Prime Sums of Primes

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Abstract

We present a variety of prime-generating constructions that are based on sums of primes. The constructions come in all shapes and sizes, varying in the number of dimensions and number of generated primes. Our best result is a construction that produces 6 new primes for every starting prime.

1 Introduction

Constructions made from primes have fascinated mathematicians for many decades due to the beauty of their design. A number of such constructions have been proposed, such as: prime magic squares [4, 9], prime arrays [8] and primes in arithmetic progressions [1, 2].

In this paper we investigate some new prime-generating constructions that are based on sums of primes. Our constructions come in two flavours: *standard* and *recursive*. In standard constructions new primes are generated as the sum of primes used in the construction. Recursive constructions generate new primes, which in turn generate further primes. The recursion terminates when no more primes can be generated. Typically we only use odd primes (ignore 2), forcing our sums to contain an odd number of elements. Our overall aim is to generate constructions of the largest size (*order*). If two constructions have the same order then we typically prefer the one with smallest sum of elements (*weight*). To find all the constructions we use a variant of the randomised hill-climbing algorithm. For small constructions we were able to find the optimal solutions (smallest weight) by using a brute force method.

We describe the following standard constructions: prime vectors (Section 2), cyclic prime vectors (Section 2.1), Goldbach squares (Section 6) and prime matrices (Section 7). We describe the following recursive constructions: prime tuples (Section 3), prime stairs (Section 4), prime pyramids (Section 4.1) and prime cylinders (Section 5).

2 Prime Vectors

Definition 2.1. A prime vector of order n is an array of distinct primes $P = (p_0, p_1, \ldots, p_{n-1})$, such that every sum of an odd number of consec-

utive elements is also prime. In other words

$$\sum_{0 \le k \le 2L} P(i+k) \quad \text{is prime for} \\ \forall \ i \ \text{such that} \ 0 \le i \le i+2L \le n.$$
(1)

In the above definition, i is the index of the first prime in each sum, while (2L + 1) is the number of terms in each sum.¹ For a given n there are $\lfloor (n-1)^2/4 \rfloor$ sums. Consider a prime vector of order 5: (3, 11, 5, 7, 17). Its every element is prime, as well as, every sum of an odd number of consecutive elements:

$$3 + 11 + 5 = 19, \quad 11 + 5 + 7 = 23, \\ 5 + 7 + 17 = 29, \quad 3 + 11 + 5 + 7 + 17 = 43.$$
(2)

We used a variant of hill-climbing to find prime vectors (see Algorithm 1). We start with a random array of distinct primes and then perform various mutations, such as swapping two primes or replacing one prime with a new one. If the mutation improves the score then we keep it, otherwise we revert it. The score measures the number of "incorrect" (composite) sums that the array generates. Hence we want to minimise this score. Using this algorithm we were able to obtain a prime vector of order 23 that generates 121 primes²:

(239, 131, 109, 181, 83, 43, 41, 223, 53, 233, 271, 103, 269, 71, 19, 47, 241, 23, 277, 199, 281, 29, 37).

For small orders it is possible to obtain multiple solutions. In such cases we choose the solution with the smallest *weight* - sum of all elements. In fact, this allows us to define an *optimal prime vector*:

Definition 2.2. A prime vector is *optimal* if its weight is the lowest possible.

For $n \leq 14$ we were able to find the optimal prime vectors (see Table 1). To achieve this we used a brute force algorithm. This algorithm iterates through every permutation of n distinct odd primes whose weight is below the best known weight. If a permutation forms a prime vector then the best known weight is updated and the array is printed out. The algorithm terminates when there are no more permutations whose weight is less than the best known weight. Table 1 also shows the running time of this algorithm.

For n > 14 we used Algorithm 1 to find the upper bounds on the minimal weight (see Table 2). To obtain the lower bound we used sequences from the OEIS [7]. For odd n the weight must be a prime, so we used sequence A068873 - smallest prime which is a sum of n distinct primes. For even n we used sequence A071148 - sum of the first n odd primes.

2.1 Cyclic Prime Vectors

We can also introduce a *cyclic prime vector* and define its optimality in a similar fashion:

¹If L = 0 then we have a singleton rather than a sum.

²Prime vectors of smaller orders are sub-arrays of this array.

1	$bestScore \leftarrow \infty$
2	$S^* \leftarrow$ random set of n distinct primes
3	
4	while True
5	$S \leftarrow S^*$
6	$\operatorname{mutate}(S)$
7	$score \leftarrow score(S)$
8	if $score < bestScore$
9	$bestScore \leftarrow score$
10	$S^* \leftarrow S$
11	$\operatorname{print}(S^*)$
12	end
13	end

n	Prime Vector	Weight	Time
1	(2)	2	
2	(3,5)	8	
3	(3, 5, 11)	19	
4	(3, 5, 11, 7)	26	
5	(3, 11, 5, 7, 17)	43	
6	(3, 11, 5, 7, 17, 13)	56	
7	(3, 17, 23, 7, 11, 13, 5)	79	
8	(3, 11, 17, 13, 29, 19, 5, 7)	104	
9	(7, 17, 13, 23, 11, 3, 29, 5, 19)	127	
10	$\left(3,7,19,11,13,23,31,5,37,17 ight)$	166	
11	$\left(3,23,41,19,11,13,17,7,5,31,53 ight)$	223	17s
12	(7, 41, 19, 11, 23, 3, 5, 29, 13, 47, 43, 17)	258	8m
13	(13, 53, 7, 23, 11, 3, 29, 5, 19, 17, 43, 47, 37)	307	73m
14	(17, 43, 47, 13, 29, 5, 3, 23, 11, 19, 41, 7, 53, 37)	348	14h

Table 1: Optimal prime vectors for $n \leq 14$, their weight and the time required to compute them. Computation times less than 1 second are not shown.

n	15	16	17	18	19	20	21	22	23
lower bound	379	438	499	566	643	710	809	872	983
upper bound	443	522	641	888	983	1430	1627	1824	3203

Table 2: Best bounds on the minimal weight of prime vectors for $15 \le n \le 23$.

Definition 2.3. A cyclic prime vector of order n is a prime vector P of order n with the additional property that prime sums can span from the end to the start of the array. In other words

$$\sum_{0 \le k \le 2L} P((i+k) \mod n) \text{ is prime for} \forall i \text{ such that } 0 \le i < n \text{ and} \forall L \text{ such that } 0 \le 2L < n.$$
(3)

For a given n there are $(n-2)(2n-1+(-1)^n)/4$ sums. For example the cyclic prime vector (5, 7, 17, 13, 11) generates the following 6 sums:

$$5 + 7 + 17 = 29, \quad 7 + 17 + 13 = 37,$$

$$17 + 13 + 11 = 41, \quad 13 + 11 + 5 = 29,$$

$$11 + 5 + 7 = 23, \quad 5 + 7 + 17 + 13 + 11 = 53.$$

(4)

Cyclic prime vectors differ from normal prime vectors in a few key ways. Every cyclic prime vector is also a normal prime vector, but the opposite may not be the case. Unlike normal prime vectors, cyclic prime vectors can be permuted without affecting their prime sums. Also we cannot easily generate cyclic prime vectors as sub-arrays of larger cyclic prime vectors. Due to the cyclic requirement, cyclic prime vectors require more prime sums for the same order, making them significantly harder to find.

Using the brute force algorithm described above we were able to find the optimal cyclic prime vectors for $n \leq 10$ (see Table 3). The computation for the optimal cyclic prime vector of order 11 was still running after 4 days, so it is not shown. It is interesting to note that the weight for n = 9 is smaller than the weight for n = 8. Using an algorithm similar to Algorithm 1 we found cyclic prime vectors up to order 14 (see Table 4). The largest array generates 84 primes.

n	Cyclic Prime Vector	Weight	Time
1	(2)	2	
2	(3,5)	8	
3	(3, 5, 11)	19	
4	(5, 7, 17, 19)	48	
5	(5, 7, 17, 13, 11)	53	
6	(5, 29, 7, 11, 19, 37)	108	
7	(5, 7, 17, 13, 29, 31, 11)	113	
8	(11, 17, 43, 47, 13, 19, 29, 31)	210	
9	(7, 17, 13, 11, 19, 41, 29, 37, 23)	197	9s
10	(11, 19, 23, 47, 31, 53, 43, 67, 89, 127)	510	2m

Table 3: Optimal cyclic prime vectors for $n \leq 10$, their weight and the time required to compute them. Computation times less than 1 second are not shown.

3 Prime Tuples

Definition 3.1. A prime tuple of order n (odd) with length k is an array of distinct odd primes $(p_0, p_1, \ldots, p_{k-1})$, such that every term after the

n	Cyclic Prime Vector	Weight
11	(23, 73, 17, 13, 71, 19, 11, 193, 59, 137, 67)	683
12	(73, 47, 43, 137, 97, 59, 151, 239, 31, 163, 89, 131)	1260
13	(41, 43, 23, 73, 71, 53, 13, 173, 151, 59, 127, 263, 283)	1373
14	(73, 179, 41, 97, 43, 197, 199, 173, 379, 311, 131, 37, 29, 421)	2310

Table 4: Smallest (by weight) cyclic prime vectors found for $11 \le n \le 14$.

n-th term is the sum of the previous n terms. In other words

$$p_i = \sum_{q=i-n}^{i-1} p_q, \quad \forall i \ge n.$$
(5)

Note it is sufficient to use the first n terms to represent a prime tuple, since the remaining terms can be generated via sums of previous terms. We seek to find prime tuples of order n such that their length is greatest. For example, here is a prime tuple of order 7 with length 25 - the longest we have found:

(**157, 379, 487, 109, 13, 7, 271**, 1423, 2689, 4999, 9511, 18913, 37813, 75619, 150967, 300511, 598333, 1191667, 2373823, 4728733, 9419653, 18763687, 37376407, 74452303, 148306273).

The first 7 terms are shown in bold. The *weight* of a prime tuple of order n is the sum of its first n terms. When two tuples of the same order have the same length, then we prefer the one with the smaller weight.

Table 5 shows the best prime tuples that we found for $n \leq 19$. We have used a brute force approach to prove that the prime tuples for $n \in \{3, 5, 9, 11\}$ are optimal. We notice that for $n \mod 6 = 3$ and $n \mod 6 = 5$ the optimal prime tuples have length 2n + 1 and must contain a 3.

n	Prime Tuple	Length	Weight
3	(3, 13, 7)	7	23
5	$(17, 3, 19, 7, 13) \ (17, 5, 11, 23, 3)$	11	59
7	(157, 379, 487, 109, 13, 7, 271)	25	1423
9	(11, 47, 17, 23, 41, 5, 3, 13, 19)	19	179
11	(43, 7, 19, 13, 3, 17, 11, 5, 29, 41, 23)	23	211
13	(53, 137, 11, 17, 41, 227, 47, 101, 83, 5, 149, 263, 29)	34	1163
15	(29, 5, 23, 11, 41, 47, 89, 17, 71, 3, 7, 13, 37, 19, 79)	31	491
17	$(5, 47, 53, 11, 17, 41, 89, 3, 61, \\43, 97, 19, 13, 7, 37, 31, 73)$	35	647
19	$\begin{array}{c}(89,227,29,17,5,251,269,107,101,197,\\41,191,173,179,47,53,71,11,23)\end{array}$	43	2081

Table 5: Best prime tuples found for $n \leq 19$.

4 Prime Stairs

Definition 4.1. A prime stair of order $n \ge 3$ is a $\lceil \frac{n}{2} \rceil \times n$ matrix P such that every element P(r, c) at row r > 0 and column c is a distinct prime and each new row is generated from the previous row as follows:

$$P(r,c) := P(r-1,c-1) + P(r-1,c) + P(r-1,c+1).$$
(6)

For a given r > 0 we must have $c \in [r, n - r - 1]$. For a given *n* there are $\lfloor (n-1)^2/4 \rfloor$ sums. As a shorthand we can represent a prime stair of order *n* via its first (top) row only, i.e., using an array of length *n*. For example, the prime stair (13, 17, 7, 5, 11, 3, 23) looks like this:

```
\begin{array}{c} 13, 17, 7, 5, 11, 3, 23 \\ 37, 29, 23, 19, 37 \\ 89, 71, 79 \\ 239 \end{array}
```

The weight of a prime stair is defined as the sum of all elements in the first row. We were able to find the optimal prime stairs for $n \leq 11$ (see Table 6). The computation for the optimal prime stair of order 12 was still running after 4 days, so it is not shown. Using an algorithm similar to Algorithm 1 we found prime stairs up to order 15 (see Table 7). The largest stair generates 49 primes.

n	Prime Stair	Weight	Time
3	(3, 5, 11)	19	
4	(7, 5, 11, 3)	26	
5	(7, 13, 11, 5, 3)	39	
6	(7, 17, 13, 11, 5, 3)	56	
7	(13, 17, 7, 5, 11, 3, 23)	79	
8	(5, 17, 31, 11, 19, 7, 3, 13)	106	
9	(29, 23, 37, 13, 11, 5, 7, 19, 17)	161	8s
10	(5, 29, 7, 17, 13, 11, 37, 23, 19, 41)	202	3m
11	(7, 17, 19, 5, 73, 11, 13, 23, 31, 29, 41)	269	2.5h

Table 6: Optimal prime stairs for $n \leq 11$, their weight and the time required to compute them. Computation times less than 1 second are not shown.

n	Prime Stair	Weight
12	(37, 13, 17, 23, 7, 67, 5, 59, 19, 61, 29, 11)	348
13	(29, 19, 11, 127, 89, 7, 17, 37, 5, 31, 23, 43, 41)	479
14	(53, 17, 67, 29, 13, 5, 19, 149, 31, 101, 79, 11, 7, 43)	624
15	(433, 139, 491, 97, 89, 163, 29, 7, 5, 61, 17, 79, 263, 541, 83)	2497

Table 7: Best (by weight) prime stairs found for $12 \le n \le 15$.

4.1 Prime Pyramids

Similarly we can define a 3D version of the prime stair that we will call a *prime pyramid*:

Definition 4.2. A prime pyramid of order $n \ge 3$ is a $\lceil \frac{n}{2} \rceil \times n \times n$ matrix P such that every element P(k, r, c) at level k > 0, row r and column c is a distinct prime and each new level is generated from the previous level as follows:

$$P(k,r,c) := \sum_{-1 \le dr \le 1} \sum_{-1 \le dc \le 1} P(k-1,r+dr,c+dc).$$
(7)

For a given k > 0 we must have $r, c \in [k, n-k-1]$. For a given n there are n(n-1)(n-2)/6 sums. As a shorthand we can represent a prime pyramid of order n via its first (bottom) level only, i.e., using a $n \times n$ array. For example, Table 8 shows a prime pyramid of order 5:

Level 0	Level 1	Level 2

73	11	67	71	53
101	41	43	79	83
13	3	31	7	23
17	61	37	5	29
97	89	19	59	47

Table 8: Prime pyramid of order 5.

The *weight* of a prime pyramid is the sum of all elements in its first level. We were able to find all the optimal prime pyramids up to order 8 (see Table 9). We also found an order 9 prime pyramid with a weight of 27325, but its optimality is not confirmed (see Table 10).

n	Prime Pyramid								
3	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$								
4	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	458							
5	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	1159							
6	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	2582							
7	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	5115							
8		9204							

Table 9: Optimal prime pyramids for $3 \le n \le 8$.

11	79	349	461	433	859	683	587	631
367	31	593	167	331	307	277	577	743
311	67	191	151	281	47	101	619	439
389	761	613	229	173	607	13	43	271
421	563	241	557	317	337	673	751	113
73	71	127	137	163	193	661	23	181
409	571	691	61	83	251	179	233	877
467	53	227	59	89	373	401	37	149
19	547	809	521	131	41	659	503	491

Table 10: Prime pyramid of order 9 with weight 27325.

5 Prime Cylinders

Definition 5.1. A prime cylinder of order n with k layers is a $n \times k$ matrix P of odd primes, such that for every c and r > 0: P(r,c) = P(r-1,c-1) + P(r-1,c) + P(r-1,c+1).

Note that the columns wrap around and hence the term 'cylinder'. For example here is a prime cylinder of order 4 and 6 layers - the best found so far:

 $1091, 3001, 271, 257\\4349, 4363, 3529, 1619\\10331, 12241, 9511, 9497\\32069, 32083, 31249, 29339\\93491, 95401, 92671, 92657\\281549, 281563, 280729, 278819$

Since all the values below the first layer can be generated from previous values, a prime cylinder can be described using its first layer only. So the above prime cylinder would be described as (1091, 3001, 271, 257). The *weight* of a prime cylinder is the sum of values in its first layer. When multiple prime cylinders have the same order and number of layers, then we prefer the one with the smaller weight. Prime cylinders were originally introduced in [5], but were limited to n = 4. Here we investigate other values of n. It turns out that prime cylinders of odd orders cannot have more than two layers, so we focus on prime cylinders of even orders. Table 11 shows the best prime cylinders found for $n \leq 12$.

6 Goldbach Squares

The famous Goldbach conjecture states that

n	Prime Cylinder	Layers	Weight
4	(1091, 3001, 257, 271)	6	4620
6	(163, 1109, 307, 1163, 109, 1307)	6	4158
8	(67, 541, 23, 137, 109, 193, 389, 431)	5	1890
10	(19, 17, 7, 107, 43, 23, 13, 71, 79, 101)	4	480
12	(11, 29, 31, 79, 53, 5, 109, 43, 47, 41, 61, 139)	4	648

Table 11: Best prime cylinders found for $n \leq 12$.

Every even integer greater than 2 can be expressed as the sum of two primes.

Although the conjecture has been verified up to 4×10^{18} [3], a proof still remains elusive. Here we investigate a problem related to the Goldbach conjecture: can we place primes into a square such that every even number is generated as the sum of two adjacent cells? This puzzle has been explored in [6]. More formally we have:

Definition 6.1. A *Goldbach square* of order n is a $n \times n$ matrix of odd primes (not necessarily unique) such that the sum of any two adjacent cells is one of the even numbers from 6 to 4 + 4n(n-1) inclusive and every even number in this range appears exactly once.

For example, here is a Goldbach square of order 3:

7	5	3
17	11	3
3	7	19

The sums across rows are:

$$7 + 5 = 12, \quad 5 + 3 = 8,$$

$$17 + 11 = 28, \quad 11 + 3 = 14,$$

$$3 + 7 = 10, \quad 7 + 19 = 26.$$

(8)

The sums down columns are:

$$7 + 17 = 24, \quad 17 + 3 = 20,$$

$$5 + 11 = 16, \quad 11 + 7 = 18,$$

$$3 + 3 = 6, \quad 3 + 19 = 22.$$
(9)

Notice that every even number from 6 to 28 appears exactly once. If there are multiple Goldbach squares for a given n then we prefer the one with the smallest sum of cells (*weight*). Tables 12 and 13 show the best Goldbach squares that we found for $n \leq 10$.

n			(Gold	lbac	h S	qu	are				Weight
2	$ \begin{array}{cccc} 5 & 7 \\ 3 & 3 \end{array} $											18
3	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$										75	
4			1	5 17 3 5	11 19 29 23	3	.3 1 1 3	5 7 3 3				208
5			5 31 11 3 3	7 31 37 47 17	4 3	3 3	1' 5; 2! 2! 3	3 9 9	5 3 23 17 7	-		499
6	-	7 41 37 61 19 5	79 17 29 7 10 13	7)	13 71 23 47 17 3	11 4 5 5 2 2 3	3 9 3 3	11 31 59 3 10 5		3 19 13 31 7 5		1078
7	$ \begin{array}{ c c c c c c c c c c c c c c c c c c c$	111 177 677 155 3 37	1	19 109 37 71 17 107 29	9 61 61 11 149 149 7 13		83 19 73 83 3 137 7		41 17 59 89 29 55 7	7)))	53 53 3 7 109 3 43	2077

Table 12: Goldbach squares for $2 \le n \le 7$.

n							G	oldl	oac	h S	qua	are								Weight		
8		$ \begin{array}{r} 17 \\ 17 \\ 3 \\ 23 \\ 41 \\ 149 \\ 71 \\ 11 \\ 11 \\ 1 1 $		$ \begin{array}{r} 17 \\ 3 \\ 23 \\ 41 \\ 149 \end{array} $		17 3 23 41 149 71		1	57 47 31 7 13 11 3 81	1: 7: 3: 1: 1: 1: 1: 1: 5: 4:	3 1 1 1 1	43 79 61 14 3 83 31 15	3) 2 9 3	37 79 12 19 3 13 59 47) 7) 3	31 97 37 89 12' 47 3 10	7)	67 12 73 13 13 17 17 7 7 31	7 9 7 9			3766
9	9 2: 89 2: 10 13 89 10 73		101 59 107 139 23 89 101 73 113		59 191 107 29 139 79 23 173 89 113 101 179		1)) 3 3 9	13 43 19 13 37 29 5 37 151	43 107 19 3 13 13 37 31 29 227 5 3 37 223		7 97 7 31 7 31 8 11 8 11 127 127 3 31		131 101 173 89 7 7 73 67		113 163 109 127 7 23 29 89		59 1 13 7 139 1 151 1 13 0 41 4 17 7		7. 7. 13. 6. 4. 19. 7.	49 73 7 37 37 51 43 99 71 07		6187
10	10 1 3 8 19 9 25 7	7	$ \begin{array}{c} 1\\22\\8\\22\\7\\14\\1\\8\\10\\16\end{array} $	27 3 23 9 49 7 9 9 9	101 23 47 11 73 179 11 255 31 183)	71 251 79 163 151 97 31 101 191 89	4 ; 4 ; 1; 2	7 3 7 3	1 31 17 7 7 1 1 1 2	9 13 3 73 7 9 1	5 3 18 7, 12 10 3 4 4 5	3 3 3 3 2 7 7 3	89 211 133 138 133 233 53 244 19 55	.1 39 31 31 33 3 11 9	59 29 12 3 22 7 3 6 16 4	9 27 7 27 1 7 7 7 33	19 4' 13 63 9' 4' 33 77 55 55	7 11 1 7 7 1	9212		

Table 13: Goldbach squares for $8 \le n \le 10$.

7 Prime Matrices

Definition 7.1. A *prime matrix* of order n is a $n \times n$ matrix P of odd primes, such that the sum of every odd number of elements in any straight line is prime. More formally, we have

$$\sum_{0 \le k \le 2L} P(r + kd_r, c + kd_c) \text{ is prime for}$$

$$\forall r, c \text{ such that } 0 \le r, c < n \text{ and}$$

$$\forall d_r, d_c \text{ such that } (d_r, d_c) \in \{(0, 1), (1, 0), (1, 1)\} \text{ and}$$

$$\forall L \text{ such that } 0 \le 2L < n \text{ and}$$

$$r + 2Ld_r < n \text{ and } c + 2Ld_c < n.$$
(10)

We were able to find prime matrices up to order 7. For $n \leq 4$ we found optimal (smallest weight) prime matrices. The results can be seen in Table 14. The lower bound on the optimal weight is the sum of the first n^2 odd primes.

n				Pri	me	Mat	rix				Weight	Lower Bound		
3	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$											127		
4	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$										438			
5	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$							1403	1159					
6		$\begin{array}{ c c c c c c c c c c c c c c c c c c c$		2 4 1 1	113 109 229 41 419 349 13 71 137 307 31 5		$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$		227 211 53 277 29 241		5796	2582		
7	547 719 691 827 53 457 347 463 163 167 367 677 1223 1087		10 50 60 3 70	.17 03 03 83 13 87 23	13 7 30 10 10		1201 373 419 647 127 193 227		29 131 73 337 1217 653 487	397 37 557 1433 643 13 887	25891	5115		

Table 14: Prime matrices for $3 \le n \le 7$.

8 Conclusion and Future Work

We have investigated a number of constructions that generate primes via the sum of primes. Some constructions are more efficient than others at generating primes. We can define a construction's *efficiency* as the number of primes it generates divided by the number of primes used to construct the construction. Table 8 shows the greatest efficiency achieved by each construction sorted from highest to lowest:

Construction	Order n	Efficiency		
Cyclic Prime Vector	14	6		
Prime Vector	23	5.26		
Prime Cylinder	4 and 6	5		
Prime Matrix	7	4		
Prime Stair	15	3.27		
Prime Tuple	7	2.57		
Prime Pyramid	9	1.04		

Many questions remain unresolved:

- What are the optimal prime vectors for $15 \le n \le 23$?
- Is there a prime vector of order 24 ?
- What are the optimal cyclic prime vectors for $11 \le n \le 14$?
- Is there a cyclic prime vector of order 15 ?
- What are the optimal prime tuples for n = 7, 13, 19 ?
- What are the optimal prime stairs for $12 \le n \le 15$?
- Is there a prime stair of order 16 ?
- What is the optimal prime pyramid of order 9?
- Is there a prime pyramid of order 10 ?
- What are the optimal prime cylinders for $n \leq 12$?
- Is there a Goldbach square of order 11 ?
- What are the optimal prime matrices for $5 \le n \le 7$?
- Is there a prime matrix of order 8 ?

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