Nonexistence of Efficient Dominating Sets in the Cayley Graphs Generated by Transposition Trees of Diameter 3

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Abstract

Let d,n be positive integers such that $d<n$, and let X_n^d be a Cayley graph generated by a transposition tree of diameter d. It is known that every X_n^d with $d<3$ splits into efficient dominating sets. The main result of this paper is that X_n^3 does not have efficient dominating sets.

Keywords: Cayley graph, efficient dominating set, sphere packing.

2010 Mathematics Subject Classification: 05C69; 05C70; 05C12.

1. INTRODUCTION AND PRELIMINARIES

Cayley graphs are very important for their useful applications (cf. [10]), including to automata theory (cf. [11, 12]), interconnection networks (cf. [1, 2, 4, 5, 6]) and coding theory (cf. [3, 4]).

Let $0<d<n$ in $\mathbb{Z}$, and let X_n^d be a Cayley graph generated by a transposition tree of diameter d. In [4], it was shown that every X_n^d with $d<3$ splits into efficient dominating sets. In the present work, the following result is proved.

Theorem 1.1. *Let $3<n$. Then no X_n^3 has efficient dominating sets.*

The rest of this section is devoted to some preliminaries. Let $0<n\in\mathbb{Z}$ and let $I_n=\{1,2,\dots,n\}$. Let S_n be the group of permutations $\sigma=\binom{1\cdots\cdots n}{\sigma_1\cdots\sigma_n}:I_n\rightarrow I_n$ with $\sigma(i)=\sigma_i$, for every $i\in I_n$, where $\{\sigma_1,\dots,\sigma_n\}=I_n$. Any such σ will simply be denoted $\sigma=\sigma_1\cdots\sigma_n$. Let $e=12\cdots n$ stand for the identity of S_n. Let $\mathcal{C}\subseteq S_n\setminus\{e\}$ be such that $\sigma\in\mathcal{C}\Leftrightarrow\sigma^{-1}\in\mathcal{C}$. The *Cayley graph* $X=X(S_n,\mathcal{C})$ of S_n with *connection set* $\mathcal{C}$ is the graph $X=(S_n,E)$, where $gh\in E\Leftrightarrow h=\sigma g$ with $\sigma=hg^{-1}\in\mathcal{C}$. If one such σ equals σ^{-1}, then we say that $gh\in E$ has *color* σ and write $\sigma=(gh)$, that is the cycle notation of transposition (gh).

Note that X is connected if and only if $\mathcal{C}$ is a generating set for S_n [7] Lemma 3.7.4. Note also that a set of transpositions (gh) from S_n generates S_n if and only if the graph with edges gh is connected [7] Lemma 3.10.1.

1.1. Transpositions, Domination and Packing.

Let τ be a connected graph with vertex set I_n and let $\mathcal{C} = \mathcal{C}_\tau$ be composed by the transpositions $\sigma = (gh)$, where gh runs over the edges of τ. Then $\sigma = \sigma^{-1}$ for each $\sigma \in \mathcal{C}_\tau$. This yields $X(S_n, \tau) = X(S_n, \mathcal{C}_\tau)$ as an edge-colored graph via the *color set* $\mathcal{C}_\tau$ with a 1-factorization into the 1-factors F_{gh} of σ-colored edges. Now, τ is called the *transposition graph* of $X(S_n, \tau)$ [5, 6].

For domination and packing in Cayley graphs, the terminology of [8] is used. A *stable subset* $J \subseteq S_n$ (i.e. a set of nonadjacent vertices) with each vertex of $S_n \setminus J$ adjacent in the Cayley graph X to just one vertex of J is an *efficient dominating set* (or *E-set*) of X. The 1-*sphere* with *center* $g \in S_n$ is the subset $\{h \in S_n \,|\, \delta(g,h) \leq 1\}$, where δ is the graph distance of X. Every E-set in X is the set of centers of the 1-spheres in a *perfect sphere packing* (as in [9], page 109) of X. Let X' be a proper subgraph of X (X' specified in Subsection 1.4). Let $\mathcal{S}$ be a perfect 1-sphere packing of X'. Then an $\mathcal{S}$-*sphere* is the union of a 1-sphere of $\mathcal{S}$ with its neighbors in $S_n \setminus V(X')$. The union of two 1-spheres centered at adjacent vertices is said to be a *double-sphere* in X. A collection of pairwise disjoint 1-spheres (resp., $\mathcal{S}$-spheres and double-spheres) in X is said to be a *1-sphere packing* of X (resp., a *special packing* of X, to be used in Section 7). It may happen that X has a packing $\mathcal{T}$ by the $\mathcal{S}$-spheres, see Figure 1 below.

Given a packing $\mathcal{S}$ of 1-spheres in X whose union has cardinality $\alpha|S_n| = \alpha n!$, $(0 < \alpha \leq 1)$, we say that the set J of centers of the 1-spheres of $\mathcal{S}$ is an α-*efficient dominating set* (or α-E-set) of X, in which case we may denote (by abuse of notation) the induced subgraph $X[J]$ by J. Note that a 1-E-set is an E-set, and viceversa.

1.2. Transposition Trees of Diameter less than 3.

[7] Theorem 3.10.2 implies $\mathcal{C}_\tau$ is a minimal generating set for $S_n \Leftrightarrow \tau$ is a tree. We take $\tau = \tau^d$ to be a diameter-d tree and denote $X_n^d = X(S_n, \tau^d)$. Let $\tau^{d_1} = \tau^0 = (I_1, \emptyset)$. Let $\tau^{d_n} = K_{1,n-1}$ with $d_n = 2$ if $n > 2$ and $d_n = 1$ if $n = 2$. Let $n > 0$. By assuming $1 \in I_n = V(\tau^{d_n})$ of degree $n-1$, it is seen that $S_n = V(X_n^{d_n})$ splits into E-sets ξ_i^1, $(i \in I_n)$, formed by those $\sigma \in S_n$ with $\sigma_1 = i$ [2]. In this terms [4] showed that if $n > 1$ then for each $i \in I_n$, it holds that $X_n^{d_n} - \xi_i^1$ is the disjoint union of $n-1$ copies ξ_i^j of $X_{n-1}^{d_{n-1}}$, where ξ_i^j is induced by all $\sigma \in S_n$ with $\sigma_j = i$ and $j \in I_n \setminus \{1\}$. Using this, we prove below that no X_n^3 has E-sets, with related developments.

1.3. Transposition Trees of Diameter 3.

A diameter-3 tree τ^3 has two vertices of degrees r, t larger than 1 and joined by an edge ϵ. Then $n = r + t$. We write $\tau^3 = \tau^3_{r,t}$ and take: **(i)** r and $r^* = r + 1$

as the vertices of $\tau^3_{r,t}$ of degrees r and t so that $\epsilon = rr^*$; **(ii)** $1, \ldots, r-1$ (resp., $r^*+1, \ldots, n$) as the neighbors of r (resp., r^*) in $\tau^3_{r,t}$. (This vertex numbering is modified from Section 8 on). Edge pairs in $\tau^3_{r,t}$ induce copies of both: **(A)** the disjoint union $2K_2 = 2P_2$ of two paths of length 1; **(B)** the path P_3 of length 2. Using two-color alternation in $X^3_{r,t} = X(S_n, \tau^3_{r,t})$, the edge pairs (A) (resp., (B)) determine 4-cycles (resp., 6-cycles). The subgraphs of $X^3_{r,t}$ induced by the $\binom{n}{r}$ cosets of $S_r \times S_t$ in S_n are the components of the subgraph $X^3_{r,t} \setminus F_\epsilon$ of $X^3_{r,t}$, see Subsection 1.1. These components are copies of a cartesian product $\Pi^t_r = X^{d_r}_r \Box X^{d_t}_t$ with: **(a)** $d_r = d_t = 2$, if $\min(r,t) > 2$; **(b)** $d_r = 2 = d_t + 1$, if $r > t = 2$; **(c)** $d_r = d_t = 1$, if $r = t = 2$.

If an α-E-set J of $X^3_{r,t}$ is equivalent in all copies of Π^t_r in $X^3_{r,t}$, both J and its associated 1-sphere packing are said to be *uniform*. There is no uniform α-E-set in $X^3_{2,2}$, see Figure 1 below. Theorem 6.1 will show that if $4 < n = r+t$, then uniform α-E-sets of $X^3_{r,t}$ have $\alpha \leq \frac{n}{rt} < 1$. Theorems 10.1–10.2 and Corollary 10.3 certify that such an upper bound $\frac{n}{rt}$ can only be attained by uniform α-E-sets that intersects each copy of Π^t_r in a product $J' \times J''$ of E-sets $J' \subset X^{d_r}_r$, $J'' \subset X^{d_t}_t$. These results will establish Theorem 1.1 as all α-E-sets in the graphs $X^3_{r,t}$ happen with $\alpha < 1$. Our plan for this task is sketched after the following example.

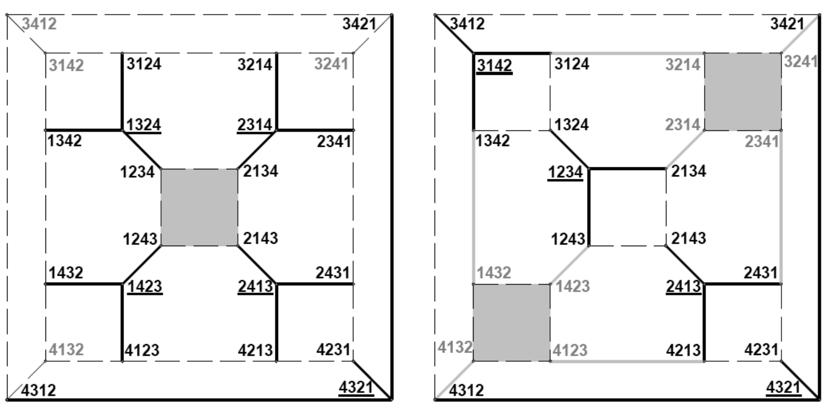

Figure 1. Representations of a (5/6)- and a (2/3)-E-set of $X^3_{2,2}$

Every α-E-set in $X^3_{2,2}$ avoids at least one of the six copies of Π^2_2 in $X^3_{2,2}$. See the two instances of α-E-sets in $X^3_{2,2}$ in Figure 1, with each avoided copy of Π^2_2 bounding a solid-gray square. On the left, the edges incident to a (5/6)-E-set are in thick trace. (In expressing n-tuples in S_n commas and parentheses are ignored). On the right, (to be compared with the construction in Section 7 and initiating the inductive construction of Section 8), a 1-sphere packing $\mathcal{S}$ of $X^3_{2,2}$ is shown that covers $16 = (2/3)4!$ vertices, with underlined black 1-sphere centers. The 1-spheres of $\mathcal{S}$, forming a (2/3)-E-set, induce the edges in thick black trace. Of the other edges, those colored $(23) = (\epsilon)$, induced by the $\mathcal{S}$-spheres, forming a $\mathcal{T}$ as in Subsection 1.1, are in thick light-gray. The eight vertices in the $\mathcal{S}$-spheres of $\mathcal{T}$ not in the 1-spheres of $\mathcal{S}$ are light-gray (in contrast with the remaining vertices, in black) and span two 4-cycles bounding solid gray squares as cited above.

1.4. A Subgraph of the Cayley Graph

In proving Theorems 10.1–10.2, we use that if $r = t > 1$ then in each copy of Π_r^t (see Subsection 1.3) a partition of $S_r = V(X_r^2)$ into E-sets (see Subsection 1.2) can be combined by concatenation with a corresponding partition of $A_t = V(X_t^2[A_t])$, where A_t has index 2 in S_t. Then connected subgraph $X' = X'_{r,r}$ induced by 2^r of the $\binom{n}{r}$ copies of Π_r^r in $X_{r,r}^3$ has an E-set J, where X' is the largest subgraph of $X_{r,r}^3$ with a perfect 1-sphere packing. Also, $V(X')$ is a subgroup of S_n containing the E-set J as a subgroup. Theorem 10.1 implies that J, whose associated 1-sphere packing has locally maximum density, cannot be extended to an E-set of $X_{r,r}^3$. If $r = t > 2$, then J extends to a maximum nonuniform α-E-set of $X_{r,r}^3$ with a largest $\alpha > \frac{n}{r^2}$ such that $\alpha < 1$. Corollary 10.3 allows to extend this case of $X_{r,r}^3$ to the case of $X_{r,t}^3$ $(r > t > 2)$, via puncturing restriction. This allows the completion of the proof of Theorem 6.1.

Remark 1.2. A conjecture in [4] says that no E-set of X_n^d exists if $d > 2$. Remark 1 [3] says that a proof of this conjecture in Theorem 5 [4], fails. This can be corrected for $d > 2$ by restricting to either $n = 4$ or n a prime $n > 4$, proved in [3] for path graphs τ^d. It can be proved for any tree τ^d by [4] Lemma 6, that generalizes the decomposition of $X_{r,t}^3 \setminus F_\epsilon$ in Subsection 1.3.

2. JOHNSON GRAPHS

Let $2 < r < n-1$ in $\mathbb{Z}$. Let $\Gamma_n^r = (V, E)$ be the edge-colored graph with $V = \{r\text{-subsets of } I_n\}$ and $tu \in E \Leftrightarrow t \cap u$ is an $(r-1)$-subset, said to be the *color* of tu. Note that Γ_n^r is the Johnson graph $J(n, r, r-1)$ [7]. A subgraph Ψ of Γ_n^r is *tight* if each two of its edges incident to a common vertex have the $(r-1)$-subsets representing their colors sharing exactly $r-2$ elements of I_n. A tight subgraph of Γ_n^r is *exact* if the vertices u, v, w of each $P_3 = uvw$ in Ψ involve $r+2$ elements of I_n, that is: $|u \cup v \cup w| = r+2$. Exact spanning subgraphs Φ_n^r in Γ_n^r are applied in Sections 4–6 to packing 1-spheres into $X_{r,t}^3$.

An exact cycle in Γ_5^3 is $\psi_5 = (345, 234, 123, 512, 451)$ (or in reverse, $\psi_5^{-1} = (321, 432, 543, 154, 215)$), where each triple $a_0a_1a_2$ acquires the element a_0 among those absent in the preceding triple and loses the element a_2 among those present in the following triple, with 3-strings taken cyclically mod 5. This is also expressed as a *condensed cycle* (or *CC*) of triples $\psi_5 = (12345)$, (resp., $\psi_5^{-1} = (54321)$), whose successive composing triples yield corresponding successive terms of the original form of ψ_5, (resp., ψ_5^{-1}). We can take an exact $\Phi_5^3 \in \{\{\psi_5, \psi'_5\}, \{\psi_5^{-1}, \psi_5'^{-1}\}, \{\psi_5, \psi_5'^{-1}\}, \{\psi_5^{-1}, \psi'_5\}\}$, where

$$\begin{array}{ll} \psi_5 = (345, 234, 123, 512, 451) = (12345), & \psi'_5 = (135, 413, 241, 524, 352) = (13524), \\ \psi_5^{-1} = (321, 432, 543, 154, 215) = (54321), & \psi_5'^{-1} = (142, 314, 531, 253, 425) = (53142), \end{array} \tag{1}$$

are expressed as cycles of triples in Γ_5^3 and as their respective CCs.

If 3 divides n, some r-subsets do not form part of any cycle of a tight 2-factor Φ_n^3. For example, the triples 246 and 135 are in no such a cycle (of length necessarily at least 4), in particular in any tight Φ_6^3. This is solved via the treatment of Section 3, or by defining Φ_6^3 to be constituted by a Hamilton cycle ψ_6 of Γ_6^3 expressed as follows. If $w = a_0a_1a_2$ and u are two contiguous triples in ψ_6 with w preceding u, then a_0 and a_1 coincide with the last two elements of u. We append to w a subindex 1 or 2 according to whether a_0 and a_1 have their order reversed or preserved in u, respectively, with a_0 as the sole element absent in the triple preceding w in ψ_6. One such ψ_6 is expressible as:

$$\begin{array}{r} \psi_6 = (321_2, 432_2, 543_2, 654_2, 165_2, 216_1, 412_1, 314_2, 531_1, 235_1, \\ 632_2, 163_2, 416_2, 541_1, 245_1, 642_2, 364_1, 563_2, 256_2, 125_1) \end{array} \tag{2}$$

$$\begin{array}{r} = (321, 432, 543, 654, 165, 261, 421, 314, 513, 253, \\ 632, 163, 416, 541, 245, 642, 346, 563, 256, 125), \end{array} \tag{3}$$

where display (3) is as display (2) but without the subindices 1 or 2.

Now, tight 2-factors Φ_7^3 and Φ_8^3 in terms of CCs can be expressed respectively:

$$\begin{array}{l} \{(1234567), (1357246), (1473625), (12457134672356)\}, \\ \{(12345678), (1357)(2468), (14725836), (1245782356813467), (1256)(2367)(3478)(1458)\}, \end{array} \tag{4}$$

both exemplifying the definition of bipermutation, in Section 3 below, the first being exact, the second not, because of the presence of non-exact 4-cycles; thus, this Φ_8^3 needs to be modified into an exact Φ_8^3, see Theorem 3.1.

3. CONDENSED NOTATION

In CC notation, if $3 \not| n$ then a tight 2-factor Φ_n^3 of Γ_n^3 can be seen as a collection of objects each of which is either: **(a)** a permutation ϕ_i of I_n written in cycle notation with empty fixed-point set, where $i \in I_n$ with $\lceil \frac{n-2}{2} \rceil \geq i$ is a constant increment mod $n(\equiv 0)$ from each entry of ϕ_i to the subsequent one, or **(b)** a generalization $\phi_{i,j}$ of ϕ_i that we call a *bipermutation*, where $i, j \in I_n$ with $\lfloor \frac{n-2}{2} \rfloor \geq j > i$ are alternate increments mod n in the composing CCs, with each element of I_n in such $\phi_{i,j}$ present twice (as noncontiguous entries).

Each permutation or bipermutation as in (a) or (b) above is said to be a μ-permutation (or μP) of respective multiplicity $\mu = 1$ or $\mu = 2$. Thus, a tight 2-factor Φ_n^3 of Γ_n^3 can be considered as a family of μPs. In these, for each $i \in I_n$

the triples of contiguous entries one of which is i are the classes of a partition $\mathcal{P}_i$ of the set of vertices of Γ_n^3 that as triples contain i. (For example, Φ_5^3 below yields $\mathcal{P}_1 = \{\{451, 512, 123\}, \{241, 413, 135\}\}, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5$). Any such ϕ_i (resp., $\phi_{i,j}$) is formed by $gcd(i,n)$ (resp., $gcd(i+j,n)$) CCs of length $n/gcd(i,n)$ (resp., $2n/gcd(i+j,n)$). Examples of Φ_n^3 (or auxiliary Ψ_n^3) are:

Φ_5^3=$\{\phi_1$=(12345), ϕ_2=(13524)$\}$, (μ=1,1);
Ψ_6^3=$\{\phi_1$=(123456), ϕ_2=(135)(246), $\phi_{1,2}$=(1245)(2356)(3461)$\}$, (μ=1,1,2);
Φ_7^3=$\{\phi_1$=(1234567), ϕ_2=(1357246), ϕ_3=(1473625), $\phi_{1,2}$=(12457134672356)$\}$, (μ=1,1,1,2);
Φ_8^3=$\{\phi_1$=(12345678), ϕ_2=(1357)(2468), ϕ_3=(14725836), $\phi_{1,2}$=(1245782356813467),
$\phi_{1,3}$=(1256)(2367)(3478)(4581)$\}$, (μ=1,1,1,2,2);
Ψ_9^3=$\{\phi_1$=(123456789), ϕ_2=(135792468), ϕ_3=(147)(258)(369), ϕ_4=(159483726),
$\phi_{1,2}$=(124578)(235689)(346791), $\phi_{1,3}$=(125691458934782367),
$\phi_{2,3}$=(136824793581469257)$\}$, (μ=1,1,1,2,2,2).

For each integer $i > 0$, let $A(i) = (A_{i,j} \,|\, j \in I_\ell)$ be the sequence of length $\ell = \lfloor \frac{i+2}{3} \rfloor$ defined as shown here (vertically, to produce columns (B_1^1), (B_2^1, B_2^2), (B_3^1, B_3^2, B_3^3), $(B_4^1, B_4^2, B_4^3, B_4^4)$, $\ldots$) and then horizontally):

$A(1)$=(1), $A(4)$=(2,2), $A(7)$=(4,3,3), $A(10)$=(5,5,4,4), $A(13)$=(7,6,6,5,5), $A(16)$=(8,8,7,7,6,6),
$A(2)$=(1), $A(5)$=(3,2), $A(8)$=(4,4,3), $A(11)$=(6,5,5,4), $A(14)$=(7,7,6,6,5), $A(17)$=(9,8,8,7,7,6),
$A(3)$=(2), $A(6)$=(3,3), $A(9)$=(5,4,4), $A(12)$=(6,6,5,5), $A(15)$=(8,7,7,6,6), $A(18)$=(9,9,8,8,7,7),

for $i = 1, 2, \ldots, 18$, and then via $(B_\ell^k)^T = (A_{3\ell-2,k}, A_{3\ell-1,k}, A_{3\ell,k})$, where $1 < \ell$ and $k \in I_\ell$, by starting with B_ℓ^ℓ, (e.g. $B_6^6 = (6,6,7)$), then continuing with $B_\ell^{\ell-1}$ (e.g. $B_6^5 = (6,7,7)$) and so on, by *descending* induction:

$$\begin{array}{lcccl} B_\ell^\ell & = & & & (\ell, \ell \qquad, \ell+1); \\ B_\ell^{\ell-1} & = & & & (\ell, \ell+1, \ell+1); \\ B_\ell^{k-2} & = & B_\ell^k & + & (1, \qquad 1, \qquad 1), \text{ for } k = \ell, \ell-1, \ldots, 4, 3. \end{array}$$

If $n = i+2$, then these $A(i)$ provide CCs in tight 2-factors Φ_n^3 (or in auxiliary families Ψ_n^3 if 3 divides n) as follows. By letting $\phi_{0,j} = \phi_j$ for $j = 1, \ldots A_{i,1}$, it is seen that $A(i)$ encodes (via CCs) $A_{i,k} - k + 1$ μPs, namely $\phi_{k-1,k}, \ldots, \phi_{k-1,A_{i,k}}$ of multiplicity $\mu = 1$ if $k = 1$ and $\mu = 2$ otherwise, unless $k = 1$ and $3|\,n$, in which case $A(i)$ encodes $A_{i,1} - 1$ permutations, namely $\phi_1, \ldots, \phi_{\frac{n}{3}-1}, \phi_{\frac{n}{3}+1}, \ldots, \phi_{A_{i,1}}$, since now $\phi_{\frac{n}{3}}$ is composed by $\frac{n}{3}$ isolated triples. Thus, the $|V(\Gamma_n^3)|$ triples do not form a tight 2-factor Φ_n^3 if and only if $n = 3\kappa$ with $\kappa \in \mathbb{Z}$, because then ϕ_κ has κ isolated triples. This is fixed as follows, (where we also write $\phi_{j,j} = \phi_j$ when applicable). If $\kappa > 1$, then ϕ_κ and $\phi_{1,\kappa-1}$ are modified into a tight cycle ϕ' in Γ_n^3, shown for $\kappa = 2$ (where $\phi_{1,\kappa-1} = \phi_{1,1} = \phi_1$) as $([135], 156, 654, 543, 432, [246], 612, 123) = ([135], 156432, [246], 6123)$, with bracketed isolated triples and the rest in CC notation. For $\kappa > 2$, the following concatenating rows are given by *ascending* induction via the alternate increments 1 and $k-1$, from the second to the last row, while the first row is *descending*, yielding a tight cycle:

$$\begin{array}{l}([1(\kappa{+}1)(2\kappa{+}1)],1(2\kappa{+}2)(2\kappa{+}1)(\kappa{+}2)(\kappa{+}1)21(2\kappa{+}2),\\ \;[2(\kappa{+}2)(2\kappa{+}2)],23(\kappa{+}2)(\kappa{+}3)(2\kappa{+}2)(2\kappa{+}3)23,\;\cdots\\ \;[i(\kappa{+}i)(2\kappa{+}i)],i(i{+}1)(\kappa{+}i)((\kappa{+}i{+}1)(2\kappa{+}i)(2\kappa{+}i{+}1)i(i{+}1),\;\ldots\\ \;[\kappa(2\kappa)(3\kappa)],\kappa(\kappa{+}1)(2\kappa)(2\kappa{+}1)(3\kappa)1\kappa(\kappa{+}1)).\end{array}$$

Theorem 3.1. *If $5 \leq n \in \mathbb{Z}$ is odd then the Johnson graph $\Gamma_n^3 = J(n,3)$ has an exact 2-factor Φ_n^3 the length of whose cycles is at least 5.*

Proof. The proof arises from the previous arguments. ■

Clearly, Theorem 3.1 holds for n odd. The 4-cycles of Φ_n^3, appearing just for n even, can be modified altogether in order to yield exact cycles of length at least 5. The tool we use is adequate for the applications of Section 4. If $n = 2m$ with $m \in \mathbb{Z}$, then a tight 2-factor Φ_n^3 of Γ_n^3 has $\binom{m}{2}$ 4-cycles. They participate of those $\phi_{i,m-i}$ for which $0 < i \leq \frac{m}{2}$. We would like to modify those $\phi_{i,m-i}$ for which $0 < i < \frac{m}{2}$ towards an exact 2-factor Φ_n^3 by replacing it by cycles that continue the following top and bottom patterns:

$$\begin{array}{llll}\ldots[j(j{+}i)(j{+}m), & (j{+}i)(j{+}m)(j{+}m{+}i)],[(j{+}i)(j{+}2i)(j{+}i{+}m), & & (j{+}2i)(j{+}i{+}m)(j{+}2i{+}m)]\ldots\\ \ldots[(j{+}i)(j{+}m)(j{+}m{+}i),(j{+}m)(j{+}m{+}i)j], & & [(j{+}2i)(j{+}i{+}m)(j{+}2i{+}m),(j{+}i{+}m)(j{+}2i{+}m)(j{+}i)],\ldots & \end{array}$$

where $1 \leq j \leq n$. For example, $\phi_{1,2}$ for Φ_6^3 can be modified into

$$\begin{array}{llll}([124,245],[235,356],[346,461]) & \text{or condensed} & ([1245],[2356],[3461]) & \text{(where square brackets surround}\\ ([451,512],[562,623],[613,134]) & \text{in CC as:} & ([4512],[5623],[6134]), & \text{vertices in a common 4-cycle).}\end{array}$$

If $i = \frac{m}{2} \in \mathbb{Z}$, we would like to modify $\phi_{i,i}$ and $\phi_{\lfloor \frac{i}{2} \rfloor, \lceil \frac{i}{2} \rceil}$ together towards an exact 2-factor Φ_n^3. Here, we have two cases, the second one set between parentheses, where $m = 2i = 2p$: $x = \frac{p}{2} \in \mathbb{Z}$ (resp., $x \in \{\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil\} \cap 2\mathbb{Z}$, otherwise). In the first case, Φ_n^3 can be modified towards an exact 2-factor by replacing $\phi_{p,p}$ and $\phi_{x,x}$ by one or more cycles that continue the following pattern, where $1 \leq j \leq p$

$$\begin{array}{ll}\ldots([j(j{+}p)(j{+}3p)(j{+}2p)], & (j{+}3p)(j{+}5x)(j{+}2p)(j{+}3x),\\ \;[(j{+}5x)(j{+}7x)(j{+}3x)(j{+}x)], & (j{+}7x)j(j{+}x)(j{+}2x)(j{+}3x),\ldots\end{array}.$$

For example, $\phi_{2,2}$ and $\phi_{1,1}$ for Φ_8^3 are modified into $([1375], 7654, [6842], 8123)$ $([3517], 1876, [8624], 2345)$. Of the remaining case, $x \in \{\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil\} \cap 2\mathbb{Z}$, we offer example (C) in Section 5, using $\phi_{3,3}$ and $\phi_{1,2}$ for Φ_{12}^3.

4. APPLICATION TO SPHERE PACKING

The exact 2-factors above combine with the decomposition of $X_{r,t}^3 \setminus F_\epsilon$ into copies of Π_r^t in Subsection 1.3. In preparation for Theorem 6.1, we provide two examples.

First, $X_{3,2}^3 \setminus F_\epsilon$, (where $(34) = (\epsilon)$), splits into ten copies of $\Pi_3^2 = X_3^2 \square X_2^1$. Each 2×6 array in Figure 2 shows one such copy, composed by: **(i)** two copies of X_3^2 (shown as contiguous rows), i.e. two 6-cycles (obtained in the upper-left corner, by concatenating 45 or 54 to each entry of $(312, \xi_3^1, 321, \xi_2^2, 123, \xi_1^1, 132, \xi_3^2,$

31245 32145 12345 13245 23145 21345
31254 32154 12354 13254 23154 21354

34251 32451 42351 43251 23451 24351
34215 32415 42315 43215 23415 24315

34512 35412 45312 43512 53412 54312
34521 35421 45321 43521 53421 54321

14523 15423 45123 41523 51423 54123
14532 15432 45132 41532 51432 54132

12534 15234 25134 21534 51234 52134
12543 15243 25143 21543 51243 52143

15342 13542 53142 51342 31542 35142
15324 13524 53124 51324 31524 35124

14325 13425 43125 41325 31425 34125
14352 13452 43152 41352 31452 34152

14253 12453 42153 41253 21453 24153
14235 12435 42135 41235 21435 24135

54231 52431 42531 45231 25431 24531
54213 52413 42513 45213 25413 24513

53214 52314 32514 35214 25314 23514
53241 52341 32541 35241 25341 23541

Figure 2. A uniform (5/6)-E-set in $X^3_{3,2}$ via an exact Φ^3_5

TABLE I

$X(123)\square X(456)$	321456	123546	$X(236)\square X(145)$	632145	236514
$X(234)\square X(156)$	432516	324156	$X(136)\square X(245)$	163425	631245
$X(345)\square X(126)$	543612	435216	$X(146)\square X(235)$	416523	164325
$X(456)\square X(123)$	654123	546312	$X(145)\square X(236)$	541236	415623
$X(156)\square X(234)$	165234	651423	$X(245)\square X(136)$	245613	542136
$X(126)\square X(345)$	216435	162534	$X(246)\square X(135)$	642315	246513
$X(124)\square X(356)$	412356	214635	$X(346)\square X(125)$	364512	643215
$X(134)\square X(256)$	314526	413256	$X(356)\square X(124)$	563214	365412
$X(135)\square X(246)$	531246	315426	$X(256)\square X(134)$	256134	562314
$X(235)\square X(146)$	235614	532146	$X(125)\square X(346)$	125346	251634

$231, \xi^1_2, 213, \xi^2_1)$, with edges represented by the copies ξ^i_j of X^1_2, using Subsection 1.2); **(ii)** six column-wise copies of X^1_2; **(iii)** six 4-cycles given by contiguous columns. The five copies of Π^2_3 on the left of the figure are in ordered correspondence with the terms of the 5-cycle $\psi_5^{-1} = (321, 432, 543, 154, 215)$ in display (1): the black vertices in each of the five copies of Π^2_3 determine two 1-spheres with the two dark-gray vertices in the subsequent copy of Π^2_3, where: **(a)** the top copy of Π^2_3 is taken to be subsequent to the bottom copy; **(b)** the center of each such 1-sphere is underlined; **(c)** one of the two underlined vertices in each copy of Π^2_3 starts with the triple given by a corresponding term in ψ_5; and **(d)** the remaining vertices are light-gray. For example, a 1-sphere here is given by the underlined-black vertex 32145 (forming part of the product $J = \xi^3_1 \times \xi^4_4$ of E-sets in $\Pi^2_3 = X^2_3 \square X^1_2)$ in the top copy of Π^2_3, its black neighbors 12345, 31245 and 32154 and the dark-gray vertex 32415 in the subsequent copy of Π^2_3. Similarly, the five copies of Π^2_3 on the right of Figure 2 are linked to the 5-cycle

321456 123456 132456 231456 213456 312456 632145 236145 263145 362145 326145 623145
321546 123546 132546 231546 213546 312546 632415 236415 263415 362415 326415 623415
321645 123645 132645 231645 213645 312645 632514 236514 263514 362514 326514 623514
321465 123465 132465 231465 213465 312465 632154 236154 263154 362154 326154 623154
321564 123564 132564 231564 213564 312564 632451 236451 263451 362451 326451 623451
321654 123654 132654 231654 213654 312654 632541 236541 263541 362541 326541 623541

432516 234516 243516 342516 324516 423516 163425 361425 316425 613425 631425 136425
432615 234615 243615 342615 324615 423615 163245 361245 316245 613245 631245 136245
432165 234165 243165 342165 324165 423165 163542 361542 316542 613542 631542 136542
432561 234561 243561 342561 324561 423561 163452 361452 316452 613452 631452 136452
432651 234651 243651 342651 324651 423651 163254 361254 316254 613254 631254 136254
432156 234156 243156 342156 324156 423156 163524 361524 316524 613524 631524 136524

543612 345612 354612 453612 435612 534612 416523 614523 641523 146523 164523 461523
543216 345216 354216 453216 435216 534216 416253 614253 641253 146253 164253 461253
543126 345126 354126 453126 435126 534126 416352 614352 641352 146352 164352 461352
543621 345621 354621 453621 435621 534621 416532 614532 641532 146532 164532 461532
543261 345261 354261 453261 435261 534261 416235 614235 641235 146235 164235 461235
543162 345162 354162 453162 435162 534162 416325 614325 641325 146325 164325 461325

654123 456123 465123 564123 546123 645123 541236 145236 154236 451236 415236 514236
654213 456213 465213 564213 546213 645213 541326 145326 154326 451326 415326 514326
654312 456312 465312 564312 546312 645312 541623 145623 154623 451623 415623 514623
654132 456132 465132 564132 546132 645132 541263 145263 154263 451263 415263 514263
654231 456231 465231 564231 546231 645231 541362 145362 154362 451362 415362 514362
654321 456321 465321 564321 546321 645321 541632 145632 154632 451632 415632 514632

165234 561234 516234 615234 651234 156234 245613 542613 524613 425613 452613 254613
165324 561324 516324 615324 651324 156324 245163 542163 524163 425163 452163 254163
165423 561423 516423 615423 651423 156423 245361 542361 524361 425361 452361 254361
165243 561243 516243 615243 651243 156243 245631 542631 524631 425631 452631 254631
165342 561342 516342 615342 651342 156342 245136 542136 524136 425136 452136 254136
165432 561432 516432 615432 651432 156432 245316 542316 524316 425316 452316 254316

216435 612435 621435 126435 162435 261435 642315 246315 264315 462315 426315 624315
216345 612345 621345 126345 162345 261345 642135 246135 264135 462135 426135 624135
216543 612543 621543 126543 162543 261543 642531 246531 264531 462531 426531 624531
216453 612453 621453 126453 162453 261453 642351 246351 264351 462351 426351 624351
216354 612354 621354 126354 162354 261354 642153 246153 264153 462153 426153 624153
216534 612534 621534 126534 162534 261534 642513 246513 264513 462513 426513 624513

412356 214356 241356 142356 124356 421356 364512 463512 436512 634512 643512 346512
412536 214536 241536 142536 124536 421536 364152 463152 436152 634152 643152 346152
412635 214635 241635 142635 124635 421635 364251 463251 436251 634251 643251 346251
412365 214365 241365 142365 124365 421365 364521 463521 436521 634521 643521 346521
412563 214563 241563 142563 124563 421563 365125 463125 436125 634125 643125 346125
412653 214653 241653 142653 124653 421653 364215 463215 436215 634215 643215 346215

314526 413526 431526 134526 143526 341526 563214 365214 356214 653214 635214 536214
314256 413256 431256 134256 143256 341256 563124 365124 356124 653124 635124 536124
314652 413652 431652 134652 143652 341652 563421 365421 356421 653421 635421 536421
314562 413562 431562 134562 143562 341562 563241 365241 356241 653241 635241 536241
314265 413265 431265 134265 143265 341265 563142 365142 356142 653142 635142 536142
314625 413625 431625 134625 143625 341625 563412 365412 356412 653412 635412 536412

531246 135246 153246 351246 315246 513246 256134 652134 625134 526134 562134 265134
531426 135426 153426 351426 315426 513426 256314 652314 625314 526314 562314 265314
531624 135624 153624 351624 315624 513624 256413 652413 625413 526413 562413 265413
531264 135264 153264 351264 315264 513264 256143 652143 625143 526143 562143 265143
531462 135462 153462 351462 315462 513462 256341 652341 625341 526341 562341 265341
531642 135642 153642 351642 315642 513642 256431 652431 625431 526431 562431 265431

235614 532614 523614 325614 352614 253614 125346 521346 512346 215346 251346 152346
235164 532164 523164 325164 352614 253164 125436 521436 512436 215436 251436 152436
235461 532461 523461 325461 352461 253461 125634 521634 512634 215634 251634 152634
235641 532641 523641 325641 352641 253641 125364 521364 512364 215364 251364 152364
235146 532146 523146 325146 352146 253146 125463 521463 512463 215463 251463 152463
235416 532416 523416 325416 352416 253416 125643 521643 512643 215643 251643 152643

Figure 3. A uniform (2/3)-E-set in $X^3_{3,3}$ via an exact Φ^3_6

$\psi'_5 = (135, 413, 241, 524, 352)$. As a result, the underlined vertices yield a (2/3)-E-set.

Second, $X^3_{3,3}$ admits a (2/3)-E-set, for instance by means of the Hamilton cycle ψ_6 in displays (2)-(3), illustrated in Figure 3 where each 6×6 array stands for the disposition of vertices in an embedding of a copy of Π^3_3 in a torus. There are $20 = \binom{6}{3} = \binom{n}{r}$ such copies. They are in ordered correspondence with the terms of ψ_6 (clarified below) starting with the ten 6×6 arrays on the left of the figure followed by the remaining ten to their right. In each of these twenty arrays, (call it Y), we select a product $J = J' \times J''$ of E-sets J' and J'' of X^2_3, with the four degree-5 vertices of J underlined, two of them starting with the triple of a corresponding term in ψ_6. The members of the (graph-theoretical) open neighborhoods they define are shown as 16 black vertices in Y and four dark-gray vertices in the 6×6 array Y' that follows Y. Remaining vertices are light-gray. For example, the product J in the upper-left 6×6 array Y in Figure 3 (with $J' = \xi^3_1$ and $J'' = \xi^4_4$, using Subsection 1.2) is given by the underlined-black vertices 321456, 231456, 321465 and 231465, and their corresponding black neighbors together with the four dark-gray vertices in Y'.

Figure 3 is encoded in Table I, having each copy of Π^3_3 denoted on the left as $X(Y)\Box X(Z)$, where Y and Z are respectively the common initial and terminal triples of the composing vertices, followed by one of the underlined-black vertices and then by one of its dark-gray vertices.

5. CYCLIC ORDERED PARTITIONS

No exact 2-factor Φ^4_6 exists. This is remedied in example (D) below. On the other hand, an exact 2-factor Φ^4_7 is given by the CCs ϕ_1, ϕ_2, ϕ_3, that we equalize to the respective *cyclic ordered partitions* (or COPs) $1114 = \phi_1$, $2221 = \phi_2$, $1213 = \phi_3$ of the integer 7 (associated with the successive difference triples 111, 222, 333 of quadruples) and by alternating the quadruples in the COPs

1123={1235,2346,3457,4561,5672,6713,7124}, 2113={1345,2456,3567,4671,5712,6123,7234},

into the exact CC $(1235, 1345, 4561, 4671, 7124, 7234, 3457, 3567, 6713, 6123, 2346, 2456, 5672, 5712)$. On the other hand, Γ^5_7 has COPs $11113 = \phi_1$, $11122 = \phi_2$ and $11212 = \phi_3$, yielding easily an exact 2-factor Φ^5_7.

To be resolved in example (E) below, the COPs for Γ^4_8 are: $1115 = \phi_1$, $2222 = \phi_2$, $2123 = \phi_3$, 1313, 1124, 1133, 1214, 1223, 2114, 2213. In detail:

1115={1234,2345,3456,4567,5678,6781,7812,8123}; 2222={1357,2468};
2123={1346,2457,3568,4671,5782,6813,7124,8235}; 1313={1256,2367,3478,4581};
1124={1235,2346,3457,4568,5681,6782,7813,8124}; 1133={1236,2347,3458,4561,5672,6783,7814,8125};
1214={1245,2356,3467,4578,5681,6712,7823,8134}; 1223={1246,2357,3468,4571,5682,6713,7824,8135};
2114={1345,2456,3567,4678,5781,6812,7123,8234}; 2213={1356,2467,3578,4681,5712,6823,7134,8245}.

We obtain the following exact cycles (apart from ϕ_1 and ϕ_3) by alternating the COP pairs $\{1124, 1133\}$, $\{1214, 1223\}$ and $\{2114,\ 2213\}$:

(1235,1236,2346,2347,3457,3458,4568,4561,5671,5672,6782,6783,7813,7814,8124,8125),
(1245,1246,6712,6713,3467,3468,8134,8135,5681,5682,2356,2357,7823,7824,4578,4571),
(1345,1356,3567,3578,5781,5712,7123,7134)(2456,2467,4678,4681,6812,6823,8234,8245).

A uniform E-set in $X^3_{4,4}$ is obtained from this in example(E) below.

Exact spanning subgraphs of largest degree 3 in Γ^r_n whose components are unicyclic caterpillars, (i.e. graphs for which the removal of its pendant vertices makes them cyclic) will be called *nests*. Then, a nest leads to a uniform α-E-set with $\alpha = \frac{n}{rt}$. For example: **(A)**, the nest of Γ^3_5 formed by the CC (12345) plus the edges $(132, 135)$, $(423, 421)$, $(354, 352)$, $(415, 413)$ and $(251, 254)$ leads to a uniform (5/6)-E-set; **(B)** by modifying Ψ^3_6 deleting the isolated triples 135 and 246 and adding to the CC (123456) the edges $(123, 135)$ and $(456, 246)$, again a (2/3)-E-set in $X^3_{3,3}$ is obtained; **(C)** In Γ^3_{12}, we modify $\phi_{3,3}$ and $\phi_{1,2}$ by inserting into the following three 8-cycles of $\phi_{1,2}$ the edge quadruples following them in the following display:

(134679ac),(134,147),(467,47a),(79a,7a1),(ac1,a14);
(24578ab1),(245,258),(578,58b),(8ab,8b2),(b12,b25);
(35689bc2),(356,369),(689,69c),(9bc,9c3),(c23,c36);

the resulting nest yields a (4/9)-E-set in $X^3_{3,9}$; **(D)** In Γ^4_6, the COPs 1113 and 1122 alternate into the exact 12-cycle $(1234, 1235, 2345, 2346, 3456, 3451, 4561, 4562, 5612, 5613, 6123, 6124)$; a nest is obtained by attaching edges with pendant vertices in the COP $1212 = \{1245, 2356, 3461\}$, say edges $(1235, 1245)$, $(3451, 3461)$ and $(5613, 2356)$; this leads to a uniform (1/4)-E-set in $X^3_{4,2}$; an alternate nest of Γ^4_6 is formed by the 5-cycles

$$(12345)|6 = (1236, 2346, 3456, 4516, 5126)$$
$$(62413)|5 = (6245, 2415, 4135, 1365, 3625)$$

plus the edges $(6245, 1246)$, $(2415, 1234)$, $(4135, 5234)$, $(1365, 1346)$, $(3625, 5123)$; **(E)** the exact cycles in Γ^4_8 cited above are taken into a nest by adding to them the following edges ending in the COPs 1313 and 2222: (1245,1256), (6712,2367), (3467,3478), (8134,4581), (8135,1357), (5682,2468); this leads to a (1/2)-E-set in $X^3_{4,4}$; **(F)** In a likewise fashion, we obtain a nest of Γ_{10} formed by two 10 cycles for each of COPs 1117 and 3331 , six unicyclic caterpillars (two over a 20-cycle and four over a 10-cycle) for COP pairs $\{1126, 1135\}$, $\{1216, 1315\}$, $\{1252, 1522\}$, $\{1324, 1342\}$ with pendant edges to the respective COP pairs $\{1162, 1432\}$, $\{1234, 1225\}$, $\{1243, 1153\}$, $\{1144, 2233\}$, and five unicyclic caterpillar over a 4-cycle obtained by alternating the COPs 2323 and 1414 with the COP 1423, together with opposing pendant edges to the COP 2226; this leads to a (2/5)-E-set in $X^3_{5,5}$.

Motivated by these six examples, we conjecture that every Γ^r_n contains a nest Φ^r_n whose subgraph resulting from the removal of its pendant vertices is the disjoint union of n- and $2n$-cycles. This would yield a uniform α-E-set in $X^3_{r,t}$ with $\alpha = \frac{n}{rt}$. At the moment, we just know that if $1 < t$, $2 < r$ and $n = r + t$, then $\frac{n}{rt}$ is an upper bound of those α for which there is a uniform α-E-set. This will be used in Theorem 6.1, below.

For $n > 4$, exact non-spanning subgraphs of Γ^r_n yield $\alpha < \frac{n}{rt}$. To exemplify this, we reselect the centers of disjoint 1-spheres in Figures 2 and 3 by taking all vertices in a copy of Π^t_r as dark-gray and its neighbors via F_ϵ underlined-black, then setting as dark-gray enough vertices at distance 2 from underlined-black vertices, traversing F_ϵ to set underlined-black vertices in all copies of Π^t_r. One can select more than one copy of Π^t_r to be completely dark-gray, e.g. those copies containing vertices 123456 and 654321 in $X^3_{3,3}$ and proceed as above until the twenty copies of Π^t_r have underlined-black vertices, but the value of α in such cases is still less than $\frac{n}{rt}$.

6. UNIFORM SPHERE PACKING

Assume $4 < n = r + t$, where $r, t \in \mathbb{Z}$. Then each copy Π' of $\Pi^t_r = X^{d_r}_r \Box X^{d_t}_t$ in $X^3_{r,t}$, where $d_r, d_t \in \{1, 2\}$, has $r!t!$ vertices. In view of Sections 8-10 below (showing that covering one Π' with 1-spheres of a perfect sphere packing $\mathcal{S}$ of $X^3_{r,t}$ prevents the E-set associated to $\mathcal{S}$ for being uniform), it arises from Sections 4-5 that uniform α-E-sets J in $X^3_{r,t}$ must have $\alpha \le \frac{n}{rt}$, since their intersection with each Π' is contained at most in a product of E-sets. If $\alpha = \frac{n}{rt}$ then each $\Pi' \cap J$ equals $J' \times J''$. Here, J' and J'' are E-sets in $X^{d_r}_r$ and $X^{d_t}_t$ of the forms ξ^r_i $(1 \le i < r)$ and $\xi^{r^*}_j$ $(r^* < j \le n)$ respectively, (instead of ξ^1_i with $1 < i \le n$, as in Subsection 1.2). Let $N[J' \times J'']$ be the union of the 1-spheres centered at the vertices of $J' \times J''$. Then $\Pi' - N[J' \times J'']$ is the disjoint union of $(r-1)(t-1)$ copies of Π^{t-1}_{r-1}. Also, each Π' intersects J in $(r-1)!(t-1)!$ vertices. These are the centers of pairwise disjoint 1-spheres, yielding a total of $(r-1)!(t-1)!n$ vertices in all those spheres. This way, $\frac{n}{rt}n!$ vertices of $X^3_{r,t}$ become covered by pairwise disjoint 1-spheres in $X^3_{r,t}$. This together with the outcome of the penultimate paragraph of Section 1 yields a maximal imperfect uniform 1-sphere packing of $X^3_{r,t}$. Such a packing ensures the nonexistence of E-sets of $X^3_{r,t}$ via the arguments of Theorems 10.1–10.2 and Corollary 10.3 below.

Theorem 6.1. *Let $4 < n = r + t$, $(, r, t \in \mathbb{Z})$. Then, there are at most $\frac{n}{rt}n!$ vertices in the union of 1-spheres of an imperfect uniform 1-sphere packing of $X^3_{r,t}$. This ensures the nonexistence of E-sets of $X^3_{r,t}$.*

7. LOCALLY MAXIMUM PACKING DENSITY

The techniques used in this and four following sections lead to what we may call *locally maximum packing density*, for we will pack as many 1-spheres as possible in each copy of Π_r^t according to the decomposition of $X_{r,t}^3 \setminus F_\epsilon$ in Subsection 1.3.

A 1-sphere packing $\mathcal{S}$ of $X_{3,3}^3$ is indicated in Figure 4, containing eight 6×6 arrays (in the fashion of Figure 3). In each such array in Figure 4, the black 6-tuples represent centers of 1-spheres in $\mathcal{S}$. There are two such centers in the first, (resp., third), [resp., fifth] row, namely in columns 1 and 4, (resp. 3 and 6), [resp., 5 and 2]. Each dark-gray 6-tuple stands for a vertex adjacent to one of the said 1-sphere centers in a different copy of Π_3^3 via transposition $(\epsilon) = (34)$. There are two of these dark-gray 6-tuples in the second, (resp., fourth), [resp., sixth] row of each 6×6 array, namely in columns 2 and 5, (resp., 4 and 1), [resp., 6 and 3]. This divides the black and dark-gray 6-tuples in each 6×6 array into three 2×2 sub-arrays obtained from the diagonal black 6-tuples by transpositions (12) and (56) and their composition. The left of Figure 5 represents, with the same 6-tuple colors of Figure 4, its upper-left copy of Π_3^3, namely $X(123)\square X(456)$.

TABLE II

$X(123)\square X(456)$	123456	213456	312564	132564	231645	321645
$X(214)\square X(365)$	214365	124365	421653	241653	142536	412536
$X(326)\square X(154)$	326154	236154	632541	362541	263415	623415
$X(135)\square X(246)$	135246	315246	513462	153462	351624	531624
$X(246)\square X(135)$	246135	426135	624351	264351	462513	642513
$X(154)\square X(326)$	154326	514326	415263	145263	541632	451632
$X(365)\square X(214)$	365214	635214	536142	356142	653421	563142
$X(456)\square X(123)$	456123	546123	645231	465231	564312	645312

Table II lists on its leftmost column the copies of Π_3^3 of Figure 4, followed to their right by three pertaining pairs of 6-tuples encodable as $(a_{i,1}, a_{i,2}, a_{i,3})$, where $i \in I_8$. For instance, $a_{1,1} = \{123456, 213456\}$, $a_{1,2} = \{312564, 132564\}$, etc. Consider the following pairs of pairs of black 6-tuples in the main diagonals of the eight 6×6 arrays in Figure 4 related by the permutation (12)(34)(56):

$$\begin{array}{l} \{a_{1,1},a_{2,1}\}, \{a_{1,2},a_{4,1}\}, \{a_{1,3},a_{3,1}\}, \{a_{2,2},a_{5,1}\}, \{a_{2,3},a_{6,2}\}, \{a_{3,2},a_{7,1}\}, \\ \{a_{3,3},a_{5,2}\}, \{a_{4,2},a_{6,1}\}, \{a_{4,3},a_{7,2}\}, \{a_{5,3},a_{8,2}\}, \{a_{6,3},a_{8,1}\}, \{a_{7,3},a_{8,3}\}. \end{array} \tag{5}$$

The eight copies of Π_3^3 in Figure 4 induce a subgraph $X' = X'_{3,3}$ in $X_{3,3}^3$ (see right of Figure 5) whose vertex set admits a partition into 48 1-spheres around the black 6-tuples, with a partial total of 288 vertices. Moreover, X' has an E-set J formed by the black 6-tuples, encoded in the pairs of display (5). Consider the vertices of the remaining twelve copies of Π_3^3 in $X_{3,3}^3$ at distance 2 from a

123456	132456	231456	**213456**	312456	321456	**246135**	264135	462135	**426135**	624135	642135
123546	**132546**	231546	213546	**312546**	321546	246315	**264315**	462315	426315	**624315**	642315
123645	132645	**231645**	213645	312645	**321645**	246513	264513	**462513**	426513	624513	**642513**
123465	132465	231465	**213465**	312465	321465	**246153**	264153	462153	**426153**	624153	642153
123564	**132564**	231564	213564	**312564**	321564	246351	**264351**	462351	426351	**624351**	642351
123654	132654	**231654**	213654	312654	**321654**	246531	264531	**462531**	426531	624531	**642531**
214365	241365	142365	**124365**	421365	412365	**154326**	145326	541326	**514326**	415326	451326
214635	**241635**	142635	124635	**421635**	412635	154236	**145236**	541236	514236	**415236**	451236
214536	241536	**142536**	124536	421536	**412536**	154632	145632	**541632**	514632	415632	**451632**
214356	241356	142356	**124356**	421356	412356	**154362**	145362	541362	**514362**	415362	451362
214653	**241653**	142653	124653	**421653**	412653	154263	**145263**	541263	514263	**415263**	451263
214563	241563	**142563**	124563	421563	**412563**	154623	145623	**541623**	514623	415623	**451623**
326154	362154	263154	**236154**	632154	623154	**365214**	356214	653214	**635214**	536214	563214
326514	**362514**	263514	236514	**632514**	623514	365124	**356124**	653124	635124	**536124**	563124
326415	362415	**263415**	236415	632415	**623415**	365421	356421	**653421**	635421	536421	**563421**
326145	362145	263145	**236145**	632145	623145	**365241**	356241	653241	**635241**	536241	563241
326541	**362541**	263541	236541	**632541**	623541	365142	**356142**	653142	635142	**536142**	563142
326451	362451	**263451**	236451	632451	**623451**	365412	356412	**653412**	635412	536412	**563412**
135246	153246	351246	**315246**	513246	531246	**456123**	465123	564123	**546123**	645123	654123
135426	**153426**	351426	315426	**513426**	531426	456213	**465213**	564213	546213	**645213**	654213
135624	153624	**351624**	315624	513624	**531624**	456312	465312	**564312**	546312	645312	**654312**
135264	153264	351264	**315264**	513264	531264	**456132**	465132	564132	**546132**	645132	654132
135462	**153462**	351462	315462	**513462**	531462	456231	**465231**	564231	546231	**645231**	654231
135642	153642	**351642**	315642	513642	**531642**	456321	465321	**564321**	546321	645321	**654321**

Figure 4. Local maximum packing density in $X^3_{3,3}$

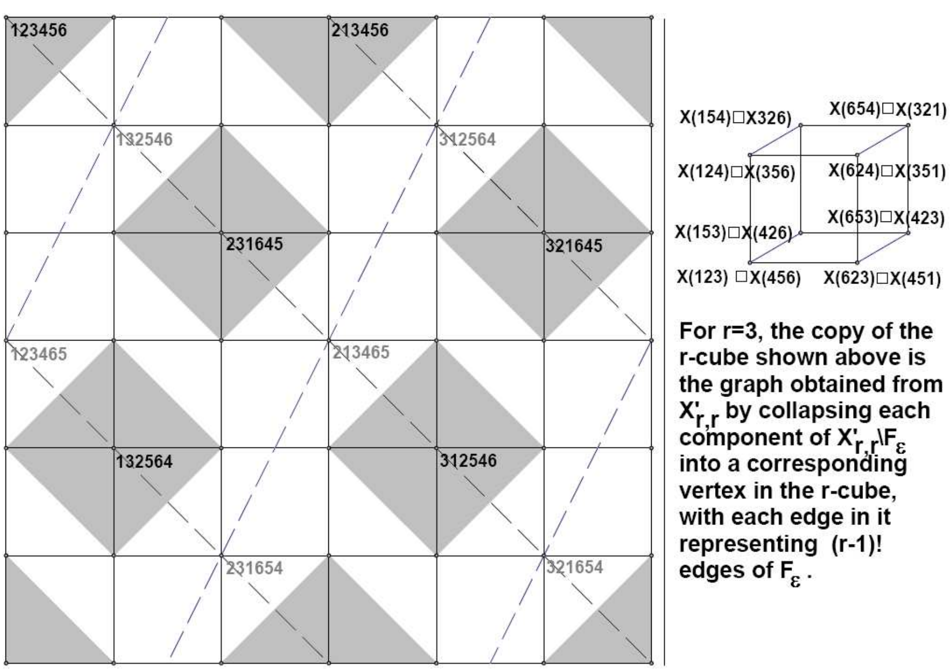

Figure 5. Embedding of $X(123)\Box X(456)$ in a torus and a representation of $X'_{3,3}$

center of a 1-sphere among the cited 48. There are 192 such vertices. Each of the remaining 240 vertices in these twelve copies is at distance 3 from the center of one of the 48 1-spheres.

Table III allows to select 24 centers of pairwise disjoint 1-spheres to cover half of those 240 vertices: just choose one 1-sphere center per pair of two 6-tuples in each box in the table. There are 144 vertices in the selected 24 1-spheres. In sum, we obtain $\frac{3}{5}6!$ vertices of $X^3_{3,3}$ packed into $72 = 48 + 24$ 1-spheres.

Let us apply the definitions of double-sphere and $\mathcal{S}$-sphere in Subsection 1.1 with $X = X^3_{3,3}$ and $X' = X'_{3,3}$. By adding to each 1-sphere Σ in the above packing of X' the end-vertices of the (ϵ)-colored edges departing from Σ, where $(\epsilon) = (34)$, a corresponding $\mathcal{S}$-sphere Σ' is obtained enlarging Σ. On the other hand, the 24 1-spheres selected above can be extended into 24 double-spheres, which forms a double-sphere packing. A transformation of the 1-sphere packing $\mathcal{S}$ in Figure 3 into a perfect special packing of $X^3_{r,t}$ is obtained by enlarging the 48 1-spheres that pack perfectly $X'_{3,3}$ into corresponding $\mathcal{S}$-spheres by addition of the 192 vertices not in $X'_{3,3}$ and at distance 2 from the centers of the 48 1-spheres. The reader may compare this with the $\mathcal{S}$-sphere packing of $X^3_{2,2}$ suggested on the right of Figure 1.

TABLE III

$X(162)\square X(534)$ $X(165)\square X(234)$	162534 165234	612534 615234	162543 165243	612543 615243
$X(163)\square X(425)$ $X(164)\square X(325)$	163425 164325	613425 614325	163452 164352	613452 614352
$X(256)\square X(134)$ $X(251)\square X(634)$	256134 251634	526134 521634	256143 523416	526143 521643
$X(431)\square X(652)$ $X(436)\square X(152)$	431652 436152	341652 346152	431625 436125	341652 346152
$X(432)\square X(516)$ $X(435)\square X(216)$	432516 435216	342516 345216	432561 435261	342561 345261
$X(524)\square X(613)$ $X(526)\square X(413)$	524361 523461	254361 523461	524316 523416	254316 523416

Selecting instead 24 centers of 1-spheres to be the neighbors via the transposition (23) (or (13)) of the 24 centers allowed above by means of Table III leaves room to selecting additional 24 centers of 1-spheres in the six still untouched copies of Π^3_3. The selection of the 24 new centers of 1-spheres in those six copies must be done via the transposition (45) (or (46)). This yields a packing of $X^3_{3,3}$ by 96 1-spheres comprising $576 = \frac{4}{5}|V(X^3_{3,3})|$ vertices. Observe that the 96 corresponding centers are obtained by modifying the original 1-sphere centers both adjacently and alternatively, idea to be generalized in Theorems 10.2.

8. RENUMBERING TREE VERTICES

In generalizing the locally maximum packing density of Section 7, we found it convenient to modify the order of vertices of the tree $\tau^3_{r,t}$ in the paragraph in

Section 1 containing items (i)-(ii) by letting: **(i′)** 1 and $r^* = r+1$ denote the vertices of respective degrees r and t in $\tau^3_{r,t}$ so that $\epsilon = 1r^*$; **(ii′)** $2, \ldots, r$ (resp., $r^*+1, \ldots, n$) denote the vertices adjacent to vertex 1 (resp., r^*) in $\tau^3_{r,t}$.

Assuming this modification, we pass to exemplify it via Figure 5, on whose top a representation of the copy $X(12)\square X(34)$ of Π^2_2 is given that presents, before and after (symbol) $\square$, the copies of K_2 forming $X(12)$ and $X(34)$, respectively. Similar representations can be given for $X(32)\square X(14)$, $X(14)\square X(32)$ and $X(34)\square X(12)$, forming with $X(12)\square X(34)$ a subgraph $X'_{2,2}$ of $X^3_{2,2}$ preceding in spirit the subgraph $X'_{3,3}$ of $X^3_{3,3}$ in Section 7. The two remaining squares $X(13)\square X(24)$ and $X(24)\square X(13)$ coincide with those whose interiors are shaded in light-gray color in Figure 1 and form a second subgraph $X''_{2,2}$ of $X^3_{2,2}$.

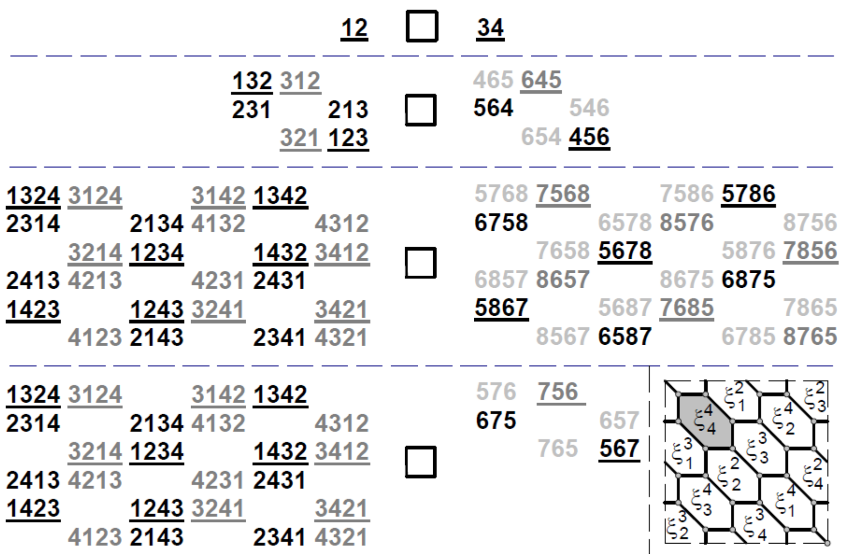

Figure 6. Interpretations of Π^2_2, Π^3_3, Π^4_4 and Π^3_4

Subsequently in Figure 6, a similar representation of the cartesian product $X(123)\square X(456)$ is given that shows, before and after $\square$, the 6-cycles $X(123)$ and $X(456)$, respectively, by disposing adjacent vertices contiguously: horizontally, vertically and diagonally between upper-left and lower-right. Here, the black centers of the three 1-spheres in the main diagonal of the 6×6 array representing $X(123)\square X(456)$ as in Figure 4 (but with the vertex order assumed above in this section) are recovered by: **(A)** taking a partition of $V(X(123))$ into the E-sets $\xi^1_1, \xi^1_2, \xi^1_3$ (see Subsection 1.2) given by: **(i)** underlined-black color for $\xi^1_1 = \{123, 132\}$, **(ii)** (not underlined) black color for $\xi^1_2 = \{213, 231\}$ and **(iii)** underlined-dark-gray color for $\xi^1_3 = \{312, 321\}$; **(B)** assigning the three colors of (A) respectively to the even-parity vertices in $X(456)$ as follows: **(i)** $456 \in \xi^4_4$,

(ii) $564 \in \xi_5^4$ and **(iii)** $645 \in \xi_6^4$, while the odd-parity vertices, namely 465, 546 and 654, shown in light-gray, do not intervene; **(C)** concatenating the vertices of $X(123)$ and $X(456)$ having a common color.

Now, we embed each copy of X_4^2 into a torus, as in the lower-right corner of Figure 6, with its copies ξ_i^j, ($j \in \{2,3,4\}$; $i \in I_4$), of X_3^2 disposed as above into their place. This way, the previous representation of $X(123)\Box X(456)$ is extended to Π_4^4 as in the lower two instances of Figure 6, where the shown cartesian products can be denoted $X(1234)\Box X(5678)$ and $X(1234)\Box X(567)$, this one obtained by restricting, i.e. puncturing $X(1234)\Box X(5678)$.

TABLE IV

$X_{3,3}^3\backslash F_\epsilon=$	$20(\Pi_3^3)$	$3/3=$	1	$5!/20$	$=6$	$3!3!/6=6$
$X_{4,3}^3\backslash F_\epsilon=$	$35(\Pi_4^3)$	$12/3=$	4	$6!/35$	$\notin\mathbb{Z}$	
$X_{4,4}^3\backslash F_\epsilon=$	$70(\Pi_4^4)$	$12/4=$	3	$7!/70$	$=72$	$4!4!/72=8$
$X_{5,3}^3\backslash F_\epsilon=$	$56(\Pi_5^3)$	$60/3=$	20	$7!/56$	$=90$	$5!3!/90=8$
$X_{5,4}^3\backslash F_\epsilon=$	$126(\Pi_5^4)$	$60/4=$	15	$8!/126$	$=320$	$5!4!/320=9$
$X_{6,3}^3\backslash F_\epsilon=$	$84(\Pi_6^3)$	$360/3=$	120	$8!/84$	$=480$	$6!3!/480=9$
$X_{5,5}^3\backslash F_\epsilon=$	$252(\Pi_5^5)$	$60/5=$	12	$9!/252$	$=1440$	$5!5!/1440=10$
$X_{6,4}^3\backslash F_\epsilon=$	$210(\Pi_6^4)$	$360/4=$	90	$9!/210$	$=1728$	$6!4!/1728=10$
$X_{7,3}^3\backslash F_\epsilon=$	$120(\Pi_7^3)$	$2520/3=$	840	$9!/120$	$=3024$	$7!3!/3024=10$
$X_{6,5}^3\backslash F_\epsilon=$	$462(\Pi_6^5)$	$360/5=$	72	$10!/462$	$\notin\mathbb{Z}$	
$X_{7,4}^3\backslash F_\epsilon=$	$330(\Pi_7^4)$	$2520/4=$	630	$10!/330$	$\notin\mathbb{Z}$	
$X_{8,3}^3\backslash F_\epsilon=$	$165(\Pi_8^3)$	$20160/3=$	13440	$10!/165$	$\notin\mathbb{Z}$	

In the third case of Figure 6, the coloring used for $X(123)\Box X(456)$ above is extended with a fourth color: (not underlined) dark-gray. On the left of $\Box$, the colors correspond to the E-sets ξ_i^1, where $i \in I_4$. On the right of $\Box$, the even-parity 4-tuples are given the same color i when their intersection with an E-set of the partition $\{\xi_j^5; j = 5,6,7,8\}$ starts with $j = i+4$. As mentioned, the situation for $X(1234)\Box X(567)$ can be considered a restriction of that of $(1234)\Box X(5678)$. We may write $X(567) = (567, \xi_7^7, 657, \xi_5^6, 756, \xi_5^6, 6576, \xi_7^6, 675, \xi_5^7, 765, \xi_6^6)$.

In a typical cartesian product $\Pi_r^t = X_r^2 \Box X_t^2$, where $2 < t \leq r$, we notice that: **(A)** the subset Q of vertices of the copy $X(r^* \cdots n)$ of X_t^2, where $r^* = r+1$, which as t-tuples have the same parity as the t-tuple $r^* \cdots n$ has a partition into t subsets Q_i with the t-tuples in Q_i starting at $(r+i)$, for every $i \in I_t$; **(B)** the vertex set of the copy $X(1 \cdots r)$ of X_r^2 has a partition into the r E-sets ξ_j^1 for every $j \in I_r$; **(C)** it eases treatment to consider the n-tuples obtained by concatenating every r-tuple in ξ_i^1 with every t-tuple in Q_i, for every $i \in I_t$.

The convenience of this new vertex numbering is that to obtain a maximal number of disjoint 1-sphere centers in the copies of Π_r^t, say $X(1 \cdots r)\Box X(r^* \cdots n)$, we can order both factors of these products in the same direction, resulting in transpositions between the first entry of either an initial r- or a terminal t-tuple with any of the remaining entries of that tuple, plus the transposition of both first entries. We concatenate initial r-tuples and terminal t-tuples whenever they have the same color (as in the instances of Figure 6), where the color set of the second factor in the product must coincide with, or be contained in, the color set of the first factor, considering that the second coloring here is given on the elements of the alternate subgroup $A_t \subset S_t$ while the first coloring is taken from a partition of S_r into E-sets.

A list of some cases of $X^3_{r,t}$ is considered in Table IV, that contains in each line: **(a)** a presentation of $X^3_{r,t} \setminus F_\epsilon$ as the union of $\binom{n}{r}$ copies of Π^t_r; **(b)** the number of even-parity vertices in X^2_t that start with a specific entry, i.e. the quotient of $|A_t|$ by the number of E-sets in a vertex partition of X^2_t into E-sets; **(c)** the largest possible number of centers of pairwise disjoint 1-spheres in a copy of Π^t_r as in the locally maximum packing density approach used since section 7, obtained as the number of vertices of $X^3_{r,t}$ having a common initial entry divided by the number of copies of Π^t_r in $X^3_{r,t}$, i.e. the quotient $(n-1)!/\binom{n}{r} = \frac{r!t!}{n}$; if this number is an integer, we proceed to fill the rightmost column; **(d)** a verification of the packing condition of 1-spheres induced by an E-set in X^3_n, that is: only if n divides $r!t!$, or equivalently, only if n is non-prime larger than 4; we note that this verification will hold even if $d > 3$.

9. FURTHER EXAMPLES

Two lines of Table IV are developed here into 1-sphere packings of corresponding graphs $X^3_{r,r}$ in preparation for Theorems 10.1–10.2.

For the third line in Table IV, given $z \in I_8$ denote $z' = z \pm 4 \in I_8$ and $\{z, z'\} = \mathbf{z}$. There are 16 copies of Π^4_4 in $X^3_{4,4}$ of the form $\Pi^4_4 = X(abcd)\Box X(a'b'c'd')$, where $\mathbf{a} = \{a, a'\} = \{1, 5\} = \mathbf{1}$, $\mathbf{b} = \{b, b'\} = \{2, 6\} = \mathbf{2}$, $\mathbf{c} = \{c, c'\} = \{3, 7\} = \mathbf{3}$ and $\mathbf{d} = \{d, d'\} = \{4, 8\} = \mathbf{4}$. The subgraph $X'_{4,4}$ induced by these 16 copies has an E-set J, (like the J with $|J| = 8 \times 6 = 48$, listed in Table II for $X^3_{3,3}$, but now with $|J| = 16 \times 72 = 1152$). Let H^{zw}_{xy} be a copy of Π^4_4 in $X^3_{4,4}$ of the form

$$\Pi^4_4 = X(xx'zw)\Box X(yy'z'w'), \tag{6}$$

with x, y, z, w respectively in $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$, where $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$. There are 48 such copies, inducing in $X^3_{4,4}$ a subgraph $X''_{4,4}$ disjoint from $X'_{4,4}$. In expressing each vertex of $X^3_{r,r}$ we set a dot separating the two halves of its representing $2r$-tuple. This is used in saying that J is at distance 2 from a vertex subset $W^{zw}_{xy} \subset H^{zw}_{xy}$ given by $W^{zw}_{xy} = \{xabc.ydef \mid \{a, b, c\} = \{x', z, w\}; \{d, e, f\} = \{y', z', w'\}\}$. Here, $|W^{zw}_{xy}| = 144$ yields a total of 6912 vertices in all 48 copies of

Π_4^4, viewed as in (6). Thus, they conform a vertex subset W with $|W| = 6912$. Moreover, W induces 1728 4-cycles, each the cartesian product of two copies of K_2 (each copy of K_2 being a component of the graph induced by the union of two E-sets in a corresponding vertex partition). Also, $V(X''_{4,4}) \setminus W$ induces similar 4-cycles in each H_{xy}^{zw}. As an example, consider the induced 4-cycle

(7) $\quad C = (2315.6748,\ 3215.6748,\ 3215.7648,\ 2315.7648) \subset H_{14}^{23} \setminus W_{14}^{23}.$

At most one vertex of C can be added to J in trying to extend it to an E-set of $X_{4,4}^3$. Thus, there does not exist an E-set of $X_{4,4}^3$ both containing $X'_{4,4}$ and covering all the vertices of $X''_{4,4}$. However, an extension $J \cup J'$ of J exists, where J' in $X''_{4,4}$ is composed by the centers of disjoint 1-spheres, reaching to a maximum covering of a third of $V(X''_{4,4})$, namely with $|V(X''_{4,4})|/3 = 48 \times 576/3 = 27648/3 = 9216$ vertices, leaving 18432 vertices of $V(X''_{4,4})$ out of those 1-spheres. Moreover, the 6912 vertices of W, being at distance 2 from J, cannot be members of any α-E-set $J \cup J'$. Away from them (in fact at distance 3 from J) and in $X''_{4,4}$, the vertices in a product of E-sets of $X(1237)\square X(5648)$ can be arranged as in the following 6×6 array:

(8)

2137.6548	2173.6548	2317.6548	2371.6548	2731.6548	2713.6548
2137.6584	2173.6584	2317.6584	2371.6584	2731.6584	2713.6584
2137.6458	2173.6458	2317.6458	2371.6458	2731.6458	2713.6458
2137.6485	2173.6485	2317.6485	2371.6485	2731.6485	2713.6485
2137.6854	2173.6854	2317.6854	2371.6854	2731.6854	2713.6854
2137.6845	2173.6845	2317.6845	2371.6845	2731.6845	2713.6845

encodable as a *concatenation product*: 2(137).6(548), where $2(137) = \{2137, 2173, 2317, 2371, 2713, 2731\}$ and $6(548) = \{6548, 6584, 6458, 6485, 6854, 6845\}$. In this notation, consider the following quadruples related via transpositions (13) and (14), twice each:

(9)
15(26;37,48)={1(237).5(648), 5(237).1(648), 1(637).5(248), 5(637).1(248)} and
34(26;17,18)={3(217).5(648), 5(237).4(618), 3(617).5(248), 5(637).4(218)},

To extend the treatment of Section 7 leading to a double-sphere packing (and then to a 1-sphere packing) in $X'_{3,3}$ a collection of double-spheres in $X''_{4,4}$ is taken whose centers are the 8-tuples in the quadruples (as in the top row of (9)):

(10)
15(26;37,48), 15(37;48,26), 15(48;26,37), 26(15;48,37), 26(37;15,48), 26(48,37,15),
37(15;26,48), 37(26;48,15), 37(48;15,26), 48(15;37,26), 48(26;15,37), 48(37;26,15).

They complete a packing of $X''_{4,4}$ covering 12096 vertices of $X_{4,4}^3 \setminus X'_{4,4}$ with the set of centers of the composing 1-spheres forming a subset J' of $V(X''_{4,4})$. By adjacency modifications as in the bottom row of (9) (to the data in the corresponding top row; compare Section 7, Table III), we obtain a packing of $X''_{4,4}$ by 1-spheres centered at the modified 8-tuples (forming a set again denoted J'), namely:

(11)
34(26;17,18), 42(37;18,16), 23(48;16,17), 34(15;28,27), 41(37;25,28), 13(48,27,25),
24(15;36,38), 41(26;38,35), 12(48;35,36), 32(15;47,46), 21(37;46,45), 13(26;45,47).

This totals $|V(X''_{4,4})|/2 = 24 \times 576 = 13824$ vertices of $X^3_{4,4} \setminus X'_{4,4}$. There are $6 = 70 - 16 - 48$ copies of Π^4_4 in $X^3_{4,4}$ disjoint from J. They are of the form

$$\Pi^4_4 = X(xx'yy')\Box X(zz'ww'), \tag{12}$$

for example two of them reached from the vertices of C shown in (7) by traversing the (ϵ)-colored edges, where $(\epsilon) = (15)$, yielding:

$$\begin{array}{ll} 6315.2748 \in X(1536)\Box X(4827), & 6215.3748 \in X(1526)\Box X(4837), \\ 7215.3648 \in X(1527)\Box X(4863), & 7315.2648 \in X(1537)\Box X(4826), \end{array}$$

where the second and fourth copies of Π^4_4 are as in display (12). There is a total of 432 4-cycles in $V(X''_{4,4})$ having each two alternate vertices with their neighbors via (ϵ)-colored edges in copies as in display (12). The neighbors of J' in $X^3_{4,4}\setminus(X'_{4,4}\cup X''_{4,4})$ corresponding to the 8-tuples in the quadruples in displays (10) and (11) are represented by the vertices in the following corresponding quadruples. First, let $(51, 26; 37, 48)$ stand for:

$$\{5(137).2(648),\ 6(237).5(148),\ 5(137).6(248),\ 2(637).5(148)\},$$

composed by two products of E-sets in each of $X(1357)\Box X(2468)$ and $X(2367)\Box X(1458)$. With this notation, the quadruples in question are:

$$\begin{array}{l} (51,26;37,48),\ (62,15,48,37),\ (62,37;15,48),\ (51,48;26,37),\ (73,15;26,48),\ (51,37;48,26), \\ (84,37;26,15),\ (73,48;15,26),\ (84,15;37,26),\ (73,26;48,15),\ (84,26;15,37),\ (62,48;37,15). \end{array}$$

This implies that all products of E-sets in these six copies are at distance two from vertices of J', which does not reach to an E-set of $X^3_{4,4}$ larger than $J \cup J'$. Thus, $J \cup J'$ is a $\frac{4}{7}$-E-set of $X^3_{4,4}$ and this is a maximum E-set of $X^3_{4,4}$.

For the seventh line in Table IV, given $z \in I_{10}$ denote $z' = z \pm 5 \in I_{10}$ and $\mathbf{z} = \{z, z'\}$. There are 32 copies of Π^5_5 of the form $\Pi^5_5 = X(abcde)\Box X(a'b'c'd'e')$, where $\mathbf{a} = \{a, a'\} = \{1, 6\} = \mathbf{1}$, $\mathbf{b} = \{b, b'\} = \{2, 7\} = \mathbf{2}$, $\mathbf{c} = \{c, c'\} = \{3, 8\} = \mathbf{3}$, $\mathbf{d} = \{d, d'\} = \{4, 9\} = \mathbf{4}$ and $\mathbf{e} = \{e, e'\} = \{5, 10\} = \mathbf{5}$. The subgraph $X'_{5,5}$ induced by these 32 copies has an E-set J with $|J| = 14400$ that also dominates the subset of vertices $y_1b_2b_3b_4b_5.y_5d_2d_3d_4d_5$ with $\{b_2, b_3, b_4, b_5\} = \{y'_1, y_2, y_3, y_4\}$ and $\{d_2, d_3, d_4, d_5\} = \{y'_5, y'_2, y'_3, y'_4\}$ in each of the 160 copies of Π^5_5 of the form $\Pi^5_5 = X(y_1y'_1y_2y_3y_4)\Box X(y_5y'_5y'_2y'_3y'_4)$ in $X^3_{5,5}$ with $y_z \in \mathbf{y}_z$ for $z \in I_5$ and $\{\mathbf{y}_z; z \in I_5\} = \{\mathbf{z}; z \in I_5\}$. Let H^{zuv}_{xy} be a copy of Π^5_5 in $X^3_{5,5}$ of the form

$$\Pi^5_5 = X(xx'zuv)\Box X(yy'z'u'v'), \tag{13}$$

with x, y, z, u, v taken respectively in $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}$, where $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}\} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$. There are 160 such copies, inducing in $X^3_{5,5}$ a subgraph $X''_{5,5}$ disjoint from $X'_{5,5}$. Note that J is at distance 2 from a vertex subset $W^{zuv}_{xy} \subset H^{zuv}_{xy}$ given by $W^{zuv}_{xy} = \{xabcd.ydefg \,|\, \{a, b, c, d\} = \{x', z, u, v\}; \{d, e, f, g\} = \{y', z', u', v'\}\}$ with

$|W_{xy}^{zuv}| = 2880$ that yields a total of 6912 vertices in all 160 copies of Π_5^5 as in display (13), forming a vertex subset W with $|W| = 460800$. Then, W induces 115200 4-cycles. Each such 4-cycle is the cartesian product of two copies of K_2 (each copy of K_2 being a component of the graph induced by the union of two E-sets in a corresponding vertex partition). Also, $V(X''_{5,5}) \setminus W$ induces similar 4-cycles in each H_{xy}^{zuv}. Let C be one such 4-cycle. At most one vertex of C can be added to J in trying to extend it to an E-set of $X_{5,5}^3$. This means that there does not exist an E-set of $X_{5,5}^3$ containing $X'_{5,5}$ and covering all the vertices of $X''_{5,5}$. In fact, the 460800 vertices of W, being at distance 2 from J, cannot be members of any E-set J' enlarging J. With a notation similar to that in display (9), let 1(23)45 stand for the following octad of products of E-sets as in display (8), where $a = 10$:

$$\begin{array}{c}\{1(2349).6(785a),\ 6(2349).1(785a),\ 1(7349).6(285a),\ 6(7349).1(285a),\\ 1(2849).6(735a),\ 6(2849).1(735a),\ 1(7849).6(235a),\ 6(7849).1(235a)\}\end{array} \tag{14}$$

Let the subset J' of $V(X_{5,5}^3)$ be given by the 10-tuples in the following 20 octads:

1(23)45, 2(34)51, 3(45)12, 4(51)23, 5(12)34,1(45)32, 2(51)43, 3(12)54, 4(23)15, 5(34)21,
1(35)24, 2(41)35, 3(52)41, 4(13)52, 5(24)13,1(24)53, 2(35)14, 3(41)25, 4(52)31, 5(13)42.

The set J' allows only induced double-spheres in $X''_{5,5}$. For example, the left entry of a each pair in display (14) is related to the corresponding right entry by means of transposition $(\epsilon) = (16)$. A modification of the composing 10-tuples allows a different J' by applying transpositions (14) and $(6a)$ alternatively to their left and right halves. This modified J' covers just 64×14400 vertices and is maximal. There are $60 = 252 - 32 - 160$ copies of Π_5^5 in $X_{5,5}^3$ disjoint from J. They are of the form $\Pi_5^5 = X(x_0x'_0x_1x'_1x_5)\Box X(x_2x'_2x_3x'_3x'_5)$, two of them reached from alternate vertices of a 4-cycle C as above by traversing (ϵ)-colored edges, where $(\epsilon) = (16)$. Let $[62, 71; 67, 21]$ stand for the octad

$$\begin{array}{c}\{6(1349).2(785a),\ 7(2349).1(685a),\ 6(135a).7(2849),\ 2(735a).1(6849),\\ 6(1849).2(735a),\ 7(2849).1(635a),\ 6(185a).7(2349),\ 2(785a).1(6349)\}\end{array}$$

of products of E-sets in copies of Π_5^5. With this notation, the collection of octads $\{[62, 71; 67, 21], [63, 81, 68, 31], [71, 62; 76, 12], [73, 82; 78, 32], [81, 63; 86, 13],$ $[82, 73; 87, 23]\}$ have their composing 10-tuples as the elements of a subset J'' of $V(X_{5,5}^3)$ contained in the union of the 60 copies above, where J'' is the set of centers of disjoint 1-spheres which also are disjoint from the 1-spheres centered at the 10-tuples in $J \cup J'$. We note that $J \cup J' \cup J''$ is a $\frac{2}{9}$-E-set of $X_{5,5}^3$ and that this result is best possible.

10. NONUNIFORM SPHERE PACKING

Let $r > t > 1$. If $z, z' \in I_n$ with $|z - z'| = r$, we denote $\mathbf{z} = \{z, z'\}$. There are 2^r copies of Π_r^r of the form $\Pi_r^r = X(a_1a_2\cdots a_r)\Box X(a'_1\ a'_2\cdots a'_r)$, where

$\mathbf{a_i} = \{a_i, a'_i\} = \{i, r+i\} = \mathbf{i}$, for $i \in I_r$. The subgraph $X'_{r,r}$ induced by these copies has an E-set J as exemplified in Sections 7–9 that also dominates a subset $\{y_1 b_2 \cdots b_r y_r d_2 \ \cdots d_r \,|\, \{b_2, \ldots, b_r\} = \{y'_1, y_2, \ldots y_{r-1}\}; \{d_2, \ldots, d_r\} = \{y'_r, y'_2, \ldots, y'_{r-1}\}\}$ in each copy of Π^r_r of the form $\Pi^r_r = X(y_1 y'_1 y_2 \cdots y_{r-1})\Box$ $(y_5 y'_r y'_2 \ \cdots y'_{r-1})$ in $X^3_{r,r}$ with $y_z \in \mathbf{y}_z$ for $z \in I_5$ and $\{\mathbf{y}_z \,|\, z \in I_r\} = \{\mathbf{z} \,|\, z \in I_r\}$. The $\binom{2r}{r}$ copies of Π^r_r in $X^3_{r,r}$ are of the following types:

$$\begin{array}{rcl}
X(a_1a_2 \cdots a_r) & \Box & X(a'_1a'_2 \cdots a'_r); \\
X(a_1a'_1a_3a_4 \cdots a_r) & \Box & X(a_2a'_2a'_3a'_4 \cdots a'_r); \\
X(a_1a'_1a_2a'_2a_5a_6 \cdots a_r) & \Box & X(a_3a'_3a_4a'_4a'_5a'_6 \cdots a'_r); \\
\cdots & & \cdots \\
X(a_1a'_1 \cdots a_ka'_ka_{2k+1}a_{2k+2} \cdots a_r) & \Box & X(a_{k+1}a'_{k+1} \cdots a_{2k}a'_{2k}a'_{2k+1}a_{2k+2} \cdots a'_r); \\
\cdots & & \cdots .
\end{array} \tag{15}$$

Let $X'_{r,r}, X''_{r,r}, X'''_{r,r}, \ldots, X^{(k^*)}_{r,r}$ be the subgraphs induced respectively by the types in the first, second, third, $\ldots$, k^*-th lines of display (15), where $k^* = k+1$. The number of times each $X^{(k^*)}_{r,r}$ occurs in $X^3_{r,r}$ is given by the sequence A051288 [13] as a number triangle T each of whose terms $T(r,k)$, read by rows ($r \geq 0$; $k = 0, 1, \ldots, \lfloor r/2 \rfloor$), $T(r,k)$, is the number of paths of r upsteps U and r downsteps D with exactly k subpaths UUD. In fact, $T(r,k) = \binom{r}{2k} 2^{r-2k} \binom{2k}{k}$. The left of Table V illustrates T, where each row of values $T(r,k)$ adds up to $\Sigma_r = \binom{2r}{r}$. Note F_ϵ have edges only between contiguous subgraphs $X^{(k)}_{3,3}$ and $x^{(k^*)}_{3,3}$, for $k = 0, 1, \ldots, \lfloor r/2 \rfloor$.

TABLE V

$r\,;k$	0	1	2	3	$\cdots$	Σ_r	$r\,;k$	0	1	2	3	$\cdots$	Σ'_r
2	4	2			$\cdots$	6	2	4	0			$\cdots$	4
3	8	12			$\cdots$	20	3	8	8			$\cdots$	16
4	16	48	6		$\cdots$	70	4	16	24	0		$\cdots$	40
5	32	160	60		$\cdots$	252	5	32	64	24		$\cdots$	120
6	64	480	360	20	$\cdots$	924	6	64	160	120	0	$\cdots$	344
7	128	1344	1680	280	$\cdots$	3432	7	128	384	480	80	$\cdots$	1072
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

In continuation to our approach in Sections 7–9. the right of Table V gives the number, say $S(r,k) \leq T(r,k)$, of vertices covered at best by a maximum α-E-set J. The intersection of such J and each copy of Π^r_r in $X^3_{r,r} - X'_{r,r}$ must be a product of E-sets of X^2_r. Each row in the table adds up to the sum Σ'_r. To start with, we select products of E-sets in the copies of Π^r_r in $X''_{r,r}$ as in Section 9; then take the vertices of those products as centers of 1-spheres in their pertaining copies of Π^r_r. These appear in pairs of adjacent vertices in $X^3_{r,r}$ yielding a packing $\mathcal{S}''$ by double-spheres whose centers form a subset J^*. By displacing the vertices of J^* via alternate adjacency in the two components X^2_r of each copy of Π^r_r in $X''_{r,r}$ as in the examples in Section 9, we replace $\mathcal{S}''$ by a 1-sphere packing $\mathcal{S}'$ containing $(2r) \times ((r-1)!)^2$ vertices of the $(r!)^2$ vertices of each copy of Π^r_r in

$X''_{r,r}$, a proportion of $2/r$ of the vertices of $X''_{r,r}$. At best, the same proportion is maintained in the remaining $X''', \ldots, X^{(k^*)}, \ldots$, starting by choosing 1-spheres in the copies of Π_r^r in $X'''_{r,r}$ avoiding the neighbors (via F_ϵ) of the 1-spheres in $\mathcal{S}'$ and then using "exact" paths in Johnson graphs as in Section 2, at most combined as in the proof of Theorem 10.2 below.

Let us see that no 1-sphere in $X'''_{r,r}$ can touch $X''_{r,r}$. We can select the vertices in the product $a_r(a_1a'_1a_2a'_2a_5a_6\cdots a_{r-1}).a'_r(a_3a'_3a_4a'_4a'_5a'_6\cdots a'_{r-1})$ of E-sets in its corresponding copy U of Π_r^r as centers of pairwise disjoint 1-spheres in $X'''_{r,r}$. This amounts to $((r-1)!)^2$ vertices that determine $2(r-1)((r-1)!)^2$ vertices in U. But the transposition $(1r^*)$, where $r^* = r+1$, still yields edges in $X'''_{r,r}$, a contradiction. For example, the copy $U = X(16275)\square X(3748a)$ contains the product $5(1627).a(3849)$, which possesses vertex $51627.a3849$, adjacent to $15627.a3849$ in U. The edge colored with $(\epsilon) = (16)$ takes this via adjacency to vertex $a5627.13849$, belonging to the product $a(5276).1(3849)$ in its corresponding copy, still in $X'''_{r,r}$.

Theorem 10.1. *If $n = 2r \geq 4$, where $r \in \mathbb{Z}$, then an α-E-set of $X^3_{r,r}$ intersecting a copy of Π_r^r with locally maximum packing density, that is in $\frac{1}{2}((r-1)!)^2$ vertices, has $\alpha < 1$. Thus, no E-set exists in $X^3_{3,3}$.*

Proof. To be maximum, the α-E-set in the statement must intersect 2^r copies of Π_r^r each in $\frac{1}{2}((r-1)!)^2$ vertices. Those copies induce the subgraph $X'_{r,r}$. Then the argument previous to the statement leads to the claim. ■

Each column $k = 0, 1, 2, \ldots$ in T yields a subsequence $a_r^k = T(r,k)$ with $a_{2k}^k = \binom{2k}{k}$ and successively $a_{r^*}^k = 2a_r^k r^*/(r-3)$. This yields a total of $\binom{2k}{k}2^{r-2k}\binom{r}{2k}$ copies of Π_r^r in $X'''_{r,r}$. This total for $k = 2$ is $3(2^{r-3})\binom{r}{4}$ copies of Π_r^r in $X'''_{r,r}$.

The cardinality of the set J'' of vertices in $X''_{r,r}$ that are at distance two from J is $2^r((r!)^2 - r!(r-1)!)/r2^{r-1} = 2(r-1)((r-1)!)^2$. Adding to this the $2^{r-1}r(r!)^2$ vertices in J' yields $2(r-1)((r-1)!)^2 + 2^{r-1}r(r!)^2$ vertices covered by $J' \cup J''$ in $X''_{r,r}$. These are not enough to cover all vertices of $X''_{r,r}$. Again, this shows that no perfect 1-sphere packing exists that covers $X''_{r,r}$.

Theorem 10.2. *If $n = 2r \geq 4$, where $r \in \mathbb{Z}$, then:* **(a)** *a connected subgraph $X'_{r,r}$ induced in $X^3_{r,r}$ by the disjoint union of 2^r copies of Π_r^r has a perfect 1-sphere packing $\mathcal{S}$ and its associated E-set;* **(b)** *$\mathcal{S}$ cannot be extended to a perfect 1-sphere packing of $X^3_{r,r}$;* **(c)** *at best, a maximum nonuniform 1-sphere packing $\mathcal{S}'$ of $X^3_{r,r}$ is obtained as an extension of $\mathcal{S}$ that yields an α-E-set of $X^3_{r,r}$ with $\alpha = \Sigma'_r/\Sigma_r = (2^r + \frac{2}{r}P_r)/\binom{2r}{r}$, where $P_r = \binom{2r}{r} - 2^r$ if r is odd and $P_r = \binom{2r}{r} - 2^r - \binom{r}{r/2}$ if r is even. By Theorem 10.1, this value of α is an $\alpha < 1$.*

Proof. Items (a) and (b) arise respectively starting this section and from Theorem 10.1. Apart from the 2^r copies of Π_r^r in $X'_{r,r}$ there are in $X^3_{r,r}$ still: $\binom{2r}{r} - 2^r$

copies of Π_r^r if r is odd and $\binom{2r}{r} - 2^r - \binom{r}{r/2}$ copies of Π_r^r if r is even. At best, in these copies we could select products U of the form $a(b_2 \cdots b_r).a'(c_2 \cdots c_r)$ formed by E-sets $a(b_2 \cdots b_r)$ and $a'(c_2 \cdots c_r)$. The cardinality of each such U is $((r-1)!)^2$, its vertices as centers of 1-spheres pairwise disjoint in their copies of Π_r^r with F_ϵ allowing perhaps the formation of pairwise disjoint double-spheres. As in the final discussion in Section 7 (presented with our initial notation, as in Table III), we could displace adjacently and alternatively the 1-sphere centers in the first and second components X_r^2 of Π_r^r. This can modify those double 1-spheres into pairwise disjoint 1-spheres which cover at best $2r((r-1)!)^2$ vertices of $X_{r,r}^3$. The number of times that $(r!)^2$ appears at most in the vertex counting of the resulting nonuniform packing of $X_{r,r}^3$ is $2^r + 2rP_r((r-1)!)^2/(r!)^2 = 2^r + 2rP_r/r^2 = 2^r + \frac{2}{r}P_r$. Thus, an α-E-set of $X_{r,r}^3$ has $\alpha < (2^r + \frac{2}{r}P_r)/\binom{2r}{r}$. ■

As in the bottom example of Figure 6, the general case of $X_{r,t}^3$ with $r \geq t$ can be considered a restriction, if necessary, of the one of $X_{r,r}^3$ by means of the puncturing technique mentioned in Section 8. This way, we get the following.

Corollary 10.3. *Let $r > t > 1$. A maximum nonuniform 1-sphere packing of $X_{r,t}^3$ exists that yields an α-E-set of $X_{r,t}^3$ with $\alpha \leq \frac{\Sigma_t'}{\Sigma_r}$, where $\Sigma_t' = (2^t + \frac{2}{t}P_t)$ and $\Sigma_r = \binom{2r}{r}$ with $P_t = \binom{2t}{t} - 2^t$ if t is odd and $P_t = \binom{2t}{t} - 2^t - \binom{t}{t/2}$ if t is even.*

11. FURTHER DEVELOPMENT

Consider a graph $X_n^d = X(S_n, \tau_n^d)$, where $d \geq 3$, with a 1-factor F_ϵ of X_n^d corresponding to a non-leaf edge ϵ of τ_n^d. According to Lemma 6 [4], $X_n^d \setminus F_\epsilon$ is the disjoint union of $\binom{n}{r}$ copies of $X_r^{d_r} \Box X_t^{d_t}$ (induced by the cosets of the subgroup $S_r \times S_t$). Moreover, the removal of ϵ from X_n^d yields two trees τ^{d_r} and τ^{d_t} of orders r and t, respectively, with $r + t = n$, so each such copy of $X_r^{d_r} \Box X_t^{d_t}$ in X_n^d has regular degree $\delta = n - 1 = (r-1) + (t-1) + 1$ and any 1-sphere in X_d^r contains just $\delta + 1 = n$ vertices in each such copy.

Theorem 11.1. *Let $d \geq 3$ and let $n = 4$ or n be a prime $n > 4$. Then, X_n^d does not have E-sets or perfect 1-sphere packings.*

Proof. Let $n > 4$. An E-set in X_n^d must intersect each copy $X(x_1 x_2 \cdots x_r) \Box X(x_{r^*} x_{r^*+1} \cdots x_n)$ of $X_r^{d_r} \Box X_t^{d_t}$, where $r^* = r + 1$, in a constant number of vertices, involving: **(i)** all vertices of $X(x_{r^*}, x_{r^*+1}, \ldots, x_n)$ with a common parity (even or odd) and starting at a common entry; **(ii)** the partition of the vertex set of $X(x_1, x_2, \ldots, x_r)$ into E-sets, the vertices of each E-set starting at a common entry. Items (i) and (ii) must be combined as exemplified in Section 8, which allows the largest number of centers of pairwise disjoint 1-spheres in a 1-sphere packing of X_n^d per copy of $X_r^{d_r} \Box X_t^{d_t}$. The resulting maximum must be an integer, but if n is a prime with $n > 4$, then there is not exact divisibility allowing it, as shown in the third column of Table IV. Taking into account the situations discussed for $n = 4$ in relation to Figure 1, the statement follows. ■

Despite Theorem 3.10.2 [7], asserting that J_τ is a minimal generating set for $S_n \Leftrightarrow \tau$ is a tree, we notice that Section V in [3] insures an E-set J^0 in $X(S_5, C_5)$. This also contrasts with our shown nonexistence of E-sets in $X(S_5, \tau^4$ (Theorem 11.1), contained in $X(S_5, C_5)$. To compare with this situation, let the vertex set of such J^0 be composed by the following permutations:

a_0=12345;	b_0=13524;	c_0=14253;	d_0=15432;
a_1=23451;	b_1=35241;	c_1=42531;	d_1=54321;
a_2=34512;	b_2=52413;	c_2=25314;	d_2=43215;
a_3=45123;	b_3=24135;	c_3=53142;	d_3=32154;
a_2=34512;	b_2=52413;	c_2=25314;	d_2=43215;
a_3=45123;	b_3=24135;	c_3=53142;	d_3=32154;
a_4=51234;	b_4=41352;	c_4=31425;	d_4=21543.

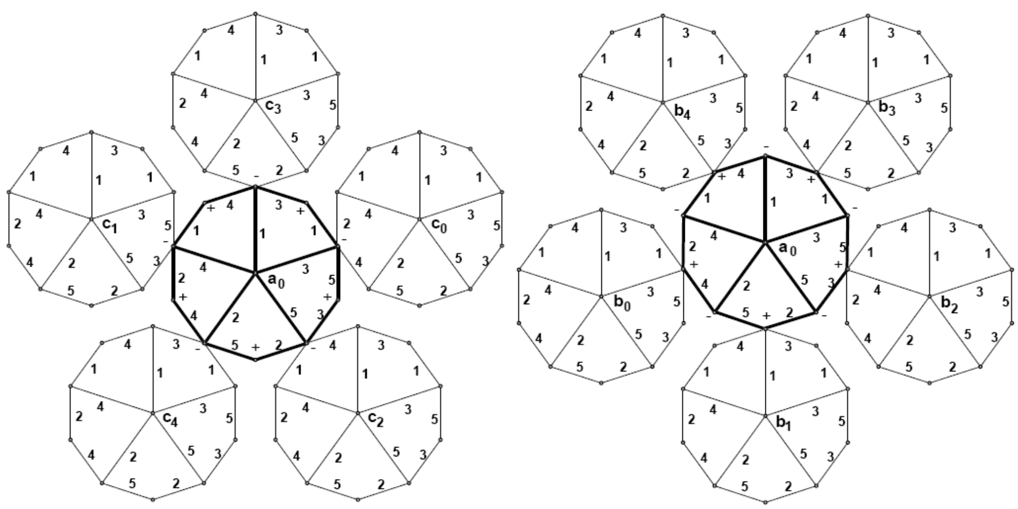

Figure 7. The ten 10-cycles incident to a^0 in $X(S_5, C_5)$

Each vertex x_i of $X(S_5, C_5)$, where $x = a, b, c, d$ and $i \in \mathbb{Z}_5$, is associated to a 10-cycle x^i of $X(S_5, C_5)$ with alternate vertices w adjacent to x_i so that each of the five remaining vertices w' induces a 4-cycle together with x_i and the two neighbors of x_i in x^i. Each vertex of an x^i here is incident to exactly one other 10-cycle y^j, where $x \neq y$ in $\{a, b, c, d\}$ and $j \in \mathbb{Z}_5$. For example, $x^0 = a^0$ can be represented as $(c^3, b^3, c^0, b^2, c^2, b^1, c^4, b^0, c^1, b^4)$, where the first and remaining odd entries represent the vertices adjacent to a_0, each vertex v of form either w or w' belonging too, to the only remaining 10-cycle incident to x^0 at v, as shown in the two graph representations in Figure 7 to be superposed for a view of $X(S_5, C_5)$ containing a^0 and the ten 10-cycles incident to a^0 in $X(S_5, C_5)$.

Table VI shows the representations of the twenty x^i in terms of their successive incident y^j, each column headed by the symbol z^- (resp., z^+) with $z \in I_5$ representing the transposition of the edge $z(z+1)$ joining the corresponding vertex x'_i (resp., y'_j) in x^i (resp., y^j) to v, where 1 must take the place of $5+1$.

TABLE VI

v	1^-	2^+	3^-	4^+	5^-	1^+	2^-	3^+	4^-	5^+	v	1^-	2^+	3^-	4^+	5^-	1^+	2^-	3^+	4^-	5^+
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
a^0	c^3	b^3	c^0	b^2	c^2	b^1	c^4	b^0	c^1	b^4	c^0	d^3	a^3	d^0	a^2	d^2	a^1	d^4	a^0	d^1	a^4
a^1	c^0	b^1	c^2	b^0	c^4	b^4	c^1	b^3	c^3	b^2	c^1	d^0	a^1	d^2	a^0	d^4	a^4	d^1	a^3	d^3	a^2
a^2	c^2	b^4	c^4	b^3	c^1	b^2	c^3	b^1	c^0	b^0	c^2	d^2	a^4	d^4	a^3	d^1	a^2	d^3	a^1	d^0	a^0
a^3	c^4	b^2	c^1	b^1	c^3	b^0	c^0	b^4	c^2	b^3	c^3	d^4	a^2	d^1	a^1	d^3	a^0	d^0	a^4	d^2	a^3
a^4	c^1	b^0	c^3	b^4	c^0	b^3	c^2	b^2	c^4	b^1	c^4	d^1	a^0	d^3	a^4	d^0	a^3	d^2	a^2	d^4	a^1
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
b^0	a^3	d^3	a^0	d^2	a^2	d^1	a^4	d^0	a^1	d^4	d^0	b^3	c^3	b^0	c^2	b^2	c^1	b^4	c^0	b^1	c^4
b^1	a^0	d^1	a^2	d^0	a^4	d^4	a^1	d^3	a^3	d^2	d^1	b^0	c^1	b^2	c^0	b^4	c^4	b^1	c^1	b^3	c^2
b^2	a^2	d^4	a^4	d^3	a^1	d^2	a^3	d^1	a^0	d^0	d^2	b^2	c^4	b^4	c^3	b^1	d^2	b^3	c^3	b^0	c^0
b^3	a^4	d^2	a^1	d^1	a^3	$d)$	a^0	d^4	a^2	d^3	d^2	b^4	c^2	b^1	c^1	b^3	c^0	b^0	c^4	b^2	c^3
b^4	a^1	d^0	a^3	d^4	a^0	d^3	a^2	d^2	a^4	d^1	d^4	b^1	c^0	b^3	c^4	b^0	c^3	b^2	c^2	b^4	c^1

The following facts are verified: **(A)** the vertex set of $X(S_5, C_5)$ admits a partition onto E-sets J^i, one of them being J^0, with remaining E-sets J^i obtained by successive translation along the edges colored with $i \in I_5$; this yields a covering map $\phi_5 : X(S_5, C_5) \to K_6$, where $V(K_6) = \{0, 1, 2, 3, 4, 5\}$ and $\phi_5^{-1}(0) = J^0$; **(B)** the edge set of $X(S_5, C_5)$ admits a 1-factorization into the colors of I_5; eliminating one of these colors leaves twenty vertices (one per 10-cycle) not dominated by J^0; in that case, each other J^i has its own twenty vertices not dominated; and this establishes a partition of S_5 into subsets of vertices not dominated by each of the J^i in $X(S_5, C_5)$; **(C)** superposing the two parts (left and right) of Figure 7 with common $[a^0 \cup a_0]$ produces a graph with 111 vertices, so that the remaining nine vertices of $X(S_5, C_5)$ are a_i, for $i = 1, 2, 3, 4$, and d_j, for $j \in \mathbb{Z}_5$.

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