

NEW ESTIMATES FOR SOME FUNCTIONS DEFINED OVER PRIMES

CHRISTIAN AXLER

ABSTRACT. In this paper we first establish new explicit estimates for Chebyshev's ϑ -function. Applying these new estimates, we derive new upper and lower bounds for some functions defined over the prime numbers, for instance the prime counting function $\pi(x)$, which improve the currently best ones. Furthermore, we use the obtained estimates for the prime counting function to give two new results concerning the existence of prime numbers in short intervals.

1. INTRODUCTION

Let $\pi(x)$ denotes the number of primes not exceeding x . Since there are infinitely many primes, we have $\pi(x) \rightarrow \infty$ for $x \rightarrow \infty$. In 1793, Gauß [18] stated a conjecture concerning the asymptotic behaviour for the prime counting function $\pi(x)$, namely

$$(1.1) \quad \pi(x) \sim \operatorname{li}(x) \quad (x \rightarrow \infty),$$

where the *logarithmic integral* $\operatorname{li}(x)$ defined for every real $x \geq 0$ as

$$(1.2) \quad \operatorname{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right\}.$$

The asymptotic formula (1.1) was proved by Hadamard [20] and, independently, by de la Vallée-Poussin [37] in 1896, and is known as the *Prime Number Theorem*. In his later paper [38], where he proved the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\operatorname{Re}(s) = 1$, de la Vallée-Poussin also estimated the error term in the Prime Number Theorem by showing

$$(1.3) \quad \pi(x) = \operatorname{li}(x) + O(x \exp(-c_0 \sqrt{\log x})),$$

where c_0 is a positive absolute constant. The work of Korobov [23] and Vinogradov [39] implies the currently best error term, namely that there is a positive absolute constant c_1 so that

$$(1.4) \quad \pi(x) = \operatorname{li}(x) + O\left(x \exp\left(-c_1 (\log x)^{3/5} (\log \log x)^{-1/5}\right)\right).$$

The computation of the prime counting function $\pi(x)$ for large values of x is a difficult problem (the latest record was $\pi(10^{25}) = 176\,846\,309\,399\,143\,769\,411\,680$ and is due to Büthe, Franke and Kleinjung [4]). Since the asymptotic formula (1.4) is not very meaningful with regard to the computation of $\pi(x)$ for some fixed x , we are interested to find new explicit estimates for the prime counting function. In order to do this, we first need to establish the following result on Chebyshev's ϑ -function

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

which improves several known estimates for this function.

Theorem 1.1 (See Theorem 2.4). *For every $x \geq 19\,035\,709\,163$, we have*

$$\vartheta(x) > x - \frac{0.15x}{\log^3 x},$$

and for every $x > 1$, we have

$$\vartheta(x) < x + \frac{0.15x}{\log^3 x}.$$

Date: May 18, 2017.

2010 Mathematics Subject Classification. Primary 11N05; Secondary 11A41.

Key words and phrases. Chebyshev's ϑ -function, prime counting function, primes in short intervals.

In 2000, Panaitopol [28, p. 55] gave another asymptotic formula for the prime counting function by showing that for each positive integer m , we have

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \dots - \frac{k_m}{\log^m x}} + O\left(\frac{x}{\log^{m+2} x}\right),$$

where the positive integers k_1, \dots, k_m are defined by the recurrence formula

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \dots + (m-1)!k_1 = m \cdot m!.$$

In Section 3, we use the inequalities obtained in Theorem 1.1 to find among others the following explicit estimates for the prime counting function, which improve the current best estimates for $\pi(x)$.

Theorem 1.2 (See Theorem 3.2). *For every $x \geq 49$, we have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.15}{\log^2 x} - \frac{12.85}{\log^3 x} - \frac{71.3}{\log^4 x} - \frac{463.2275}{\log^5 x} - \frac{4585}{\log^6 x}}.$$

Theorem 1.3 (See Theorem 3.8). *For every $x \geq 19\,033\,744\,403$, we have*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.85}{\log^2 x} - \frac{13.15}{\log^3 x} - \frac{70.7}{\log^4 x} - \frac{458.7275}{\log^5 x} - \frac{3428.7225}{\log^6 x}}.$$

In Section 4, we apply these new estimates for the prime counting function to derive two new result concerning the existence of prime numbers in short intervals. The origin of this problem is Bertrand's postulate, which states that for each positive integer n there is a prime number p with $n < p \leq 2n$. We give the following both refinements.

Theorem 1.4 (See Theorem 4.1). *For every $x \geq 6\,034\,256$ there is a prime number p , such that*

$$x < p \leq x \left(1 + \frac{0.087}{\log^3 x}\right),$$

and for every $x > 1$ there is a prime number p , such that

$$x < p \leq x \left(1 + \frac{198.2}{\log^4 x}\right).$$

In Section 5 and Section 6, we use Theorem 1.1 to derived some upper and lower bounds for the prime functions

$$\sum_{p \leq x} \frac{1}{p}, \quad \sum_{p \leq x} \frac{\log p}{p}, \quad \prod_{p \leq x} \left(1 - \frac{1}{p}\right),$$

which improves Dusart's [13] estimates for these functions.

2. NEW ESTIMATES FOR CHEBYSHEV'S ϑ -FUNCTION

The prime counting function $\pi(x)$ and Chebyshev's ϑ -function are connected by the identities

$$(2.1) \quad \pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt$$

and

$$(2.2) \quad \vartheta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt,$$

which hold for every $x \geq 2$ (see, for instance, Apostol [1, Theorem 4.3]). In order to find new estimates for the prime counting function, we first derive some new upper and lower bounds for Chebyshev's ϑ -function and then use (2.1). Using (2.2), it is easy to see that the Prime Number Theorem (1.1) is equivalent to

$$(2.3) \quad \vartheta(x) \sim x \quad (x \rightarrow \infty).$$

By proving the existence of a zero-free region for the Riemann zeta-function, de la Vallée-Poussin [38] was able to bound the error term in (2.3) by

$$(2.4) \quad \vartheta(x) = x + O(x \exp(-c_2 \sqrt{\log x})),$$

where c_2 is a positive absolute constant. The asymptotic formula (2.4) implies that for every positive integer k there is a positive real number η_k and a real $x_0(k) \geq 2$ so that

$$(2.5) \quad |\vartheta(x) - x| < \frac{\eta_k x}{\log^k x}$$

for every $x \geq x_0(k)$. The work of Korobov [23] and Vinogradov [39] imply the best known error term in (2.3) namely

$$\vartheta(x) = x + O\left(x \exp\left(-c_3 \log^{3/5} x (\log \log x)^{-1/5}\right)\right),$$

where c_3 is an absolute positive constant. Under the assumption that the Riemann hypothesis is true, von Koch [22] deduced the improved asymptotic formula

$$\vartheta(x) = x + O(\sqrt{x} \log^2 x).$$

A precise version of this was given by Schoenfeld [32, Theorem 10]. He found under the assumption that the Riemann hypothesis is true that

$$(2.6) \quad |\vartheta(x) - x| < \frac{1}{8\pi} \sqrt{x} \log^2 x$$

for every $x \geq 599$. In 2014, Büthe [5, p. 2495] found that the inequality (2.6) holds unconditionally for every x such that $599 \leq x \leq 1.4 \cdot 10^{25}$ by using the following lemma.

Lemma 2.1 (Büthe, [5]). *Let $T > 0$ be such that the Riemann hypothesis holds for every $0 < \text{Im}(\rho) \leq T$. Then, under the condition $4.92\sqrt{x/\log x} \leq T$, the following estimates hold:*

- (a) $|\vartheta(x) - x| < \frac{1}{8\pi} \sqrt{x} \log^2 x$ for every $x \geq 599$,
- (b) $|\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x$ for every $x \geq 2657$.

In the following proposition we also make use of Lemma 2.1 to increase the number $1.4 \cdot 10^{25}$ in Büthe's result on (2.6).

Proposition 2.2. *The inequality (2.6) holds unconditionally for every x such that $599 \leq x \leq 5.5 \cdot 10^{25}$.*

Proof. Let $N(T)$ be the number of complex zeros ρ of the Riemann zeta function $\zeta(s)$ satisfying $0 < \text{Im}(\rho) \leq T$. Trudgian [35, Corollary 1] found that $N(T)$ is bounded by

$$(2.7) \quad N(T) \leq \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + 0.112 \log T + 0.278 \log \log T + 2.51 + \frac{0.2}{T}$$

for every $T \geq e$. Setting $T_0 = 4768099715087$, we use (2.7) to get

$$(2.8) \quad N(T_0) \leq 2 \cdot 10^{13};$$

i.e. there are at most $2 \cdot 10^{13}$ complex zeros ρ of the Riemann zeta function $\zeta(s)$ satisfying $0 < \text{Im}(\rho) \leq T_0$. By [19], the first $2 \cdot 10^{13}$ zeros of the Riemann zeta function satisfy the Riemann hypothesis. Together with (2.8), we obtain that the Riemann hypothesis holds for every complex zeros ρ such that $0 < \text{Im}(\rho) \leq T_0$. Now, we set $x_0 = 5.5 \cdot 10^{25}$ to get $4.92\sqrt{x_0/\log x_0} \leq T_0$ and it remains to apply Lemma 2.1. \square

In the direction of (2.5), Dusart found in [11, Theorem 5.2] and [13, Theorem 4.2] the following explicit estimates for the distance between x and $\vartheta(x)$.

Lemma 2.3 (Dusart, [11], [13]). *We have*

$$|\vartheta(x) - x| < \frac{\eta_k x}{\log^k x}$$

for every $x \geq x_0(k)$ with

k	1	2	2	3	3	4
η_k	0.001	0.2	0.01	1	0.5	151.3
$x_0(k)$	908 994 923	3 594 641	7 713 133 853	89 967 803	767 135 587	2

In the following theorem, we find the corresponding value x_0 for the case $k = 3$ and $\eta_3 = 0.15$. In the proof, we use explicit estimates for Chebyshev's ψ -function, which is defined by

$$\psi(x) = \sum_{p^m \leq x} \log p.$$

Theorem 2.4. *For every $x \geq 19035709163 = p_{841508302}$, we have*

$$(2.9) \quad \vartheta(x) > x - \frac{0.15x}{\log^3 x},$$

and for every $x > 1$, we have

$$(2.10) \quad \vartheta(x) < x + \frac{0.15x}{\log^3 x}.$$

Proof. First, we check that the inequality

$$(2.11) \quad |\vartheta(x) - x| < \frac{0.15x}{\log^3 x}$$

holds for every $x \geq e^{32}$. By Dusart [12, Corollary 1.2], we have

$$(2.12) \quad |\vartheta(x) - x| < \frac{\sqrt{8}}{\sqrt{\pi\sqrt{R}}} x (\log x)^{1/4} e^{-\sqrt{(\log x)/R}}$$

for every $x \geq 3$, where $R = 5.69693$. Since $g(x) = (\log x)^{13/4} e^{-\sqrt{(\log x)/R}}$ is a monotonic decreasing function for every $x \geq e^{169R/4}$, we get that

$$|\vartheta(x) - x| < \frac{\sqrt{8}}{\sqrt{\pi\sqrt{R}}} g(e^{5000}) \frac{x}{\log^3 x} < \frac{0.148x}{\log^3 x}$$

for every $x \geq e^{5000}$. Using [13, Corollary 4.5], we get

$$(2.13) \quad |\vartheta(x) - x| < \left(\frac{(1 + 1.47 \cdot 10^{-7})b_i^3}{\sqrt{e^{b_i}}} + \frac{1.78b_i^3}{\sqrt[3]{e^{2b_i}}} + \varepsilon_i b_{i+1}^3 \right) \frac{x}{\log^3 x}$$

for $e^{b_i} \leq x \leq e^{b_{i+1}}$, where b_i and the corresponding ε_i are given in Table 5.2 of [13]. Substituting $b_{31} = 1500, b_{32} = 2000, b_{33} = 2500, b_{34} = 3000, b_{35} = 3500, b_{36} = 4000, b_{37} = 4500$ and the corresponding values of ε_i in (2.13), we obtain that the inequality (2.11) also holds for every $e^{1500} \leq x \leq e^{5000}$. From Tables 6.4 and 6.5 of [11], it follows that the inequality (2.11) holds for every x such that $e^{32} \leq x < e^{1500}$.

So, to prove that (2.9) holds for every $x \geq 19\,035\,709\,163$, it remains to deal with the case where $19\,035\,709\,163 \leq x \leq e^{32}$. By Büthe [3, Theorem 2], we have $\vartheta(t) \geq t - 1.95\sqrt{t}$ for every t such that $1\,423 \leq t \leq 10^{19}$. Since $0.15\sqrt{t} > 1.95\log^3 t$ for every $t \geq 34\,485\,879\,392$, we get that the inequality (2.9) holds also for every x such that $34\,485\,879\,392 \leq x \leq e^{32}$. In addition, Büthe [3, p. 13] found that $-0.8 \leq (t - \psi(t))/\sqrt{t} \leq 0.81$ for every t such that $100 \leq t \leq 5 \cdot 10^{10}$. Now, we use Lemma 1 of [3] to get

$$\vartheta(t) \geq t - 1.81\sqrt{t} - 0.8t^{1/4} - 1.03883(t^{1/3} + t^{1/5} + 2t^{1/13} \log t)$$

for every t such that $10\,000 \leq t \leq 5 \cdot 10^{10}$. Since $t^{1/5} + 2t^{1/13} \log t \leq t^{1/3}$ for every $t \geq 783\,674$, we get

$$(2.14) \quad \vartheta(t) \geq t - 1.81\sqrt{t} - 0.8t^{1/4} - 2 \cdot 1.03883t^{1/3}$$

for every t such that $783\,674 \leq t \leq 5 \cdot 10^{10}$. Now, we notice that $0.15t/\log^3 t \geq 1.81\sqrt{t} + 0.8t^{1/4} + 2 \cdot 1.03883t^{1/3}$ for every $t \geq 29\,946\,085\,320$. Hence, by (2.14), the inequality (2.9) is fulfilled for every x such that $29\,946\,085\,320 \leq x \leq 34\,485\,879\,392$ as well. To prove that the inequality (2.9) is also valid for every x such that $19\,035\,709\,163 \leq x < 29\,946\,085\,320$, we set $f(x) = x(1 - 0.15/\log^3 x)$. Since f is a strictly increasing function on $(1, \infty)$, it suffices to check with a computer that $\vartheta(p_n) > f(p_{n+1})$ for every positive integer n such that $\pi(19\,035\,709\,163) \leq n \leq \pi(29\,946\,085\,320)$.

Now, we show that (2.10) for every $x > 1$. Using the inequality (2.11), it suffices to prove that (2.10) holds for every x such that $1 < x < e^{32}$. For this, we use another result of Büthe [3, Theorem 2]. He found that $\vartheta(x) < x$ for every x such that $1 \leq x \leq 10^{19}$, which clearly implies that the inequality (2.10) holds for every x such that $1 < x < e^{32}$. \square

Remark. In [2, Proposition 3.2] it is shown that $\vartheta(x) > x - 0.35x/\log^3 x$ for every $x \geq e^{30}$. Using Theorem 2.4, we get that this inequality also holds for every x such that $19\,035\,709\,163 \leq x \leq e^{30}$. A computer check gives that the inequality

$$\vartheta(x) > x - \frac{0.35x}{\log^3 x}$$

holds for every $x \geq 1\,332\,492\,593$.

In the next proposition, we give a slightly improvement of Lemma 2.3 for the case $k = 4$, which improves the inequality (2.9) for every $x \geq e^{666+2/3}$.

Proposition 2.5. *For every $x \geq 70\,111$, we have*

$$(2.15) \quad |\vartheta(x) - x| < \frac{100x}{\log^4 x}.$$

Proof. Let $R = 5.69693$. We use (2.12) to get that $|\vartheta(x) - x| < 100x/\log^4 x$ for every $x \geq e^{6000}$. Similarly to the proof of Theorem 2.4, we check with Table 5.2 of [13] that the inequality (2.15) holds for every x such that $e^{1000} \leq x \leq e^{6000}$ as well. From Tables 6.4 and 6.5 of [11], it follows that the required inequality holds for every x such that $e^{23} \leq x < e^{1000}$. Finally, we obtain that $1/\log^3 t < 100/\log^4 t$ for every t such that $1 < t \leq e^{100}$ and so, Lemma 2.3 implies the validity of the required inequality for every x such that $89\,967\,803 \leq x \leq e^{23}$. To prove that the inequality (2.15) is also fulfilled for every x such that $70\,111 \leq x < 89\,967\,803$, we set $f(x) = x(1 - 100/\log^4 x)$. Since f is a strictly increasing function for every $x > 1$, it is enough to check with a computer that $\vartheta(p_n) > f(p_{n+1})$ for every positive integer n such that $\pi(70\,111) \leq n \leq \pi(89\,967\,803)$. \square

3. NEW ESTIMATES FOR THE PRIME COUNTING FUNCTION

Under the assumption that the Riemann hypothesis is true, von Koch [22] deduced a remarkable refinement of error term in the Prime Theorem, which is given by

$$\pi(x) = \text{li}(x) + O(\sqrt{x} \log x).$$

A precise version of Koch's result is due to Schoenfeld [32, Corollary 1]. He found under the assumption that the Riemann hypothesis is true that the inequality

$$(3.1) \quad |\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x$$

holds for every $x \geq 2\,657$. In 2014, Büthe [5, p. 2495] showed that the inequality (3.1) holds unconditionally for every x such that $2\,657 \leq x \leq 1.4 \cdot 10^{25}$. The following proposition gives a slightly improvement of Büthe's result.

Proposition 3.1. *The inequality (3.1) holds unconditionally for every x such that $2\,657 \leq x \leq 5.5 \cdot 10^{25}$.*

Proof. Similar to the proof of Proposition 2.2. \square

Now, let k be a positive integer and let $\eta_k, x_1(k)$ be positive real numbers with $x_1(k) \geq 2$ so that

$$|\vartheta(x) - x| < \frac{\eta_k x}{\log^k x}$$

for every $x \geq x_1(k)$. Together with (2.1), we get

$$(3.2) \quad J_{k, -\eta_k, x_1(k)}(x) \leq \pi(x) \leq J_{k, \eta_k, x_1(k)}(x)$$

for every $x \geq x_1(k)$, where

$$J_{k, \eta_k, x_1(k)}(x) = \pi(x_1(k)) - \frac{\vartheta(x_1(k))}{\log x_1(k)} + \frac{x}{\log x} + \frac{\eta_k x}{\log^{k+1} x} + \int_{x_1(k)}^x \left(\frac{1}{\log^2 t} + \frac{\eta_k}{\log^{k+2} t} \right) dt.$$

The function $J_{k, \eta_k, x_1(k)}$ was already introduced by Rosser and Schoenfeld [31, p.81] (for the case $k = 1$) and Dusart [11, p. 9]. In this section, we use (3.2) and the estimates for Chebyshev's ϑ -function obtained in the previous section to establish new explicit estimates for the prime counting function $\pi(x)$.

3.1. New upper bounds for the prime counting function. First we recall that Panaitopol [28, p. 55] gave the asymptotic formula

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \dots - \frac{k_m}{\log^m x}} + O\left(\frac{x}{\log^{m+2} x}\right),$$

where m is a positive integer and k_1, \dots, k_m are defined by the recurrence formula

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \dots + (m-1)!k_1 = m \cdot m!.$$

For instance, we have

$$\pi(x) = \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{13}{\log^3 x} - \frac{71}{\log^4 x} - \frac{461}{\log^5 x} - \frac{3441}{\log^6 x}} + O\left(\frac{x}{\log^8 x}\right).$$

In this direction, Theorem 2.4 implies the following upper bound for the prime counting function.

Theorem 3.2. *For every $x \geq 49$, we have*

$$(3.3) \quad \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.15}{\log^2 x} - \frac{12.85}{\log^3 x} - \frac{71.3}{\log^4 x} - \frac{463.2275}{\log^5 x} - \frac{4585}{\log^6 x}}.$$

Proof. Let $x_1 = 10^{15}$, let $f(x)$ be given by the right-hand side of (3.3), and let $r(x)$ be the denominator of $f(x)$. By (3.2) and Theorem 2.4, we get $\pi(x) \leq J_{3,0.15,x_1}(x)$ for every $x \geq x_1$. In the first step of the proof, we compare $f(x)$ with $J_{3,0.15,x_1}(x)$. In order to prove that the function $g(x) = f(x) - J_{3,0.15,x_1}(x)$ is positive for every $x \geq x_1$, we need to show that $g(x_1) > 0$ and that the derivative of g is positive for every $x \geq x_1$. By Dusart [11, Table 6.2], we have $\vartheta(x_1) \geq 999\,999\,965\,752\,660$. Further, $\pi(x_1) = 29\,844\,570\,422\,669$ and so we compute that $g(x_1) \geq 3 \cdot 10^6$. To show that the derivative of g is positive for every $x \geq x_1$, we set

$$\begin{aligned} h_1(y) = & 1119.6775y^{11} - 38212.4575y^{10} - 13858.278375y^9 - 45007.842875y^8 - 189106.352125y^7 \\ & - 865668.98286875y^6 - 4248412.96105y^5 - 21029165.2496875y^4 - 47509.2738384375y^3 \\ & - 246389.1037096875y^2 - 1241825.47125y \end{aligned}$$

and compute that $h_1(y) > 0$ for every $y \geq 34.525$. Therefore, we get that the inequality $g'(x) = (h_1(\log x) + 9460001.25)/(r^2(x) \log^{17} x) > 0$ holds for every $x \geq x_1$. So, $f(x) - J_{3,0.15,x_1}(x) = g(x) > 0$ for every $x \geq x_1$ and, by (3.2), we conclude that the inequality (3.3) holds for every $x \geq x_1$.

In the second step, we check (3.3) for every x such that $1\,095\,698 \leq x \leq 10^{15}$ by comparing $f(x)$ with the logarithmic integral $\text{li}(x)$. For this, we set

$$\begin{aligned} h_2(y) = & 0.15y^{11} - 0.75y^{10} + 0.75y^9 - 0.195y^8 + 1118.8525y^7 - 38220.7675y^6 - 13920.74325y^5 \\ & - 45874.13675y^4 - 183890.7415y^3 - 868400.71675625y^2 - 4247796.175y - 21022225. \end{aligned}$$

Then, it is easy to see that $h_2(y) \geq 0$ for every $y \geq 12.2714$. Hence, for every $x \geq 213\,502$, we have $f'(x) - \text{li}'(x) = h_2(\log x)/(r^2(x) \log^{13} x) \geq 0$. In addition, we have $f(1\,095\,698) - \text{li}(1\,095\,698) > 0$. Hence $f(x) > \text{li}(x)$ for every $x \geq 1\,095\,698$. Now we use a result of Büthe [3, Theorem 2], namely that

$$(3.4) \quad \pi(x) < \text{li}(x)$$

for every x such that $2 \leq x \leq 10^{19}$, to obtain that the required inequality holds for x such that $1\,095\,698 \leq x \leq 10^{15}$ as well. To deal with the case where $101 \leq x \leq 1\,095\,698$, we notice that $f(x)$ is strictly increasing for every x such that $101 \leq x \leq 1\,095\,698$. So we check with a computer that $f(p_n) > \pi(p_n)$ for every positive integer n such that $\pi(101) \leq n \leq \pi(1\,095\,698) + 1$. A computer check for smaller values of x completes the proof. \square

We obtain the following weaker but more compact upper bounds.

Corollary 3.3. *We have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{a_1}{\log x} - \frac{a_2}{\log^2 x} - \frac{a_3}{\log^3 x} - \frac{a_4}{\log^4 x} - \frac{a_5}{\log^5 x}}$$

for every $x \geq x_0$, where

a_2	1	1	1	1	1.15
a_2	3.15	3.15	3.15	3.69	0
a_3	12.85	12.85	14.21	0	0
a_4	71.3	80.43	0	0	0
a_5	540.59	0	0	0	0
x_0	32	22	14	10 031 975 087	38 284 442 297

Proof. We only show that the inequality

$$(3.5) \quad \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.15}{\log^2 x} - \frac{12.85}{\log^3 x} - \frac{71.3}{\log^4 x} - \frac{540.59}{\log^5 x}}$$

holds for every $x \geq 32$. The proofs of the remaining inequalities are similar to the proof of (3.5) and we leave the details to the reader. For every $x \geq 5.5 \cdot 10^{25}$, Theorem 3.2 implies the validity of (3.5). Denoting the right-hand side of (3.5) by $f(x)$, we set $g(x) = f(x) - \text{li}(x) - \sqrt{x} \log x / (8\pi)$. We compute that $g(10^{14}) > 10^6$ and $g'(x) \geq 0$ for every $x \geq 10^{14}$. Hence $f(x) \geq \text{li}(x) + \sqrt{x} \log x / (8\pi)$ for every $x \geq 10^{14}$. Now we apply Proposition 3.1 to get that the inequality (3.5) also holds for every x such that $10^{14} \leq x \leq 5.5 \cdot 10^{25}$ as well. A comparison with $\text{li}(x)$ shows that $f(x) > \text{li}(x)$ for every $x \geq 4\,560\,187$. From (3.4) follows that the inequality (3.5) also holds for every x such that $4\,560\,187 \leq x \leq 10^{14}$. To verify that $f(x) > \pi(x)$ holds for every x such that $67 \leq x \leq 4\,560\,187$, it suffices to check that $f(p_n) > \pi(p_n)$ for every positive integer n such that $\pi(67) \leq n \leq \pi(4\,560\,187) + 1$, since $f(x)$ is strictly increasing for every $x \geq 67$. We conclude by direct computation. \square

In [2, Theorem 1.3], the present author purports that the inequality

$$(3.6) \quad \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.35}{\log^2 x} - \frac{12.65}{\log^3 x} - \frac{71.7}{\log^4 x} - \frac{466.1275}{\log^5 x} - \frac{3489.8225}{\log^6 x}}$$

holds for every $x \geq e^{3.804}$. But the proof of this inequality in its present form is not correct. There is a mistake in the first part of the proof, where it is claimed that the inequality (3.6) holds for every $x \geq 10^{14}$. Fortunately, this incorrectness will be fixed by Theorem 3.2.

Corollary 3.4. *For every $x \geq e^{3.804}$, we have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.35}{\log^2 x} - \frac{12.65}{\log^3 x} - \frac{71.7}{\log^4 x} - \frac{466.1275}{\log^5 x} - \frac{3489.8225}{\log^6 x}}.$$

Proof. The proof in [2] that the inequality (3.6) holds for every x such that $e^{3.804} \leq x \leq 10^{14}$ is still correct and it suffices to consider the remaining case $x \geq 10^{14}$. In this case the required inequality follows directly from Theorem 3.2. \square

Using Proposition 2.5, we get the following upper bound for the prime counting function, which improves the inequality (3.3) for every sufficiently large values of x .

Proposition 3.5. *For every $x \geq 41$, we have*

$$(3.7) \quad \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{113}{\log^3 x}}.$$

Proof. The proof is similar to the proof of Theorem 3.2 and we leave the details to the reader. We denote the right-hand side of (3.7) by $f(x)$ and let $x_1 = 10^{15}$. Comparing $f(x)$ with $J_{4,100,x_1}(x)$, we get, by using $f(x) > J_{4,100,x_1}(x)$ holds for every $x \geq 10^{15}$. Then, by (3.2) and Proposition 2.5, that $f(x) > \pi(x)$ for every $x \geq 10^{15}$. Next, we compare $f(x)$ with $\text{li}(x)$ and obtain that the desired inequality holds for every x such that $e^7 \leq x \leq 10^{15}$ as well. A direct computation for smaller values of x completes the proof. \square

Integration of parts in (1.3) implies that for every positive integer m , we have

$$(3.8) \quad \pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \dots + \frac{(m-1)!x}{\log^m x} + O\left(\frac{x}{\log^{m+1} x}\right).$$

In this direction, we get the following upper bound for the prime counting function.

Proposition 3.6. *For every $x > 1$, we have*

$$(3.9) \quad \pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6.15x}{\log^4 x} + \frac{24.15x}{\log^5 x} + \frac{120.75x}{\log^6 x} + \frac{724.5x}{\log^7 x} + \frac{6601x}{\log^8 x}.$$

Proof. We set $x_1 = 10^{15}$. Further, let $f(x)$ be the right-hand side of (3.9). A comparison with $J_{3,0.15,x_1}(x)$ shows that $f(x) > J_{3,0.15,x_1}(x)$ for every $x \geq 10^{15}$. By (3.2) and Theorem 2.4, we get that $f(x) > \pi(x)$ for every $x \geq x_1$. Next, we compare $f(x)$ with $\text{li}(x)$ and get that $f(x) > \text{li}(x)$ for every $x \geq 1\,509\,412$. Together with (3.4), we obtain that $f(x) > \pi(x)$ for every x such that $1\,509\,412 \leq x \leq 10^{15}$ as well. It remains to deal with the case where $1 < x \leq 1\,509\,412$. Since $f(x)$ is a strictly increasing function for every $x \geq 47$, it suffices to check that $f(p_n) > \pi(p_n)$ for every positive integer n such that $\pi(47) \leq n \leq \pi(1\,509\,412) + 1$. For smaller values of x , we conclude by direct computation. \square

Remark. Using Proposition 2.5, instead of Theorem 2.4, in the proof of Proposition 3.6, we get similarly that the inequality

$$\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{133x}{\log^5 x}$$

holds for every $x > 1$.

We get the following weaker but more compact upper bound for the prime counting function.

Corollary 3.7. *For every $x \geq 27\,777\,762\,891$, we have*

$$\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2.3x}{\log^3 x}.$$

Proof. From Proposition 3.6 follows that the required inequality holds for every $x \geq 5.1 \cdot 10^{10}$. Denoting the right-hand side of the desired inequality by $f(x)$, we get that $f(x) > \text{li}(x)$ for every $x \geq 33\,272\,003\,003$. Together with (3.4), we conclude the proof for every $x \geq 33\,272\,003\,003$. For every positive integer n such that $\pi(27\,777\,762\,917) \leq n \leq \pi(33\,272\,003\,003)$, we check that $f(p_n) \geq \pi(p_n)$. Since f is an increasing function for every $x \geq 7$, we get that $f(x) > \pi(x)$ for every x such that $27\,777\,762\,917 \leq x < 33\,272\,003\,003$. A direct computer check for small values of x completes the proof. \square

3.2. New lower bounds for the prime counting function. In this subsection, we give new lower bounds for the prime counting function, which improve the currently best known lower bound given in [2, Theorem 1.4], namely

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.65}{\log^2 x} - \frac{13.35}{\log^3 x} - \frac{70.3}{\log^4 x} - \frac{455.6275}{\log^5 x} - \frac{3404.4225}{\log^6 x}}$$

for every $x \geq 1\,332\,479\,531$.

Theorem 3.8. *For every $x \geq 19\,033\,744\,403$, we have*

$$(3.10) \quad \pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.85}{\log^2 x} - \frac{13.15}{\log^3 x} - \frac{70.7}{\log^4 x} - \frac{458.7275}{\log^5 x} - \frac{3428.7225}{\log^6 x}}.$$

Proof. Let $x_1 = 5 \cdot 10^9$. Further, let $f(x)$ be the right-hand side of (3.10) and let $r(x)$ be the denominator of $f(x)$. To prove that the function $g(x) = J_{3,-0.15,x_1}(x) - f(x)$ is positive for every $x \geq x_1$, we need to show that $g(x_1) > 0$ and that the derivative of g is positive for every $x \geq x_1$. By Dusart [11, Table 6.2], we have $\vartheta(x_1) \leq 4\,999\,906\,576$. Combined with $\pi(x_1) = 234\,954\,223$, we compute that $g(x_1) > 18.955$. To show that the derivative of g is positive for every $x \geq x_1$, we set

$$h(y) = 28\,930y^{10} + 11\,393y^9 + 37\,131y^8 + 151\,211y^7 + 697\,310y^6 + 3\,145\,306y^5 + 11\,749\,355y^4 \\ - 34\,521y^3 - 158\,992y^2 - 347\,857y + 5\,290\,262.$$

Clearly, we have $h(y) > 0$ for every $y \geq \log(x_1)$. Hence, $g'(x)r^2(x)\log^{19}x \geq h_1(\log x) \geq 0$ for every $x \geq x_1$. So, $J_{3,-0.15,x_1}(x) - f(x) = g(x) > 0$ for every $x \geq x_1$. Using (3.2) and Theorem 2.4, we get that required inequality for every $x \geq 19\,035\,709\,163$. To deal with the remaining case where $19\,033\,744\,403 \leq x < 19\,035\,709\,163$, we note that $f(x)$ is increasing for every $x \geq 91$. So we check with a computer that $\pi(p_n) > f(p_{n+1})$ for every positive integer n such that $\pi(19\,033\,744\,403) \leq n \leq \pi(19\,035\,709\,163)$. \square

In the next corollary, we establish some weaker lower bounds for the prime counting function.

Corollary 3.9. *We have*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{a_1}{\log x} - \frac{a_2}{\log^2 x} - \frac{a_3}{\log^3 x} - \frac{a_4}{\log^4 x} - \frac{a_5}{\log^5 x}}$$

for every $x \geq x_0$, where

a_1	1	1	1	1	1
a_2	2.85	2.85	2.85	2.85	0
a_3	13.15	13.15	13.15	0	0
a_4	70.7	70.7	0	0	0
a_5	458.7275	0	0	0	0
x_0	11 532 441 449	7 822 207 951	1 331 532 233	38 099 531	468 049

Proof. By comparing each right-hand side with the right-hand side of (3.10), we see that each inequality holds for every $x \geq 19\,033\,744\,403$. For smaller values of x we use computer. \square

Now, we apply Proposition 2.5 to obtain the following result, which refines Theorem 3.8 for all sufficiently large values of x .

Proposition 3.10. *For every $x \geq 19\,423$, we have*

$$(3.11) \quad \pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} + \frac{87}{\log^3 x}}.$$

Proof. Let $x_1 = 10^6$ and denote the right-hand side of (3.11) by $f(x)$. A comparison with $J_{4,-100,x_1}(x)$ gives that $J_{4,-100,x_1}(x) > f(x)$ for every $x \geq 10^6$. Now we use (3.2) and Proposition 2.5 to get that $\pi(x) > f(x)$ for every $x \geq 10^6$. To prove that the inequality (3.11) is also valid for every x such that

$19423 \leq x < 10^6$, it suffices to check with a computer that $\pi(p_n) > f(p_{n+1})$ for every positive integer n such that $\pi(19423) \leq n \leq \pi(10^6)$, since f is a strictly increasing function on the interval $(1, \infty)$. \square

The asymptotic expansion (3.8) implies that the inequality

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \dots + \frac{(n-1)!x}{\log^n x}$$

holds for all sufficiently large values of x . The best explicit result in this direction was given in [2, Theorem 1.2], namely that

$$(3.12) \quad \pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{5.65x}{\log^4 x} + \frac{23.65x}{\log^5 x} + \frac{118.25x}{\log^6 x} + \frac{709.5x}{\log^7 x} + \frac{4966.5x}{\log^8 x}$$

for every $x \geq 1\,332\,450\,001$. A consequence of Theorem 3.8 is the following refinement of (3.12).

Proposition 3.11. *For every $x \geq 19\,027\,490\,297$, we have*

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{5.85x}{\log^4 x} + \frac{23.85x}{\log^5 x} + \frac{119.25x}{\log^6 x} + \frac{715.5x}{\log^7 x} + \frac{5008.5x}{\log^8 x}.$$

Proof. Let $U(x)$ denotes the right-hand side of the required inequality and let $R(y) = U(y) \log y / y$. Further, we set $S(y) = (y^7 - y^6 - y^5 - 2.85y^4 - 13.15y^3 - 70.7y^2 - 458.7275y - 3428.7225) / y^6$. Then $S(y) > 0$ for every $t > 3.79$ and $y^{13}R(y)S(y) = y^{14} - T(y)$, where $T(y) = 11137.2625y^6 + 19843.008375y^5 + 63112.7025y^4 + 252925.911y^3 + 1091195.634375y^2 + 475078.76325y + 17172756.64125$. By Theorem (3.8),

$$\pi(x) > \frac{x}{S(\log x)} > \frac{x}{S(\log x)} \left(1 - \frac{T(\log x)}{\log^{14} x} \right) = U(x)$$

for every $x \geq 19\,033\,744\,403$. So it remains to deal with the case where $19\,027\,490\,297 \leq x \leq 19\,033\,744\,403$. Since $U(x)$ is a strictly increasing function for every $x \geq 44$, it suffices to check with a computer that $\pi(p_n) > U(p_{n+1})$ for every positive integer n such that $\pi(19\,027\,490\,297) \leq n \leq \pi(19\,033\,744\,403)$. \square

4. ON THE EXISTENCE OF PRIME NUMBERS IN SHORT INTERVALS

Bertrand's postulate states that for each positive integer n there is a prime number p with $n < p \leq 2n$, and was proved, for instance, by Chebyshev [7] and by Erdős [14]. In the following, we note some of the remarkable improvements of Bertrand's postulate. The first result is due to Schoenfeld [32, Theorem 12]. He found that for every $x \geq 2\,010\,759.9$ there is a prime number p with $x < p < x(1 + 1/16\,597)$. In 2003, Ramaré and Saouter [30, Theorem 3] found that for every $x \geq 10\,726\,905\,041$ there is a prime number p so that $x < p \leq x(1 + 1/28\,313\,999)$. Further, they [30, Table 1] gave a table of sharper results, which hold for large x . In 2014, Kadiri and Lumley [21, Table 2] found a series of improvements. For instance, they showed that for every $x \geq e^{150}$ there is a prime number p such that $x < p < x(1 + 1/2\,442\,159\,713)$. In 1998, Dusart [10, Théorème 1.9] proved that for every $x \geq 3\,275$ there exists a prime number p such that $x < p \leq x(1 + 1/(2 \log^2 x))$ and then, in 2010, reduced the interval himself [11, Proposition 6.8] by showing that for every $x \geq 396\,738$ there is a prime number p satisfying $x < p \leq x(1 + 1/(25 \log^2 x))$. In 2016, Trudgian [36, Corollary 2] proved that for every $x \geq 2\,898\,242$ there exists a prime number p with

$$(4.1) \quad x < p \leq x \left(1 + \frac{1}{111 \log^2 x} \right).$$

Recently, Dusart [13, Corollary 5.5] improved Trudgian's result by showing that for every $x \geq 468\,991\,632$ there exists a prime number p such that

$$(4.2) \quad x < p \leq x \left(1 + \frac{1}{5\,000 \log^2 x} \right).$$

In [2, Theorem 1.5], it is shown that for every $x \geq 58\,837$ there is a prime number p such that $x < p \leq x(1 + 1.1817/\log^3 x)$. In [13, Proposition 5.4], Dusart refined the last result by showing that for every $x \geq 89\,693$ there exists a prime number p such that

$$(4.3) \quad x < p \leq x \left(1 + \frac{1}{\log^3 x} \right).$$

In the following theorem, we improve (4.3) on the one hand by decreasing the coefficient of the $1/\log^3 x$ term and on the other hand by increasing the exponent of the $\log x$ term. In order to do this, we use some estimates for the prime counting function obtained in Section 3.

Theorem 4.1. *For every $x \geq 6\,034\,256$ there is a prime number p , such that*

$$x < p \leq x \left(1 + \frac{0.087}{\log^3 x}\right),$$

and for every $x > 1$ there is a prime number p , such that

$$(4.4) \quad x < p \leq x \left(1 + \frac{198.2}{\log^4 x}\right).$$

Proof. Similar to the proof of Theorem 2.4, we get that

$$(4.5) \quad |\vartheta(x) - x| < \frac{0.043x}{\log^3 x}$$

for every $x \geq e^{40}$. Setting $f(x) = 0.087/\log^3 x$, we use (4.5) to get that

$$\vartheta(x + xf(x)) - \vartheta(x) > \frac{x}{\log^3 x} \left(0.001 - \frac{0.003741}{\log^3 x}\right) \geq 0$$

for every $x \geq e^{40}$, which implies that for every $x \geq e^{40}$ there is a prime number p satisfying $x < p \leq x(1 + 0.087/\log^3 x)$. From (4.2) it is clear that the claim follows for every x such that $468\,991\,632 \leq x \leq e^{40}$. To deal with the case where $156\,007 \leq x \leq 468\,991\,632$, we check with a computer that the inequality $p_n(1 + 0.087/\log^3 p_n) > p_{n+1}$ holds for every positive integer n such that $\pi(6\,034\,393) \leq n \leq \pi(468\,991\,632)$. Finally, we notice that $\pi(x(1 + 0.087/\log^3 x)) > \pi(x)$ for every x such that $6\,034\,256 \leq x < 6\,034\,393$, which completes the proof of the first part.

We define $g(x) = 198.2/\log^4 x$. To show the second part, we first note that

$$(4.6) \quad |\vartheta(x) - x| < \frac{99.07x}{\log^4 x}$$

for every $x \geq e^{25}$. The proof of this inequality is quite similar to the proof of Proposition 2.5 and we leave the details to the reader. Using (4.6), we obtain that

$$\vartheta(x + xg(x)) - \vartheta(x) > \frac{x}{\log^4 x} \left(0.06 - \frac{19635.674}{\log^4 x}\right) \geq 0$$

for every $x \geq e^{25}$. Analogously to the proof of the first part, we check with a computer that for every $1 < x < e^{25}$ there is a prime p so that $x < p \leq x(1 + 198.2/\log^4 x)$. \square

By using (4.1), Dudek [9, Theorem 3.6] purports to prove that for every positive integer $m \geq 4.971 \cdot 10^9$ there exists a prime number between n^m and $(n+1)^m$ for all $n \geq 1$. In fact, he showed the slightly weaker lower bound $m \geq 4.97117 \cdot 10^9$. Applying (4.4) to Dudek's proof, we get the following refinement.

Proposition 4.2. *Let $m \geq 3\,239\,773\,013$. Then there is a prime between n^m and $(n+1)^m$ for all $n \geq 1$.*

Proof. Let $m \geq M_0$, where $M_0 = 3\,239\,773\,013$. First, we set $x = n^m$ in (4.4) to get that there is a prime p so that

$$(4.7) \quad n^m \leq p < n^m \left(1 + \frac{198.2}{\log^4(n^m)}\right)$$

for every $n \geq 2$. We have

$$(4.8) \quad n^m \left(1 + \frac{198.2}{\log^4(n^m)}\right) \leq n^m + mn^{m-1}$$

if and only if $198.2n/\log^4 n \leq m^5$. Setting $n_0(m) = \max\{x \in \mathbb{N} \mid 198.2x/\log^4 x \leq m^5\}$, we get $n_0(m) \geq n_0(M_0) \geq 4.18498732 \cdot 10^{53}$. Now, we apply (4.8) to (4.7) to get that there is a prime p so that

$$(4.9) \quad n^m \leq p < n^m + mn^{m-1}$$

for every $2 \leq n \leq n_0(m)$. By the binomial theorem, we have $n^m + mn^{m-1} \leq (n+1)^m$. So, (4.9) implies that there is a prime between n^m and $(n+1)^m$ for every $2 \leq n \leq n_0(m)$. On the other hand, Dudek [9, p. 42] showed that for every positive integer $t \geq 1000$ there is a prime between n^t and $(n+1)^t$ for every $n \geq n_1(t)$, where $n_1(t) = \exp(1000 \exp(19.807)/t)$. Therefore

$$n_1(m) = \exp\left(\frac{1000 \exp(19.807)}{m}\right) \leq \exp\left(\frac{1000 \exp(19.807)}{M_0}\right) \leq 4.1849871 \cdot 10^{53}.$$

Since $n_1(m) \leq n_0(m)$, we conclude the proof for every $n \geq 2$. The remaining case $n = 1$ is clear. \square

5. ON ESTIMATES OF TWO SUMS OVER PRIMES

In this section, we give some refined estimates for the sums

$$\sum_{p \leq x} \frac{1}{p}, \quad \sum_{p \leq x} \frac{\log p}{p},$$

where p runs over primes not exceeding x .

5.1. On the sum of the reciprocals of all prime numbers not exceeding x . In 1737, Euler [15] proved that the sum of the reciprocals of all prime numbers diverges. In particular, this result implies that there are infinitely many primes. Further, Euler [15, Theorema 19] and later Gauss [17] stated that the sum of the reciprocals of all prime numbers not exceeding x grows like $\log \log x$. In 1874, Mertens [26, p. 52] used several results of Chebyshev's papers [6], [7] to find that $\log \log x$ is the right order of magnitude for the sum of the reciprocals of all prime numbers p not exceeding x by showing that

$$(5.1) \quad \sum_{p \leq x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right).$$

Here, B denotes the Mertens' constant (see [33]) and is defined by

$$(5.2) \quad B = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) = 0.2614972128476427837554268386 \dots,$$

where $\gamma = 0.577215664901532860606512090082402431 \dots$ denotes the Euler-Mascheroni constant. In 1962, Rosser and Schoenfeld [31, p. 74] derived a remarkable identity, which connects the sum of the reciprocals of all prime numbers not exceeding x with Chebyshev's ϑ -function, namely

$$(5.3) \quad \sum_{p \leq x} \frac{1}{p} = \log \log x + B + \frac{\vartheta(x) - x}{x \log x} - \int_x^\infty \frac{(\vartheta(y) - y)(1 + \log y)}{y^2 \log^2 y} dy.$$

Together with (2.4), they [31, p. 68] refined the error term in Mertens' result (5.1) by giving

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B + O(\exp(-a\sqrt{\log x})).$$

Using (5.3) and explicit estimates for Chebyshev's ϑ -function, Rosser and Schoenfeld [31, Theorem 5] were able to find

$$\log \log x + B - \frac{1}{2 \log^2 x} < \sum_{p \leq x} \frac{1}{p} < \log \log x + B + \frac{1}{2 \log^2 x},$$

where the left-hand side inequality is valid for every $x > 1$ and the right-hand side inequality holds for every $x \geq 286$. After some remarkable improvements, the currently best known estimates for the sum of the reciprocals of all prime numbers not exceeding x are due to Dusart [13, Theorem 5.6]. He used (5.3) together with (2.5) to get that

$$(5.4) \quad \left| \sum_{p \leq x} \frac{1}{p} - \log \log x - B \right| \leq \frac{\eta_k}{k \log^k x} + \frac{(k+2)\eta_k}{(k+1) \log^{k+1} x}$$

for every $x \geq x_0(k)$. Then he [13, Theorem 5.6] applied Lemma 2.3 with $k = 3$ and $\eta_3 = 0.5$, and get

$$(5.5) \quad -\frac{1}{5 \log^3 x} \leq \sum_{p \leq x} \frac{1}{p} - \log \log x - B \leq \frac{1}{5 \log^3 x}$$

for every $x \geq 2278383$. Following Dusart's proof of (5.5), we obtain the following slightly refinements of these estimates by using Theorem 2.4.

Proposition 5.1. *We have*

$$-\frac{1}{20 \log^3 x} - \frac{3}{16 \log^4 x} \leq \sum_{p \leq x} \frac{1}{p} - \log \log x - B \leq \frac{1}{20 \log^3 x} + \frac{3}{16 \log^4 x},$$

where the left-hand side inequality holds for every $x > 1$ and the right-hand side inequality is valid for every $x \geq 46909074$.

Proof. We use (5.4) and Theorem 2.4 to get that these inequalities hold for every $x \geq 19\,035\,709\,163$. To verify that the left-hand side inequality holds for every x such that $2 \leq x \leq 19\,035\,709\,163$ as well, we check with a computer that for every positive integer $n \leq \pi(19\,035\,709\,163)$,

$$\sum_{k \leq n} \frac{1}{p^k} \geq \log \log p_{n+1} + B - \frac{1}{20 \log^3 p_{n+1}} - \frac{3}{16 \log^4 p_{n+1}}.$$

Clearly, the left-hand side inequality holds for every x such that $1 < x < 2$. A similar calculation shows that the right-hand side inequality holds for every x such that $46\,909\,074 \leq x \leq 19\,035\,709\,163$ as well. \square

5.2. On another sum over all prime numbers not exceeding x . In 1857, de Polignac [29, part 3] stated without proof that $\log x$ is the right asymptotic behaviour for the sum

$$(5.6) \quad \sum_{p \leq x} \frac{\log p}{p},$$

where p runs over primes not exceeding x . A rigorous proof for this was given by Mertens [26, p. 49] in 1874. He showed that

$$(5.7) \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

In 1909, Landau [24, §55] was able to precise (5.7) by finding

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + E + O(\exp(-\sqrt[4]{\log x})),$$

where E is a constant (see [34]) defined by

$$E = -\gamma - \sum_p \frac{\log p}{p(p-1)} = -1.332582275733220881765828776071027748838459 \dots$$

Rosser and Schoenfeld [31, p. 74] connected the sum in (5.6) with Chebyshev's ϑ -function by showing

$$(5.8) \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + E + \frac{\vartheta(x) - x}{x} - \int_x^\infty \frac{\vartheta(y) - y}{y^2} dy.$$

Using their own explicit estimates for Chebyshev's ϑ -function, they [31, Theorem 6] found

$$\log x + E - \frac{1}{2 \log x} < \sum_{p \leq x} \frac{\log p}{p} < \log x + E + \frac{1}{2 \log x},$$

where the left-hand side inequality is valid for every $x > 1$ and the right-hand side inequality holds for every $x \geq 319$. In 2010, Dusart [11, Theorem 6.11] utilized (5.8) and 2.5 to get that the inequality

$$(5.9) \quad \left| \sum_{p \leq x} \frac{\log p}{p} - \log x - E \right| \leq \frac{\eta_k}{(k-1) \log^{k-1} x} + \frac{\eta_k}{\log^k x}$$

holds for every $x \geq x_0(k)$. Then he [12, Theorem 5.7] applied Lemma 2.3 with $k = 3$ and $\eta_3 = 0.5$ to (5.9) and obtained the current best estimates for the sum given in (5.6), namely

$$-\frac{0.3}{\log^2 x} < \sum_{p \leq x} \frac{\log p}{p} - \log x - E < \frac{0.3}{\log^2 x}$$

for every $x \geq 912\,560$. Now, (5.9) and Theorem 2.4 imply the following refinement.

Proposition 5.2. *We have*

$$(5.10) \quad -\frac{3}{40 \log^2 x} - \frac{3}{20 \log^3 x} \leq \sum_{p \leq x} \frac{\log p}{p} - \log x - E \leq \frac{3}{40 \log^2 x} + \frac{3}{20 \log^3 x},$$

where the left-hand side inequality is valid for every $x > 1$ and the right-hand side inequality holds for every $x \geq 30\,972\,320$.

Proof. From (5.9) and Theorem 2.4, it follows that the required inequalities (5.10) holds for every $x \geq 19\,035\,709\,163$. Similarly to the proof of Proposition 5.1, we use a computer to check the desired inequalities for smaller values of x . \square

6. REFINED ESTIMATES FOR A PRODUCT OVER PRIMES

The asymptotic formula (5.1) implies that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right)$$

and in this direction, Rosser and Schoenfeld [31, Theorem 7] found that

$$(6.1) \quad \frac{e^{-\gamma}}{\log x} \left(1 - \frac{1}{2 \log^2 x}\right) < \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \left(1 + \frac{1}{2 \log^2 x}\right),$$

where the left-hand side inequality is valid for every $x \geq 285$ and the right-hand side inequality holds for every $x > 1$. After several improvements, the sharpest known estimates for this product are due to Dusart [12, Theorem 5.9]. Following Rosser's and Schoenfeld's proof of (6.1), Dusart used (5.4) and Lemma 2.3 with $k = 3$ and $\eta_k = 0.5$ to find

$$\frac{e^{-\gamma}}{\log x} \left(1 - \frac{0.2}{\log^3 x}\right) < \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \left(1 + \frac{0.2}{\log^3 x}\right)$$

for every $x \geq 2278382$. We use the same method combined with Proposition 5.1 to obtain the following

Proposition 6.1. *For every $x \geq 46909038$, we have*

$$(6.2) \quad \prod_{p \leq x} \left(1 - \frac{1}{p}\right) > \frac{e^{-\gamma}}{\log x} \left(1 - \frac{1}{20 \log^3 x} - \frac{3}{16 \log^4 x}\right),$$

and for every $x > 1$, we have

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \left(1 + \frac{0.07}{\log^3 x}\right).$$

Proof. First, let $x \geq 46909074$ and $S = \sum_{p > x} (\log(1 - 1/p) + 1/p) = -\sum_{k=2}^{\infty} \sum_{p > x} 1/kp^k$. Using the right-hand side inequality of Proposition 5.1 and the definition (5.2) of B , we have

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) > \frac{e^{-\gamma}}{\log x} \exp\left(-S - \frac{1}{20 \log^3 x} - \frac{3}{16 \log^4 x}\right).$$

Now we use the inequality $e^t \geq 1 + t$, which holds for every real t , and the fact that $S < 0$ to obtain that the required inequality (6.2) holds for every $x \geq 46909074$. We conclude with a computer check.

Analogously, we use the left-hand side inequality of Proposition 5.1 to get

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \exp\left(-S + \frac{1}{20 \log^3 x} + \frac{3}{16 \log^4 x}\right)$$

for every $x > 1$. By Rosser and Schoenfeld [31, p. 87], we have $-S < 1.02/((x-1)\log x)$ for every $x > 1$. Since

$$\frac{1}{20 \log^3 x} + \frac{3}{16 \log^4 x} + \frac{1.02}{(x-1)\log x} \leq \frac{0.06}{\log^3 x} \leq \log\left(1 + \frac{0.07}{\log^3 x}\right)$$

for every $x \geq 1.4 \cdot 10^8$, we get that the desired upper bound holds for every $x \geq 1.4 \cdot 10^8$. We conclude by direct computation. \square

ACKNOWLEDGEMENT

I would like to express my great appreciation to Jan Büthe and Marc Deléglise for the computation of several special values of Chebyshev's ϑ -function. I also would like to thank Marco Aymone for pointing out a mistake in the proof of Proposition 6.1, which has been corrected in this version.

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