arXiv:1703.07268v1 [math.NT] 21 Mar 2017

The Stern Sequence and Moments of Minkowski's Question Mark Function

Roland Bacher

March 22, 2017

Abstract[1]*: We use properties of the Stern Sequence for numerical computations of moments* $\int_0^1 t^n d?(t)$ *associated to Minkowski's Question Mark function.*

1 Introduction

Minkowski's question mark function $x \longmapsto ?(x)$ and its inverse function, Conway's box function $x \longmapsto \Box(x)$, are related to continued fraction expansions, transcendence properties and probabilistic distributions of rationals in the Calkin-Wilf tree. Denjoy proved apparently that $?(x)$ is monotonic continuous and singular (derivable on a set of full measure with zero derivative on this set), see [4]. Using a functional equation satisfied by $?(x)$, Alkauskas investigated the sequence $m_0, m_1, \ldots, m_n = \int_0^1 x^n d?(s)$ of moments of the probability density $d?$ in a series of articles. Denoting by $\Box(y)$ the reciprocal function, known as *Conway's Box function*, of the increasing homeomorphism $? : [0,1] \longrightarrow [0,1]$, the substitution $t = ?(x)$ (with $d?(x) = dt$ and $x = \Box(t)$) yields

$$m_n = \int_0^1 (\Box(t))^n dt \ . \tag{1}$$

In the present paper we link these moments to the Stern sequence (which underlies the Calkin-Wilf tree) $s(0) = 0, s(1) = 1, s(2n) = s(n), s(2n+1) = s(n) + s(n+1),\ n \geq 1$. This gives new proofs for many results of Alkauskas, see for example [1], [2], [3]. It also leads to the discovery of some new properties.

The sequel of the paper is organized as follows:

Section 2 links the Stern sequence with Conway's Box function $\Box$ appearing in (1).

[1]Keywords: Minkowski's Question Mark Function, Conway's Box Function, Stern sequence, Farey Sequence, Continued Fraction. Math. class: Primary: 11A55, Secondary: 11B57.

Section 3 recalls properties of Minkowski's question mark function.

Section 4 lists a few well-known identities among binomial coefficients and elements of the Stern sequence for later use.

Section 5 presents a set of linear relations obtained by considering Riemann sums for $\int_0^1 \Box(x)^n dx$. These relations differ from the relations found by Alkauskas: they are perhaps slightly simpler but more interestingly, a crude spectral analysis of the underlying linear operator T is easy. T has a unique eigenvector $(m_0, m_1, m_2, \dots)$ of eigenvalue 1. All other eigenvalues belong to the closed complex disc of radius 1/2. The maximal error of the associated algorithm is thus roughly halved at each iteration.

Section 6 discusses a different set of Riemann sums which leads to linear relations used by Alkauskas.

We extend in Section 7 the moment-function $n \longmapsto m_n$ to an entire function $z \longmapsto m_z$ for $z \in \mathbb{C}$.

A computation of the derivative of this function at 0 to high accuracy suggests the conjectural identities

$$\log 2 = \sum_{n=1}^{\infty} \frac{m_n}{2n} \left(1 + \frac{1}{2^{n-1}}\right) = \sum_{n=1}^{\infty} \frac{m_n}{n} \left(\frac{1}{2^n} - (-1)^n\right)$$

given in Section 8.

Section 9 introduces a third type of Riemann sums, particularly well suited for asymptotic computations. The resulting asymptotic formula

$$m_n \sim \sum_{j=0}^{\infty} \frac{(\log 2)^j}{j!} m_j \sum_{h=2}^{\infty} \frac{1}{2^h} \left(1 - \frac{1}{h}\right)^n \tag{2}$$

is the object of Section 10. It is more complicated but experimentally more accurate than Alkauskas's asymptotic formula given in [3]. Alkauskas's formula can however be deduced from (2) by a simple application of Laplace's method.

Section 10.2 derives a second asymptotic formula related to (2) by a finer subdivision in the underlying Riemann sum. Since this should lead to slightly more accurate results, we consider (admittedly in a not completely rigorous way) in Section 10.3 the difference between the two formulae as a measure of accuracy for (2).

Section 11 is devoted to values m_{-n} of moments at negative integers. This leads to a sequence of identities among $m_0, m_1, m_2, \dots$. The two initial identities are

$$\sum_{j=0}^{\infty} m_j = \frac{5}{2} \qquad \text{and} \qquad \sum_{j=1}^{\infty} j\, m_j = m_2 + \frac{11}{2}\ .$$

Finally, Section 12 discusses the starting point of this work: asymptotics for $\prod_{j=2^n}^{2^{n+1}} s_j$ allowing to compute some geometric means for values of the Stern sequence.

2 Conway's box function

We denote by $\mathcal{D} = \mathbb{Z}[1/2] \cap [0,1]$ the subset of all rational dyadic numbers in $[0,1]$. The restriction to $\mathcal{D}$ of *Conway's Box function* $\square$ is recursively defined as follows: $\square(0) = 0, \square(1) = 1$ and

$$\square\left(\frac{2m+1}{2^{n+1}}\right) = \frac{a+c}{b+d}$$

if $\square\left(\frac{m}{2^n}\right) = \frac{a}{b}$ and $\square\left(\frac{m+1}{2^n}\right) = \frac{c}{d}$ where a, b, respectively c, d, are coprime natural numbers. The values $\square(m/16)$ for $m = 0, \ldots, 16$ are:

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\square\left(\frac{m}{16}\right)$	$\frac{0}{1}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{1}{1}$

Values of $\square$ for arguments in $\mathcal{D}$ are easy to compute as follows: We define the *Stern-sequence* $s(0), s(1), s(2), \ldots$ recursively by $s(0) = 0, s(1) = 1$, $s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$. Its first coefficients are given by

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$s(n)$	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4
n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$s(n)$	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5

The main tool used in this paper is the following simple observation which defines $\square$ on $\mathcal{D}$ in terms of the Stern-sequence:

Proposition 2.1. *We have*

$$\square\left(\frac{m}{2^n}\right) = \frac{s(m)}{s(2^n + m)}$$

for all natural integers m, n such that $0 \leq m \leq 2^n$.

We leave the easy proof to the reader. $\square$

Since $\frac{a}{b} < \frac{c}{d}$ with $bd > 0$ implies $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$, the function $\square$ is strictly increasing. Induction on k shows

$$\square\left(\frac{m}{2^n} \pm \frac{1}{2^{n+k}}\right) = \frac{ka+c}{kb+d}, \ k \geq 1 \tag{3}$$

if $\square\left(\frac{m}{2^n}\right) = \frac{a}{b}$ and $\square\left(\frac{m}{2^n} \pm \frac{1}{2^n}\right) = \frac{c}{d}$ (with a, b and c, d pairs of coprime natural numbers). In particular, $\square$ extends to a strictly increasing continuous function (still denoted) $\square : [0,1] \longrightarrow [0,1]$. Since

$$\lim_{k\to\infty} \frac{\square\left(\frac{m}{2^n} \pm \frac{1}{2^{n+k}}\right) - \square\left(\frac{m}{2^n}\right)}{\pm 2^{-n-k}} = \frac{2^{n+k}}{b(kb+d)} = \infty,$$

the function $\square$ has a vertical tangent at dyadic arguments.

Proposition 2.2. *We have*

$$\Box(x) = 1 - \Box(1-x)\ . \tag{4}$$

Proof. Continuity of Conway's Box function implies that it is enough to prove Proposition 2.2 for all dyadic rationals of the form $\frac{m}{2^n}$. This is done by induction using the trivial identity $\frac{a+c}{b+d} = 1 - \frac{b-a+d-c}{b+d}$. □

Corollary 2.3. *The function* $x \longmapsto 2\Box\left(x - \frac{1}{2}\right) - 1$ *is symmetric.*

Thus we have

$$0 = \int_0^1 (\Box(x) - 1/2)^{2n+1} dx = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-2)^{k-2n-1} m_k \tag{5}$$

for every odd natural number $2n+1$. This can be restated as:

Corollary 2.4. *For all* $n \geq 0$ *we have the identity*

$$m_{2n+1} = \qquad \frac{1}{2^{2n+1}} \sum_{k=0}^{2n} (-2)^k \binom{2n+1}{k} m_k\ . \tag{6}$$

In particular, m_{2n+1} *is a* $\mathbb{Z}\left[\frac{1}{2}\right]$ *−linear combination of* $m_0, m_2, m_4, \ldots, m_{2n}$.

3 Minkowski's question mark function

Given an irrational real number x in $(0,1)$ with continued fraction expansion given by

$$x = [0; a_1, a_2, a_3, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}},$$

Minkowski's question mark function is defined by

$$?(x) = -2 \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{a_1 + \cdots + a_k}}\ . \tag{7}$$

Proposition 3.1. *Minkowski's question mark function is an increasing homeomorphism of* $[0,1]$ *such that* $\Box \circ ?(x) = ? \circ \Box(x) = x$.

Proof (given for the sake of self-containedness). Since $\Box$ is an increasing homeomorphism of $[0,1]$, it is enough to prove that $\Box \circ ?(x) = x$ for every rational number x in $[0,1]$. We show this by induction on the length n of the continued fraction expansion $x = [0; a_1, a_2, \ldots, a_n]$ of x. The result clearly holds for $n = 0$ (corresponding to $x = 0$) and for $n = 1$ (corresponding to the inverse of a non-zero natural integer). Writing $\frac{p_k}{q_k} = [0; a_1, \ldots, a_k]$ we have

$$?\left(\frac{p_{n-1}}{q_{n-1}}\right) = -2 \sum_{j=1}^{n-1} \frac{(-1)^j}{2^{a_1 + \cdots + a_j}} = \frac{m}{2^{a_1 + \cdots + a_{n-1} - 1}}$$

for a suitable natural number m. We also have

$$?\left(\frac{p_{n-2}}{q_{n-2}}\right) = ?\left(\frac{p_{n-1}}{q_{n-1}}\right) - \frac{(-1)^n}{2^{a_1+\cdots+a_{n-1}-1}}.$$

Using the induction hypothesis $\frac{p_k}{q_k} = \Box \circ ?\left(\frac{p_k}{q_k}\right)$ for $k < n$ and applying (3) to

$$?\left(\frac{p_n}{q_n}\right) = \frac{m}{2^{a_1+\cdots+a_{n-1}-1}} - \frac{(-1)^n}{2^{a_n}2^{a_1+\cdots+a_{n-1}-1}}$$

we get

$$\Box \circ ?\left(\frac{p_n}{q_n}\right) = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n} \ .$$

□

The graph of ? is well-known to behave in a self-similar way as shown by the following well-known result:

Proposition 3.2. *We have*

$$?(1-x) = 1 - ?(x) \tag{8}$$

and

$$?\left(\frac{x}{1+x}\right) = \frac{1}{2}?(x) \tag{9}$$

for all $x \in [0,1]$.

Proof. Identity (8) follows from Proposition 2.2 and Proposition 3.1. Identity 9 follows from the Definition (7) applied to $\frac{x}{1+x} = \frac{1}{1+1/x} = [0; 1 + a_1, a_2, a_3, \ldots]$. □

The aim of this paper is to study the moments

$$m_z = \int_0^1 \Box(t)^z dt = \int_0^1 x^z d?(x)$$

of the probability measure $d?$ associated to the distribution function $?(x) = \int_0^x d?(t)$. The inequalities

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 - \frac{1}{n}\right)^j \leq m_j \leq 2\sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 - \frac{1}{n}\right)^j$$

coming from the evaluation $\Box(1 - 2^{-m}) = 1 - \frac{1}{m+1}$, and the trivial upper bound $\left|\binom{-z-j+1}{j}\right| \leq \binom{|z|+j-1}{j} \leq (|z|+j)^{|z|}$ show that $z \mapsto m_z$ is an entire function of $\mathbb{C}$.

The function m_z is also given by the expression

$$m_z = \sum_{k=0}^{\infty} \binom{z+k-1}{k} \gamma(k+z) m_k$$

(see (24)) where $\gamma(z)$ is the entire function defined by

$$\gamma(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{(1+n)^z} \ .$$

We give the series expansion of the entire function $z \mapsto m_z$ at $z = 0$ and study the asymptotics of m_z for real $z \longrightarrow \pm\infty$.

Proposition 3.3. *We have the identities*

$$m_z = \sum_{j=0}^{\infty} \binom{z}{j} (-1)^j m_j = \sum_{j=0}^{\infty} \binom{-z+j-1}{j} m_j$$

where $\binom{z}{j} = \frac{z(z-1)(z-2)\cdots(z-j+1)}{j!}$.

The main contribution to m_{-k} given by Proposition 3.3 corresponds to indices j such that $\frac{k+j}{j} e^{-\sqrt{(\log 2)/j}} \sim 1$ yielding $j \sim \frac{k^2}{\log 2}$.

Thus we have for example

$$m_{-1} = \sum_{n=0}^{\infty} m_n$$

$$m_{-2} = \sum_{n=0}^{\infty} (n+1) m_n$$

$$m_{-3} = \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2) m_n$$

and more generally

$$m_{-n} = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} m_k \ .$$

Proof of Proposition 3.3. Proposition 2.2 implies the equalities

$$m_z = \int_0^1 \square(t)^z dt = \int_0^1 (1-\square(t))^z dt = \sum_{j=0}^{\infty} \binom{z}{j} (-1)^j m_j = \sum_{j=0}^{\infty} \binom{-z+j-1}{j} m_j$$

which hold for all $z \in \mathbb{C}$ since $\square(t) \in (0,1)$ for $t \in (0,1)$. $\square$

4 A few useful identities

Almost all results of this paper are based on a few trivial identities, recorded in this Section for later use.

4.1 Binomial coefficients

Lemma 4.1. *We have the series expansion*

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k \tag{10}$$

for x in the open complex unit-disc.

Proof. Apply the equality $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$ (where $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$) to Newton's identity $(1+(-x))^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k}(-x)^k$ or use induction on n. ☐

Remark 4.2. *Lemma 4.1 has the following nice combinatorial proof: $\frac{1}{(1-x)^n}$ is the generating series for colouring Easter eggs with n different colours (or, equivalently, for the number of monomials in n commuting variables). The k-th coefficient is thus given by $\binom{k+n-1}{n-1}$.*

Lemma 4.3. *We have*

$$\sum_{l=k}^{j} \binom{j}{l}\binom{l}{k} x^l = \binom{j}{k} x^k (x+1)^{j-k}$$

In particular, for $x = \frac{-1}{2}$ we get

$$\sum_{l=k}^{j} \binom{j}{l}\binom{l}{k} \frac{1}{(-2)^l} = \binom{j}{k} \frac{(-1)^k}{2^j}$$

Proof. Compare the coefficients $\binom{j}{l}\binom{l}{k}$ and $\binom{j}{k}\binom{j-k}{l-k}$ of x^l of both sides. ☐

4.2 Identities for the Stern sequence

We recall that the Stern sequence $s : \mathbb{N} \longrightarrow \mathbb{N}$ is recursively defined by $s(0) = 0, s(1) = 1, s(2n) = s(n)$ and $s(2n+1) = s(n) + s(n+1)$ for $n \geq 1$.

Proposition 4.4. *For all $n \geq 0$ and for all r such that $0 \leq r \leq 2^n$, the Stern sequence satisfies the identities*

$$s(2^n + r) = s(2^n - r) + s(r), \tag{11}$$

$$s(2^n + r) = s(2^{n+1} - r), \tag{12}$$

$$s(r) = 2s(2^n + r) - s(3 \cdot 2^n + r). \tag{13}$$

Proof. The identities hold for $n = 0$ and $r \in \{0, 1\}$. Since $s(2m) = s(m)$ they hold for r even by induction. For odd $r = 2t + 1 < 2^{n+1}$, we sum the identities corresponding to $(n-1, t)$ and $(n-1, t+1)$ which hold by induction. The definition $s(2m+1) = s(m)+s(m+1)$ and induction implies the identities for odd r. □

The main idea of this paper is to apply Lemma 4.1 to the trivial identities

$$\frac{\alpha s + \beta S}{\gamma s + \delta S} = \frac{\alpha}{\delta} \frac{s}{S} \frac{1}{\left(1 + \frac{\gamma}{\delta} \frac{s}{S}\right)} + \frac{\beta}{\delta} \frac{1}{\left(1 + \frac{\gamma}{\delta} \frac{s}{S}\right)}, \tag{14}$$

$$\frac{\alpha s + \beta S}{\gamma s + \delta S} = \frac{\alpha}{\gamma} + \left(\frac{\beta}{\delta} - \frac{\alpha}{\gamma}\right) \frac{1}{\left(1 + \frac{\gamma}{\delta} \frac{s}{S}\right)}, \tag{15}$$

$$\frac{\alpha s + \beta S}{\gamma s + \delta S}, = \frac{\beta}{\delta} + \left(\frac{\alpha}{\delta} - \frac{\beta\gamma}{\delta^2}\right) \frac{s}{S} \frac{1}{\left(1 + \frac{\gamma}{\delta} \frac{s}{S}\right)}. \tag{16}$$

5 A simple set of linear equations for $m_{\mathbb{N}}$

Theorem 5.1. *The sequence $m_0 = 1, m_1 = \frac{1}{2}, m_2, \ldots$ of moments defined by $m_n = \int_0^1 \square(t)^n dt$ (see (1)) satisfies the equalities*

$$m_n = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \phi_{2k} \tag{17}$$

where

$$\phi_n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{m_{n+k}}{2^{n+k}} \ . \tag{18}$$

Remark 5.2. *Since the increasing function*

$$k \longmapsto \frac{\binom{n+k-1}{k}}{2^{n+k}} \frac{2^{n+k+1}}{\binom{n+k}{k+1}} = 2\frac{k+1}{n+k}$$

(for $k > 0$ and n a fixed natural integer) equals 1 for $k = n - 2$ and since the moments m_n are slowly decreasing, the main contribution to ϕ_n comes asymptotically from summands with indices k roughly equal to n.

The main contribution to ϕ_n is thus given by moments of the form m_{2n+l} with l an element of $\mathbb{Z}$ of small absolute value.

Similarly, the main contribution to m_n in Formula (17) corresponds asymptotically to indices $k \sim n/4$. and involves thus mainly moments of the form m_{n+l} for l a small integer.

Theorem 5.1 is an immediate consequence of the following result.

Proposition 5.3. *For all $n \in \mathbb{N}$ we have the identities*

$$\int_0^{1/2} \Box(t)^n dt = \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi_k$$

and

$$\int_{1/2}^{1} \Box(t)^n dt = \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \phi_k$$

with ϕ_k defined by Formula (18).

Lemma 5.4. *We have*

$$\phi_n = 2 \int_{1/2}^{1} (2\Box(t) - 1)^n dt \ . \tag{19}$$

for ϕ_n defined by Formula (18).

Corollary 2.3 shows that Lemma 5.4 can be restated as $\phi_n = \int_0^1 |2\Box(t) - 1|^n \, dt$.

Proof of Lemma 5.4. Proposition 2.1 and the definition of Riemann sums show that we have

$$\begin{aligned} 2\int_{1/2}^{1} (2\Box(t) - 1)^n dt &= 2 \lim_{l \to \infty} \frac{1}{2^{l+1}} \sum_{r=0}^{2^l} \left(2\frac{s(2^l + r)}{s(2^{l+1} + 2^l + r)} - 1 \right)^n \\ &= \lim_{l \to \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(\frac{2s(2^l + r) - s(3 \cdot 2^l + r)}{s(3 \cdot 2^l + r)} \right)^n \ . \end{aligned}$$

Using (13) we get

$$2\int_{1/2}^{1} (2\Box(t) - 1)^n dt = \lim_{l \to \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(\frac{s(r)}{2s(2^l + r) - s(r)} \right)^n \tag{20}$$

or equivalently

$$2\int_{1/2}^{1} (2\Box(t) - 1)^n dt = \lim_{l \to \infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(\frac{s(r)}{2s(2^l + r)} \right)^n \left(\frac{1}{1 - \frac{s(r)}{2s(2^l+r)}} \right)^n \ .$$

Applying (10) we have

$$2\int_{1/2}^{1} (2\Box(t) - 1)^n dt = \sum_{j=0}^{\infty} \binom{n+k-1}{k} \frac{m_{n+k}}{2^{n+k}}$$

which ends the proof. $\Box$

Proof of Proposition 5.3. Using Lemma 5.4 we have

$$\frac{1}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\phi_k = \frac{1}{2^n}\int_{1/2}^{1}\sum_{k=0}^{n}\binom{n}{k}(1-2\Box(t))^k dt$$
$$= \int_{1/2}^{1}(1-\Box(t))^n dt$$

which equals $\int_0^{1/2}\Box(t)^n dt$ by (4). This proves the first equality.

The proof of the second equality is similar and left to the reader. $\Box$

5.1 Spectral properties

Theorem 5.1 expresses the moment-vector $(m_0, m_1, m_2, \dots)$ as a fixed point of a continuous linear operator T acting on the vector space $l^\infty(\mathbb{R})$ of real bounded sequences. We study here a few spectral properties of T. They imply in particular uniqueness of the fixed point $(m_0, m_1, \dots)$ satisfying $m_0 = 1$.

We denote by $l^\infty = l^\infty(\mathbb{R})$ the real Banach space of bounded sequences with norm $\parallel v \parallel_\infty = \sup_{n\in\mathbb{N}}|v_n|$ for $v = (v_0, v_1, \dots)$ in l^∞. We set

$$\parallel U \parallel = \sup_{v\in l^\infty,\ \|v\|_\infty=1} \parallel U(v) \parallel$$

for the norm $\parallel U \parallel$ of an endomorphism $U \in \mathrm{End}(l^\infty)$. Similarly, we consider the norm

$$\parallel L \parallel = \sup_{v\in l^\infty,\ \|v\|_\infty=1} |L(v)|$$

of a continuous linear form $L : l^\infty \longrightarrow \mathbb{R}$.

Formulae (17) and (18) suggest to consider the sequence of operators

$$v = (v_0, v_1, \dots) \longmapsto T_n(v) = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\sum_{l=0}^{\infty}\binom{2k+l-1}{l}\frac{v_{2k+l}}{2^{2k+l}}\ . \quad (21)$$

Proposition 5.5. *Formula (21) defines continuous linear forms $T_0, T_1, T_2, \dots$ of norm $\parallel T_0 \parallel = 1$ and $\parallel T_n \parallel = \frac{1}{2}$ for $n \geq 1$.*

We define an endomorphism $T : l^\infty \longrightarrow l^\infty$ of the vector-space l^∞ by setting $T = (T_0, T_1, T_2, \dots)$. Proposition 5.5 and $T_0(v_0, v_1, \dots) = v_0$ imply the following result:

Corollary 5.6. *The restriction of the linear operator $T = (T_0, T_1, T_2, \dots)$ to the subspace l_0^∞ formed by all bounded sequences $(v_0, v_1, v_2, \dots)$ starting with $v_0 = 0$ yields an endomorphism of l_0^∞ whose spectrum is contained in $\{z \in \mathbb{C} \mid |z| \leq \frac{1}{2}\}$.*

In particular, the linear map

$$v \longmapsto T(v) = (T_0(v), T_1(v), \dots)$$

defines a bounded linear operator of l^∞ *which has a unique eigenvector of eigenvalue* 1 *of the form* $\left(1, \frac{1}{2}, \dots\right)$.

The coordinates $(m_0, m_1, \dots) = \left(1, \frac{1}{2}, \dots\right)$ of the unique eigenvector of eigenvalue 1 of T are of course the moments $m_n = \int_0^1 x^n d?x$ of the density function associated to Minkowski's question-mark function ?.

Proof of Proposition 5.5. For $v \in l^\infty$ such that $\parallel v \parallel_\infty \leq 1$, we have

$$|T_n(v)| \leq \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{1}{2^{2k}} \sum_{j=0}^{\infty} \binom{2k-1+j}{j} \frac{1}{2^j}$$

with equality if and only if v is (up to a sign) the vector $\mathbf{1} = (1, 1, 1, \dots)$ with all coefficients equal to 1.

Applying (10) we have thus

$$\begin{aligned} \parallel T_n \parallel &= |T_n(\mathbf{1})| \\ &= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{1}{2^{2k}} \left(\frac{1}{1-\frac{1}{2}}\right)^{2k} \\ &= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \\ &= \frac{1}{2^n} \left(\frac{(1+1)^n + (1-1)^n}{2}\right) \\ &= \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{2} & \text{if } n \geq 1 \end{cases} \end{aligned}$$

which completes the proof. □

Remark 5.7. *Laplace's method shows that the coefficient*

$$\frac{1}{2^n} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{m-1}{2k-1} \frac{1}{2^m}$$

of v_m *in* T_n *given by Formula (21) is asymptotically equal to*

$$\frac{1}{2} \frac{1}{\sqrt{2\pi n \mu (1+\mu)}} \left(\frac{((1+\mu)/2)^{1+\mu}}{\mu^\mu}\right)^n$$

for $\mu = \frac{m}{n}$ having a bounded logarithm. This coefficient is asymptotically maximal for $\mu = 1$ and decays exponentially fast otherwise. We have

$$\lim_{n\to\infty} n \int_0^\infty \frac{1}{2} \frac{1}{\sqrt{2\pi n\mu(1+\mu)}} \left(\frac{((1+\mu)/2)^{1+\mu}}{\mu^\mu}\right)^n d\mu = \frac{1}{2}$$

in agreement with Proposition 5.5.

Remark 5.8. *The linear operator T has an unbounded eigenvector of eigenvalue $\frac{1}{2}$ given by $w = (0, 0, 2, 3, 4, 5, 6, 7, \dots)$ as can be seen as follows: We have $T_0(w) = T_1(w) = 0$. For $n \geq 2$, Formula (21) with $w = (0, 0, 2, 3, 4, 5, 6, \dots)$ boils down to*

$$T_n(w) = \frac{1}{2^n} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} \sum_{l=0}^{\infty} \binom{2k+l-1}{l} \frac{2k+l}{2^{2k+l}}.$$

Computing the derivative $2k\frac{x^{2k-1}}{(1-x)^{2k+1}}$ of $\left(\frac{x}{1-x}\right)^{2k}$ at $x = \frac{1}{2}$ either directly or using the series expansion (10) given by Lemma 4.1 we get the identity

$$4k = \sum_{l=0}^{\infty} \binom{2k+l-1}{l} \frac{2k+l}{2^{2k+l}}\ .$$

For $n \geq 2$ we have thus

$$\begin{aligned} T_n(w) &= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 4k \\ &= \frac{1}{2^n} \left((1+x)^n + (1-x)^n\right)' \big|_{x=1} \\ &= \frac{n}{2}. \end{aligned}$$

5.2 Computational aspects

Theorem 5.1 is useful for computing numerical approximations of the first moments $m_0, m_1, \dots, m_N$ of Minkowski's question mark function.

This can be done by computing an approximation $(\tilde{m}_0, \tilde{m}_1, \dots, \tilde{m}_N)$ of the unique attracting fixed point $(\tilde{m}_0, \tilde{m}_1, \dots)$ of the form $(1, \dots)$ of the linear operator $T \circ \pi_N$ where $\pi_N : l^\infty \longrightarrow l^\infty$ is the projection defined by

$$\pi_N(x_0, x_1, \dots, x_N, x_{N+1}, \dots) = (x_0, x_1, \dots, x_N, 0, 0, 0, \dots).$$

The error $|\tilde{m}_i - m_i|$ is of order $O(m_{N+1}) = O\left(N^{1/4} e^{-2\sqrt{N \log 2}}\right)$, see Formula (42).

Since the distance to the fixed point is essentially divided by 2 under each iteration of $T \circ \pi_N$, the complexity of the resulting algorithm is roughly of order $O\left(\sqrt{N/\log 2}N^2\right)$ if aiming at maximal accuracy.

More precisely, the algorithm can be implemented as follows:

010 $\tilde{m}_0 := 1$,
020 For $n = 1, 2, 3, \ldots, N$ do:
030    $\tilde{m}_n := 0$,
040 End of loop over n,
050 Iterate the following loop:
060   For $n = 0, 2, 4, \ldots, 2\lfloor N/2 \rfloor$ do:
070      $b := \frac{1}{2^n}$,
080      $\tilde{\phi}_n := 0$,
090   For $k = 0, 1, 2, \ldots, N - n$ do:
100        $\tilde{\phi}_n := \tilde{\phi}_n + b\tilde{m}_{n+k}$,
110        $b := \frac{n+k}{2(k+1)} b$,
120   End of loop over k,
130   End of loop over n,
140   For $n = 1, 2, 3, \ldots, N$ do:
150      $b := \frac{1}{2^n}$,
160      $\tilde{m}_n := 0$,
170   For $k = 0, 1, 2, \ldots, \lfloor N/2 \rfloor$ do:
180        $\tilde{m}_n := \tilde{m}_n + b\tilde{\phi}_{2k}$,
190        $b := \frac{(n-2k-1)(n-2k)}{(2k+1)(2k+2)} b$,
200   End of loop over k,
210   End of loop over n,
220 End of outer loop (starting at 050).

Comments:

1. Computations should be done over the real numbers with sufficient accuracy (maximal achievable accuracy is of order $O(m_{N+1})$, see Section 10 for estimations).

2. The range and increment of the loop-variable n in line 060 is due to the fact that $m_1, \ldots, m_N$ depend only on $\phi_0, \phi_2, \phi_4, \ldots, \phi_{2\lfloor N/2 \rfloor}$ in Formula (17).

3. Instructions 070 and 150 need a loop in many programming languages.

4. The variable b in line 070, 100, 110 corresponds to the factor $\binom{n+k-1}{k}\frac{1}{2^{n+k}}$ in Formula (18).

5. The variable b in line 150, 180,190 corresponds to the factor $\frac{1}{2^n}\binom{n}{2k}$ in Formula (17).

6. Maximal possible accuracy is achieved by iterating the outer loop (instructions 060-210) roughly $2\sqrt{N/\log 2}$ times, see Corollary 10.2.

7. Using a known sequence of good approximations for $m_1, \dots, m_N$ instead of 0 when initializing $\tilde{m}_1, \dots, \tilde{m}_N$ (instruction 030) decreases the number of useful (i.e. leading to significantly better precision) iterations for the outer loop.

8. A progressive increase of N (starting from some small initial value) during the iteration of the outer loop yields a small speedup.

6 Formulae of Alkauskas

Theorem 5.1 is based on Riemann sums for the integral

$$A = 2\int_{1/2}^{1} (2\square(t) - 1)^n dt$$

obtained by subdividing the interval $\left[\frac{1}{2}, 1\right]$ into 2^l sub-intervals of equal length $\frac{1}{2^{l+1}}$.

In this section we give a new proof of some formulae obtained by Alkauskas by considering the infinite subdivision

$$[0,1] = \{0\} \cup \dots \left[\frac{1}{2^h}, \frac{1}{2^{h-1}}\right] \cup \left[\frac{1}{2^{h-1}}, \frac{1}{2^{h-2}}\right] \cup \dots \cup \left[\frac{1}{4}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{1}{1}\right]$$

suggested by the easy evaluations $\square\left(\frac{1}{2^h}\right) = \frac{1}{h+1}$.

Theorem 6.1. *We have*

$$m_n = \sum_{h=1}^{\infty} \frac{1}{2^h} \frac{1}{(h+1)^n} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{m_k}{(h+1)^k} \tag{22}$$

and

$$m_n = \sum_{h=1}^{\infty} \frac{1}{2^h} \frac{1}{h^n} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{m_k}{(-h)^k} \ . \tag{23}$$

Remark 6.2. *From a computational point of view it is perhaps useful to rewrite the formulae of Theorem 6.1 as*

$$m_n = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \gamma_{k+n} m_k \tag{24}$$

and

$$m_n = \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (-1)^k c_{k+n} m_k \tag{25}$$

where

$$\gamma_n = \sum_{k=1}^{\infty} \frac{1}{2^k(k+1)^n} = 2\mathrm{Li}_n\left(\frac{1}{2}\right) - 1$$

and

$$c_n = \sum_{k=1}^{\infty} \frac{1}{2^k k^n} = \mathrm{Li}_n\left(\frac{1}{2}\right)$$

where $\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$ *for x in the open complex unit-disc.*

Formula (22) (or (24)) should be preferred over (23) (or (25)). It converges faster (under iteration) and positivity of all coefficients ensures numerical stability.

Precomputing (and storing) the constants γ_k and using (24) needs only twice as much memory but provides a significant speed-up.

Formula (25) has been used by Alkauskas for numerical computations of the first values of m_n, see Appendix A3 of [1] or Proposition 5 of [2].

Since $\gamma_n \sim \frac{1}{2^{n+1}}$ for large n, the arguments of Remark 5.2 show that the main contribution to m_n in Formula (24) corresponds asymptotically to summands $k \sim n$ involving $m_{n-a}, \ldots, m_{n+a}$.

Proposition 6.3. *Setting*

$$I_h(n) = \int_{2^{-h}}^{2^{-h+1}} \square(t)^n dt \ . \tag{26}$$

we have

$$I_h(n) = \frac{1}{2^h} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{m_k}{(h+1)^{k+n}} \tag{27}$$

and

$$I_h(n) = \frac{1}{2^h} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (-1)^k \frac{m_k}{h^{k+n}} \ . \tag{28}$$

Lemma 6.4. *We have*

$$\square\left(\frac{1}{2^h} + \frac{r}{2^{h+l}}\right) = \frac{s(2^l+r)}{(h+1)s(2^l+r) - s(r)} \tag{29}$$

and

$$\square\left(\frac{1}{2^h} + \frac{r}{2^{h+l}}\right) = \frac{s(2^l+r)}{hs(2^l+r) + s(2^l-r)} \tag{30}$$

for $0 \leq r \leq 2^l$.

Remark 6.5. *More generally, if*

$$\square\left(\frac{q}{2^h}\right)=\frac{a}{b} \text{ and } \square\left(\frac{q+1}{2^h}\right)=\frac{c}{d}$$

with $(a,b)\in\mathbb{N}^2$ *and* $(c,d)\in\mathbb{N}^2$ *pairs of relatively prime natural numbers, then*

$$\begin{aligned}\square\left(\frac{q}{2^h}+\frac{r}{2^{h+l}}\right) &= \frac{as(2^l+r)+(c-a)s(r)}{bs(2^l+r)+(d-b)s(r)} \\ &= \frac{cs(2^l+r)+(a-c)s(2^l-r)}{ds(2^l+r)+(b-d)s(2^l-r)}\end{aligned}$$

for $l\in\mathbb{N}$ *and for* r *such that* $0\le r\le 2^l$*. One can then apply (14), (15), (16) (or a similar identity) with* $S=(2^l+r),\ s=s(r)$ *in order to get Riemann sums for* $\int_{a/b}^{c/d}\square(t)^n dt$*.*

Proof of Lemma 6.4. An induction on h establishes the formula for $l=0$ (and $r\in\{0,1\}$).

An induction on l (for constant h) ends the proof. ☐

Proof of Proposition 6.3. We have

$$I_h(n)=\frac{1}{2^h}\lim_{l\to\infty}\frac{1}{2^l}\sum_{r=0}^{2^l}\square\left(\frac{1}{2^h}+\frac{r}{2^{h+l}}\right)^n .$$

By (29) we have

$$\begin{aligned}I_h(n) &= \frac{1}{2^h}\lim_{l\to\infty}\frac{1}{2^l}\sum_{r=0}^{2^l}\left(\frac{s(2^l+r)}{(h+1)s(2^l+r)-s(r)}\right)^n \\ &= \lim_{l\to\infty}\frac{1}{2^{l+h}}\sum_{r=0}^{2^l}\frac{1}{(h+1)^n}\left(\frac{1}{1-\frac{s(r)}{(h+1)s(2^l+r)}}\right)^n\end{aligned}$$

and (10) implies now

$$\begin{aligned}I_h(n) &= \lim_{l\to\infty}\frac{1}{2^{l+h}}\sum_{r=0}^{2^l}\frac{1}{(h+1)^n}\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\left(\frac{s(r)}{(h+1)s(2^l+r)}\right)^k \\ &= \frac{1}{2^h}\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\frac{m_k}{(h+1)^{k+n}} .\end{aligned}$$

This proves the first equality.

The second equality follows from (12) applied to (30) yielding the identities

$$I_h(n) = \frac{1}{2^h} \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(\frac{s(2^l+r)}{hs(2^l+r)+s(r)} \right)^n$$

$$= \lim_{l\to\infty} \frac{1}{2^{l+h}} \sum_{r=0}^{2^l} \frac{1}{h^n} \left(\frac{1}{1+\frac{s(r)}{hs(2^l+r)}} \right)^n$$

$$= \frac{1}{2^h} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} (-1)^k \frac{m_k}{h^{k+n}} \ .$$

□

Proof of Theorem 6.1. Follows from $m_n = \sum_{h=1}^{\infty} I_h(n)$ where $I_h(n)$ is evaluated using Proposition 6.3. □

7 Holomorphicity of m_x

Theorem 7.1. *(i) The map $n \longmapsto m_n$ extends to an entire function $x \longmapsto m_x$.*

(ii) The series expansion of $x \longmapsto m_x$ at $x=0$ is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=n}^{\infty} c_{n,k} m_k \tag{31}$$

where

$$\sum_{k=n}^{\infty} c_{n,k} x^k = (\log(1-x))^n \ . \tag{32}$$

Equivalently, the numbers $c_{n,k}$ are given by the equality

$$c_{n,k} = (-1)^k \frac{n!}{k!} s(k,n) \tag{33}$$

where the numbers $s(k,m)$ defined by $\sum_{m=0}^{k} s(k,m)x^m = x(x-1)(x-2)\cdots(x-k+1)$ are Stirling numbers of the first kind.

Remark 7.2. *The rational numbers $c_{n,k}$ defined by (32) are given by the recursive formulae $c_{0,0} = 1,\ c_{0,k} = 0$ if $k > 0$ and*

$$c_{n+1,k} = -\sum_{j=1}^{k-1} \frac{c_{n,k-j}}{j} \ , n > 0 \ .$$

They are also defined by the equality

$$c_{n,k} = (-1)^n \sum_{a_1,\dots,a_n \geq 1,\ a_1+\cdots+a_n=k} \frac{1}{a_1 \cdot a_2 \cdots a_n} \ .$$

Proof of Theorem 7.1. Extending formula (26) by considering

$$I_h(x) = \int_{2^{-h}}^{2^{-h+1}} e^{x \log(\square(t))} dt$$

for arbitrary $x \in \mathbb{C}$ (where $\log(\square(t)) \in \mathbb{R}$ denotes the usual logarithm of the strictly positive real number $\square(t)$), the inequalities

$$\frac{1}{h+1} = \square(2^{-h}) \leq \square(t) \leq \square(2^{-h+1}) = \frac{1}{h},\ t \in [2^{-h}, 2^{-h+1}]$$

show

$$|I_h(x)| \leq \frac{1}{2^h} \max_{t \in [\frac{1}{1+h}, \frac{1}{h}]} |t^x| \leq \frac{(1+h)^{|x|}}{2^h} .$$

This implies

$$\left| \sum_{h=1}^{\infty} I_h(x) \right| \leq \sum_{h=1}^{\infty} \frac{(1+h)^{|x|}}{2^h} < \infty .$$

The map $x \longmapsto m_x = \sum_{h=1}^{\infty} I_h(x)$ defines thus an entire function which coincides with m_x for $x \in \mathbb{N}$.

Using the symmetry $\square(x) = 1 - \square(x)$ we have

$$m_x = \lim_{l \to \infty} \frac{1}{2^l} \sum_{k=1}^{2^l} \left(\frac{s(k)}{s(2^l + k)} \right)^x = \lim_{l \to \infty} \frac{1}{2^l} \sum_{k=0}^{2^l - 1} \left(1 - \frac{s(k)}{s(2^l + k)} \right)^x .$$

The $n-$th derivative of m_x at $x = 0$ evaluates thus to

$$\lim_{l \to \infty} \frac{1}{2^l} \sum_{k=0}^{2^l - 1} \left(\log \left(1 - \frac{s(k)}{s(2^l + k)} \right) \right)^n$$
$$= \lim_{l \to \infty} \frac{1}{2^l} \sum_{k=0}^{2^l - 1} \left(- \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{s(k)}{s(2^l + k)} \right)^j \right)^n$$

which proves formula (31). $\square$

Remark 7.3. *Holomorphicity of $x \longmapsto m_x$ can also be proved using Proposition 3.3.*

8 Two conjectural relations

The derivative of the holomorphic function $x \longmapsto m_x$ (see Theorem 7.1) is given by

$$- \sum_{n=1}^{\infty} \frac{m_n}{n} \sim -0.79242512859548911819121151529989139888941278204 38$$

at the origin $x = 0$. It coincides experimentally with the number

$$-2\left(\log 2 - \sum_{n=1}^{\infty} \frac{m_n}{n2^n}\right)$$

leading to the following conjectural identity.

Conjecture 8.1. *We have*

$$\log 2 = \sum_{n=1}^{\infty} \frac{m_n}{2n}\left(1 + \frac{1}{2^{n-1}}\right) . \tag{34}$$

A variation is given by

Conjecture 8.2.

$$\log 2 = \sum_{n=1}^{\infty} \frac{m_n}{n}\left(\frac{1}{2^n} - (-1)^n\right) . \tag{35}$$

9 A third set of formulae

In this section we consider the partition

$$[0,1] \setminus \{1\} = \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right] \cup \left[\frac{3}{4}, \frac{7}{8}\right] \cup \left[\frac{7}{8}, \frac{15}{16}\right] \cup \ldots .$$

The resulting identities, well suited for computing asymptotics, are given by the following result:

Theorem 9.1.

$$\begin{aligned} m_n &= \frac{1}{2}\sum_{j=0}^{\infty} \binom{n+j-1}{j}(-1)^j m_{n+j} \\ &+ \sum_{h=2}^{\infty} \frac{1}{2^h}\left(\frac{h-1}{h}\right)^n \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{h(h-1)}\right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j}(-1)^j \frac{m_{k+j}}{h^j} \end{aligned}$$

and

$$m_n = \sum_{h=1}^{\infty} \frac{1}{2^h}\left(\frac{h}{h+1}\right)^n \sum_{k=0}^{n} \binom{n}{k}\left(\frac{-1}{h(h+1)}\right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j} \frac{m_{k+j}}{(h+1)^j} .$$

Remark 9.2. *Only terms of order $h \sim \sqrt{n/\log 2} + O\left(n^{1/4}\right)$ yield large contributions to the first sum of the formulae in Theorem 9.1. Corresponding terms of the second sum (over k) for such contributions decay exponentially fast. Terms of the third sum (over j) decay also exponentially fast for fixed $h > 1$ and for k small.*

We set

$$J_h(n) = \int_{1-2^{-h+1}}^{1-2^{-h}} \square(t)^n dt \ . \tag{36}$$

Proposition 9.3. *We have for all $h \in \mathbb{N},\ h \geq 1$ the identities*

$$J_h(n) = \frac{1}{2^h}\left(\frac{h-1}{h}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{h(h-1)}\right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j} (-1)^j \frac{m_{k+j}}{h^j} \tag{37}$$

and

$$J_h(n) = \frac{1}{2^h}\left(\frac{h}{h+1}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{h(h+1)}\right)^k \sum_{j=0}^{\infty} \binom{k-1+j}{j} \frac{m_{k+j}}{(h+1)^j} \ . \tag{38}$$

Observe that (37) boils down to

$$J_1(n) = \frac{1}{2}\sum_{j=0}^{\infty} \binom{n+j-1}{j} (-1)^j m_{n+j} \tag{39}$$

for $h = 1$.

Proof of Proposition 9.3. Identity (11) of Proposition 4.4 implies

$$J_h(n) = \int_{2^{-h}}^{2^{-h+1}} (1-\square(t))^n \, dt \ .$$

Using Formula (30) of Lemma 6.4 we get

$$\begin{aligned}
J_h(n) &= \frac{1}{2^h} \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(1 - \frac{s(2^l+r)}{hs(2^l+r)+s(2^l-r)}\right)^n \\
&= \frac{1}{2^h} \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(1 - \frac{s(2^l+r)}{hs(2^l+r)+s(r)}\right)^n \\
&= \lim_{l\to\infty} \frac{1}{2^{h+l}} \sum_{r=0}^{2^l} \left(\frac{h-1}{h} + \frac{1}{h^2}\left(\frac{s(r)}{s(2^l+r)+\frac{s(r)}{h}}\right)\right)^n \\
&= \lim_{l\to\infty} \frac{1}{2^{h+l}} \left(\frac{h-1}{h}\right)^n \sum_{r=0}^{2^l}\sum_{k=0}^{n} \binom{n}{k} \frac{1}{h^k(h-1)^k} \left(\frac{s(r)}{s(2^l+r)+\frac{1}{h}s(r)}\right)^k \ .
\end{aligned}$$

Using the identity

$$\left(\frac{s(r)}{s(2^l+r)+\frac{s(r)}{h}}\right)^k = \left(\frac{s(r)}{s(2^l+r)}\right)^k \left(\frac{1}{1+\frac{1}{h}\frac{s(r)}{s(2^l+r)}}\right)^k$$
$$= \sum_{j=0}^{\infty} \binom{k-1+j}{j} \frac{(-1)^j}{h^j} \left(\frac{s(r)}{s(2^l+r)}\right)^{k+j}$$

obtained by applying formula (10), we get the first equation.

Starting with

$$J_h(n) = \frac{1}{2^h} \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(1 - \frac{s(2^l+r)}{(h+1)s(2^l+r)-s(r)}\right)^n$$
$$= \lim_{l\to\infty} \frac{1}{2^{h+l}} \sum_{r=0}^{2^l} \left(\frac{h}{h+1} - \frac{1}{(h+1)^2}\left(\frac{s(r)}{s(2^l+r)-\frac{s(r)}{h+1}}\right)\right)^n$$
$$= \lim_{l\to\infty} \frac{1}{2^{h+l}} \left(\frac{h}{h+1}\right)^n \sum_{r=0}^{2^l}\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{h^k(h+1)^k} \left(\frac{s(r)}{s(2^l+r)-\frac{s(r)}{h+1}}\right)^k .$$

and finishing as above yields the second identity. □

Proof of Theorem 9.1. Follows from Proposition 9.3 applied to the obvious identity $m_n = \sum_{h=1}^{\infty} J_h(n)$. □

10 Asymptotics

We set

$$\lambda = \sum_{n=0}^{\infty} \frac{(\log 2)^n}{n!} m_n . \qquad (40)$$

Numerically, λ is approximately equal to

1.42815984554560290424313465212729430726822547802532544939052972 .

Theorem 10.1. *For every strictly positive ϵ there exists a natural integer N such that*

$$\left| m_n - \lambda \sum_{h=2}^{\infty} \frac{1}{2^h}\left(\frac{h-1}{h}\right)^n \right| \le \epsilon\, m_n$$

if $n \ge N$.

The error given by the asymptotic approximation

$$m_n \sim \lambda \sum_{h=2}^{\infty} \frac{1}{2^h}\left(1-\frac{1}{h}\right)^n \tag{41}$$

in Theorem 10.1 is surprisingly small, see Section 10.3.

Corollary 10.2. *We have*

$$m_n \sim \lambda \frac{n^{1/4}}{(\log 2)^{3/4}} \sqrt{\frac{\pi}{2}} e^{-2\sqrt{n \log 2}} \tag{42}$$

for $n \to \infty$.

Corollary 10.2 is of course equivalent to Theorem 1 in [3]. The constant λ defined by (40) is related to the constant

$$c_0 = \int_0^1 2^t \left(1 - \frac{1}{2}?(t)\right) dt = \frac{1}{\log 2} - \frac{1}{2}\int_0^1 2^t ?(t) dt$$

in Theorem 1 of [3] by

$$\lambda = c_0 2 \log 2$$

and satisfies the following additional identities:

Proposition 10.3. *We have*

$$\lambda = 2\sum_{n=0}^{\infty} \frac{(-\log 2)^n}{n!} m_n = \frac{4}{3}\sum_{n=0}^{\infty} \frac{(\log 2)^{2n}}{(2n)!} m_{2n} = 4\sum_{n=0}^{\infty} \frac{(\log 2)^{2n+1}}{(2n+1)!} m_{2n+1} \ .$$

Observe that the constant λ appears also in the asymptotic expression $\lambda \frac{n!}{(\log 2)^{n+1}}$ for m_{-n}, see [2] or Proposition 11.8.

Remark 10.4. *A computation of λ with high precision needs only relatively few initial values of $m_0, m_1, m_2, \ldots$. I ignore however a direct approach for accurately computing only the first few values of $m_2, m_3, m_4, \ldots$.*

Proposition 10.5. *We have*

$$\lim_{n\to\infty} \left(\frac{n^{1/4}}{(\log 2)^{3/4}} \sqrt{\frac{\pi}{2}} e^{-2\sqrt{n \log 2}}\right)^{-1} \left(\sum_{h=2}^{\infty} \frac{1}{2^h}\left(1-\frac{1}{h}\right)^n\right) = 1 \ .$$

Proof of Proposition 10.5. We apply Laplace's method to $\sum_{h=2}^{\infty} \frac{1}{2^h}\left(1-\frac{1}{h}\right)^n$.

The derivative

$$f_n'(x) = \frac{1}{2^x}\left(1-\frac{1}{x}\right)^n \frac{(n + x(1-x)\log 2)}{x(x-1)}$$

of the function $f_n(x) = \frac{1}{2^x}\left(1-\frac{1}{x}\right)^n$ has roots given by the solutions of $x^2 - x = \frac{n}{\log 2}$.

Assuming x real and positive, the positive root of f_n' is given by

$$\rho = \frac{1+\sqrt{1+4n/\log 2}}{2} = \sqrt{\frac{n}{\log 2}} + \frac{1}{2} + \frac{1}{8}\sqrt{\frac{\log 2}{n}} + O\left(\frac{1}{\sqrt{n}^3}\right)$$

and we have

$$f_n(\rho) = \frac{1}{\sqrt{2}} e^{-2\sqrt{n\log 2}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) .$$

A straightforward computation shows

$$f_n''(\rho) = \frac{1}{\sqrt{2}} e^{-2\sqrt{n\log 2}} \left(-2\frac{\sqrt{\log 2}^3}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right) . \tag{43}$$

Applying Laplace's method

$$\int_2^\infty f_n(h)dh \sim f_n(\rho) \int_{-\infty}^\infty e^{-\frac{f_n''(\rho)}{f_n(\rho)}t^2/2} dt$$
$$= \sqrt{\frac{2\pi f_n(\rho)^3}{-f_n''(\rho)}}$$

to the integral approximation $\int_2^\infty f_n(h)dh$ of $\sum_{h=2}^\infty f_n(h)$ we get the result. □

Proposition 10.6. *For every $\epsilon > 0$ there exists a natural integer A such that*

$$0 \le m_n - \sum_{h \in \lfloor \sqrt{n/\log 2} - A n^{1/4} \rfloor}^{\lfloor \sqrt{n/\log 2} + A n^{1/4} \rfloor} J_h(n) < \epsilon m_n$$

for all n large enough with $J_h(n) = \int_{1-2^{-h+1}}^{1-2^{-h}} \square(t)^n dt$ given by (36).

Proof. The easy evaluation $\square\left(1 - \frac{1}{2^h}\right) = 1 - \frac{1}{h+1}$ for $h \in \mathbb{N}$ shows

$$1 - \frac{1}{h} \le \square(t) \le 1 - \frac{1}{h+1}$$

for $t \in \left[1 - \frac{1}{2^{h-1}}, 1 - \frac{1}{2^h}\right]$ and we have

$$\frac{1}{2^h}\left(1 - \frac{1}{h}\right)^x \le J_h(x) \le \frac{1}{2^h}\left(1 - \frac{1}{h+1}\right)^x$$

for real positive x. Since the unique positive root of the logarithmic derivative

$$\frac{df/dh}{f} = -\log 2 + \frac{x}{h(h-1)}$$

with respect to h of $f = \frac{1}{2^h}\left(1-\frac{1}{h}\right)^x$ is given by $h \sim \sqrt{x/\log 2}$ for large x, the decay of the function

$$s \longrightarrow \frac{1}{2^{\sqrt{x/\log 2+sx^{1/4}}}}\left(1-\frac{1}{\sqrt{x/\log 2+sx^{1/4}}}\right)^x$$

is exponentially fast in $|s|$ for large x. This implies the result. □

Proof of Theorem 10.1. Setting

$$\tilde{J}_h(n) = 2^h\left(\frac{h}{h-1}\right)^n J_h(n)$$

formula (37) of Proposition 9.3 shows the identities

$$\begin{aligned}\tilde{J}_h(n) &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{h^k(h-1)^k}\sum_{j=0}^{\infty}\binom{k-1+j}{j}(-1)^j\frac{m_{k+j}}{h^j}\\ &= \sum_{k=0}^{n}\frac{(\log 2)^k}{k!}\frac{\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)}{\left(\frac{h(h-1)}{n}\log 2\right)^k}\sum_{j=0}^{\infty}\binom{k-1+j}{j}(-1)^j\frac{m_{k+j}}{h^j}\ .\end{aligned}$$

For k fixed and for $h = \sqrt{n/2\log 2} + O(n^{1/4})$ we have

$$\lim_{n\to\infty}\frac{\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)}{\left(\frac{h(h-1)}{n}\log 2\right)^k} = 1$$

and we get the asymptotics

$$\tilde{J}_h(n) \sim \sum_{k=0}^{\infty}\frac{(\log 2)^k}{k!}m_k = \lambda$$

for $h = \sqrt{n/\log 2} + O\left(n^{1/4}\right)$.

Proposition 10.6 shows now

$$\begin{aligned} m_n &\sim_\epsilon \sum_{h=\lfloor\sqrt{n/\log 2}-An^{1/4}\rfloor}^{\lfloor\sqrt{n/\log 2}+An^{1/4}\rfloor} J_h(n)\\ &\sim_\epsilon \sum_{h=\lfloor\sqrt{n/\log 2}-An^{1/4}\rfloor}^{\lfloor\sqrt{n/\log 2}+An^{1/4}\rfloor}\frac{1}{2^h}\left(\frac{h-1}{h}\right)^n\tilde{J}_h(n)\\ &\sim_\epsilon \lambda\sum_{h=2}^{\infty}\frac{1}{2^h}\left(\frac{h-1}{h}\right)^n\end{aligned}$$

for $n\to\infty$ and fixed A (depending on ϵ) with $a \sim_\epsilon b$ denoting $|a-b| < \epsilon a$ for arbitrary small ϵ if n is large enough. □

Proof of Proposition 10.3. Working with formula (38) we get the asymptotics

$$m_n \sim \sum_{h=1}^{\infty} \frac{1}{2^h} \left(\frac{h}{h+1} \right)^n \sum_{k=0}^{\infty} \frac{(-\log 2)^k}{k!} m_k$$
$$= 2 \sum_{k=2}^{\infty} \frac{1}{2^h} \left(\frac{h-1}{h} \right)^n \sum_{k=0}^{\infty} \frac{(-\log 2)^k}{k!} m_k$$

which imply the first equality by comparing with Theorem 10.1. The two other identities are easy consequences. □

Proof of Corollary 10.2. Follows from Theorem 10.1 and Proposition 10.5. □

10.1 Asymptotic formula for ϕ_n

Using similar techniques, we get the asymptotic approximation

$$\phi_n \sim 2\lambda \sum_{h=3}^{\infty} \frac{1}{2^h} \left(1 - \frac{2}{h} \right)^n \tag{44}$$

(where λ is given by (40)) for $\phi_n = 2 \int_{1/2}^{1} (2\square(t) - 1)^n \, dt$, see Formula (19) in Lemma 5.4. The relative error seems again to be of order $O\left(\phi_n^{5/4}\right)$ and has again (suitably normalized) a more or less periodic behaviour as a function of $\sqrt{n}$.

Using Laplace's method for the right side of (44) we get the simpler and less accurate expression

$$\phi_n \sim \lambda \frac{(2n)^{1/4}\sqrt{\pi}}{\log(2)^{3/4}} e^{-2\sqrt{2n \log 2}} \; . \tag{45}$$

10.2 A second asymptotic formula

The motivation for this section is the estimation of the order of the error in the asymptotic approximation (41).

A refinement of the Riemann sum underlying Formula (41) should yield a slightly more accurate approximation for m_n. The order of the difference between the two formulae should be a measure for the accuracy of (41).

We subdivide the interval underlying the integral $J_h(n)$ defined by (36) into two intervals of equal lengths. We have $J_h(n) = A_h(n) + B_h(n)$ where

$$A_h(n) = \int_{1-2^{-h+1}}^{1-3\cdot 2^{-h-1}} \square(t)^n dt \text{ and } B_h(n) = \int_{1-3\cdot 2^{-h-1}}^{1-2^{-h}} \square(t)^n dt \; .$$

We have

$$\begin{aligned}
A_h(n) &= \frac{1}{2^{h+1}} \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(1 - \frac{s(2^l+r)+s(r)}{hs(2^l+r)+(h+1)s(r)}\right)^n \\
&= \frac{1}{2^{h+1}} \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=0}^{2^l} \left(\frac{2h-1}{2h+1} - \frac{s(r)}{(2h+1)^2(s(2^l+r)-\frac{h+1}{2h+1}s(r))}\right)^n \\
&= \frac{1}{2^{h+1}} \left(\frac{2h-1}{2h+1}\right)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(4h^2-1)^k} \lim_{l\to\infty} \frac{2^k}{2^l} \sum_{r=0}^{2^l} \left(\frac{s(r)}{2s(2^l+r)-\frac{2h+2}{2h+1}s(r)}\right)^k
\end{aligned}$$

which yields

$$\lim_{h\to\infty} A_h(n) = \frac{1}{2^{h+1}} \left(\frac{2h-1}{2h+1}\right)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(4h^2-1)^k} 2^k \phi_k$$

by Identity (20).

For $h = \sqrt{n/\log 2} + O(n^{1/4})$ we have thus

$$A_h(n) \sim \frac{1}{2^{h+1}} \left(\frac{2h-1}{2h+1}\right)^n \sum_{k=0}^{\infty} \frac{(-\log 2)^k}{2^k\ k!} \phi_k\ .$$

A similar calculation shows

$$B_h(n) \sim \frac{1}{2^{h+1}} \left(\frac{2h-1}{2h+1}\right)^n \sum_{k=0}^{\infty} \frac{(\log 2)^k}{2^k\ k!} \phi_k$$

for $h = \sqrt{n/\log 2} + O(n^{1/4})$.

We get thus for large n and $h = \sqrt{n/\log 2} + O(n^{1/4})$ the approximation

$$J_h(n) \sim \frac{1}{2^h} \sum_{k=0}^{\infty} \frac{(\log 2)^{2k}}{2^{2k}\ (2k)!} \phi_{2k}\ .$$

Setting

$$\rho = \sum_{k=0}^{\infty} \frac{(\log 2)^{2k}}{2^{2k}\ (2k)!} \phi_{2k} \tag{46}$$

we have asymptotically

$$m_n \sim \rho \sum_{h=1}^{\infty} \frac{1}{2^h} \left(1 - \frac{2}{2h+1}\right)^n\ .$$

Using Laplace's method we get the asymptotic approximation

$$\sum_{h=1}^{\infty} \frac{1}{2^h} \left(1 - \frac{2}{2h+1}\right)^n \sim \frac{n^{1/4}\sqrt{\pi}}{(\log 2)^{3/4}} e^{-2\sqrt{n\log 2}}.$$

This shows

$$m_n \sim \rho \frac{n^{1/4}\sqrt{\pi}}{(\log 2)^{3/4}} e^{-2\sqrt{n \log 2}} \tag{47}$$

and implies the identity

$$\rho = \frac{\lambda}{\sqrt{2}} \tag{48}$$

as can be seen by comparing the two asymptotic approximations (42) and (47) of m_n.

The asymptotic formula

$$m_n \sim \lambda \sum_{h=1}^{\infty} \frac{1}{2^{h+1/2}} \left(1 - \frac{1}{h+1/2}\right)^n \tag{49}$$

should thus be slightly better than (41), see Figure 1 in Section 10.3.

10.3 An estimation for the error of the asymptotic formulae

Setting

$$x \longmapsto S_x(n) = \sum_{h=1}^{\infty} \frac{1}{2^{h+x}} \left(1 - \frac{1}{h+x}\right)^n, \tag{50}$$

the asymptotic formulae (41) and (49) can be rewritten as $m_n \sim \lambda S_0(n)$ and $m_n \sim \lambda S_{1/2}(n)$. Since $x \longmapsto S_x(n)$ is almost 1-periodic (for small positive x and huge fixed n) and oscillates experimentally around the exact value of the integral

$$S_\int(n) = \int_1^\infty \frac{1}{2^t} \left(1 - \frac{1}{t}\right)^n dt, \tag{51}$$

it is tempting to rescale the errors $m_n - S_x(n)$ by the inverse of the factor

$$\kappa(n) = \sqrt{\Big(S_0(n) - S_\int(n)\Big)^2 + \Big(S_{1/4}(n) - S_\int(n)\Big)^2} \tag{52}$$

given by the "amplitude" of the almost 1-periodic function $x \longmapsto S_x(n) - S_\int(n)$.

The sequence $S_\int(n)$ of integrals is easy to compute recursively: We have the initial values

$$
\begin{aligned}
S_\int(0) &= \int_1^\infty \frac{dt}{2^t} = \frac{1}{2\log 2} \\
S_\int(1) &= \int_1^\infty \frac{1}{2^t}\left(1-\frac{1}{t}\right) dt \\
&= \frac{1}{2\log 2} + \int_{-\infty}^{-\log 2} \frac{e^t}{t} dt? \\
&= \frac{1}{2\log 2} - \int_{\log 2}^{\infty} \frac{e^{-t}}{t} dt? \\
&= \frac{1}{2\log 2} + \mathrm{Ei}(-\log 2)
\end{aligned}
$$

(where $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral) and integration by parts yields the recursion relation

$$
S_\int(n) = \left(2 + \frac{\log 2}{n-1}\right) S_\int(n-1) - S_\int(n-2). \tag{53}
$$

The normalized errors

$$
E_0(n) = = \frac{1}{\kappa(n)}\left(m_n - \lambda \sum_{h=2}^{\infty} \frac{1}{2^h}\left(1-\frac{1}{h}\right)^n\right), \tag{54}
$$

$$
E_{1/2}(n) = \frac{1}{\kappa(n)}\left(m_n - \lambda \sum_{h=1}^{\infty} \frac{1}{2^{h+1/2}}\left(1-\frac{1}{h+1/2}\right)^n\right), \tag{55}
$$

$$
E_\int(n) = \frac{1}{\kappa(n)}\left(m_n - \lambda \int_1^\infty \frac{1}{2^t}\left(1-\frac{1}{t}\right)^n dt\right) \tag{56}
$$

are depicted in Figure 1 representing the points $(\sqrt{n}, E_0(n))$, $\left(\sqrt{n}, E_{1/2}(n)\right)$ and $\left(\sqrt{n}, E_\int(n)\right)$ for n in $\{100, \ldots, 400\}$. Points on the smallest sinusoidal curve are associated to $E_\int$, points on the sinusoidal curve of intermediate size to $E_{1/2}$ and points on the largest curve to E_0. In all three cases the error seems to be close to a damped periodic function of $\sqrt{n}$ of local amplitude $O(\kappa(n))$.

Remark 10.7. *The existence of the linear recurrence relation (53) implies the existence of asymptotic recurrence relations (given by the same formula) for the sequences m_n and $S_x(n)$.*

The asymptotic linear recurrence formula for m_n can be improved into an affine asymptotic formula using ideas of the next Section.

Remark 10.8. *It would be interesting to understand the asymptotic behaviour of the amplitude $\kappa(n)$ given by Formula (52). (The number $\kappa(n)$ is*

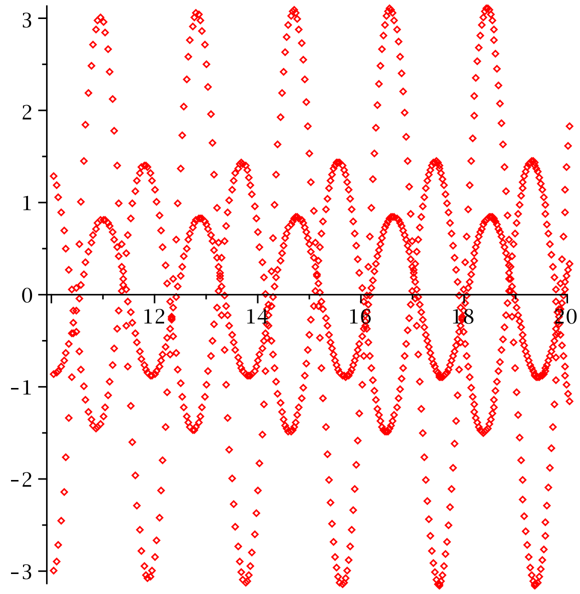

Figure 1: Error of asymptotical formulae

essentially the error term in Euler-MacLaurin's summation formula.) For moderate values of n *it seems to be comparable to*

$$\frac{\sqrt{n}}{\log n \ \log\log n} e^{-9/2\sqrt{n\log 2}}$$

which implies $\kappa(n) < m_n^{9/4}$. *The accuracy of the asymptotic formulae* $m_n \sim \lambda S_*(n)$ *(for* $* = 0, 1/2$ *and* $\int$*) is thus surprisingly high.*

10.4 Increasing accuracy

The behaviour of the error-terms $E_*(n)$ occurring in the previous Section suggests to try an asymptotic formula of the form

$$m_n \sim \lambda S_\int(n) + a\left(S_0(n) - S_\int(n)\right) + b\left(S_{1/4}(n) - S_\int(n)\right) \qquad (57)$$

with λ defined by (40) and $S_*(n)$ as in the previous Section. Experimentally, such a formula seems to exist with

$$a \sim -.52190105634043253677472587344 6,$$
$$b \sim -.14875585176359533863462893319 3.$$

The term $\lambda S_\int(n)$ is of course the principal contribution and plays the role of Formula (41) or (49). The two remaining terms $a\left(S_0(n) - S_\int(n)\right)$ and

$b\left(S_{1/4}(n) - S_{\int}(n)\right)$ sum up to a fairly regular (damped) oscillatory contribution of much lesser size. More precisely, its local amplitude should be asymptotically equal to $\frac{\lambda\sqrt{a^2+b^2}}{\kappa(n)}$ with $\kappa(n)$ given by Formula (52).

10.5 An improved algorithm

Accurate asymptotic approximations can be used for improving the algorithm given in Section 5.2. Indeed, the cutoff at N induces large relative errors for the last values of $\tilde{\phi}_n$. It is thus natural to compute $\tilde{\phi}_0, \tilde{\phi}_2, \ldots, \tilde{\phi}_{2\lfloor N/2 \rfloor}$ using the first $N+1$ values $\tilde{m}_0, \ldots, \tilde{m}_N$ and $M - N$ additional values $\tilde{m}_{N+1}, \ldots, \tilde{m}_M$ given by asymptotic approximations of $m_{N+1}, \ldots, m_M$ (for $M > N$ a suitable integer depending on N and on the accuracy of the chosen approximation).

We illustrate this by modifying the algorithm of Section 5.2 using high-level instructions in order to involve the asymptotic approximation (41) (the approximation (42) is of much lesser interest):

Add the lines

005 Precompute (and store) sufficiently accurate values $\tilde{S}(n)$ of $S_0(n)$ (or, slightly better, of $S_{\int}(n)$) for $n = N+1, \ldots, M$.

051 Compute $\tilde{\lambda} := \sum_{n=0}^{M} \frac{(\log 2)^n}{n!} \tilde{m}_n$,

052 Set $\tilde{m}_n = \lambda \tilde{S}(n)$ for $n = N+1, \ldots, M$.

at the obvious locations.

Replace 090 by

090 For $k = 0, 1, 2, \ldots, M - n$ do:

The resulting algorithm can easily be modified in order to work with other asymptotic approximations. The author used mainly (57) (this needs precomputations of approximations for $S_{\int}(n), S_0(n), S_{1/4}(n)$ with n in $\{N+1, \ldots, M\}$).

Concerns using an algorithm based on a conjectural formula can be avoided by checking the final data using a single iteration of (the main loop in) the original algorithm (described in Section 5.2) with a sufficiently high value $N' > N$ (with missing values replaced by their (conjecturally very accurate) approximations). The obtained data are exact up to an absolute error bounded by $\max(|\epsilon|, m_{N'+1})$ with ϵ denoting the maximal modification of $\tilde{m}_1, \ldots, \tilde{m}_{N'}$ during the final checking-run.

The improved version has smaller memory requirement and a much better running time : The (conjectural) accuracy of the used approximation should more than double the number of achievable correct digits for a given value of N. In order to achieve the same accuracy, the original algorithm has to be run with N multiplied by more than 4 which multiplies the running time of the main loop by more than $16 = 4^2$.

11 Values of m at negative integers

Proposition 11.1. *The equality*

$$m_{-n} = m_n + \sum_{k=0}^{n-1} \binom{n}{k} (m_{-k} + m_k) \tag{58}$$

holds for $n \in \mathbb{N}$ a natural integer.

Remark 11.2. *The generalization*

$$m_z = \frac{1}{2} \sum_{k=0}^{\infty} \binom{-z}{k} (m_{-k} + m_k)$$

of Proposition 11.1 fails for arbitrary complex values of z. Indeed, Proposition 11.1 is based on the identity $(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k$ for arbitrary $x \in \mathbb{R}$ which breaks down if $-z$ is not in $\mathbb{N}$.

Proof of Proposition 11.1. We have for $n \in \mathbb{N}$

$$m_{-n} = \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=1}^{2^l} \left(\frac{s(2^l + r)}{s(r)} \right)^n$$

Using (11) we have

$$\begin{aligned}
m_{-n} &= \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=1}^{2^l} \left(\frac{s(r) + s(2^l - r)}{s(r)} \right)^n \\
&= \lim_{l\to\infty} \frac{1}{2^l} \sum_{r=1}^{2^l} \left(1 + \frac{s(2^l - r)}{s(r)} \right)^n \\
&= \lim_{l\to\infty} \frac{1}{2^l} \left(\sum_{r=1}^{2^{l-1}} \left(1 + \frac{s(2^l - r)}{s(r)} \right)^n + \sum_{r=2^{l-1}+1}^{2^l} \left(1 + \frac{s(2^l - r)}{s(r)} \right)^n \right) \\
&= \lim_{l\to\infty} \frac{1}{2^l} \left(\sum_{r=1}^{2^{l-1}} \left(1 + \frac{s(2^l - r)}{s(r)} \right)^n + \sum_{r=0}^{2^{l-1}-1} \left(1 + \frac{s(r)}{s(2^l - r)} \right)^n \right)
\end{aligned}$$

Using (12) we have thus

$$\begin{aligned}
m_{-n} &= \lim_{l\to\infty} \frac{1}{2^l} \left(\sum_{r=1}^{2^{l-1}} \left(1 + \frac{s(2^{l-1} + r)}{s(r)} \right)^n + \sum_{r=1}^{2^{l-1}} \left(1 + \frac{s(r)}{s(2^{l-1} + r)} \right)^n \right) \\
&= \lim_{l\to\infty} \frac{1}{2^l} \sum_{k=0}^{n} \binom{n}{k} \sum_{r=1}^{2^{l-1}} \left(\left(\frac{s(2^{l-1} + r)}{s(r)} \right)^k + \left(\frac{s(r)}{s(2^{l-1} + r)} \right)^k \right) \\
&= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} (m_{-k} + m_k)
\end{aligned}$$

which implies the result. □

11.1 Matrices relating $m_{-\mathbb{N}}$ and $m_{\mathbb{N}}$

Identity (58) of Proposition 11.1 implies the existence of infinite lower diagonal triangular unipotent matrices $A, B = A^{-1}$ with integral coefficients such that

$$\begin{pmatrix} m_0 \\ m_{-1} \\ m_{-2} \\ \vdots \end{pmatrix} = A \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \end{pmatrix} \text{ and } \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \end{pmatrix} = B \begin{pmatrix} m_0 \\ m_{-1} \\ m_{-2} \\ \vdots \end{pmatrix} .$$

The first few rows and columns of the matrices $A, B = A^{-1}$ are

$$\begin{pmatrix} 1 &&&& \\ 2 & 1 &&& \\ 6 & 4 & 1 && \\ 26 & 18 & 6 & 1 & \\ 150 & 104 & 36 & 8 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 &&&& \\ -2 & 1 &&& \\ 2 & -4 & 1 && \\ -2 & 6 & -6 & 1 & \\ 2 & -8 & 12 & -8 & 1 \end{pmatrix}$$

and their coefficients are described by the following result.

Proposition 11.3. *Let $\sigma_n,\ n \in \mathbb{Z}$ be a sequence (with values in a commutative ring containing 1) indexed by the set $\mathbb{Z}$ of all integers such that*

$$\sigma_{-n} - \sigma_n = \sum_{k=0}^{n-1} \binom{n}{k} (\sigma_{-k} + \sigma_k) .$$

Then

$$\sigma_{-i} = \sum_{j=0}^{i} \alpha_{i,j} \sigma_j$$

$$\sigma_i = \sum_{j=0}^{i} \beta_{i,j} \sigma_{-j}$$

for all $i \in \mathbb{N}$ where $\alpha_{i,j}, \beta_{i,j},\ 0 \le i, j$ are integers given by the formulae

$$\alpha_{i,j} = \binom{i}{j} \sum_{h=1}^{\infty} \frac{h^{i-j}}{2^h} \tag{59}$$

and

$$\beta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 2(-1)^{i+j}\binom{i}{j} & \text{otherwise.} \end{cases}$$

In particular, the matrices A and B with coefficients $\alpha_{i,j}, \beta_{i,j},\ 0 \le i, j$ are mutually inverse lower triangular unipotent integral matrices.

Proof. We have $\alpha_{i,i} = 1$ as required and the matrix A is clearly lower triangular. The proof is now by induction on the row-index i of the coefficients $\alpha_{i,j}$ for A. Equation (58) of Proposition 11.1 shows that we have

$$\alpha_{i+1,j} = \sum_{l=0}^{i} \binom{i+1}{l} (\alpha_{l,j} + \delta_{l,j})$$

for $i + 1 \geq j$, where $\delta_{l,j} = 1$ if $l = j$ and $\delta_{l,j} = 0$ otherwise.

We get

$$\begin{aligned}
\alpha_{i+1,j} &= \binom{i+1}{j} - \binom{i+1}{i+1}\binom{i+1}{j}\sum_{h=1}^{\infty}\frac{h^{i+1-j}}{2^h} \\
&\quad + \sum_{h=1}^{\infty}\sum_{k=0}^{i+1}\binom{i+1}{k}\binom{k}{j}\frac{h^{k-j}}{2^h} \\
&= \binom{i+1}{j} - \binom{i+1}{j}\sum_{h=1}^{\infty}\frac{h^{i+1-j}}{2^h} + \sum_{h=1}^{\infty}\frac{\binom{i+1}{j}}{2^h}\sum_{k=0}^{i+1}\binom{i+1-j}{k-j}h^{k-j} \\
&= \binom{i+1}{j}\left(1 - \sum_{h=1}^{\infty}\frac{h^{i+1-j}}{2^h} + \sum_{h=1}^{\infty}\frac{(h+1)^{i+1-j}}{2^h}\right) \\
&= \binom{i+1}{j}\sum_{h=1}^{\infty}\frac{h^{i+1-j}}{2^h} .
\end{aligned}$$

This implies the formula for the coefficients of A.

We prove the formula for the coefficients of the inverse matrix $B = A^{-1}$ by computing the product AB. We have

$$\begin{aligned}
\sum_{k=j}^{i}\alpha_{i,k}\beta_{k,j} &= 2\sum_{k=j}^{i}\alpha_{i,k}(-1)^{k+j}\binom{k}{j} - \alpha_{i,j} \\
&= 2\sum_{h=1}^{\infty}\sum_{k=j}^{i}\frac{h^{i-k}}{2^h}\binom{i}{k}\binom{k}{j}(-1)^{k+j} - \alpha_{i,j} \\
&= \sum_{h=1}^{\infty}\frac{(-1)^j h^i}{2^{h-1}}\sum_{k=j}^{i}\binom{i}{k}\binom{k}{j}\frac{1}{(-h)^k} - \alpha_{i,j} .
\end{aligned}$$

Identity 4.3 of Lemma 4.3 shows that this simplifies to

$$\sum_{h=1}^{\infty}\frac{(h-1)^{i-j}}{2^{h-1}}\binom{i}{j} - \alpha_{i,j} .$$

This equals 1 if $i = j$ and 0 for $i > j$ by definition of $\alpha_{i,j}$. □

The sum appearing in (59) defines natural integers having a recursive definition:

Proposition 11.4. *The natural integers*

$$\gamma_n = \sum_{h=1}^{\infty} \frac{h^n}{2^h}$$

(appearing in (59)) have the recursive definition $\gamma_0 = 1$ *and*

$$\gamma_n = 1 + \sum_{j=0}^{n-1} \binom{n}{j} \gamma_j$$

for $n \geq 1$.

The sequence of integers $\gamma_0, \gamma_1, \dots$ starts as

$$1, 2, 6, 26, 150, 1082, 9366, 94586, 1091670, \dots \ ,$$

see sequence A629 of [5].

Proof of Proposition 11.4. We have

$$\begin{aligned}\gamma_n &= \frac{1}{2} + \frac{1}{2} \sum_{h=1}^{\infty} \frac{(h+1)^n}{2^h} \\ &= \frac{1}{2} + \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \sum_{h=1}^{\infty} \frac{h^n}{2^h} \\ &= \frac{1}{2} + \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \gamma_j\end{aligned}$$

which implies the result. □

Remark 11.5. *Lower triangular matrices with lower triangular coefficients* $\gamma_{i,j}$ *of the form* $\binom{i}{j} c_{i-j}$ *for some sequence* $c_0, c_1, \dots$ *form a commutative algebra. Indeed, the map associating to such a matrix with coefficients* $\binom{i}{j} c_{i-j}$ *the formal exponential power series* $\sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$ *defines an isomorphism of algebras onto the algebra of formal exponential power series (with product given by the obvious "bilinear"extension of* $\frac{x^i}{i!} \frac{x^j}{j!} = \binom{i+j}{i} \frac{x^{i+j}}{(i+j)!}$*). The easy equality* $\sum_{n=0}^{\infty} \beta_{n,0} \frac{x^n}{n!} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n!} = 2e^{-x} - 1$ *shows thus the identity*

$$\sum_{n=0}^{\infty} \gamma_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_{n,0} \frac{x^n}{n!} = \frac{1}{2e^{-x} - 1} \ .$$

Proposition 11.1 and Proposition 11.3 imply the following result:

Corollary 11.6. *We have*

$$m_{-n} = \sum_{k=0}^{n} \binom{n}{k} \gamma_{n-k} m_k \tag{60}$$

(with γ_n defined by Proposition 11.4) and

$$m_n = m_{-n} + 2 \sum_{k=0}^{n-1} (-1)^{n+k} \binom{n}{k} m_{-k} \tag{61}$$

for all n in $\mathbb{N}$.

Corollary 11.6 is better suited than Proposition 3.3 for computing values $m_{-\mathbb{N}}$ using $m_{\mathbb{N}}$. It involves only finitely many terms of $m_{\mathbb{N}}$ with coefficients which are decreasing. (The main contribution to m_{-n} given by the formula of Proposition 3.3 corresponds to summands indexed by integers close to $\frac{n^2}{\log 2}$.)

Combining Formula (60) of Corollary 11.6 with Proposition 3.3 we get:

Corollary 11.7. *We have for all n in $\mathbb{N}$ the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \gamma_{n-k} m_k = \sum_{j=0}^{\infty} \binom{n+j-1}{j} m_j \ . \tag{62}$$

Corollary 11.7 yields

$$\begin{aligned}
2m_0 + m_1 &= \sum_{j=0}^{\infty} m_j, \\
6m_0 + 4m_1 + m_2 &= \sum_{j=0}^{\infty} (j+1) m_j, \\
26m_0 + 18m_1 + 6m_2 + m_3 &= \sum_{j=0}^{\infty} \binom{j+2}{2} m_j, \\
150m_0 + 104m_1 + 36m_2 + 8m_3 + m_4 &= \sum_{j=0}^{\infty} \binom{j+3}{3} m_j.
\end{aligned}$$

Using the easy evaluation $m_0 = 1, m_1 = \frac{1}{2}$, the case $n = 1$ yields the nice evaluation

$$\sum_{j=0}^{\infty} m_j = \frac{5}{2} \tag{63}$$

which can be used as an accuracy-check for numerical computations.

Similarly, using $n = 2$, we get the identity $m_2 + 8 = \sum_{j=0}^{\infty} (j+1) m_n$. Subtraction of (63) yields

$$\sum_{j=1}^{\infty} j \ m_j = m_2 + \frac{11}{2}. \tag{64}$$

11.2 Asymptotics for m_{-n}

Proposition 11.8. *We have*

$$\lim_{n\to\infty} m_{-n}\frac{(\log 2)^{n-1}}{n!} = \lambda$$

for $\lambda = \sum_{k=0}^{\infty} \frac{(\log 2)^k}{k!} m_k$ *given by (40).*

The following easy result is probably well-known:

Lemma 11.9. *We have*

$$\lim_{n\to\infty} \frac{(\log 2)^{n+1}}{n!} \sum_{k=1}^{\infty} \frac{k^n}{2^k} = 1\ .$$

Proof of Lemma 11.9. We apply Laplace's method to $\int_1^\infty x^n 2^{-x} dx \sim \sum_{k=1}^\infty k^n 2^{-k}$.

The derivative

$$\frac{d}{dx}\left(x^n e^{-x\log 2}\right) = \left(\frac{n}{x} - \log 2\right) x^n e^{-x\log 2}$$

of $x^n 2^{-x}$ has a unique strictly positive root at $\frac{n}{\log 2}$ and second derivative $-\frac{(\log 2)^2}{n}\left(\frac{n}{\log 2}\right)^n e^{-n}$ at the critical point $x = \frac{n}{\log 2}$ corresponding to the maximum $\left(\frac{n}{\log 2}\right)^n e^{-n}$ of the function $x \longmapsto x^n 2^{-x}$.

Laplace's method yields thus the asymptotics

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{k^n}{2^k} &\sim \left(\frac{n}{\log 2}\right)^n e^{-n} \int_{-\infty}^{\infty} e^{-\frac{(\log 2)^2}{2n}x^2} dx \\
&= \sqrt{\frac{2\pi n}{(\log 2)^2}} \left(\frac{n}{\log 2}\right)^n e^{-n} \\
&= \frac{1}{(\log 2)^{n+1}} \sqrt{2\pi n}\frac{n^n}{e^n} \\
&\sim \frac{n!}{(\log 2)^{n+1}}
\end{aligned}$$

where the last asymptotic equivalence follows from Stirling's formula $n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}$. □

Proof of Proposition 11.8. Using Corollary 11.6 and the asymptotics $\sum_{k=1}^\infty \frac{k^n}{2^k} \sim \frac{n!}{(\log 2)^{n+1}}$ given by Lemma 11.9, we get the asymptotics

$$\begin{aligned}
m_{-n} &= \sum_{k=0}^{n} \alpha_{n,k} m_k \sim \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{(\log 2)^{n-k+1}} m_k \\
&= \frac{n!}{(\log 2)^{n+1}} \sum_{k=0}^{n} \frac{(\log 2)^k}{k!} m_k \sim \lambda \frac{n!}{(\log 2)^{n+1}}
\end{aligned}$$

with λ given by (40). □

12 Geometric means for the Stern sequence

It is an easy exercise to compute the arithmetic mean $\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}} s(j)$.

The following result gives asymptotics for the geometric mean:

Theorem 12.1. *There exists a real constant β such that*

$$\lim_{n\to\infty} e^{-(n\alpha+\beta)2^n} \prod_{j=2^n}^{2^{n+1}} s(j) = 1$$

where

$$\begin{aligned}\alpha &= \log 2 - \sum_{j=1}^{\infty} \frac{m_j}{j2^j} \\ &\sim .39621256429774455909560575764994569944470639102190\ .\end{aligned}$$

Remark 12.2. *The constant α is involved in the Hausdorff dimension of growth points for $?(x)$, see Kinney or Alkauskas. See also Conjecture 8.1 for a conjectural manifestation of α.*

I am not aware of the existence of an efficient method for computing the value of $\beta \sim -.0851895$ with high precision.

Lemma 12.3. *Given an increasing function $\varphi : [0,1] \longrightarrow [0,1]$ and a strictly positive natural integer N we have*

$$\left| \int_0^1 \varphi(t)dt - \frac{1}{N}\sum_{k=0}^{N-1} \frac{\varphi\left(\frac{k}{N}\right) + \varphi\left(\frac{k+1}{N}\right)}{2} \right| \leq \frac{1}{2N}\ .$$

Proof. The error of the trapezoidal rule

$$\int_a^b \varphi(t)dt \sim (b-a)\frac{\varphi(a)+\varphi(b)}{2}$$

is bounded by $\left|\frac{(b-a)(\varphi(b)-\varphi(a))}{2}\right|$ if φ is monotonous. □

Proof of Theorem 12.1. We consider

$$\begin{aligned} S(n) &= \frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}} \log(s(k)) \\ &= \frac{1}{2^n}\sum_{k=1}^{2^n} \frac{\log(s(2^n+k-1)) + \log(s(2^n+k))}{2} \end{aligned}$$

where the second identity follows from the evaluations $s(2^n) = s(2^{n+1}) = 1$. Using (12) we get

$$\begin{aligned} S(n+1) &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} \frac{\log(s(2^{n+1}+k-1)) + \log(s(2^{n+1}+k))}{2} \\ &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^n} \frac{\log(s(2^{n+1}+k-1)) + \log(s(2^{n+1}+k))}{2} \\ &\quad + \frac{1}{2^{n+1}} \sum_{k=1+2^n}^{2^{n+1}} \frac{\log(s(2^{n+1}+k-1)) + \log(s(2^{n+1}+k))}{2} \\ &= \frac{2}{2^{n+1}} \sum_{k=1}^{2^n} \frac{\log(s(3\cdot 2^n+k-1)) + \log(s(3\cdot 2^n+k))}{2} \end{aligned}$$

and (13) yields

$$\begin{aligned} S(n+1) &= \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{\log(2s(2^n+k-1) - s(k-1)) + \log(2s(2^n+k) - s(k))}{2} \\ &= \log(2) + \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{\log(s(2^n+k-1) - \frac{1}{2}s(k-1)) + \log(s(2^n+k) - \frac{1}{2}s(k))}{2} \end{aligned}$$

Using $\frac{d^k}{dx^k} \log(u - vx) = -(k-1)! \frac{v^k}{(u-vx)^k}$ for $k \geq 1$, we get

$$S(n+1) = \log(2) + S(n) - \sum_{j=1}^{\infty} \frac{1}{j2^j} \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{1}{2} \left(\left(\frac{s(k-1)}{s(2^n+k-1)} \right)^j + \left(\frac{s(k)}{s(2^n+k)} \right)^j \right)$$

which implies

$$\lim_{n\to\infty} (S(n+1) - S(n)) = \log 2 - \sum_{j=1}^{\infty} \frac{m_j}{j2^j} = \alpha$$

by Proposition 2.1.

Lemma 12.3 shows

$$\begin{aligned} &\left| \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j2^j} \left(m_j - \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{1}{2} \left(\left(\frac{s(k-1)}{s(2^n+k-1)} \right)^j + \left(\frac{s(k)}{s(2^n+k)} \right)^j \right) \right) \right| \\ &\leq \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{2^n} \leq \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{2^{n+j}} = 2 \; . \end{aligned}$$

This proves the existence of β. $\square$

References

[1] G. Alkauskas, *Explicit Series for the dyadic Period Function and Moments*, Math. Comp. **79**, No. 269 (2010), 383–418.

[2] G. Alkauskas, *The Moments of Minkowski Question Mark Function: The Dyadic Period Function*, Glasgow Math. J. **52** (2010), 41–64.

[3] G. Alkauskas, *An asymptotic Formula for the Moments of Minkowski Question Mark Function in the Interval* $[0, 1]$, Lith. Math. J. **48**, No. 4 (2008), 357–367.

[4] A. Denjoy, *Sur une fonction réelle de Minkowski*, J. Math. Pures Appl. **17** (1938), 105–151.

[5] On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.

Roland BACHER,
Univ. Grenoble Alpes, Institut Fourier,
F-38000 Grenoble, France.

e-mail: Roland.Bacher@ujf-grenoble.fr