_0) + \sum_{i=1}^m \left(\binom{m}{i} + \binom{m}{i-1} \right) q^{m-i+1} p^i \gamma_{d-i}(t'_0) \\
&= \sum_{i=0}^{m+1} \binom{m+1}{i} q^{m+1-i} p^i \gamma_{d-i}(t'_0).
\end{aligned}$$

And summing the remaining terms, we get,

$$\begin{aligned}
A_2 &= q^{m+1}\gamma_d(t) + \sum_{i=1}^m \binom{m}{i} q^{m-i+1} p^i \gamma_{d-i}(t) + \sum_{i=0}^m \binom{m}{i} q^{m-i} p^{i+1} \gamma_{d-i-1}(t) \\
&\quad + \sum_{i=1}^m q^i \gamma_d(t) + \sum_{j=1}^m p^j \gamma_{d-j}(t) \left(\sum_{i=0}^{m-j} \binom{j+i}{i} q^i \right) \\
&= \sum_{i=1}^{m+1} q^i \gamma_d(t) + p^{m+1}\gamma_{d-m-1}(t) + \sum_{j=1}^m p^j \gamma_{d-j}(t) \left(\binom{m+1}{j} q^{m-j+1} + \sum_{i=0}^{m-j} \binom{j+i}{i} q^i \right) \\
&= \sum_{i=1}^{m+1} q^i \gamma_d(t) + \sum_{j=1}^{m+1} p^j \gamma_{d-j}(t) \left(\sum_{i=0}^{m+1-j} \binom{j+i}{i} q^i \right).
\end{aligned}$$

Taking $A_1 + A_2$ gives the inequality for the $(m+1)$ 'th iteration, and proves the claim. \square

3.2 Upper bound on γ

Setting $t_0 = t$ and $m = t$ in equation 4 and using 3, we get,

$$\frac{1 - 2q + q^{t+1}}{1 - q} \gamma_d(t) \leq \sum_{j=1}^t p^j \gamma_{d-j}(t) \left(\sum_{i=0}^{t-j} \binom{j+i}{i} q^i \right).$$

Using the induction hypothesis, for $\mu = \kappa 2^t$, we have that $\gamma_{d-c}(t) \leq (\mu p)^{d-c}$ for all p, c, d, t where $p \leq \frac{1}{\mu}$. Note that γ is a probability and since $p \leq \frac{1}{\mu}$, it is also valid when $c > d$.² Then we have,

$$\frac{1 - 2q + q^{t+1}}{1 - q} \gamma_d(t) \leq (\mu p)^d \left(\sum_{j=1}^t \sum_{i=0}^{t-j} \left(\frac{1}{\mu} \right)^j \binom{j+i}{i} q^i \right). \quad (5)$$

To get that $\gamma_d(t) \leq (\mu p)^d$, we only need to show the following lemma.

Lemma 18. For $t \geq 1$ and $\mu = 4 \cdot 2^t$,

$$\left(\sum_{j=1}^t \sum_{i=0}^{t-j} \left(\frac{1}{\mu} \right)^j \binom{j+i}{i} q^i \right) \left(\frac{1 - q}{1 - 2q + q^{t+1}} \right) \leq 1.$$

²In fact, whenever $d < t$, we can indeed get much fewer terms in the summation, and get a better bound, although asymptotically it does not make a difference.

Proof. The summations in the first term stated above can be made simpler by adding diagonally, i.e., for values for which $j + i = l$. Note that $1 \leq j + i \leq t$. Thus, setting $i = l - j$, and using the fact that $\binom{j+i}{i} = \binom{j+i}{j}$,

$$\begin{aligned}
\sum_{j=1}^t \sum_{i=0}^{t-j} \left(\frac{1}{\mu}\right)^j \binom{j+i}{i} q^i &= \sum_{j=1}^t \sum_{l=j}^t \left(\frac{1}{\mu}\right)^j \binom{l}{j} q^{l-j} \\
&= \sum_{l=1}^t \sum_{j=1}^l \left(\frac{1}{\mu}\right)^j \binom{l}{j} q^{l-j} \\
&= \sum_{l=1}^t \left(\frac{1}{\mu} + q\right)^l - \sum_{l=1}^t q^l \\
&= r \frac{1-r^t}{1-r} - q \frac{1-q^t}{1-q}
\end{aligned} \tag{6}$$

where

$$r = q + \frac{1}{\mu}. \tag{7}$$

Let

$$U = \left(r \frac{1-r^t}{1-r} - q \frac{1-q^t}{1-q} \right) \frac{1-q}{1-2q+q^{t+1}}. \tag{8}$$

We first show that for $t \geq 1$, $\frac{dU}{dp} < 0$. From equation 6, let

$$A = \sum_{l=1}^t \left(\frac{1}{\mu} + q\right)^l - \sum_{l=1}^t q^l$$

and

$$B = \frac{1-q}{1-2q+q^{t+1}} = \frac{1+p}{2p + \frac{(1-p)^{t+1}}{2^t}} = \frac{C}{D}.$$

Since $\frac{1}{\mu} + q > q$, we have that $A > 0$. Since $0 \leq p \leq 1$, we have that $B > 0$ and $D > 0$.

$$\frac{dA}{dp} = \frac{1}{2} \left(\sum_{l=1}^t l \left(q^{l-1} - \left(\frac{1}{\mu} + q\right)^{l-1} \right) \right) < 0$$

since each of the terms inside the summation is strictly less than 0. Further,

$$\begin{aligned}
\frac{dB}{dp} &= \frac{1}{D^2} \left(2p + \frac{(1-p)^{t+1}}{2^t} - (1+p) \left(2 - \frac{t+1}{2^t} (1-p)^t \right) \right) \\
&= \frac{1}{D^2} \left(2p + \frac{(1-p)^{t+1}}{2^t} - 2 - 2p + \frac{t+1}{2^t} (1-p)^t (1+p) \right) \\
&= \frac{1}{D^2} \left(\frac{(1-p)^{t+1}}{2^t} + \frac{t+1}{2^t} (1-p)^{t-1} (1-p^2) - 2 \right)
\end{aligned}$$

where in the last line, we used the fact that $t \geq 1$. Note that since the terms involving p are all decreasing in p , they are maximized for $p = 0$. Thus, in the last line

$$\frac{dB}{dp} \leq \frac{1}{D^2} \left(\frac{t+2}{2^t} - 2 \right) < 0,$$

where the last inequality used the fact that $t \geq 1$. Thus, we have that

$$\frac{dU}{dp} = A \frac{dB}{dp} + B \frac{dA}{dp} < 0.$$

Since U is a decreasing function of p , setting $p = 0$, and $\mu = \kappa 2^t$ in equation 7, we get,

$$r = \frac{1}{2} \left(1 + \frac{2}{\kappa 2^t} \right),$$

and substituting $p = 0$ in 8, we require

$$U = 2^t \left(r \frac{1-r^t}{1-r} - 1 + 2^{-t} \right) \leq 1$$

or

$$2r - r^{t+1} \leq 1$$

or

$$\frac{4}{\kappa} \leq \left(1 + \frac{2}{\kappa 2^t} \right)^{t+1}$$

which is implied for all $t \geq 1$ by setting $\kappa = 4$, and we get the required bound. \square

We have thus shown that $\gamma_d(t) \leq (4p2^t)^d$, and Theorem 1, which we reproduce here for convenience.

Theorem 19. *For any boolean function f that has a t -clipped decision tree, for a random p -restriction ρ ,*

$$\Pr_{\rho} [DT_{depth}(f|_{\rho}) \geq d] \leq (4p2^t)^d.$$

3.3 Observations from the upper bound

We make a few observations from the upper bound proof that will help us create structures for which we could prove a lower bound. In the recurrence 1 of total probability, since we used the union bound in our proof of Theorem 1, we neglected the term

$$\Pr_{\rho'} [E_{T_0, d-1} \cap E_{T_1, d-1}].$$

How worse can this term be? First we make the following observation: Without loss of generality, let

$$\phi = x_1 x_2 \dots x_t \vee \phi'$$

be a t -DNF where ϕ' is also a t -DNF. If we write ϕ as a t -clipped decision tree, the variables x_1 to x_t will be connected by 0-edges and end in a leaf. Further, the 1-edges out of each of the variables x_i will be connected to a t -clipped decision T_i . But note that each of the T_i 's is a decision tree for ϕ' . However, each T_i could have a different value of $\text{DT}_{\text{depth}}(T_i)$, due to the different variables to which they are connected. More specifically, for instance, the tree T_2 does not *know* the value of x_3 , but T_4 *knows* that that $x_3 = 0$. However, since we will consider the *maximum* of $\text{DT}_{\text{depth}}(T_i)$ over all i , if we take t to be much smaller than n , it would be safe to assume that for every i , the value of $\text{DT}_{\text{depth}}(T_i)$ is approximately the same. As a result, we would have

$$\Pr_{\rho'}[E_{T_0,d-1} | E_{T_1,d-1}] \approx 1.$$

Thus, the loss due to the union bound in this case is significant, and if we want the union bound to be tight, we would essentially want that

$$\Pr_{\rho'}[E_{T_0,d-1} \cap E_{T_1,d-1}] \approx 0.$$

However, this would require creating intricate (anti)-correlations between variables, and it is unclear how that can be done. Instead, it is possible to get,

$$\Pr_{\rho'}[E_{T_0,d-1} \cap E_{T_1,d-1}] \approx \Pr_{\rho'}[E_{T_0,d-1}] \Pr_{\rho'}[E_{T_1,d-1}],$$

by ensuring that there are no correlations between the two events, which would be true, at least if the variables in the two subtrees T_0 and T_1 are different. If we had such a case, the one-step recurrence 2 for γ would become

$$\gamma_d(t_0) \approx q\gamma_d(t_0 - 1) + q\gamma_d(t) + p\gamma_{d-1}(t_0 - 1) + p\gamma_{d-1}(t) - p\gamma_{d-1}(t_0 - 1)\gamma_{d-1}(t).$$

Further, since a $(t_0 - 1, t)$ clipped tree is also a (t, t) clipped tree for $t_0 \leq t$, we have that

$$\gamma_{d-1}(t_0 - 1) \leq \gamma_{d-1}(t),$$

and we can write

$$\gamma_d(t_0) \geq q\gamma_d(t_0 - 1) + q\gamma_d(t) + p\gamma_{d-1}(t_0 - 1) + p\gamma_{d-1}(t) - p\gamma_{d-1}^2(t).$$

But note that we expect the variables γ_{d-1} to exponentially decrease with increase in d in the final bound, and thus

$$\gamma_{d-1}^2(t) \ll \gamma_{d-1}(t),$$

and we should have

$$\gamma_d(t_0) \approx q\gamma_d(t_0 - 1) + q\gamma_d(t) + p\gamma_{d-1}(t_0 - 1) + p\gamma_{d-1}(t).$$

This intuition turns out to be correct, as we show in section 5, for functions that we define next.

Remark 20. The reason why we do not get a bound as strong as that for t -DNFs, i.e. $O(pt)^d$, in Theorem 1 is that t -DNFs have *additional* structure which t -clipped decision trees do not. Note that in setting up the recursion in equation 2, we used a union bound for *every* variable that was assigned $*$ by ρ . However, in the case of t -DNFs, if we try to write the same recursive expressions for every clause, if any variable in the clause is assigned the value 0 by ρ , the clause evaluates to 0 *even if* some other variables of the clause are assigned $*$ by ρ , and a union bound would *not* be necessary in that case. Only if every variable in the clause is assigned 1 or $*$ by ρ do we need to use a union bound for the variables in a clause, a luxury that we do not have in the case of t -clipped decision trees.

4 Tree tribes

4.1 Tree tribes

We now formally define tree tribes and their variants. A specific variant, t -clipped xor tree tribe, will be the function that will help to achieve the bounds stated in Theorem 2.

Definition 21. (*Tree tribe*) A boolean function f is called a tree tribe, denoted by Ξ , if there is a decision tree T deciding f , such that all the variables at all the vertices of T are distinct, and from every vertex of T , there is a path to a leaf labelled 0 and a path to a leaf labelled 1.

An example for a tree tribe is the OR function.

Remark 22. In the definition of Ξ , the condition of having a path to a 0-leaf and a path to a 1-leaf only ensures that no vertex is redundant, since if all the paths from some vertex x go to a 0-leaf (or 1-leaf), x can be replaced by a 0-leaf (or 1-leaf).

Given definition of a tree tribe, it is possible to derive a variety of different tree tribes by imposing additional structure, and we define the specific structure that we'll need.

Definition 23. (*Complete clipped decision trees*) Denote a complete t -clipped decision tree T on r levels by $W_t(r)$, and define it recursively as follows: $W_t(0)$ is a leaf. $W_t(r)$ consists of vertices $\{v_1, \dots, v_t\}$, a leaf denoted by v_{t+1} , and edges $e_{i,0}$ and $e_{i,1}$ for $i \in \{1, \dots, t\}$. The vertices v_1 to v_t will be said to belong to layer or level 1. Each edge $e_{i,0}$ is labelled 0 and it will be called a 0-edge, and it connects v_i and v_{i+1} . Each edge $e_{i,1}$ is labelled 1 and it will be called a 1-edge, and it connects v_i to the root of $W_t(r-1)$.

Definition 24. (*Clipped xor tree tribe*) A boolean function f is called a t -clipped xor tree tribe on r levels, denoted by $\Xi_t(r)$, if $T(f)$ is a tree tribe, and can be expressed as a complete t -clipped decision tree on r levels, in which the leaves are labelled by the parity of the edges on the path from the root to the leaf.

Definition 25. (*Level of variable x in $\Xi_t(r)$*) We can define the level formally, but the informal definition is cleaner. We define the level of some variable x in $\Xi_t(r)$ by the *recursive* step at which it was added to $\Xi_t(r)$. The variables at level 1 are x_1 to x_t , each of which is connected to a copy of $\neg\Xi_t(r-1)$ on distinct variables. The first t variables in each of the t independent copies of $\neg\Xi_t(r-1)$, a total of t^2 variables, are at level 2, and each of them is connected to a copy of $\Xi_t(r-2)$, and so on.

An example for $\Xi_2(3)$ is given in figure 1. The functions $\Xi_t(r)$ exhibit many nice properties, which we discuss next.

4.2 Properties of tree tribes

We discuss some properties and observations about Ξ and $\Xi_t(r)$. Varying the parameters t and r lead to many interesting properties.

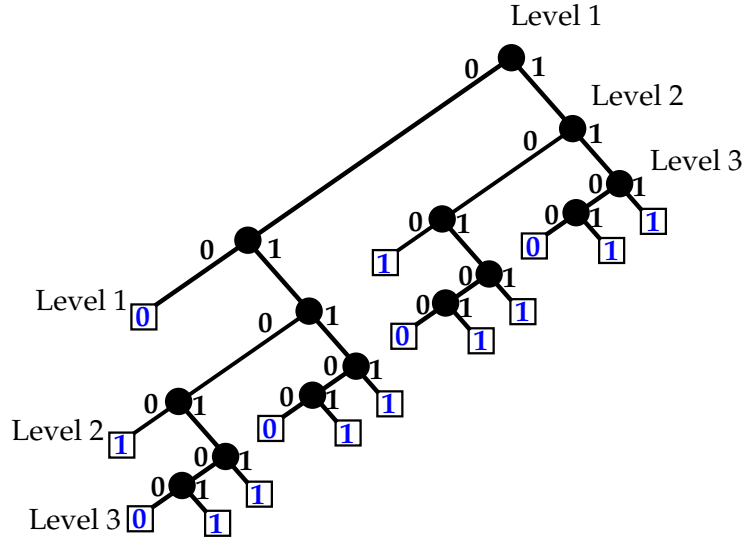


Figure 1: A 2-clipped xor tree tribe on 3 levels, or $\Xi_2(3)$. The variables at each of the vertices are distinct, and the leaves are labelled by the parity of the edges along the root to leaf path. Note that there are 2 vertices at level 1, 4 vertices at level 2, and 8 vertices at level 3.

Basic properties

The following properties of $\Xi_t(r)$ follow directly from the definitions.

1. The number of variables n in $\Xi_t(r)$ is

$$n = \sum_{i=1}^r t^i = t \frac{t^r - 1}{t - 1} \approx t^r$$

and if $t = 1$, then $n = r$.

2. All the leaves at the same level in $\Xi_t(r)$ have the same value, i.e., the odd levels have leaves evaluating to 0 and the even levels have leaves evaluating to 1.
3. $\Xi_t(r)$ can be written as a DNF of width $O\left(\frac{t}{\log t} \log n\right)$.
4. $\Xi_t(r)$ preserves its substructure. If $\neg \Xi_t(r)$ is the negation of the function $\Xi_t(r)$, then all the 1-edges out of the vertices in the 1st level of $\Xi_t(r)$ are connected to $\neg \Xi_t(r - 1)$. This is exactly the manner in which it behaves like parity or xor of variables.
5. Every vertex in $\Xi_t(r)$ is at a distance of at most $t + 1$ from a leaf that evaluates to 0, *and* at a distance of at most $t + 1$ from a leaf that evaluates to 1.

Amongst all these, we feel that observation (5) is the main reason that $\Xi_t(r)$ turns out to be resilient towards random p -restrictions. Due to (5), $\Xi_t(r)$ behaves like a *delayed* parity, or more specifically a *t-delayed* parity, since the value of a vertex can affect the function after $(t + 1)$ steps, and cause it

to evaluate to any of 0 or 1. However, it is different from parity since a variable can affect the value of the function *only* if an input string takes a path from the root to the leaf through that variable in the decision tree. This fact puts a limit on the depth to which we can prove Theorem 2.

Bias

The bias of $\Xi_t(r)$, as is immediate from equations 19 and 21 that will be derived later, is given by

$$\text{bias}(\Xi_t(r)) = \left| \frac{1 - \left(\frac{1}{2^t} - 1\right)^{r+1}}{2 - \frac{1}{2^t}} - \frac{1}{2} \right|.$$

For the case $r \gg t$, the bias is about $\frac{1}{2-2^{-t}} - \frac{1}{2}$. If $t = 1$, the bias is about $\frac{1}{6}$. As t becomes large enough, the bias goes to 0. For the case $r \ll t$, the bias is about $\frac{1}{2}$. Note that if $r = 1$, then $\Xi_t(1)$ is simply the OR function on t variables. If $r \approx t$, then the bias is about $\frac{1}{2} - \frac{t}{2^t}$. If $r \approx 2^t$, then the bias is about a small constant less than $\frac{1}{2}$.

Correlation with parity

As observed in 5, the function $\Xi_t(r)$ behaves like a t -delayed parity in a certain sense. However, if $t \geq 2$, it has correlation exactly $\frac{1}{2}$ with parity, i.e., same as that of a constant function. To see this, let \bar{x} be a bit string on which $T(\bar{x})$ evaluates to 0, taking some path from the root x_1 to some leaf l . Since all the variables of $\Xi_t(r)$ are different and $t \geq 2$, there is some variable y that is not on the path from x_1 to l , and flipping its value cannot change the value of $T(\bar{x})$, but flipping y will always change the parity of the variables. Thus the correlation of $\Xi_t(r)$ with parity is exactly $\frac{1}{2}$, when $t \geq 2$. When $t = 1$, $\Xi_1(r)$ has exactly r variables, and the correlation with parity is about $\frac{1}{2} + \frac{1}{2^r}$ which again goes to $\frac{1}{2}$ for large enough r .

Influence of variables

An interesting thing about a tree tribe Ξ in general is the influence of variables. Let a variable y be at a distance d from the root. Then note that y has no influence over the value of the function for any string \bar{x} that takes a path in Ξ from the root to a leaf not passing through y . Thus,

$$\text{Inf}(y) \leq 2^{-d}.$$

As a result, the variables in a tree tribe have exponentially decreasing influences with distance from the root.³

Fourier coefficients

The fourier coefficients of $\Xi_t(r)$ exhibit interesting combinatorics.

The case $t = 1$: For $t = 1$, we compute the fourier coefficients recursively. Let the variables of $\Xi_1(n)$ be $X = (x_1, \dots, x_n)$ taken naturally from the root x_1 . We claim the following.

³The name *tree dictators* seems more suitable due to this property, but we call them tree tribes for succinctness.

Lemma 26. Let $S \subseteq X$ be a subset of $X = \{x_1, \dots, x_n\}$, let the variables of $f = \Xi_1(n)$ be ordered naturally with x_1 as the root, and let n be some odd number. Let $j \in \{1, \dots, n\}$ be the largest index of a variable appearing in S . Then

$$\hat{f}_S = \frac{(-1)^{|S|+j}}{2^{n-1}} J_{n-j+1}$$

where J_i is the i 'th Jacobsthal number, given by

$$J_i = \frac{2^i - (-1)^i}{3}.$$

Proof. We assume the variables and leaves take values in 1 or -1 without loss of generality, corresponding to 0 to 1. If we write the multilinear expansion of f , taking products of the terms $\frac{1 \pm x_i}{2}$ from every root to leaf path, the term χ_S will appear only for the leaves that appear after the vertex x_j . Further, since j is the largest index in S , the terms after j will contribute only the value $\pm \frac{1}{2}$. Thus we get that,

$$\begin{aligned} \hat{f}_S &= \frac{(-1)^{|S|-1}}{2^{j-1}} (-1)^{j+1} \left(\frac{1}{2} + \frac{1}{4} \sum_{i=0}^{n-j-2} \left(-\frac{1}{2} \right)^i \right) \\ &= \frac{(-1)^{|S|-1}}{2^{j-1}} (-1)^{j+1} \left(\frac{1}{2} + \sum_{i=2}^{n-j} \left(-\frac{1}{2} \right)^i \right) \\ &= \frac{(-1)^{|S|-1}}{2^{j-1}} (-1)^{j+1} \frac{2}{3} \left(1 - \left(-\frac{1}{2} \right)^{n-j+1} \right) \\ &= \frac{(-1)^{|S|+j}}{2^{n-1}} J_{n-j+1} \end{aligned}$$

as required. Further properties of these numbers can be found at [oIS]. □

The case for any general t .

Lemma 27. Let S be a subset of variables in $f = \Xi_t(r)$. Let $y \in S$ be the variable which is farthest from the root of $\Xi_t(r)$. Let y be at a distance d from the root, at a distance k from the closest leaf, and at level l of $\Xi_t(r)$. Let $\alpha = 2^{-t}$ and $\beta = 2^{-t} - 1$.

If the variables in S do not lie on a path in $\Xi_t(r)$, then $\hat{f}_S = 0$. If all the variables of S lie on a path, then

$$\hat{f}_S = \pm \frac{1}{2^{k+d-1}} \left(\frac{1 - \beta^{r-l+1}}{2 - \alpha} \right).$$

Proof. We make two observations about \hat{f}_S .

1. If the variables in S do not all belong to the same path, i.e., for any two variables x and y , if their least common ancestor in $\Xi_t(r)$ is not one of x or y , then $\hat{f}_S = 0$. This follows because if we write the multilinear expansion of f for every root to leaf path, the term χ_S would never appear, since all the variables in $\Xi_t(r)$ are different and there is no path that contains both x and y .

2. The second observation is similar to the observation made in Lemma 26. Given observation (1), assume that all the variables of S lie on some path. If we write the multilinear expansion for f , any path not passing through y will not contain the term $\frac{1 \pm y}{2}$, and contribute to 0 to the coefficient of χ_S . Further, since y is the most distant variable from the root, the variables after y will contribute only the constant values $\pm \frac{1}{2}$, and y is connected to the function $\Xi_t(r-l+1)$ where the root has distance k to a leaf out of its 0 edge, and to the function $\neg \Xi_t(r-l)$ out of its 0 edge.

Let the parameter t be fixed. Let T be exactly $\Xi_t(r)$, except that the root is at a distance k from the leaf, and all other vertices are at a distance t from some leaf. If there is probability $\frac{1}{2}$ of choosing any edge out of any vertex of T , let $\mu_r(k)$ be the difference between the probability of reaching an edge labelled 1 and an edge labelled -1. Then the recurrence for μ is as follows:

$$\begin{aligned}\mu_k(r) &= \frac{1}{2}\mu_{k-1}(r) - \frac{1}{2}\mu_t(r-1) \\ \mu_0(r) &= 1.\end{aligned}$$

Solving the recurrence, we get,

$$\mu_k(r) = \frac{1}{2^k} - \left(\sum_{i=1}^k \frac{1}{2^i} \right) \mu_t(r-1)$$

and evaluating for $k = t$, we get,

$$\begin{aligned}\mu_t(r) &= \frac{1}{2^t} - \left(\sum_{i=1}^t \frac{1}{2^i} \right) \mu_t(r-1) \\ &= \alpha + \beta \mu_t(r-1) \\ &= \alpha \sum_{i=0}^{r-1} \beta^i + \beta^r \\ &= \alpha \frac{1 - \beta^r}{1 - \beta} + \beta^r.\end{aligned}$$

Substituting the value for $\mu_k(r)$, we get

$$\mu_k(r) = \frac{1}{2^k} + \left(\frac{1}{2^k} - 1 \right) \left(\alpha \frac{1 - \beta^{r-1}}{1 - \beta} + \beta^{r-1} \right).$$

Let us compute \hat{f}_S . Let $S' = S \setminus y$. In the multilinear expansion of f , summing the terms containing coefficient of $\chi_{S'}$ for all the paths passing through y , we get,

$$\pm \frac{\chi_{S'}}{2^d} \left(\frac{1+y}{2} \mu_{k-1}(r-l+1) - \frac{1-y}{2} \mu_t(r-l) \right) = \pm \frac{\chi_S}{2^{d+1}} (\mu_{k-1}(r-l+1) + \mu_t(r-l))$$

and thus

$$\begin{aligned}\hat{f}_S &= \pm \frac{1}{2^{d+1}} \left(\frac{1}{2^{k-1}} + \left(\frac{1}{2^{k-1}} - 1 \right) \mu_t(r-l) + \mu_t(r-l) \right) \\ &= \pm \frac{1}{2^{k+d}} (1 + \mu_t(r-l))\end{aligned}$$

and replacing the value of $\mu_t(r-l)$, it becomes

$$\hat{f}_S = \pm \frac{1}{2^{k+d-1}} \left(\frac{1 - \beta^{r-l+1}}{2 - \alpha} \right).$$

□

Note that the magnitude of the fourier coefficients in Lemma 27 depends only on the parameters k, l , and d for any set S . In particular, the parameters for the farthest variable from the root in S completely determines the *magnitude* of \hat{f}_S , and other variables of S only affect the *sign* of \hat{f}_S . Further, if $r \gg t$, then for paths or sets S in which the farthest vertex is close to the root, $\beta \approx 0$, and

$$|\hat{f}_S| \approx \frac{1}{2^{k+d-1}} \left(\frac{1}{2 - 2^{-t}} \right),$$

implying that all the fourier coefficients are exponentially decreasing in k and d . If we consider r and t as parameters, we get many different sequences of numbers based on the manner in which the parameters - r, t, k, l, d - are set, and these sequences might demonstrate many interesting properties of the function, but we do not explore that here.

5 Lower bound

We now prove Theorem 2 for t -clipped xor tree tribes or $\Xi_t(r)$, by induction on d . However, we cannot use $d = 0$ as the base case, and we discuss this when we do the inductive step. The base case will be $d = 1$, which we solve next. We do not optimize any constants in this section.

5.1 The case for $d = 1$

The base case $d = 1$ turns out to be the most interesting. Here, we want to evaluate the probability that the function $\Xi_t(r)$ has depth more than 1 after being hit with a random p -restriction. More specifically, we want to show the following.

$$\Pr_{\rho}[\text{DT}_{\text{depth}}(\Xi_t(r)|_{\rho}) \geq 1] \geq c_0 p 2^t.$$

Note that this cannot be true for small r , since if the function has fewer than $\sim 2^t$ variables, then the number of variables assigned $*$ will be low on average, and the event $\text{DT}_{\text{depth}}(\Xi_t(r)|_{\rho}) \geq 1$ would be extremely unlikely. Thus, we would show the following.

Lemma 28. *For some universal constants c_0 and c_p , for $r \in \Omega(2^t)$ and $0 \leq p \leq c_p 2^{-t}$, if ρ is a random p -restriction, then*

$$\Pr_{\rho}[\text{DT}_{\text{depth}}(\Xi_t(r)|_{\rho}) \geq 1] \geq c_0 p 2^t.$$

Note that in Lemma 28, we require that the *smallest* depth decision tree for $\Xi_t(r)|_{\rho}$ has depth greater than 1 with good probability, which means that *any* decision tree representing $\Xi_t(r)|_{\rho}$ must query at least one variable. This will be made possible by a simple observation.

Lemma 29. *$\text{DT}_{\text{depth}}(\Xi_t(r)|_{\rho}) \geq 1$ if and only there is a path in $\Xi_t(r)|_{\rho}$ from the root to a leaf that evaluates to 0 and to a leaf that evaluates to 1.*

Proof. Let $\text{DT}_{\text{depth}}(\Xi_t(r)|_\rho) \geq 1$. If all the paths from the root of $\Xi_t(r)|_\rho$ were to a 0-leaf (or 1-leaf), then the function is the 0 function (or 1 function), which has $\text{DT}_{\text{depth}}(\Xi_t(r)|_\rho) = 0$, a contradiction. Now assume that there is a path from the root of $\Xi_t(r)|_\rho$ to a 0-leaf and a path to a 1-leaf. This means that in the truth-table of $\Xi_t(r)|_\rho$, there is a string for which the function is 0 and a string for which the function is 1, implying that it is not equivalently the 0 or 1 function, and $\text{DT}_{\text{depth}}(\Xi_t(r)|_\rho) > 0$ or $\text{DT}_{\text{depth}}(\Xi_t(r)|_\rho) \geq 1$. \square

We define the probabilities that $\Xi_t(r)$ evaluates equivalently to 0, or 1, or has $\text{DT}_{\text{depth}} \geq 1$ respectively.

Definition 30. Let $\Xi_t(r)$ be a t -clipped xor tribe on r levels and ρ a random p -restriction. Let $P_0(r)$ be the probability that $\Xi_t(r)$ evaluates to the 0 function after being hit by a random p -restriction ρ , $P_1(r)$ the probability that $\Xi_t(r)$ evaluates to the 1 function after being hit by ρ , and $P_*(r)$ the probability that $\Xi_t(r)$ has $\text{DT}_{\text{depth}} \geq 1$ after being hit by a random restriction. More formally, letting the parameters p and t be implicit,

$$\begin{aligned} P_0(r) &= \Pr_\rho[f(\Xi_t(r)|_\rho) \equiv 0], \\ P_1(r) &= \Pr_\rho[f(\Xi_t(r)|_\rho) \equiv 1], \\ P_*(r) &= \Pr_\rho[\text{DT}_{\text{depth}}(\Xi_t(r)|_\rho) \geq 1] \\ &= 1 - P_0(r) - P_1(r). \end{aligned} \tag{9}$$

Given Lemma 29 and definition 30, our proof strategy for Lemma 28 is as follows:

Step 1: We write the exact expressions for $P_0(r)$ and $P_1(r)$ as polynomials in p , in Lemmas 31 and 33. The expressions are obtained by counting arguments and using recursion.

Step 2: In the next step, we reason about the constant coefficients in $P_0(r)$ and $P_1(r)$. We derive the expressions for $[1]P_0(r)$ and $[1]P_1(r)$ in Lemmas 34 and 36, and show the following in Lemma 35:

$$[1]P_0(r) + [1]P_1(r) = 1.$$

Step 3: In the third step, we first compute the recursive expressions for $[p]P_0(r)$ and $[p]P_1(r)$ in Lemma 37, and in Lemmas 38 and 39, show that for $r \in \Omega(2^t)$,

$$-4 \cdot 2^t \leq [p](P_0(r) + P_1(r)) \leq -\frac{1}{6}2^t.$$

Step 4: We show that higher powers of p do not substantially affect the coefficient of p . In Lemma 40, we show that for all r , for $0 \leq p \leq c_p 2^{-t}$ where $c_p = \frac{1}{420}$,

$$G_2(P_0(r) + P_1(r)) \leq 30 \cdot 2^{2t}.$$

Step 5: Using the conclusions of Steps 2,3,4, we infer Lemma 28.

We start by writing the recurrence relations for $P_0(r)$ and $P_1(r)$.

5.1.1 Recurrence relations

Note that we set independently set a variable to $*$ with probability p , and to 0 or 1 with probability $q = \frac{1}{2}(1 - p)$.

Lemma 31. *For the base case, $P_0(1)$ and $P_1(1)$ are*

$$P_0(1) = q^t, \quad (10)$$

$$P_0(1) = 1 - (1 - q)^t. \quad (11)$$

Proof. Let the tree tribe be $\Xi_t(1)$. Let the variables queried be $\{x_1, \dots, x_t\}$. Note that for this case of $r = 1$, $\Xi_t(1)$ behaves just like the OR function on the variables. Thus, after being hit by a random restriction, the tree evaluates to the 0 function only if all the variables are assigned 0 by ρ . If there is any variable assigned $*$ or 1 by ρ , the tree cannot evaluate to 0. Thus,

$$P_0(1) = q^t.$$

Computing $P_1(1)$ is simple and will illustrate observation 29. In this case, it is possible that some variables are assigned $*$ by ρ , however, the function evaluates to 1. As an example, consider the restriction such that $\rho(x_1) = *$ and $\rho(x_2) = 1$. Here, although x_1 is assigned $*$, the function evaluates to 1, specifically because the root does not have a path to a 0-leaf and a path to a 1-leaf. To compute $P_1(1)$ succinctly, let $P_{1,t}(1)$ be the probability that $\Xi_t(1)$ evaluates to 1, i.e.,

$$P_{1,t}(1) = \Pr_{\rho}[f(\Xi_t(1)|_{\rho}) \equiv 1].$$

We have made the parameter t explicit to ease the computation of $P_1(1)$. Note that $P_{1,t}(1) = P_1(1)$ and $P_{1,1}(1) = q$. The subtree connected to the 0-edge of x_1 is exactly $\Xi_{t-1}(1)$, and along the 1-edge of x_1 is the leaf labelled 1. Then, writing the recursion for $P_{1,t}(1)$, we get,

$$\begin{aligned} P_{1,t}(1) &= q + (p + q)P_{1,t-1}(1) \\ &= q \sum_{i=0}^{t-2} (p + q)^i + (p + q)^{t-1} P_{1,1}(1) \\ &= 1 - (1 - q)^t \end{aligned}$$

And thus,

$$P_0(1) = 1 - (1 - q)^t,$$

as required. □

Lemma 32. *Let the variables in the first level of $\Xi_t(r)$ be $X = \{x_1, \dots, x_t\}$, and the t children out of each of their 1-edges be Q_1, \dots, Q_t respectively. Let $Y_0 \subseteq X$ and $Y_* \subseteq X$ be subsets of variables assigned 0 and $*$ respectively by ρ where $Y_0 \cap Y_* = \emptyset$, such that the index of every variable in Y_0 and Y_* is less than j , and $\rho(x_j) = 1$. Then $\Xi_t(r)|_{\rho} \equiv 0$ if and only if $Q_i|_{\rho} \equiv 0$ for all $x_i \in Y_*$ and $Q_j|_{\rho} \equiv 0$.*

Proof. For any $i < j$ where i is the least index such that $x_i \in Y_*$, if $Q_i|_{\rho} \not\equiv 0$ and $Q_j|_{\rho} \not\equiv 1$, then $Q_i|_{\rho}$ requires a decision tree of depth at least 1, and by observation 29, this means that the root of $Q_i|_{\rho}$ has a path to both a 0-leaf and a 1-leaf. Since the variables of Q_i are all different from X and thus Y_* , this implies that x_1 has a path to some 0 and 1 leaves, and thus $\Xi_t(r)|_{\rho}$ requires a decision tree

of depth at least 1. Consider the case where each of $Q_i|_\rho \equiv 0$ or $Q_i|_\rho \equiv 1$ for all $x_i \in Y_*$. But if for some x_i , $Q_i|_\rho \equiv 1$, then the function $\Xi_t(r)|_\rho$ can evaluate to 1 when x_i is queried by some decision tree, and $\Xi_t(r)|_\rho \not\equiv 0$. In case $Y_* = \emptyset$, the claim trivially follows. Exactly by the same argument, it is required that $Q_j|_\rho \equiv 0$ for $\Xi_t(r)|_\rho \equiv 0$. \square

Lemma 33. *Let $U = P_1(r-1)$ and $V = P_0(r-1)$. Then,*

$$\begin{aligned} P_0(r) &= \sum_{k=0}^{t-1} qU (q + pU)^k + (q + pU)^t \\ &= qU \frac{1 - (q + pU)^t}{1 - (q + pU)} + (q + pU)^t \end{aligned} \quad (12)$$

and

$$\begin{aligned} P_1(r) &= \sum_{k=0}^{t-1} qV (q + pV)^k \\ &= qV \frac{1 - (q + pV)^t}{1 - (q + pV)}. \end{aligned} \quad (13)$$

Proof. Let the variables in the first level of $\Xi_t(r)$ be $X = \{x_1, \dots, x_t\}$, and the t children out of each of their 1-edges be Q_1, \dots, Q_t respectively, where note that by construction, each of the functions $Q_i = \neg \Xi_t(r-1)$. Thus, we have that for all $i \in \{1, \dots, t\}$,

$$\begin{aligned} \Pr[(Q_i|_\rho) \equiv 0] &= P_1(r-1), \\ \Pr[(Q_i|_\rho) \equiv 1] &= P_0(r-1). \end{aligned} \quad (14)$$

Note that since the variables in each Q_i are different, we have complete independence between the probabilities that they evaluate to 0 or 1, i.e.,

$$\Pr_\rho[Q_{i_1}|_\rho \equiv a_{i_1}, \dots, Q_{i_l}|_\rho \equiv a_{i_l}] = \Pr_\rho[Q_{i_1}|_\rho \equiv a_{i_1}] \dots \Pr_\rho[Q_{i_l}|_\rho \equiv a_{i_l}]$$

where $a_{i_j} \in \{0, 1\}$ for $j \in \{1, \dots, l\}$. To compute $P_0(r)$, we partition the events based on the least index $k \in \{1, \dots, t\}$ such that $\rho(x_k) = 1$, and the case that no variable x_k is assigned 1 by ρ . If k is the least index such that $\rho(x_k) = 1$, then note that for every $i \in \{0, \dots, k-1\}$, there is a possibility of having i variables assigned $*$ by ρ in $\{x_1, \dots, x_{k-1}\}$, and the rest assigned 0. To compute $P_0(r)$, we would require that for every Q_i where $i \in \{1, \dots, k-1\}$ and $\rho(x_i) = *$, $f(Q_i) \equiv 0$. Further, we would require that $f(Q_k) \equiv 0$. Summing up the probabilities by partitioning events in this manner, using 14 and claim 32, we get that,

$$\begin{aligned} P_0(r) &= \sum_{k=0}^{t-1} qP_1(r-1) \sum_{i=0}^k \binom{k}{i} q^{k-i} p^i (P_1(r-1))^i + \sum_{i=0}^t \binom{t}{i} q^{t-i} p^i (P_1(r-1))^i \\ &= \sum_{k=0}^{t-1} qP_1(r-1) (q + pP_1(r-1))^k + (q + pP_1(r-1))^t \\ &= qU \frac{1 - (q + pU)^t}{1 - (q + pU)} + (q + pU)^t. \end{aligned}$$

To compute $P_1(r)$, note that in this case, at least one of the variables in x_1, \dots, x_t must be assigned 1 by ρ , since if all the variables are assigned either 0 or $*$ by ρ , then by definition of $\Xi_t(r)$, there is always a possibility of the remaining function to evaluate to 0 if all the variables that are assigned $*$ by ρ take the value 0. All other details remain exactly the same, and we get,

$$\begin{aligned} P_1(r) &= \sum_{k=0}^{t-1} qP_0(r-1) \sum_{i=0}^k \binom{k}{i} q^{k-i} p^i (P_0(r-1))^i \\ &= \sum_{k=0}^{t-1} qP_0(r-1) (q + pP_0(r-1))^k \\ &= qV \frac{1 - (q + pV)^t}{1 - (q + pV)}. \end{aligned}$$

□

All of $P_0(r)$, $P_1(r)$ and $P_*(r)$ in equations 12, 13 and 9 are univariate polynomials in p . We reason about the constant coefficients in $P_0(r)$ and $P_1(r)$, i.e. $[1]P_0(r)$ and $[1]P_1(r)$. The straightforward proofs are given in appendix A.

5.1.2 The constant coefficients, $[1]P_0(r)$ and $[1]P_1(r)$

Lemma 34. *The recursive expressions for $[1]P_0(r)$ and $[1]P_1(r)$ are as follows:*

$$[1]P_0(1) = \frac{1}{2^t} \tag{15}$$

$$[1]P_1(1) = 1 - \frac{1}{2^t}, \tag{16}$$

and

$$[1]P_0(r) = \left(1 - \frac{1}{2^t}\right) [1]P_1(r-1) + \frac{1}{2^t}, \tag{17}$$

$$[1]P_1(r) = \left(1 - \frac{1}{2^t}\right) [1]P_0(r-1). \tag{18}$$

Proof. The proof is given as Lemma 45 in the appendix. □

Using the recurrence relations in Lemma 34, we can prove the following simple but useful fact.

Lemma 35. $[1]P_0(r) + [1]P_1(r) = 1$ and $[1]P_0(r) \geq 0, [1]P_1(r) \geq 0$.

Proof. The fact that the coefficients are always non-negative follows simply from equations 17 and 18 via straightforward induction on r . We use induction on r again to show that the coefficients sum to 1. For $r = 1$, from equations 15 and 16,

$$[1]P_0(1) + [1]P_1(1) = \frac{1}{2^t} + 1 - \frac{1}{2^t} = 1.$$

Adding equations 17 and 18, we get

$$[1]P_0(r) + [1]P_1(r) = \left(1 - \frac{1}{2^t}\right) ([1]P_1(r-1) + [1]P_0(r-1)) + \frac{1}{2^t}$$

and the induction follows. \square

We compute the exact equations for $[1]P_0(r)$ and $[1]P_1(r)$, which will be useful when we compute the coefficients of higher powers of p .

Lemma 36. *Let $\alpha = 2^{-t}$ and $\beta = 1 - \alpha$. Then the exact expressions for $[1]P_0(r)$ and $[1]P_1(r)$ are as follows. If $r = 2k + 1$,*

$$[1]P_0(r) = \frac{1 - \beta^{r+1}}{1 + \beta}, \quad (19)$$

$$[1]P_1(r) = \frac{\beta + \beta^{r+1}}{1 + \beta}, \quad (20)$$

and if $r = 2k$,

$$[1]P_0(r) = \frac{1 + \beta^{r+1}}{1 + \beta}, \quad (21)$$

$$[1]P_1(r) = \frac{\beta - \beta^{r+1}}{1 + \beta}. \quad (22)$$

Proof. The proof is given as Lemma 46 in the appendix. \square

We bound the coefficient of p in $P_0(r)$ and $P_1(r)$.

5.1.3 The coefficients of p , $[p]P_0(r)$ and $[p]P_1(r)$

Lemma 37. *The recurrence relations for $[p]P_0(r)$ and $[p]P_1(r)$ are as follows:*

$$[p]P_0(1) = -\frac{t}{2^t} \quad (23)$$

$$[p]P_1(1) = -\frac{t}{2^t}. \quad (24)$$

and

$$\begin{aligned} [p]P_1(r) &= \left(1 - \frac{1}{2^t}\right) [p]P_0(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1]P_0(r-1) \\ &\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_0(r-1)^2 \end{aligned} \quad (25)$$

$$\begin{aligned} [p]P_0(r) &= \left(1 - \frac{1}{2^t}\right) [p]P_1(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1]P_1(r-1) \\ &\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_1(r-1)^2 - \frac{t}{2^t} + \frac{2t}{2^t} [1]P_1(r-1). \end{aligned} \quad (26)$$

Proof. The proof is given as Lemma 47 in the appendix. \square

Given the recurrence relations for $[p]P_0(r)$ and $[p]P_1(r)$, although it is possible to compute the exact expressions for them, we will only need specific bounds on them, which we find next.

Lemma 38. For every $r \geq 1$, $-2 \cdot 2^t \leq [p]P_0(r) \leq 0$ and $-2 \cdot 2^t \leq [p]P_1(r) \leq 0$.

Proof. The proof is given as Lemma 48 in the appendix. \square

The next lemma is important, since it gives the coefficient of p that we need for the lower bound of Lemma 28.

Lemma 39. For $r \in \Omega(2^t)$, $[p](P_0(r) + P_1(r)) \leq -c_1 2^t$ where $c_1 = \frac{1}{6}$.

Proof. Define

$$g(r) = [p]P_0(r) + [p]P_1(r)$$

and

$$u(r) = [1]P_1(r).$$

Note that we have $\alpha = 2^{-t}$ and $\beta = 1 - \alpha$. Adding equations 26 and 25, and using Lemma 35, we get,

$$\begin{aligned} g(r) &= \left(1 - \frac{1}{2^t}\right) g(r-1) + \left(\frac{t+2}{2^t} - 2\right) - \frac{t}{2^t} + \frac{2t}{2^t} [1]P_1(r-1) \\ &\quad + 2 \left(1 - \frac{t+1}{2^t}\right) ([1]P_1(r-1))^2 + [1]P_0(r-1)^2 \\ &= \beta g(r-1) + \left(\frac{2}{2^t} - 2\right) + \frac{2t}{2^t} [1]P_1(r-1) \\ &\quad + 2 \left(1 - \frac{t+1}{2^t}\right) (1 - 2[1]P_1(r-1) + 2[1]P_1(r-1)^2) \\ &= \beta g(r-1) - 2\alpha t + (6\alpha t + 4\alpha - 4)u_{r-1} + (4 - 4\alpha - 4\alpha t)u_{r-1}^2 \\ &= \beta g(r-1) + C_0 + C_1 u_{r-1} + C_2 u_{r-1}^2 \end{aligned}$$

where $C_0 = -2\alpha t$, $C_1 = 6\alpha t + 4\alpha - 4$, and $C_2 = 4 - 4\alpha - 4\alpha t$. Thus,

$$g(r) = \beta^{r-1} g(1) + C_0 \sum_{i=0}^{r-2} \beta^i + C_1 \sum_{i=0}^{r-2} \beta^i u_{r-i-1} + C_2 \sum_{i=0}^{r-2} \beta^i u_{r-i-1}^2.$$

Fix $r = 2k$ to be an even number where $k \geq 2$. The third term can be written using equations 20 and 22 as,

$$\begin{aligned} \sum_{i=0}^{r-2} \beta^i u_{r-i-1} &= \sum_{i=0}^{\frac{1}{2}(r-2)-1} \beta^{2i} (u_{r-2i-1} + \beta u_{r-2i-2}) + \beta^{r-2} u_1 \\ &= \sum_{i=0}^{\frac{1}{2}(r-2)-1} \beta^{2i} \left(\frac{\beta + \beta^{r-2i}}{1 + \beta} + \beta \frac{\beta - \beta^{r-2i-1}}{1 + \beta} \right) + \beta^{r-2} u_1 \\ &= \beta \frac{1 - \beta^{r-2}}{1 - \beta^2} + \beta^{r-1} \\ &= \beta \frac{1 - \beta^r}{1 - \beta^2} \end{aligned}$$

where in the second line we used 16 for u_1 . Similarly, for the third summation,

$$\begin{aligned}
\sum_{i=0}^{r-2} \beta^i u_{r-i-1}^2 &= \sum_{i=0}^{\frac{1}{2}(r-2)-1} \beta^{2i} (u_{r-2i-1}^2 + \beta u_{r-2i-2}^2) + \beta^{r-2} u_1^2 \\
&= \sum_{i=0}^{\frac{1}{2}(r-2)-1} \beta^{2i} \left(\left(\frac{\beta + \beta^{r-2i}}{1 + \beta} \right)^2 + \beta \left(\frac{\beta - \beta^{r-2i-1}}{1 + \beta} \right)^2 \right) + \beta^{r-2} u_1^2 \\
&= \frac{\beta^2}{1 + \beta} \sum_{i=0}^{\frac{1}{2}(r-2)-1} \beta^{2i} + \frac{\beta^{2r-1}}{1 + \beta} \sum_{i=0}^{\frac{1}{2}(r-2)-1} \beta^{-2i} + \beta^{r-2} u_1^2 \\
&= \frac{\beta^2}{1 + \beta} \frac{(1 + \beta^{r-1})(1 - \beta^r)}{1 - \beta^2}.
\end{aligned}$$

Thus the expression for $g(r)$ becomes,

$$\begin{aligned}
g(r) &= \beta^{r-1} g(1) + C_0 \frac{1 - \beta^{r-1}}{1 - \beta} + C_1 \beta \frac{1 - \beta^r}{1 - \beta^2} + C_2 \frac{\beta^2}{1 + \beta} \frac{(1 + \beta^{r-1})(1 - \beta^r)}{1 - \beta^2} \\
&= \beta^{r-1} g(1) + (-2\alpha t) \frac{1 - \beta^{r-1}}{1 - \beta} + (6\alpha t + 4\alpha - 4) \beta \frac{1 - \beta^r}{1 - \beta^2} \\
&\quad + (4 - 4\alpha - 4\alpha t) \frac{\beta^2}{1 + \beta} \frac{(1 + \beta^{r-1})(1 - \beta^r)}{1 - \beta^2} \\
&= \beta^{r-1} g(1) + A(r) + B(r), \tag{27}
\end{aligned}$$

where

$$\begin{aligned}
A(r) &= (4\alpha - 4) \beta \frac{1 - \beta^r}{1 - \beta^2} + (4 - 4\alpha) \frac{\beta^2}{1 + \beta} \frac{(1 + \beta^{r-1})(1 - \beta^r)}{1 - \beta^2} \\
&= (4 - 4/\alpha) \beta \frac{1 - \beta^r}{1 + \beta} \left(1 - \frac{\beta}{1 + \beta} (1 + \beta^{r-1}) \right) \\
&= (4 - 4/\alpha) \beta \frac{(1 - \beta^r)^2}{(1 + \beta)^2} \\
&= -\frac{4}{\alpha} \beta^2 \frac{(1 - \beta^r)^2}{(1 + \beta)^2},
\end{aligned}$$

and

$$\begin{aligned}
B(r) &= (-2\alpha t) \frac{1 - \beta^{r-1}}{1 - \beta} + (2\alpha t) \beta \frac{1 - \beta^r}{1 - \beta^2} \\
&\quad + (4\alpha t) \beta \frac{1 - \beta^r}{1 - \beta^2} + (-4\alpha t) \frac{\beta^2}{1 + \beta} \frac{(1 + \beta^{r-1})(1 - \beta^r)}{1 - \beta^2} \\
&= (-2t) \left(\frac{1 - \beta^{r-1} - \beta^r + \beta^{r+1}}{1 + \beta} \right) \\
&\quad + (4t) \beta \frac{1 - \beta^r}{1 + \beta} \left(1 - \frac{\beta}{1 + \beta} (1 + \beta^{r-1}) \right) \\
&= (-2t) \left(\frac{1 - \beta^{r-1} - \beta^r + \beta^{r+1}}{1 + \beta} \right) + 4t \beta \frac{(1 - \beta^r)^2}{(1 + \beta)^2} \\
&= \frac{2t}{(1 + \beta)^2} \left(-(1 + \beta)(1 - \beta^{r-1} - \beta^r + \beta^{r+1}) + 2\beta(1 - \beta^r)^2 \right) \\
&= \frac{2t}{(1 + \beta)^2} (\beta - 1 + 2\beta^r - 4\beta\beta^r + \beta^{r-1} + 2\beta^{2r+1} - \beta^{r+2}) \\
&\leq \frac{2t\beta^r}{(1 + \beta)^2} \left(\frac{1}{\beta} + 2\beta^{r+1} - \beta^2 \right) \\
&\leq \frac{8t\beta^r}{(1 + \beta)^2},
\end{aligned}$$

where in the first inequality, we used the facts that since $\frac{1}{2} \leq \beta = 1 - 2^{-t} \leq 1$, we would have $\beta - 1 \leq 0$ and $1 - 2\beta \leq 0$, and in the second inequality we used the same bounds for β after factoring out β^r . Replacing the values of $A(r)$ and $B(r)$ in equation 27 for $g(r)$, and using that $\beta \geq 0$ and $g(1) \leq 0$ from Lemma 37, we get that

$$\begin{aligned}
g(r) &= \beta^{r-1} g(1) + A(r) + B(r) \\
&\leq -\frac{4}{\alpha} \beta^2 \frac{(1 - \beta^r)^2}{(1 + \beta)^2} + \frac{8t\beta^r}{(1 + \beta)^2} \\
&= -\frac{3}{\alpha} \beta^2 \frac{(1 - \beta^r)^2}{(1 + \beta)^2} - \frac{1}{\alpha} \beta^2 \frac{(1 - \beta^r)^2}{(1 + \beta)^2} + \frac{8t\beta^r}{(1 + \beta)^2}
\end{aligned}$$

Taking $r = C2^t$ for some constant C , we get that

$$\beta^r = \left(1 - \frac{1}{2^t} \right)^{C2^t} \leq e^{-C},$$

and using $\alpha = 2^{-t}$ and $\frac{1}{2} \leq \beta \leq 1$, we get

$$\begin{aligned}
-\frac{1}{\alpha} \beta^2 \frac{(1 - \beta^r)^2}{(1 + \beta)^2} + \frac{8t\beta^r}{(1 + \beta)^2} &\leq -2^t \frac{1}{4} \frac{(1 - \beta^r)^2}{(1 + \beta)^2} + \frac{8t\beta^r}{(1 + \beta)^2} \\
&\leq -\frac{t}{4(1 + \beta)^2} \left((1 - \beta^r)^2 - 32\beta^r \right) \\
&\leq -\frac{t}{4(1 + \beta)^2} \left((1 - e^{-C})^2 - 32e^{-C} \right) \\
&\leq 0
\end{aligned}$$

where in the second line, we used $t \leq 2^t$, and in the last line, we used the fact that for $C = 4$, $(1 - e^{-C})^2 - 32e^{-C} \geq 0$. Thus we get that

$$\begin{aligned} g(r) &\leq -\frac{3}{\alpha} \beta^2 \frac{(1 - \beta^r)^2}{(1 + \beta)^2} \\ &\leq -3 \cdot 2^t \frac{1}{4} (1 - e^{-C})^2 \\ &\leq -\frac{1}{6} 2^t \end{aligned}$$

where in the second line, we used that for $C = 4$, $(1 - e^{-C})^2 \geq \frac{16}{18}$. \square

Thus so far, we have the following facts about the polynomial $P_0(r) + P_1(r)$, as was mentioned in Steps 2 and 3.

1. The constant coefficient is 1, i.e. $[1]P_0(r) + [1]P_1(r) = 1$, and
2. The coefficient of p is between $-4 \cdot 2^t$ and $-c_1 2^t$ for $r \in \Omega(2^t)$ and $c_1 = \frac{1}{6}$, i.e.,

$$-4 \cdot 2^t \leq [p]P_0(r) + [p]P_1(r) \leq -c_1 2^t.$$

Note that we almost have sufficient information to conclude Lemma 28, however, we need to reason that higher powers of p in $P_0(r) + P_1(r)$ cannot substantially affect the coefficient of p . It is possible that higher powers of p have coefficients with large magnitudes, and thus annihilate the mass on the coefficient of p . We show next that this cannot happen. Here we will use the fact that $p \leq c_p 2^{-t}$.

5.1.4 Bounding higher powers of p in $P_0(r) + P_1(r)$

To bound the higher powers of p , we will show that $[\uparrow p^2](P_0(r) + P_1(r))$ cannot achieve a very large value for $p \leq c_p 2^{-t}$. Thus, our aim is to show that $G_2(P_0(r) + P_1(r))$ is small.

Lemma 40. *For every $r \geq 1$, for $0 \leq p \leq p_{\max} = c_p 2^{-t}$, where $c_p = \frac{1}{420}$,*

$$G_2(P_0(r) + P_1(r)) \leq 30 \cdot 2^{2t}.$$

Proof. We will assume that

$$p \leq p_{\max} = \frac{1}{C 2^t}$$

where $C = \frac{1}{c_p}$. Using part (6) of claim 11,

$$G_2(P_0(r) + P_1(r)) \leq G_2(P_0(r)) + G_2(P_1(r))$$

and we will bound the terms individually. We will show this fact by induction. For the base case $r = 1$, using claim 12,

$$G_2(P_0(1)) \leq \max_p \frac{1}{2^t} \frac{1}{p^2} ((1 - p)^t - 1 + tp) \leq \frac{1}{2^t} \binom{t}{2} \leq 30 \cdot 2^{2t},$$

and for $P_1(1)$,

$$\begin{aligned}
G_2(P_1(1)) &\leq \max_p \left| [\uparrow p^2] \left(1 - \frac{1}{2^t} (1+p)^t \right) \right| \\
&= \max_p \left| -\frac{1}{2^t} \sum_{i=2}^t \binom{t}{i} p^{i-2} \right| \\
&\leq \frac{1}{2^t} \sum_{i=2}^t \binom{t}{i} \\
&\leq 1,
\end{aligned}$$

where we used the fact that $p \leq 1$ in the third line. Assume that for all $1 \leq z \leq r$,

$$\begin{aligned}
G_2(P_0(z)) &\leq \Gamma, \\
G_2(P_1(z)) &\leq \Gamma.
\end{aligned}$$

Reproducing equations 12 and 13 for $P_0(r)$ and $P_1(r)$, for $U = P_1(r-1)$ and $V = P_0(r-1)$,

$$\begin{aligned}
P_0(r) &= qU \sum_{i=0}^{t-1} (q+pU)^i + (q+pU)^t, \\
P_1(r) &= qV \sum_{i=0}^{t-1} (q+pV)^i.
\end{aligned}$$

Let us analyze one term of $P_0(r)$, $qU(q+pU)^k$. Note that

$$\begin{aligned}
[\uparrow p^2]qU(q+pU)^k &= [\uparrow p^2] \sum_{i=0}^k \binom{k}{i} q^{k-i+1} p^i U^{i+1} \\
&= \frac{1}{p^2} \left(qU(q+pU)^k - q^{k+1}U - kq^k pU^2 \right) + [\uparrow p^2]q^{k+1}U + [\uparrow p]kq^kU^2. \\
[\uparrow p^2]q^{k+1}U &= \frac{1}{p^2} \left(\frac{U}{2^{k+1}} \left((1-p)^{k+1} - 1 + (k+1)p \right) \right) + [\uparrow p^2] \frac{U}{2^{k+1}} - [\uparrow p] \frac{k+1}{2^{k+1}} U. \\
[\uparrow p]kq^kU^2 &= \frac{1}{p} \left(\frac{kU^2}{2^k} \left((1-p)^k - 1 \right) \right) + [\uparrow p] \frac{k}{2^k} U^2.
\end{aligned}$$

To analyse each of these terms, let the functions be as follows:

$$\begin{aligned}
h_1^k &= \frac{1}{p^2} \left(qU(q+pU)^k - q^{k+1}U - kq^k pU^2 \right) \\
h_2^k &= \frac{1}{p^2} \left(\frac{U}{2^{k+1}} \left((1-p)^{k+1} - 1 + (k+1)p \right) \right) \\
h_3^k &= \frac{1}{p} \left(\frac{kU^2}{2^k} \left((1-p)^k - 1 \right) \right) \\
h_4^k &= [\uparrow p^2] \frac{U}{2^{k+1}} \\
h_5^k &= [\uparrow p] \frac{k}{2^k} U^2 \\
h_6^k &= -[\uparrow p] \frac{k+1}{2^{k+1}} U
\end{aligned}$$

Consider the function h_1^k .

$$\begin{aligned} h_1^k &= \frac{1}{p^2} \left(qU(q + pU)^k - q^{k+1}U - kq^k pU^2 \right) \\ &= \sum_{i=2}^k \binom{k}{i} q^{k-i+1} p^{i-2} U^{i+1}. \end{aligned}$$

Since $h_1^k \geq 0$, $|h_1^k| = h_1^k$. Further, since $q \geq 0$ and $p \geq 0$, and the function above is an increasing function of $U = P_1(r-1) \leq 1$, we can set $U = 1$. This step is innocuous but crucial, since there are large cancellations that happen within the powers of p in the expression for U . Thus, we can bound,

$$\begin{aligned} \max_p |h_1^k| &\leq \sum_{i=2}^k \binom{k}{i} q^{k-i+1} p^{i-2} \\ &\leq \binom{k}{2} \frac{1}{2^{k-1}} + p \sum_{i=3}^k \binom{k}{i} q^{k-i+1} p^{i-3} \\ &\leq \binom{k}{2} \frac{1}{2^{k-1}} + p \sum_{i=3}^k \binom{k}{i} \\ &\leq \binom{k}{2} \frac{1}{2^{k-1}} + \frac{1}{C} \frac{2^k}{2^i} \end{aligned}$$

where in the second line, we used $q \leq \frac{1}{2}$, in the third line, we used $q \leq 1$ and $p \leq 1$, and in the last line we used $p \leq \frac{1}{C2^i}$. By a similar argument, noting that $(1-p)^{k+1} - 1 + (k+1)p \geq 0$ for $k \geq 0$ and $p \in [0, 1]$, we get that $|h_2^k| = h_2^k$, and setting $U = 1$ and using point (6) in claim 12, we get that,

$$\max_p |h_2^k| \leq \binom{k+1}{2} \frac{1}{2^{k+1}}.$$

For h_3^k , note that $(1-p)^k - 1 \leq 0$ for $k \geq 0$ and $p \in [0, 1]$, and thus $|h_3^k| = -h_3^k$. Again setting $U = 1$ and using point (5) in claim 12,

$$\max_p |h_3^k| \leq \frac{k^2}{2^k}.$$

Using the induction hypothesis for h_4^k , we get that

$$\max_p |h_4^k| \leq \frac{\Gamma}{2^{k+1}}.$$

To bound the maximum of h_5^k and h_6^k , we can write,

$$\begin{aligned} U &= P_1(r-1) \\ &= \alpha_0 + \alpha_1 p + p^2 Q \end{aligned}$$

where α_0 and α_1 are constants and Q is a polynomial in p , such that

$$\begin{aligned} 0 &\leq |\alpha_0| \leq 1 \\ |\alpha_1| &\leq 2 \cdot 2^t \\ \max_p |Q| &\leq \Gamma \end{aligned}$$

which follow respectively from Lemmas 35, 38 and the induction hypothesis. Thus, we can write,

$$\begin{aligned} [\uparrow p]U &= \alpha_1 + pQ \\ G_1(U) = \max_p |[\uparrow p]U| &\leq |\alpha_1| + p_{\max}\Gamma = B_1, \end{aligned} \quad (28)$$

and

$$\begin{aligned} [\uparrow p]U^2 &= (2\alpha_0\alpha_1) + p(\alpha_1^2 + \alpha_0Q) + p^2(2\alpha_1Q) + p^3(Q^2) \\ \max_p |[\uparrow p]U^2| &\leq 2|\alpha_1| + p_{\max}(|\alpha_1|^2 + \Gamma) + p_{\max}^2(2|\alpha_1|\Gamma) + p_{\max}^3\Gamma^2 = B_2. \end{aligned} \quad (29)$$

Thus,

$$\begin{aligned} \max_p |h_5^k| &\leq B_2 \frac{k}{2^k}, \\ \max_p |h_6^k| &\leq B_1 \frac{k+1}{2^{k+1}}. \end{aligned}$$

Summing up all the h^k for different values of k gives the maximum of $|[\uparrow p^2]P_0(r)|$, except for the last term $(q + pU)^t$, which we analyse now, in exactly the same manner as the functions analyzed above. ⁴

$$[\uparrow p^2](q + pU)^t = h_1^t + h_2^t + h_3^t + h_4^t + h_5^t + h_6^t$$

where

$$\begin{aligned} h_1^t &= \frac{1}{p^2} \left((q + pU)^t - q^t - tq^{t-1}pU \right) \\ h_2^t &= \frac{1}{p^2} \left(\frac{1}{2^t} \left((1-p)^t - 1 + tp \right) \right) \\ h_3^t &= \frac{1}{p} \left(\frac{tU}{2^{t-1}} \left((1-p)^t - 1 \right) \right) \\ h_4^t &= [\uparrow p^2] \frac{1}{2^t} \\ h_5^t &= -[\uparrow p] \frac{t}{2^t} \\ h_6^t &= [\uparrow p] \frac{t}{2^{t-1}} U. \end{aligned}$$

⁴The fact that the factor qU does not appear in the last term, i.e., the last term is not $qU(q + pU)^t$ but only $(q + pU)^t$ is the main reason our calculations work out. And note that this factor does not appear in the last term exactly because the tree is t -clipped.

Taking bounds on them in exactly the same manner as shown before, we get that,

$$\begin{aligned}
\max_p |h_1^t| &\leq \binom{t}{2} \frac{1}{2^{t-2}} + \frac{1}{C} \\
\max_p |h_2^t| &\leq \binom{t}{2} \frac{1}{2^t} \\
\max_p |h_3^t| &\leq \frac{t^2}{2^{t-1}} \\
\max_p |h_4^t| &= 0 \\
\max_p |h_5^t| &= 0 \\
\max_p |h_6^t| &\leq B_1 \frac{t}{2^{t-1}}.
\end{aligned}$$

Note that,

$$G_2(P_0(r)) = \max_p |[\uparrow p^2]P_0(r)| \leq \sum_{j=1}^6 \sum_{k=0}^t \max_p |h_j^k|.$$

Summing the first three terms over all k and bounding them using the Lemma 12, we get,

$$\begin{aligned}
\sum_{k=0}^t \max_p |h_1^k| &\leq 2 \sum_{k=0}^{t-1} \binom{k}{2} \frac{1}{2^k} + \sum_{k=0}^{t-1} \frac{1}{C} \frac{2^k}{2^t} + \binom{t}{2} \frac{1}{2^{t-2}} + \frac{1}{C} \leq 4 + \frac{2}{C} + \frac{2t^2}{2^t} \\
\sum_{k=0}^t \max_p |h_2^k| &\leq \sum_{k=0}^{t-1} \binom{k+1}{2} \frac{1}{2^{k+1}} + \binom{t}{2} \frac{1}{2^t} \leq 2 + \frac{\frac{1}{2}t^2}{2^t} \\
\sum_{k=0}^t \max_p |h_3^k| &\leq \sum_{k=0}^{t-1} \frac{k^2}{2^k} + \frac{t^2}{2^{t-1}} = 2 \sum_{k=0}^{t-1} \binom{k}{2} \frac{1}{2^k} + \sum_{k=0}^{t-1} \frac{k}{2^k} + \frac{t^2}{2^{t-1}} \leq 6 + \frac{2t^2}{2^t}
\end{aligned}$$

and thus,

$$\begin{aligned}
D_0 &= \sum_{k=0}^t \max_p |h_1^k| + \sum_{k=0}^t \max_p |h_2^k| + \sum_{k=0}^t \max_p |h_3^k| \\
&\leq 12 + \frac{2}{C} + \frac{9t^2}{2 \cdot 2^t} \\
&\leq 20,
\end{aligned}$$

where in the last line, we assumed that $C \geq 1$. For the fourth term, we get,

$$D_1 = \sum_{k=0}^{t-1} \max_p |h_4^k| \leq \Gamma \sum_{k=0}^{t-1} \frac{1}{2^{k+1}} = \Gamma \left(1 - \frac{1}{2^t}\right).$$

For the last two terms, again bounding using Lemma 12, we get,

$$D_2 = \max_p \sum_{k=0}^{t-1} h_5^k \leq 2B_2,$$

and

$$D_3 = \max_p \sum_{k=0}^t h_6^k \leq B_1 \left(2 + \frac{t}{2^{t-1}} \right) \leq 3B_1.$$

We fix the constants now. Let

$$\Gamma = H2^{2t}$$

where H is some constant, and recall that we had set

$$p_{\max} = \frac{1}{C2^t}.$$

With these constants, the values of B_1 and B_2 can be bound as follows, using Lemma 38 for bounding α_1 :

$$\begin{aligned} B_1 &= |\alpha_1| + p_{\max}\Gamma \\ &\leq 2 \cdot 2^t + \frac{H}{C}2^t \end{aligned}$$

and

$$\begin{aligned} B_2 &= 2|\alpha_1| + p_{\max}(|\alpha_1|^2 + \Gamma) + p_{\max}^2(2|\alpha_1|\Gamma) + p_{\max}^3\Gamma^2 \\ &\leq 4 \cdot 2^t + \frac{1}{C}2^t + \frac{H}{C}2^t + \frac{2H}{C^2}2^t + \frac{H^2}{C^3}2^t. \end{aligned}$$

To show the induction, we need to show that

$$D_0 + D_1 + D_2 + D_3 \leq \Gamma$$

or

$$20 + \Gamma \left(1 - \frac{1}{2^t} \right) + 2B_2 + 3B_1 \leq \Gamma$$

which is equivalent to

$$20 + 14 \cdot 2^t + \frac{5H}{C}2^t + \frac{2}{C}2^t + \frac{4H}{C^2}2^t + \frac{2H^2}{C^3}2^t \leq H2^t$$

which after taking $2^{-t}20 \leq 10$ is implied by

$$24 + \frac{5H}{C} + \frac{2}{C} + \frac{4H}{C^2} + \frac{2H^2}{C^3} \leq H.$$

The equation stated above is satisfied by $H = 30$ and $C = 420$, and it proves the claim. Note that this gives

$$c_p = \frac{1}{C} = \frac{1}{420}$$

in Lemma 2. The proof is exactly the same for $P_1(r)$ and we omit it. □

5.1.5 Lower bound

We finally have enough tools to prove the lower bound for the $d = 1$ case. We restate Lemma 28 here for convenience.

Lemma 41. 28 For some universal constants c_0 and c_p , for $r \in \Omega(2^t)$ and $0 \leq p \leq c_p 2^{-t}$, if ρ is a random p -restriction, then

$$\Pr_{\rho}[DT_{\text{depth}}(\Xi_t(r)|_{\rho}) \geq 1] \geq c_0 p 2^t.$$

Proof. Note that the probability that the decision tree with r levels has depth more than or equal to 1, i.e. it does not become constantly 0 or 1 after being hit by a random restriction is given by

$$P_*(r) = 1 - P_0(r) - P_1(r).$$

For $r \in \Omega(2^t)$, we can write

$$\begin{aligned} P_*(r) &= 1 - [1](P_0(r) + P_1(r)) - ([p]P_0(r) + [p]P_1(r))p - ([\uparrow p^2](P_0(r) + P_1(r)))p^2 \\ &= -([p]P_0(r) + [p]P_1(r))p - ([\uparrow p^2](P_0(r) + P_1(r)))p^2 \\ &\geq c_1 2^t p - p_{\max} 2\Gamma p \\ &= p 2^t \left(\frac{1}{6} - \frac{2H}{C} \right) \\ &\geq \frac{1}{42} p 2^t \end{aligned}$$

where in the second line we used Lemma 35, and in the third line we used Lemmas 39 and 40. Note that we get $c_0 = \frac{1}{42}$. \square

5.2 Inductive step

5.2.1 The issue with taking $d = 0$ as the base case

There is an issue with using $d = 0$ as the base case for induction. Consider a decision tree T with variable x_1 as the root having subtrees T_0 and T_1 out of the 0 and 1 edges respectively. We want to lower bound the probability of the event $E_{T,1}$, the event that T has depth greater than 1 after a random restriction ρ is applied to the variables in T , i.e.

$$\Pr_{\rho}[DT_{\text{depth}}(T|_{\rho}) \geq 1].$$

In Lemma 16, we had that

$$E_{T,1} \subseteq E_{T_0,0} \cup E_{T_1,0}$$

and if x_1 was assigned $*$ by ρ , we could upper bound $\Pr_{\rho}[E_{T,1}]$ by the probabilities $\Pr_{\rho}[E_{T_0,0}]$ and $\Pr_{\rho}[E_{T_1,0}]$, both of which are 1, and there would be no error on the upper bound on $\Pr[E_{T,1}]$. However, we cannot lower bound $\Pr[E_{T,1}]$ as

$$\Pr[E_{T,1}] \geq \Pr[E_{T_0,0}] + \Pr[E_{T_1,0}] - \Pr[E_{T_0,0}]\Pr[E_{T_1,0}]$$

This is because, after applying the random restriction to T , if it assigns $*$ to x_1 , even if T_0 and T_1 had depths greater than 0 under the effect of the restriction, *only if* $T_0|_{\rho} \equiv 1$ and $T_1|_{\rho} \equiv 0$ or $T_0|_{\rho} \equiv 0$ and $T_1|_{\rho} \equiv 1$ would T have depth ≥ 1 . However, the event $E_{T_0,0}$ does not say whether T_0 evaluates to 0 or 1, and thus we cannot use $d = 0$ as the base case.

5.2.2 Lower bound for $d \geq 2$

Doing the inductive step recursively is tricky. This is because we already need about $\sim 2^t$ levels in the tree for the base case to work. Thus, any induction must take care of the fact that once the number of levels are too few, the base case would not hold.

To write the recurrence, let $\gamma_d(r)$ be the probability that $\Xi_t(r)$ has depth greater than or equal to d .

Lemma 42. *Let $\mu = 1 - \gamma_{d-1}(r-1)$. Then,*

$$\gamma_d(r) \geq \sum_{k=1}^{t-1} q \sum_{i=1}^k \binom{k}{i} q^{k-i} p^i (1 - \mu^{i+1}) + \sum_{i=0}^t \binom{t}{i} q^{t-i} p^i (1 - \mu^i) + \sum_{k=0}^{t-1} q^{k+1} \gamma_d(r-1).$$

Proof. Consider the left most branch that takes the root to a leaf, containing variables $X = \{x_1, \dots, x_t\}$ where x_1 is the root, and denote $X_{i,j} = \{x_i, \dots, x_j\}$ for $i \leq j$. We will count again by partitioning based on when the first 1 appears. The counting happens in three steps.

1. Let k be the least index such that $\rho(x_k) = 1$, and let $S \subseteq X_{1,k-1}$ be the variables assigned * by ρ , where $|S| \neq 0$. In this case, if any subtree out of the 1 edge of any variable in S has depth greater than $d-1$ when restricted to ρ , it would imply that $T|_\rho$ has depth at least d . In fact, since $|S| \neq 0$, even if the subtree out of the 1 edge of x_k has depth greater than $d-1$, it would imply that $T|_\rho$ has depth greater than d . Thus, at least one of the $|S| + 1$ subtrees, all of which are uncorrelated, should have depth greater than $d-1$.

2. Let the set $S \subseteq X_{1,t}$ be the variables assigned * by ρ , where $|S| \neq 0$, and no variable is assigned 1. In this case, the are $|S|$ subtrees, at least one of which must have depth greater than $d-1$ for $T|_\rho$ to have depth greater than d .

3. If no variables are assigned * by ρ , then if k is the least index such that $\rho(x_k) = 1$, the subtree out of the 1 edge of x_k must have depth greater than d .

Writing out the sums for each of the three partitions as mentioned above and summing over least index $k \in \{0, \dots, t-1\}$ such that $x_{k+1} = 1$ gives the recurrence. \square

Lemma 43. *For $r \in \Omega(d2^t)$, $\gamma_d(r) \geq (c_0 p 2^t)^d$.*

Proof. Let $\Delta = c_0 p 2^t$ and $\mu_j = 1 - \gamma_{d-1}(r-j)$. The expression for $\gamma_d(r)$ from Lemma 42 is reproduced here.

$$\gamma_d(r) \geq \sum_{k=1}^{t-1} q \sum_{i=1}^k \binom{k}{i} q^{k-i} p^i (1 - \mu_1^{i+1}) + \sum_{i=0}^t \binom{t}{i} q^{t-i} p^i (1 - \mu_1^i) + \sum_{k=0}^{t-1} q^{k+1} \gamma_d(r-1).$$

Let

$$\begin{aligned} A(r-1) &= \sum_{k=1}^{t-1} q \sum_{i=1}^k \binom{k}{i} q^{k-i} p^i (1 - \mu_1^{i+1}), \\ B(r-1) &= \sum_{i=0}^t \binom{t}{i} q^{t-i} p^i (1 - \mu_1^i), \\ \theta &= \sum_{k=0}^{t-1} q^{k+1} \end{aligned}$$

Case $t \geq 2$: For this case, we will consider the terms $A(r-1)$ and $\theta\gamma_d(r-1)$. Expanding the expression for m terms, we get,

$$\begin{aligned}
\gamma_d(r) &\geq A(r-1) + \theta\gamma_d(r-1) \\
&\geq \sum_{i=1}^m \theta^{i-1} A(r-i) + \theta^m \gamma_d(r-m) \\
&\geq \sum_{i=1}^m \theta^{i-1} A(r-i)
\end{aligned} \tag{30}$$

where we used the fact that $\gamma_d(r-m) \geq 0$. For $1 \leq i \leq m-1$, let $\gamma_{d-1}(r-i) \geq \Delta^{d-1}$. Then we have,

$$\begin{aligned}
A(r-j) &= \sum_{k=1}^{t-1} q \sum_{i=1}^k \binom{k}{i} q^{k-i} p^i (1 - \mu_j^{i+1}) \\
&\geq \sum_{k=1}^{t-1} q \sum_{i=1}^k \binom{k}{i} q^{k-i} p^i (1 - \mu_j) \\
&\geq \Delta^d \frac{42q}{p2^t} \sum_{k=1}^{t-1} \sum_{i=1}^k \binom{k}{i} q^{k-i} p^i
\end{aligned} \tag{31}$$

where in the second line we used $0 \leq \mu \leq 1$, and in the third line we used the inductive hypothesis, $\gamma_{d-1}(r-j) \geq \Delta^{d-1}$ for $1 \leq j \leq m$. Consider the term

$$\begin{aligned}
\frac{1}{p} \sum_{k=1}^t \sum_{i=1}^k \binom{k}{i} q^{k-i} p^i &= \frac{1}{p} \sum_{k=1}^{t-1} (p+q)^k - q^k \\
&= \frac{1}{p} \sum_{k=1}^{t-1} \frac{1}{2^k} \left((1+p)^k - (1-p)^k \right) \\
&= \frac{1}{p} \sum_{k=1}^{t-1} \frac{1}{2^k} \sum_{i=0}^k \binom{k}{i} (p^i - (-p)^i) \\
&= \frac{1}{p} \sum_{k=1}^{t-1} \frac{1}{2^k} \sum_{i=0, i \text{ is odd}}^k 2 \binom{k}{i} p^i \\
&\geq \sum_{k=1}^{t-1} \frac{2k}{2^k} \\
&= 2 \left(1 - \frac{t+1}{2^t} \right) \\
&\geq \frac{1}{2}
\end{aligned}$$

where in the first inequality, we used the fact that $p \geq 0$ and picked only the largest term, used Lemma 12 for the summation in the second last line, and used $t \geq 2$ in the last line. Substituting the above expression in 31, we can write

$$A(r-j) \geq \Delta^d \frac{21q}{2^t},$$

and substituting it in equation 30, we get ,

$$\gamma_d(r) \geq \Delta^d \frac{21q}{2^t} \left(\frac{1 - \theta^m}{1 - \theta} \right). \quad (32)$$

Note that

$$(1 - p)^t \geq (1 - c_p 2^{-t})^t \geq \frac{1}{2},$$

and using $q \leq \frac{1}{2}$, we get

$$\theta = \frac{q}{1 - q} (1 - q^t) \leq 1 - \frac{(1 - \frac{1}{C2^t})^t}{2^t} \leq 1 - \frac{1}{2 \cdot 2^t}.$$

Thus for $m = 2 \cdot 2^t$,

$$\theta^m \leq \frac{1}{e},$$

and further,

$$\begin{aligned} \frac{1 - \theta^m}{1 - \theta} &= (1 - \theta^m) \frac{1 - q}{1 - 2q + q^{t+1}} \\ &\geq \frac{1}{2} \frac{\frac{1}{2}}{\frac{1}{C2^t} + \frac{1}{2^{t+1}}} \\ &\geq 2^t \left(\frac{\frac{1}{4}}{\frac{1}{2} + \frac{1}{C}} \right) \\ &\geq 2^t \frac{1}{4}, \end{aligned} \quad (33)$$

and substituting it in equation 32, and noting that for $t \geq 2$, since $q \geq \frac{1}{2} - \frac{1}{4C} \geq \frac{3}{8}$, we get,

$$\gamma_d(r) \geq \Delta^d \frac{21}{2^t} \frac{3}{8} 2^t \frac{1}{4} \geq \Delta^d$$

as required.

Case $t = 1$: For the case of $t = 1$, we use the terms $B(r - 1)$ and $\theta \gamma_d(r - 1)$, and noting that for $1 \leq i \leq m - 1$,

$$B(r - i) \geq p \Delta^{d-1},$$

we get by using 33,

$$\begin{aligned} \gamma_d(r) &\geq B(r - 1) + \theta \gamma_d(r - 1) \\ &\geq \sum_{i=1}^m \theta^{i-1} B(r - i) \\ &\geq \Delta^d \left(\frac{42}{p2} \right) p \left(\frac{1 - \theta^m}{1 - \theta} \right) \\ &\geq \Delta^d \end{aligned}$$

as required. □

In the proof of Lemma 43, we use $O(2^t)$ steps to take the induction from d to $d - 1$ levels, and the final $d = 1$ case can also be carried out with $O(2^t)$ levels. Thus, the maximum depth upto which our conclusions hold is

$$d2^t \leq O(r) \leq c_d \frac{\log n}{\log t}$$

or

$$d \leq c_d \left(\frac{\log n}{2^t \log t} \right).$$

This concludes the proof of Theorem 2.

Remark 44. If we unroll the induction and see how it worked, for depth d , we find some vertex that is unset by ρ and is connected to a tree that has depth $d - 1$ under the action of ρ . For depth $d - 1$, we again find an unset variable connected to a tree of depth $d - 2$. This carries on until the last step, where we want to find a vertex that has depth greater than or equal to 1, i.e., has a path to both a 0-leaf and a 1-leaf. Thus, as described in the introduction, the whole proof essentially finds a “path with a split”.

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A Proofs from Section 5

Lemma 45. *34*The recursive expressions for $[1]P_0(r)$ and $[1]P_1(r)$ are as follows:

$$\begin{aligned} [1]P_0(1) &= \frac{1}{2^t} \\ [1]P_1(1) &= 1 - \frac{1}{2^{t'}} \end{aligned}$$

and

$$\begin{aligned} [1]P_0(r) &= \left(1 - \frac{1}{2^t}\right) [1]P_1(r-1) + \frac{1}{2^t}, \\ [1]P_1(r) &= \left(1 - \frac{1}{2^t}\right) [1]P_0(r-1). \end{aligned}$$

Proof. From claim 11, note that the operator $[1]$ is both linear and multiplicative, i.e.

$$\begin{aligned} [1](Q + R) &= [1]Q + [1]R, \\ [1]QR &= [1]Q[1]R. \end{aligned} \tag{34}$$

To compute the constant coefficients, we start by the base case, $P_0(1)$ and $P_1(1)$.

$$[1]P_0(1) = [1]q^t = \frac{1}{2^t}$$

and

$$[1]P_1(1) = [1]1 - \left([1]\frac{1+p}{2}\right)^t = 1 - \frac{1}{2^t}.$$

For the general case of any r , using only the equations 34 and applying it to 12, we get,

$$\begin{aligned}
[1]P_0(r) &= [1] \left(qU \sum_{k=0}^{t-1} (q + pU)^k + (q + pU)^t \right) \\
&= [1]q[1]U \sum_{k=0}^{t-1} ([1]q + [1]pU)^k + ([1]q + [1]pU)^t \\
&= \frac{1}{2}[1]U \sum_{k=0}^{t-1} 2^{-k} + 2^{-t} \\
&= \left(1 - \frac{1}{2^t} \right) [1]P_1(r-1) + \frac{1}{2^t}.
\end{aligned}$$

And similarly,

$$[1]P_1(r) = \left(1 - \frac{1}{2^t} \right) [1]P_0(r-1).$$

□

Lemma 46. 36 Let $\alpha = 2^{-t}$ and $\beta = 1 - \alpha$. Then the exact expressions for $[1]P_0(r)$ and $[1]P_1(r)$ are as follows. If $r = 2k + 1$,

$$\begin{aligned}
[1]P_0(r) &= \frac{1 - \beta^{r+1}}{1 + \beta}, \\
[1]P_1(r) &= \frac{\beta + \beta^{r+1}}{1 + \beta},
\end{aligned} \tag{35}$$

and if $r = 2k$,

$$\begin{aligned}
[1]P_0(r) &= \frac{1 + \beta^{r+1}}{1 + \beta}, \\
[1]P_1(r) &= \frac{\beta - \beta^{r+1}}{1 + \beta}.
\end{aligned}$$

Proof. To compute the exact equations, assume $r = 2k + 1$. Substituting equation 18 in equation 17 and using equation 10 for the base case of $r = 1$, we get

$$\begin{aligned}
[1]P_0(r) &= \left(1 - \frac{1}{2^t} \right)^2 [1]P_0(r-2) + \frac{1}{2^t} \\
&= \left(1 - \frac{1}{2^t} \right)^{2k} \frac{1}{2^t} + \frac{1}{2^t} \left(\frac{1 - \left(1 - \frac{1}{2^t} \right)^{2k}}{1 - \left(1 - \frac{1}{2^t} \right)^2} \right) \\
&= \frac{1 - \beta^{r+1}}{1 + \beta},
\end{aligned}$$

And from Lemma 35, for $r = 2k + 1$,

$$[1]P_1(r) = \frac{\beta + \beta^{r+1}}{1 + \beta}.$$

Similarly, for $r = 2k$, we get by using one step of equation 17 followed by 35,

$$[1]P_0(r) = \frac{1 + \beta^{r+1}}{1 + \beta},$$

and similarly,

$$[1]P_1(r) = \frac{\beta - \beta^{r+1}}{1 + \beta}.$$

□

Lemma 47. 37The recurrence relations for $[p]P_0(r)$ and $[p]P_1(r)$ are as follows:

$$\begin{aligned} [p]P_0(1) &= -\frac{t}{2^t} \\ [p]P_1(1) &= -\frac{t}{2^t}. \end{aligned}$$

and

$$\begin{aligned} [p]P_1(r) &= \left(1 - \frac{1}{2^t}\right) [p]P_0(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1]P_0(r-1) \\ &\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_0(r-1)^2 \\ [p]P_0(r) &= \left(1 - \frac{1}{2^t}\right) [p]P_1(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1]P_1(r-1) \\ &\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_1(r-1)^2 - \frac{t}{2^t} + \frac{2t}{2^t} [1]P_1(r-1). \end{aligned}$$

Proof. To evaluate the coefficient of p in the polynomials, first note that from the claim 11, the operator $[p]$ acts as follows on sums and products of polynomials Q and R :

$$\begin{aligned} [p](Q + R) &= [p]Q + [p]R, \\ [p]QR &= [p]Q[1]R + [1]Q[p]R. \end{aligned} \tag{36}$$

For the base case of $r = 1$,

$$[p]P_0(1) = [p]q^t = -\frac{t}{2^t}$$

and

$$[p]P_1(1) = [p](1 - (1 - q)^t) = -[p](1 - q)^t = -\frac{t}{2^t}.$$

For any general r , we first compute $[p]P_1(r)$ using equation 13.

$$\begin{aligned}
[p]P_1(r) &= [p] \left(\sum_{k=0}^{t-1} \sum_{i=0}^k \binom{k}{i} q^{k-i+1} p^i (P_0(r-1))^{i+1} \right) \\
&= \sum_{k=0}^{t-1} \sum_{i=0}^k \binom{k}{i} [p] q^{k-i+1} p^i (P_0(r-1))^{i+1} \\
&= \sum_{k=0}^{t-1} [p] q^{k+1} P_0(r-1) + \sum_{k=0}^{t-1} [1] k q^k (P_0(r-1))^2 \\
&= \sum_{k=0}^{t-1} [1] q^{k+1} [p] P_0(r-1) + \sum_{k=0}^{t-1} [p] q^{k+1} [1] P_0(r-1) + \sum_{k=0}^{t-1} [1] k q^k (P_0(r-1))^2 \\
&= \left(1 - \frac{1}{2^t}\right) [p] P_0(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1] P_0(r-1) \\
&\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1] P_0(r-1)^2
\end{aligned}$$

where the second and third lines used facts (1), (4) and (5) in claim 11, and the fourth line used the fact 36. Similarly, from equation 12, we get,

$$\begin{aligned}
[p]P_0(r) &= [p] \left(\sum_{k=0}^{t-1} \sum_{i=0}^k \binom{k}{i} q^{k-i+1} p^i (P_1(r-1))^{i+1} + \sum_{i=0}^t \binom{t}{i} q^{t-i} p^i (P_1(r-1))^i \right) \\
&= \left(1 - \frac{1}{2^t}\right) [p] P_1(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1] P_1(r-1) + 2 \left(1 - \frac{t+1}{2^t}\right) [1] P_1(r-1)^2 \\
&\quad + \sum_{i=0}^t \binom{t}{i} [p] q^{t-i} p^i (P_1(r-1))^i \\
&= \left(1 - \frac{1}{2^t}\right) [p] P_1(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1] P_1(r-1) \\
&\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1] P_1(r-1)^2 - \frac{t}{2^t} + \frac{2t}{2^t} [1] P_1(r-1).
\end{aligned}$$

□

Lemma 48. 38 For every $r \geq 1$, $-2 \cdot 2^t \leq [p]P_0(r) \leq 0$ and $-2 \cdot 2^t \leq [p]P_1(r) \leq 0$.

Proof. We reproduce equations 26 and 25 here for convenience:

$$\begin{aligned}
[p]P_1(r) &= \left(1 - \frac{1}{2^t}\right) [p] P_0(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1] P_0(r-1) \\
&\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1] P_0(r-1)^2
\end{aligned} \tag{37}$$

$$\begin{aligned}
[p]P_0(r) &= \left(1 - \frac{1}{2^t}\right) [p] P_1(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1] P_1(r-1) \\
&\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1] P_1(r-1)^2 - \frac{t}{2^t} + \frac{2t}{2^t} [1] P_1(r-1).
\end{aligned} \tag{38}$$

We first show that show $[p]P_0(r) \leq 0$ and $[p]P_1(r) \leq 0$ by induction on r . For $r = 1$, from equations 23 and 24, $[p]P_0(1) \leq 0$ and $[p]P_1(1) \leq 0$. Consider equation 37,

$$[p]P_1(r) = \left(1 - \frac{1}{2^t}\right) [p]P_0(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1]P_0(r-1) + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_0(r-1)^2.$$

For the first term, since $1 - 2^{-t} \geq 0$ and $[p]P_0(r-1) \leq 0$, the product is always non-positive. For the third term, $1 - 2^{-t}(t+1) \geq 0$, and since $0 \leq [1]P_0(r-1) \leq 1$ from Lemma 35, $[1]P_0(r-1)^2 \leq [1]P_0(r-1)$. Thus, we can write,

$$\begin{aligned} \left(\frac{t+2}{2^t} - 2\right) [1]P_0(r-1) + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_0(r-1)^2 &\leq \left(\frac{t+2}{2^t} - 2 + 2 - \frac{2(t+1)}{2^t}\right) [1]P_0(r-1) \\ &\leq 0. \end{aligned}$$

For $[p]P_0(r)$, consider equation 38,

$$\begin{aligned} [p]P_0(r) &= \left(1 - \frac{1}{2^t}\right) [p]P_1(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1]P_1(r-1) \\ &\quad + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_1(r-1)^2 - \frac{t}{2^t} + \frac{2t}{2^t} [1]P_1(r-1). \end{aligned}$$

The first term is non-positive similar to the argument above, and summing the remaining terms after using Lemma 35 for $[1]P_1(r-1)^2 \leq [1]P_1(r-1)$, we get,

$$\begin{aligned} \left(\frac{t+2}{2^t} - 2\right) [1]P_1(r-1) + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_1(r-1)^2 \\ + \frac{2t}{2^t} [1]P_1(r-1) - \frac{t}{2^t} \leq \frac{t}{2^t} [1]P_1(r-1) - \frac{t}{2^t} \leq 0. \end{aligned}$$

We now show that show that $[p]P_0(r) \geq -2 \cdot 2^t$ and $[p]P_1(r) \geq -2 \cdot 2^t$, again by induction on r . For $r = 1$, note that

$$[p]P_0(1) = [p]q^t = -\frac{t}{2^t} \geq -2 \cdot 2^t,$$

and

$$[p]P_1(1) = [p](1 - (1-q)^t) = -[p](1-q)^t = -\frac{t}{2^t} \geq -2 \cdot 2^t.$$

Consider equation 37. For $t \geq 1$, $1 - \alpha(t+1) \geq 0$, $\alpha(t+2) - 2 \leq 0$, and $0 \leq P_0(r-1) \leq 1$, and thus, we can write using the induction hypothesis,

$$\begin{aligned} [p]P_1(r) &= \beta[p]P_0(r-1) + (\alpha(t+2) - 2) [1]P_0(r-1) + 2(1 - \alpha(t+1)) [1]P_0(r-1)^2 \\ &\geq -2\beta 2^t + \alpha(t+2 - 2 \cdot 2^t) \\ &\geq -2(\beta + \alpha) 2^t \\ &= -2 \cdot 2^t. \end{aligned}$$

In equation 38, noting that $0 \leq P_1(r-1) \leq 1$, we can similarly write

$$\begin{aligned} [p]P_0(r) &= \left(1 - \frac{1}{2^t}\right) [p]P_1(r-1) + \left(\frac{t+2}{2^t} - 2\right) [1]P_1(r-1) + 2 \left(1 - \frac{t+1}{2^t}\right) [1]P_1(r-1)^2 \\ &\quad - \frac{t}{2^t} + \frac{2t}{2^t} [1]P_1(r-1) \\ &\geq -2\beta 2^t + \alpha(t+2 - 2 \cdot 2^t) - \alpha t \\ &\geq -2 \cdot 2^t \end{aligned}$$

as required. □