Motzkin Numbers: an Operational Point of View

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Abstract

The Motzkin numbers can be derived as coefficients of hybrid polynomials. Such an identification allows the derivation of new identities for this family of numbers and offers a tool to investigate previously unnoticed links with the theory of special functions and with the relevant treatment in terms of operational means. The use of umbral methods opens new directions for further developments and generalizations, which leads, e.g., to the identification of new Motzkin associated forms.

1 Introduction

The telephone numbers (T_n), also called convolution numbers, provide a very well known example of link between special numbers and special polynomials. The (T_n) can be expressed in terms of Hermite polynomials coefficients (h_s) [1]. Two of the present authors (M.A and G.D.) have recently pointed out in ref. [2] that the Padovan and Perrin numbers [3, 4] can be recognized to be associated with particular values of two variable Legendre polynomials [5].

Weinstein has discussed in [6] the connection between Motzkin numbers and a family of hybrid polynomials, and Blasiak et al. and Dattoli et al. have studied, in [7, 8], the relevant properties of Motzkin numbers.

The hybrid polynomials are indeed defined as [8]

$$P_n^{(q)}(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)!r!(r+q)!}, \tag{1}$$

and the relevant generating function reads

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(q)}(x,y) = \frac{I_q(2\sqrt{y}\, t)}{(\sqrt{y}\, t)^q} e^{xt}, \tag{2}$$

where $I_q(x)$ is the modified Bessel function of the first kind of order q.

Within the present framework, the Motzkin numbers sequence can be specified as [7]

$$\begin{aligned} m_n &= P_n^{(1)}(1,1) = \sum_{s=0}^{n} m_{n,s}, \\ m_{n,s} &= \binom{n}{s} f_s, \\ f_s &= \frac{s!}{\Gamma\left(\frac{s}{2}+2\right)\Gamma\left(\frac{s}{2}+1\right)} \left|\cos\left(s\frac{\pi}{2}\right)\right|, \end{aligned} \tag{3}$$

where the coefficients $m_{n,s}$ can be represented as the triangle reported in the following table, in which $m_{n,2}$ corresponds, in OEIS, to the sequence $A000217$, $m_{n,4}$ to $A034827$, $m_{n,6}$ to $A000910$ and so on.

According to eq. (2), the Motzkin numbers can also be defined as the coefficients of the following series expansion

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} m_n = \frac{I_1(2t)}{t} e^t. \tag{4}$$

$m_{n,s}$ **coefficients**										m_n **Motzkin**
Parameter		**s**								$\sum_{s=0}^{n} m_{n,s}$
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	
n	**0**	1								**1**
	1	1	0							**1**
	2	1	0	1						**2**
	3	1	0	3	0					**4**
	4	1	0	6	0	2				**9**
	5	1	0	10	0	10	0			**21**
	6	1	0	15	0	30	0	5		**51**
	7	1	0	21	0	70	0	35	0	**127**
	...	...	...	...	...	...	...	...	...	...

Table 1: Motzkin Numbers and their Coefficients.

In the following we will show how some progresses in the study of the relevant properties can be done by the use of a formalism of umbral nature.

2 Motzkin Numbers and Umbral Calculus

In order to simplify most of the algebra associated with the study of the properties of the Motzkin numbers and to get new relevant identities, we introduce a formalism successfully exploited elsewhere [9] based on methods of umbral nature [10].
To this aim we note that the function

$$C_q(x) = \frac{I_q(2\sqrt{x})}{(\sqrt{x})^q} = \sum_{r=0}^{\infty} \frac{x^r}{r!(q+r)!} \tag{5}$$

can be cast in the form

$$C_q(x) = \hat{c}^q \circ e^{\hat{c}x}, \tag{6}$$

where $\hat{c}$ is an umbral operator defined according to

$$\hat{c}^\mu = \frac{1}{\Gamma(\mu+1)}, \tag{7}$$

with μ not necessarily integer and real.
We define the following composition rule

$$\hat{c}^\mu \circ \hat{c}^\nu = \hat{c}^{\mu+\nu} \tag{8}$$

and we let $\hat{C} = \{\hat{c}^{\alpha}, \alpha \in \mathbb{C}\}$ denote the set of $\hat{c}$-operators. Then, the pair $(\hat{C}, \circ)$ satisfying the Abelian-group property. The mathematical foundations of the theory of $\hat{c}$-operators can be traced back to those underlying the Borel transform and have been carefully discussed in ref. [10].

The use of this formalism allows to restyle the hybrid polynomials in the form

$$P_n^{(q)}(x,y) = \hat{c}^q \circ H_n(x, \hat{c}\, y), \tag{9}$$

where

$$H_n(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)!r!} \tag{10}$$

are the two variable Hermite-Kampé de Fériét polynomials of order 2.
We can accordingly use the wealth of properties of this family of polynomials to derive further and new relations regarding those of the Motzkin numbers family.

By recalling indeed the generating function [8]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+l}(x,y) = H_l(x + 2yt, y) e^{xt+yt^2}, \tag{11}$$

we find

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} m_{n+l} = \hat{c} \circ H_l(1 + 2\hat{c}t, \hat{c}) e^{t+\hat{c}t^2}, \tag{12}$$

which, after using eqs. (8), (10), (6), finally yields

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} m_{n+l} = \mu_l(t)\; e^t, \\ &\mu_l(t) = l! \sum_{r=0}^{\lfloor \frac{l}{2} \rfloor} \frac{1}{r!} \sum_{s=0}^{l-2r} \frac{2^s}{s!(l-2r-s)!} \frac{I_{s+r+1}(2t)}{t^{r+1}}. \end{aligned} \tag{13}$$

Furthermore, the same procedure and the use of the Hermite polynomials duplication formula [11]

$$H_{2n}(x,y) = \sum_{r=0}^{n} \binom{n}{r}^2 r!\, (2y)^r \left(H_{n-r}(x,y)\right)^2, \tag{14}$$

yields the following identity for Motzkin numbers

$$
\begin{aligned}
m_{2n} &= \hat{c} \circ \sum_{r=0}^{n} r! \binom{n}{r}^2 (2\hat{c})^r \circ H_{n-r}(1, \hat{c}) \circ H_{n-r}(1, \hat{c}) = \\
&= \sum_{r=0}^{n} \binom{n}{r}^2 2^r r!(n-r)! \sum_{s=0}^{\lfloor \frac{n-r}{2} \rfloor} \frac{m_{n-r}^{(r+s+1)}}{(n-r-2s)!s!},
\end{aligned}
\tag{15}
$$

where

$$
m_n^{(q)} = P_n^{(q)}(1,1) = \hat{c}^q \circ H_n(1, \hat{c}) \tag{16}
$$

are associated Motzkin numbers [7].
The identification of Motzkin numbers as in eq. (16), along with the use of the recurrences of Hermite polynomials, yields, e.g., the identities

$$
\begin{aligned}
& m_{n+1}^{(q)} = m_n^{(q)} + 2\, n\, m_{n-1}^{(q+1)}, \\
& m_{n+p} = \sum_{s=0}^{\min[n,p]} 2^s s! \binom{p}{s} \binom{n}{s} M_{p-s,\, n-s,\, s}, \\
& M_{p,\, n,\, t} = p! \sum_{r=0}^{\lfloor \frac{p}{2} \rfloor} \frac{m_n^{(t+r+1)}}{(p-2r)!r!},
\end{aligned}
\tag{17}
$$

in which, the second identity, has been derived from the Nielsen formula for $H_{n+m}(x, y)$ [12].

3 Final Comments

In this paper we have shown that a fairly straightforward extension of the formalism put forward in ref. [7], allows non trivial progresses in the theory of Motzkin numbers. Further relations can be easily obtained by applying the method we have envisaged as, e.g.,

$$
\sum_{s=0}^{n} m_{n-s}\, m_s = 2\,(n+1)\, m_n^{(2)}, \tag{18}
$$

which represents a discrete self-convolution of Motzkin numbers.

We have also mentioned the existence of the associated Motzkin numbers

$$
m_n^{(q)} = P_n^{(q)}(1,1), \tag{19}
$$

touched on in ref. [7]. In the present context they have been introduced on purely algebraic grounds. Strictly speaking they are not integers and therefore they are not amenable for a combinatorial interpretation however, redefining them as

$$\tilde{m}_n^{(q)} = \frac{(n+q)!}{n!} P_n^{(q)}(1,1), \tag{20}$$

we obtain for $q = 2$ the sequences in OEIS ($A014531$), while for $q = 3$ the sequences ($A014532$) and so on.

A more appropriate interpretation in combinatorial terms can be obtained by following, e.g., the procedures indicated in ref. [13] and deserves further investigations, out of the scope of the present paper.

We have mentioned in the introduction the theory of telephone numbers $T(n)$ [14], whose importance in chemical Graph theory has been recently emphasized in ref. [15]. As well known, they can be expressed in terms of ordinary Hermite polynomials, however the use of the two variable extension is more effective. They can indeed be expressed as $T(n) = H_n(1, \frac{1}{2})$
.
The use of Hermite polynomials properties, like the index duplication formula, yields

$$T(2n) = \sum_{r=0}^{n} \binom{n}{r}^2 r!\, T(n-r)^2. \tag{21}$$

The use of the Hermite numbers h_s [16] allows the derivation of the following further expression

$$\begin{aligned} T(n) &= \sum_{s=0}^{n} t_{n,s}, \\ t_{n,s} &= \binom{n}{s} h_s\left(\frac{1}{2}\right), \\ h_s(y) &= y^{\frac{s}{2}} \Gamma\left(\frac{s}{2}+2\right) f_s = \frac{y^{\frac{s}{2}} s!}{\Gamma\left(\frac{s}{2}+1\right)} \left|\cos\left(s\,\frac{\pi}{2}\right)\right|. \end{aligned} \tag{22}$$

The coefficients $t_{n,s}$ of the telephone numbers can be arranged in the following triangle, in which, the numbers belonging to the column $s = 4$, $(3, 15, 45, 105, 210, 378, \ldots)$, are identified, in OEIS, with the sequence $A050534$ and the column in $s = 6$, $(15, 105, 420, 1260, 3150, \ldots)$, is just a multiple of $A00910$.

The use of the identification with two variable Hermite polynomials opens further perspectives, by exploiting indeed the polynomials (see [1] and references therein)

$$H_n^{(m)}(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{x^{n-mr} y^r}{(n-mr)!r!}, \tag{23}$$

we can introduce the following generalization of telephone numbers

$t_{n,s}$ **coefficients**										$T(n)$ **telephone numbers**
Parameter		**s**								$\sum_{s=0}^{n} t_{n,s}$
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	
n	**0**	1								**1**
	1	1	0							**1**
	2	1	0	1						**2**
	3	1	0	3	0					**4**
	4	1	0	6	0	3				**10**
	5	1	0	10	0	15	0			**26**
	6	1	0	15	0	45	0	15		**76**
	7	1	0	21	0	105	0	105	0	**232**
	...	...	...	...	...	...	...	...	...	...

Table 2: Telephone Number Coefficients

$$T_n^{(m)} = H_n^{(m)}\left(1, \frac{1}{m}\right), \tag{24}$$

with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^{(m)} = e^{t+\frac{1}{m}t^m}, \tag{25}$$

which satisfy the recurrence

$$T_{n+1}^{(m)} = T_n^{(m)} + \frac{n!}{(n-m+1)!} T_{n-m+1}^{(m)}. \tag{26}$$

In the case of $m = 3$ the numbers $T_n^{(3)} = (1, 1, 1, 3, 9, 21, 81, 351, 1233, \dots)$ are identified with OEIS $A001470$, while for $m = 4$, the series $(1, 1, 1, 1, 7, 31, 91, 211, 1681, 12097, \dots)$, corresponds to $A118934$. For $m = 5$ the associated series appears to be $A052501$ but should be more appropriately identified with the coefficients of the expansion (24), finally the sequence $m = 6$ is not reported in OEIS.

A more accurate analysis of this family of numbers and the relevant interplay with Motzkin will be discussed elsewhere.

References

[1] J. Riordan, *Introduction to combinatorial analysis*, Dover (2002), 85–86.

[2] M. Artioli and G. Dattoli, Geometric interpretation of Perrin and Padovan numbers, http://demonstrations.wolfram.com/ GeometricInterpretationOfPerrinAndPadovanNumbers/, *Wolfram Demonstrations Project*, Sept. 14, (2016).

[3] E. W. Weisstein, Perrin sequence, *Wolfram MathWorld-A Wolfram Web Resource*, mathworld.wolfram.com/PerrinSequence.html.

[4] E. W. Weisstein, Padovan sequence, *Wolfram MathWorld-A Wolfram Web Resource*, mathworld.wolfram.com/PadovanSequence.html.

[5] M. Artioli and G. Dattoli, Geometry of two-variable Legendre polynomials, *Wolfram Demonstrations Project-A Wolfram Web Resource*, demonstrations.wolfram.com/GeometryOfTwoVariableLegendrePolynomials.

[6] E. W. Weisstein, Motzkin number, *MathWorld-A Wolfram Web Resource*, http://mathworld.wolfram.com/MotzkinNumber.html.

[7] P. Blasiak, G. Dattoli, A. Horzela, K. A. Penson and K. Zhukovsky, Motzkin numbers, central trinomial coefficients and hybrid polynomials, *J. Integer Seq.* **11**, Art. 08.1.1, (2008).

[8] G. Dattoli , S. Lorenzutta , P. E. Ricci and C. Cesarano , On a family of hybrid polynomials, *J. Integral Transforms and Special Functions* **15** (2004), 485–490.

[9] G. Dattoli, E. di Palma, E. Sabia, K. Górska, A. Horzela and K. A. Penson, Operational versus umbral methods and the Borel transform, *Int. J. Appl. and Comput. Math.* (2017), DOI 10.1007/s40819-017-0315-7.

[10] S. Roman, The theory of the umbral calculus, *J. Math. Anal. Appl.* **87** (1982), 58.

[11] L. C. Andrews, *Special functions for engineers and applied mathematicians*, Macmillan, New York (1985).

[12] N. Nielsen, Recherches sur les polynomes d'Hermite, *Mathematisk-Fysiske Meddelelser* **1** (1918), 79, Det. Kgl, Danske Videnskabernes Selskab.

[13] C. Banderier, C. Krattenthaler, A. Krinik, D. Kruchinin, V. Kruchinin, D. Nguyen and M. Wallner, Explicit formulas for enumeration of lattice paths: basketball and the Kernel method, *Dev. Math.*, Springer, (2017).

[14] D. E. Knuth, The art of computer programming **3** (1973), *Sorting and Searching*, Reading, Mass.: Addison-Wesley, 65–67, MR 0445948.

[15] R. Hatz, M. Korpinen, V. Hanninen and L. Halonen, Generalized intermolecular interaction tensor applied to long-range interactions in hydrogen and coinage metal (Cu, Ag, and Au) clusters, *J. Phys. Chem. A* **119** (2015), 11729–11736.

[16] G. Dattoli, B. Germano, M. R. Martinelli and P. E. Ricci, Lacunary generating functions of Hermite polynomials and symbolic methods, *Ilirias J. of Math.* **4** (2015), 16–23 .