On certain ratios regarding integer numbers which are both triangulars and squares

Fabio Roman*

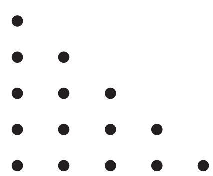
Abstract

We investigate integer numbers which possess at the same time the properties to be triangulars and squares, that are, numbers a for which do exist integers m and n such that $a = n^2 = \frac{m\cdot(m+1)}{2}$. In particular, we are interested about ratios between successive numbers of that kind. While the limit of the ratio for increasing a is already known in literature, to the best of our knowledge the limit of the ratio of differences of successive ratios, again for increasing a, is a new investigation. We give a result for the latter limit, showing that it coincides with the former one, and we formulate a conjecture about related limits.

1 Preliminaries

We recall some basic definitions from elementary number theory.

Definition 1.1. *A non-negative integer is said to be* ***triangular*** *if it can be the number of objects in a set able to form a triangle, right or equilateral.*



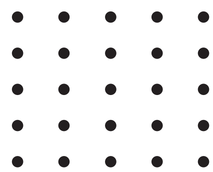
A triangular number has the form

$$T_n := \frac{n\cdot(n+1)}{2} = \binom{n+1}{2}$$

where n is a natural number.

*University of Torino, Dipartimento di Matematica, Via Carlo Alberto 10, 10123 Torino. Tel: +39 011 6702827; Fax: +39 011 6702878; e-mail: fabio.roman@unito.it

Definition 1.2. *Similarly, a non-negative integer is said to be* ***square*** *if it can be the number of objects in a set able to form a square.*



It is straightforward to say that square numbers have the form $n \cdot n = n^2$*, where* n *is a natural number.*

We can also define a generic *polygonal number* as an integer that can be the number of objects in a set able to form a regular polygon having a certain number of sides.

Definition 1.3. *A number is said to be* m***-gonal*** *if it can be the number of objects in a set able to form a regular* m*-gon. The* n*-th* m*-gonal number has the form:*

$$P_{m,n} = \frac{n[(m-2)n-(m-4)]}{2}$$

2 Basic computations

With reference to square and triangular numbers' definitions, by imposing equality, we obtain:

$$n^2 = \frac{m \cdot (m+1)}{2}$$

that can be algebrically transformed to:

$$\begin{aligned}
n^2 &= \frac{m \cdot (m+1)}{2} \\
n^2 &= \frac{m^2+m}{2} \\
2n^2 - (m^2+m) &= 0 \\
2n^2 - \left(m^2+m+\frac{1}{4}\right) &= -\frac{1}{4} \\
2n^2 - \left(m+\frac{1}{2}\right)^2 &= -\frac{1}{4} \\
8n^2 - (2m+1)^2 &= -1 \\
t^2 - 8n^2 &= 1
\end{aligned}$$

by setting at the end $t := 2m+1$.
This allows us to say that (t, n) should solve a Pell equation, assuming that t is odd; we can also set $s := 2n$, in order to write $t^2 - 2s^2 = 1$, that is the more classical Pell equation, in which we need s even.

3 A numerical approach

We treat here the problem empirically, by using a spreadsheet.
The idea is to create a table, where in the first column is listed a certain number of positive integers, in the second one the respective triangular number, in the third one its square root; in the fourth column we chop the square root at the lower integer, while in the fifth and last column, we do the difference between the third one and the fourth one, obtaining its decimal part. The first ten rows of the table give:

integers	**triangulars**	**roots**	**integer parts**	**decimal parts**
1	1	1.0000	1.0000	0.0000
2	3	1.7321	1.0000	0.7321
3	6	2.4495	2.0000	0.4495
4	10	3.1623	3.0000	0.1623
5	15	3.8730	3.0000	0.8730
6	21	4.5826	4.0000	0.5826
7	28	5.2915	5.0000	0.2915
8	36	6.0000	6.0000	0.0000
9	45	6.7082	6.0000	0.7082
10	55	7.4162	7.0000	0.4162

and so on.
It is straightforward to say that the considered triangular number is also a squadre if and only if, for its row, the fifth column is 0. Exceptions can arise due to finite arithmetic errors, but it's not the case at least for the moment, because all numbers appearing are not too large to generate machine-caused loss of precision.
By extending with the table until 65534 (if we use a spreadsheet with $2^{16} - 1 = 65535$ rows, using the first for naming columns), we can directly found some of these numbers:

integers	**triangulars**	**roots**	**integer parts**	**decimal parts**
49	1225	35.0000	35.0000	0.0000
288	41616	204.0000	204.0000	0.0000
1681	1 413 721	1189.0000	1189.0000	0.0000
9800	48 024 900	6930.0000	6930.0000	0.0000
57121	1 631 432 881	40391.0000	40391.0000	0.0000

A property that can be observed is that the ratio between two successive numbers, both triangular and square, seems to be the same, case after case. By explicit computation:

a_n	a_{n+1}/a_n
1	36.00000
36	34.02778
1225	33.97224
41616	33.97061
1 413 721	33.97056420609
48 024 900	

we can see how this ratio seems to rapidly converge to a fixed value. But we can say more of that, and this is why we kept 11 digits instead of 5 in the last step: also the ratio between differences of subsequent ratios converge to the same value. In fact:

a_n	$\frac{a_{n+1}}{a_n}$	$b_n := \frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n}$	$\frac{b_{n-1}}{b_n}$
1	36.00000		
36	34.02778	1.97222	
1225	33.97224	0.05553	35.51450
41616	33.97061	0.00163	34.01430
1 413 721	33.97056420609	0.0000480584221	33.97185
48 024 900			

So, we can create new rows in the table, by following these steps:

- we divide the latest value b_n obtained, i.e. 0.0000480584221, for the analogous from the sequence of ratios which in the table lies on its right, so 33.97185;
- we obtain 0.0000014146543, and we subtract it from the latest value available in the second column, specifically 33.97056420609;
- we obtain 33.97056279144, and we multiply it for the latest number written, 48 024 900; if what we conjectured is correct, we should obtain a number (real, not necessarily integer) well-approximating a new both triangular and square number.

In fact, we compute:

$$48\ 024\ 900 \cdot 33.97056279144 = 1\ 631\ 432\ 881.00263$$

and 1 631 432 881 is both triangular and square. That allows us to add a line into the table:

a_n	$\frac{a_{n+1}}{a_n}$	$b_n := \frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n}$	$\frac{b_{n-1}}{b_n}$
1	36.00000		
36	34.02778	1.97222	
1225	33.97224	0.05553	35.51450
41616	33.97061	0.00163	34.01430
1 413 721	33.97056420609	0.0000480584221	33.97185
48 024 900	33.97056279144	0.0000014146543	33.97185
1 631 432 881.00263			

We used a *backward* completion: by observing that the value in the last column tends to stabilize, we estimate, by accepting an error margin, that it is constant starting from the considered row, and we complete the row by calculating all values in the previous columns.
By taking account of the fact that we know the exact value of the new number both triangular and square, we can rectify the table, moving from backward completion to *forward* completion: if we know the numbers having this property, we can derive ratios and differences.

a_n	$\frac{a_{n+1}}{a_n}$	$b_n := \frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n}$	$\frac{b_{n-1}}{b_n}$
1	36.00000		
36	34.02778	1.97222	
1225	33.97224	0.05553	35.51450
41616	33.97061	0.00163	34.01430
1 413 721	33.97056420609	0.0000480584221	33.97185
48 024 900	33.97056279139	0.0000014147063	33.97060
1 631 432 881			

It seems we can approximate the limit of the left ratio, using 5 digits, to 33.97056; if we try to take the same number as the right ratio, hoping to find new numbers, we can proceed:

- $0.0000014147063/33.97056 = 0.0000000416451$;
- $33.97056279139 - 0.0000000416451 = 33.97056274974$;
- $1\ 631\ 432\ 881 \cdot 33.97056274974 = 55\ 420\ 693\ 055.99960$

Any `CAS` allows us to consider 55 420 693 056 as:

- the 332928-th triangular number: if c is the number, the algebraic equation $n^2 + n - 2c = 0$ has that value of n as positive root;
- the 235416-th square number, just by calculating its square root.

These data allows us, again, to update and rectify the table:

a_n	$\frac{a_{n+1}}{a_n}$	$b_n := \frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n}$	$\frac{b_{n-1}}{b_n}$
1	36.00000		
36	34.02778	1.97222	
1225	33.97224	0.05553	35.51450
41616	33.97061	0.00163	34.01430
1 413 721	33.97056420609	0.0000480584221	33.97185
48 024 900	33.97056279139	0.0000014147063	33.97060
1 631 432 881	33.97056274974	0.0000000416451	33.97056
55 420 693 055.99960			

a_n	$\frac{a_{n+1}}{a_n}$	$b_n := \frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n}$	$\frac{b_{n-1}}{b_n}$
1	36.00000		
36	34.02778	1.97222	
1225	33.97224	0.05553	35.51450
41616	33.97061	0.00163	34.01430
1 413 721	33.97056420609	0.0000480584221	33.97185
48 024 900	33.97056279139	0.0000014147063	33.97060
1 631 432 881	33.97056274974	0.0000000416451	33.97056
55 420 693 056			

We have in some sense *fastened* the procedure: in fact, by following what we done before, we would have taken as value in the last column 33.97060, i.e. the last value available, while we took instead 33.97056, assuming that ratios in the fourth column converge at the same quantity ratios in the second column do.
We note that we obtain an almost exact value: a rectify in the first column doesn't change anything in the others, with respect to the number of digits considered; we can also observe, as seen in the next table, that by using the same number of digits, we would obtain the same result by taking 33.97060 as right ratio, while an increment in the number of digits would likely result in a difference, in which the lower precision lies in the choice of that value.

a_n	$\frac{a_{n+1}}{a_n}$	$b_n := \frac{a_n}{a_{n-1}} - \frac{a_{n+1}}{a_n}$	$\frac{b_{n-1}}{b_n}$
1	36.00000		
36	34.02778	1.97222	
1225	33.97224	0.05553	35.51450
41616	33.97061	0.00163	34.01430
1 413 721	33.97056420609	0.0000480584221	33.97185
48 024 900	33.97056279139	0.0000014147063	33.97060
1 631 432 881	33.97056274974	0.0000000416450	33.97060
55 420 693 055.99960			

By applying again the method, we obtained another couple of numbers: 1 882 672 131 025 and 63 955 431 761 796.
We can also note that:

- if we define $c_n := \frac{b_{n-2}}{b_{n-1}} - \frac{b_{n-1}}{b_n}$, even $\frac{c_{n-1}}{c_n}$ tends to the same value; we can conjecture that it happens every time we iterate in this way, that is, if we denote $a_{1,n} := a_n,\ a_{2,n} := b_n,\ a_{3,n} := c_n$, and we define for every $i \geq 3$ a corresponding $a_{i,n} := \frac{a_{i-1,n-2}}{a_{i-1,n-1}} - \frac{a_{i-1,n-1}}{a_{i-1,n}}$, we can say that, again for every $i \geq 3$, while n tends to infinity, $\frac{a_{i,n-1}}{a_{i,n}}$ tends to the value.

- if we use more digits for the ratios, and we assume correct the conjecture, we can consider one of the ratios, call d the difference between a value and the previous one, q the recurring value of about 33.97056, and say that the subsequent difference will be approximable by $\frac{d}{q}$, the next one by $\frac{d}{q^2}$, and so on. The sum of the difference from there to infinity will be approximable by $\frac{d}{q} + \frac{d}{q^2} + \frac{d}{q^3} + \ldots = \frac{d}{q-1} = \frac{d}{32.97056}$, that allows us to obtain a gain in the relative precision of at least 32 times every single step, and at least 1000 times every two steps, that corresponds to three digits.

On the other hand, we need a certain machine precision: with 15 digits, that corresponds to a relative precision of about 2^{-52}, the standard of the `double` type, we report a loss of precision in the computation of the biggest number found before, a 14-digit integer. If we multiply that number for q, we obtain a 16-digit integer, and in general we can't exactly write a 16-digit integer as a 64-bit real value.

4 Exact approach with Pell equations

It is widely known from Pell equations' theory that, for solving:

$$t^2 - 2s^2 = 1$$

we start by write $\sqrt{2}$ as a continuous fraction, that is:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}$$

The first convergent is $\frac{3}{2}$, and $(t, s) = (3, 2)$ does in fact solve the equation, i.e. $3^2 - 2 \cdot 2^2 = 9 - 8 = 1$.
By the relation $s = 2n$, we have $n = 1$, and $n^2 = 1$, that is the first number both triangular and square.
Successive integers can be found in a traditional way, involving well-estabilished theory:

i	$(3+2\sqrt{2})^i$	t	s	m	n	n^2
1	$(3+2\sqrt{2})$	3	2	1	1	1
2	$(17+12\sqrt{2})$	17	12	8	6	36
3	$(99+70\sqrt{2})$	99	70	49	35	1225
4	$(577+408\sqrt{2})$	577	408	288	204	41616
5	$(3363+2378\sqrt{2})$	3363	2378	1681	1189	1413721
6	$(19601+13860\sqrt{2})$	19601	13860	9800	6930	48024900

and again:

i	t	s	m	n	n^2
7	114243	80782	57121	40391	1 631 432 881
8	665857	470832	332928	235416	55 420 693 056
9	3880899	2744210	1940449	1372105	1 882 672 131 025
10	22619537	15994428	11309768	7997214	63 955 431 761 796

and so on; we can generalize:

$$\begin{aligned}(t_{i-1}+s_{i-1}\sqrt{2})(3+2\sqrt{2}) &= (t_i+s_i\sqrt{2})\\ 3t_{i-1}+2\sqrt{2}t_{i-1}+3\sqrt{2}s_{i-1}+4s_{i-1} &= (t_i+s_i\sqrt{2})\\ 3t_{i-1}+4s_{i-1}+(2t_{i-1}+3s_{i-1}\sqrt{2}) &= (t_i+s_i\sqrt{2})\end{aligned}$$

and, by recurrence:

$$\begin{cases} t_i = & 3t_{i-1}+4s_{i-1} \\ s_i = & 2t_{i-1}+3s_{i-1} \end{cases}$$

4.1 Ratio limit: first ratio

By observing that $n=\frac{s}{2}$ implies $n_i^2=\frac{s_i^2}{4}$, we can express s_i as a function of s_{i-1} and not of t_{i-1}.
If we define, for every i, $t_i=k_is_i$, $\lim_{i\to+\infty}k_i=\sqrt{2}$ holds (it is straightforward to prove), and we can set $l_i:=k_i-\sqrt{2}$, so $t_i=\sqrt{2}s_i+l_is_i$, and $\lim_{i\to+\infty}l_i=0$. Now, from equations:

$$\begin{cases} s_i = 2t_{i-1}+3s_{i-1} \\ t_i = \sqrt{2}s_i+l_is_i \end{cases}$$

we obtain:

$$\begin{aligned} s_i &= 2\sqrt{2}s_{i-1}+2l_is_{i-1}+3s_{i-1}\\ s_i &= s_{i-1}(3+2\sqrt{2}+2l_i)\end{aligned}$$

from which:

$$\begin{aligned} s_i^2 &= s_{i-1}^2(3+2\sqrt{2}+2l_i)^2 \\ 4n_i^2 &= 4n_{i-1}^2(3+2\sqrt{2}+2l_i)^2 \\ n_i^2 &= n_{i-1}^2(3+2\sqrt{2}+2l_i)^2 \end{aligned}$$

and, for $i \to \infty$:

$$\begin{aligned} n_i^2 &= n_{i-1}^2(3+2\sqrt{2})^2 = n_{i-1}^2(17+12\sqrt{2}) \\ &= n_{i-1}^2(1+\sqrt{2})^4 \cong n_{i-1}^2 \cdot 33.97056 \end{aligned}$$

4.2 Ratio limit: second ratio, first method

We prove now in two ways that, if we define:

$$a_{2,j} = \frac{a_{1,j+1}}{a_{1,j}} - \frac{a_{1,j}}{a_{1,j-1}}$$

then also the ratio $a_{2,j-1}/a_{2,j}$ tends at the same value for diverging j.
Here is the first one.
We will write alternatively $a_{1,j}$ or a_j for the j-th term of the OEIS sequence `A001110` (see also [1–9] and some references therein).
We have:

$$\lim_{j\to+\infty} \frac{a_{2,j-1}}{a_{2,j}} = \lim_{j\to+\infty} \frac{\frac{a_{1,j}}{a_{1,j-1}} - \frac{a_{1,j-1}}{a_{1,j-2}}}{\frac{a_{1,j+1}}{a_{1,j}} - \frac{a_{1,j}}{a_{1,j-1}}} = \lim_{j\to+\infty} \frac{\frac{a_j}{a_{j-1}} - \frac{a_{j-1}}{a_{j-2}}}{\frac{a_{j+1}}{a_j} - \frac{a_j}{a_{j-1}}} := L_2$$

Since $a_j = \frac{s_j^2}{4}$, where s_j is the j-th value of s which is solution, for a certain value of t (namely t_j), of $t^2 - 2s^2 = 1$, we can operate a substitution, implicitely simplifying a 4 in every fraction:

$$L_2 = \lim_{j\to+\infty} \frac{\frac{s_j^2}{s_{j-1}^2} - \frac{s_{j-1}^2}{s_{j-2}^2}}{\frac{s_{j+1}^2}{s_j^2} - \frac{s_j^2}{s_{j-1}^2}}$$

Now is:

$$\begin{aligned} s_{j+1}^2 &= s_j^2 \cdot (3+2\sqrt{2}+2l_{j+1})^2 \\ &= s_{j-1}^2 \cdot (3+2\sqrt{2}+2l_{j+1})^2 \cdot (3+2\sqrt{2}+2l_j)^2 \\ &= s_{j-2}^2 \cdot (3+2\sqrt{2}+2l_{j+1})^2 \cdot (3+2\sqrt{2}+2l_j)^2 \cdot (3+2\sqrt{2}+2l_{j-1})^2 \end{aligned}$$

$$\begin{aligned} s_j^2 &= s_{j-1}^2 \cdot (3+2\sqrt{2}+2l_j)^2 \\ &= s_{j-2}^2 \cdot (3+2\sqrt{2}+2l_j)^2 \cdot (3+2\sqrt{2}+2l_{j-1})^2 \end{aligned}$$

$$s_{j-1}^2 = s_{j-2}^2 \cdot (3 + 2\sqrt{2} + 2l_{j-1})^2$$

where $l_j = t_j/s_j - \sqrt{2}$, and $l_j \to 0$ for $j \to +\infty$.
This lead to the ratios:

$$\begin{aligned} \frac{s_{j+1}^2}{s_j^2} &= (3 + 2\sqrt{2} + 2l_{j+1})^2 \\ \frac{s_j^2}{s_{j-1}^2} &= (3 + 2\sqrt{2} + 2l_j)^2 \\ \frac{s_{j-1}^2}{s_{j-2}^2} &= (3 + 2\sqrt{2} + 2l_{j-1})^2 \end{aligned}$$

We can now rewrite L_2 by using the ratios:

$$L_2 = \lim_{j\to+\infty} \frac{(3 + 2\sqrt{2} + 2l_j)^2 - (3 + 2\sqrt{2} + 2l_{j-1})^2}{(3 + 2\sqrt{2} + 2l_{j+1})^2 - (3 + 2\sqrt{2} + 2l_j)^2}$$

Now the square differences can be rewritten as a product of a sum and a difference:

$$L_2 = \lim_{j\to+\infty} \frac{(6 + 4\sqrt{2} + 2(l_{j-1} + l_j)) \cdot 2(l_{j-1} - l_j)}{(6 + 4\sqrt{2} + 2(l_j + l_{j+1})) \cdot 2(l_j - l_{j+1})}$$

Considering the fact that l_j tends to zero for diverging j, we can both approximate $6+4\sqrt{2}+2(l_{j-1}+l_j)$ and $6+4\sqrt{2}+2(l_j+l_{j+1})$ with $6+4\sqrt{2}$. Then:

$$L_2 = \lim_{j\to+\infty} \frac{(6 + 4\sqrt{2}) \cdot 2(l_{j-1} - l_j)}{(6 + 4\sqrt{2}) \cdot 2(l_j - l_{j+1})} = \lim_{j\to+\infty} \frac{l_{j-1} - l_j}{l_j - l_{j+1}}$$

and so:

$$L_2 = \lim_{j\to+\infty} \frac{l_{j-1} - l_j}{l_j - l_{j+1}} = \lim_{j\to+\infty} \frac{(k_{j-1} - \sqrt{2}) - (k_j - \sqrt{2})}{(k_j - \sqrt{2}) - (k_{j+1} - \sqrt{2})} = \lim_{j\to+\infty} \frac{k_{j-1} - k_j}{k_j - k_{j+1}}$$

where $k_j = t_j/s_j$.

$$L_2 = \lim_{j\to+\infty} \frac{\frac{t_{j-1}}{s_{j-1}} - \frac{t_j}{s_j}}{\frac{t_j}{s_j} - \frac{t_{j+1}}{s_{j+1}}} = \lim_{j\to+\infty} \frac{\frac{t_{j-1}s_j - t_j s_{j-1}}{s_j s_{j-1}}}{\frac{t_j s_{j+1} - t_{j+1} s_j}{s_j s_{j+1}}} = \lim_{j\to+\infty} \frac{(t_{j-1}s_j - t_j s_{j-1}) \cdot s_j \cdot s_{j+1}}{(t_j s_{j+1} - t_{j+1} s_j) \cdot s_j \cdot s_{j-1}}$$

By proceeding with calculations we can state:

$$L_2 = \lim_{j\to+\infty} \left(\frac{s_{j+1}}{s_{j-1}} \cdot \frac{t_{j-1}s_j - t_j s_{j-1}}{t_j s_{j+1} - t_{j+1} s_j} \right) = (3 + 2\sqrt{2})^2 \cdot \lim_{j\to+\infty} \frac{t_{j-1}s_j - t_j s_{j-1}}{t_j s_{j+1} - t_{j+1} s_j}$$

where $s_{j+1}/s_{j-1} = (s_{j+1}/s_j)\cdot(s_j/s_{j-1})$, and the limit of both factors is equal to $(3+2\sqrt{2})$.
For the remaining limit, we consider just the denominator:

$$\begin{aligned} t_j s_{j+1} - t_{j+1} s_j &= t_j(2t_j + 3s_j) - (3t_j + 4s_j)s_j = 2t_j^2 + 3s_j t_j - 3s_j t_j - 4s_j^2 \\ &= 2t_j^2 - 4s_j^2 = 2t_j^2 - 4s_j^2 = 2(t_j^2 - 2s_j^2) = 2 \cdot 1 = 2 \end{aligned}$$

where the factor in brackets is equal to 1 for every j, because (t_j, s_j) is a solution of the Pell equation $t_j^2 - 2s_j^2 = 1$. In particular the same result is obtaining by considering the numerator, because it is just the denominator with indices shifted by one. Then the ratio is constant and equal to 1; so is the limit for $j \to 0$, and:

$$L_2 = (3+2\sqrt{2})^2 = (17+12\sqrt{2}) = (1+\sqrt{2})^4$$

as we wanted to prove.

4.3 Ratio limit: second ratio, second method

We will see now an alternate way to get that result.
We know the solutions of the Pell equation to be $t_j + s_j\sqrt{2} = (3+2\sqrt{2})^j$, and also that $t_j - s_j\sqrt{2} = (3-2\sqrt{2})^j$. Observed $(3-2\sqrt{2}) = (3+2\sqrt{2})^{-1}$, and defined $\beta := (1+\sqrt{2})$, hence $(3+2\sqrt{2}) = \beta^2$, $(3-2\sqrt{2}) = \beta^{-2}$, by respectively summing and subtracting:

$$\begin{cases} t_j = \dfrac{\beta^{2j} + \beta^{-2j}}{2} \\ s_j = \dfrac{\beta^{2j} - \beta^{-2j}}{2\sqrt{2}} \end{cases}$$

This allows us to write a closed formula, from which we can generate numbers which are both triangulars and squares:

$$a_{1,j} = \frac{s_j^2}{4} = \frac{\beta^{4j} + \beta^{-4j} - 2}{32} = \frac{\alpha^j + \alpha^{-j} - 2}{32}$$

by setting $\alpha = \beta^4 = (1+\sqrt{2})^4$.
We can obtain via these calculations the well-known result:

$$\lim_{j\to+\infty} \frac{a_{1,j}}{a_{1,j-1}} = \lim_{j\to+\infty} \frac{\frac{\alpha^j+\alpha^{-j}-2}{32}}{\frac{\alpha^{j-1}+\alpha^{1-j}-2}{32}} = \lim_{j\to+\infty} \frac{\alpha^j + \alpha^{-j} - 2}{\alpha^{j-1} + \alpha^{1-j} - 2} = \lim_{j\to+\infty} \frac{\alpha^j}{\alpha^{j-1}} = \alpha$$

considering that $|\alpha| > 1$ and so other terms are trascurable for $j \to +\infty$.
In an analogue way we can compute:

$$\lim_{j\to+\infty} \frac{a_{2,j-1}}{a_{2,j}} = \lim_{j\to+\infty} \frac{\frac{a_{1,j}}{a_{1,j-1}} - \frac{a_{1,j-1}}{a_{1,j-2}}}{\frac{a_{1,j+1}}{a_{1,j}} - \frac{a_{1,j}}{a_{1,j-1}}} = \lim_{j\to+\infty} \frac{\frac{\alpha^j+\alpha^{-j}-2}{\alpha^{j-1}+\alpha^{1-j}-2} - \frac{\alpha^{j-1}+\alpha^{1-j}-2}{\alpha^{j-2}+\alpha^{2-j}-2}}{\frac{\alpha^{j+1}+\alpha^{-1-j}-2}{\alpha^j+\alpha^j-2} - \frac{\alpha^j+\alpha^j-2}{\alpha^{j-1}+\alpha^{1-j}-2}}$$

by implicitely simplifying a 32 in every fraction.
The use of standard algebra techniques gives the subsequent results.

$$\lim_{j\to+\infty}\frac{\frac{\alpha^j+\alpha^{-j}-2}{\alpha^{j-1}+\alpha^{1-j}-2}-\frac{\alpha^{j-1}+\alpha^{1-j}-2}{\alpha^{j-2}+\alpha^{2-j}-2}}{\frac{\alpha^{j+1}+\alpha^{-1-j}-2}{\alpha^j+\alpha^j-2}-\frac{\alpha^j+\alpha^j-2}{\alpha^{j-1}+\alpha^{1-j}-2}}=\lim_{j\to+\infty}\frac{\frac{(\alpha^j+\alpha^{-j}-2)\cdot(\alpha^{j-2}+\alpha^{2-j}-2)\ -\ (\alpha^{j-1}+\alpha^{1-j}-2)^2}{(\alpha^{j-1}+\alpha^{1-j}-2)\ \cdot\ (\alpha^{j-2}+\alpha^{2-j}-2)}}{\frac{(\alpha^{j+1}+\alpha^{-1-j}-2)\cdot(\alpha^{j-1}+\alpha^{1-j}-2)\ -\ (\alpha^j+\alpha^{-j}-2)^2}{(\alpha^j+\alpha^{-j}-2)\ \cdot\ (\alpha^{j-1}+\alpha^{1-j}-2)}}$$

By rearranging:

$$\lim_{j\to+\infty}\left(\frac{(\alpha^j+\alpha^{-j}-2)\cdot(\alpha^{j-2}+\alpha^{2-j}-2)\ -\ (\alpha^{j-1}+\alpha^{1-j}-2)^2}{(\alpha^{j+1}+\alpha^{-1-j}-2)\cdot(\alpha^{j-1}+\alpha^{1-j}-2)\ -\ (\alpha^j+\alpha^{-j}-2)^2}\cdot\frac{(\alpha^j+\alpha^{-j}-2)\ \cdot\ (\alpha^{j-1}+\alpha^{1-j}-2)}{(\alpha^{j-1}+\alpha^{1-j}-2)\ \cdot\ (\alpha^{j-2}+\alpha^{2-j}-2)}\right)$$

by semplifying in the right factor:

$$\lim_{j\to+\infty}\left(\frac{(\alpha^j+\alpha^{-j}-2)\cdot(\alpha^{j-2}+\alpha^{2-j}-2)\ -\ (\alpha^{j-1}+\alpha^{1-j}-2)^2}{(\alpha^{j+1}+\alpha^{-1-j}-2)\cdot(\alpha^{j-1}+\alpha^{1-j}-2)\ -\ (\alpha^j+\alpha^{-j}-2)^2}\cdot\frac{(\alpha^j+\alpha^{-j}-2)}{(\alpha^{j-2}+\alpha^{2-j}-2)}\right)$$

and by explicitely calculating the left factor:

$$\lim_{j\to+\infty}\left(\frac{\alpha^2-2\alpha^j+\alpha^{-2}-2\alpha^{-j}-2\alpha^{j-2}+2\alpha^{2-j}-2+4\alpha^{j-1}+4\alpha^{1-j}}{\alpha^2-2\alpha^{j+1}+\alpha^{-2}-2\alpha^{-1-j}-2\alpha^{j-1}+2\alpha^{1-j}-2+4\alpha^j+4\alpha^{-j}}\cdot\frac{(\alpha^j+\alpha^{-j}-2)}{(\alpha^{j-2}+\alpha^{2-j}-2)}\right)$$

Considered the fact that there is the limit operator, we can consider just the elements depending on j, in which the coefficents of them at the exponential are positive, because the others are trascurabile with respect to them, considering the operations we are doing. This finally gives:

$$\lim_{j\to+\infty}\left(\frac{-2\alpha^j-2\alpha^{j-2}+4\alpha^{j-1}}{-2\alpha^{j+1}-2\alpha^{j-1}+4\alpha^j}\cdot\frac{\alpha^j}{\alpha^{j-2}}\right)=(\alpha^{-1}\cdot\alpha^2)=\alpha$$

again, as we wanted to prove.

5 Open points

We conjecture that the result holds for every h-th ratio, $h\geq 3$, defined by:

$$a_{h,j}=\frac{a_{h-1,j+1}}{a_{h-1,j}}-\frac{a_{h-1,j}}{a_{h-1,j-1}}$$

This means that it holds:

$$\lim_{j\to+\infty}\frac{a_{h,j-1}}{a_{h,j}}=\lim_{j\to+\infty}\frac{\frac{a_{h-1,j}}{a_{h-1,j-1}}-\frac{a_{h-1,j-1}}{a_{h-1,j-2}}}{\frac{a_{h-1,j+1}}{a_{h-1,j}}-\frac{a_{h-1,j}}{a_{h-1,j-1}}}=(1+\sqrt{2})^4$$

but we are not able to either prove or disprove it, at the moment.
On the other hand, it can be investigated whether similar results can be written for other sequences of integers figurate in more than one way, like both triangular and pentagonal, both square and pentagonal, and so on.

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