

ALGEBRAS OF QUASI-PLÜCKER COORDINATES ARE KOSZUL

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ABSTRACT. Motivated by the theory of quasi-determinants, we study non-commutative algebras of quasi-Plücker coordinates. We prove that these algebras provide new examples of non-homogeneous quadratic Koszul algebras by showing that their quadratic duals have quadratic Gröbner bases.

1. INTRODUCTION

Denote $\underline{n} = \{1, 2, \dots, n\}$. Given ordered sets $I \subset J$, we denote by $J \setminus I$ the ordered set obtained by removing I , and by $J|K$ the ordered set obtained by appending an ordered set K . The set $\{j\}$ containing one element j is denoted by j .

1.1. Commutative Plücker Coordinates. For $k \leq n$ and a $k \times n$ -matrix A with commutative entries we can choose a subset $I = \{i_1, \dots, i_k\}$ of the column indices \underline{n} and consider the *Plücker coordinates*

$$(1) \quad p_I(A) := \det A(i_1, \dots, i_k),$$

using the $k \times k$ submatrix with columns corresponding to the indices in I . It is well-known (see e.g. [HP94, Chapter VII.6]) that the $p_I(A)$ satisfy GL_n -invariance, skew-symmetry with respect to commuting columns, and the *Plücker identity*

$$(2) \quad \sum_{t=1}^{k+1} (-1)^t p_{I|j_t}(A) p_{J \setminus j_t}(A) = 0,$$

for subsets $I = \{i_1, \dots, i_{k-1}\}$ and $J = \{j_1, \dots, j_{k+1}\}$ of the column indices.

For example, let $k = 2 < 3 = n$. Then we can study a commutative algebra generated by elements p_{12}, p_{13}, p_{23} , and the Plücker relations add no extra information. Letting $k = 2 < 4 = n$ and choosing $I = \{1\}$, $J = \{2, 3, 4\}$ one obtains the classical identity

$$(3) \quad p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

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For $k = 3$ and $n = 6$ we, for example, get the relations

$$(4) \quad p_{123}p_{456} - p_{124}p_{356} + p_{125}p_{346} - p_{126}p_{345} = 0,$$

$$(5) \quad p_{123}p_{245} - p_{124}p_{235} + p_{125}p_{234} = 0,$$

plus similar relations interchanging the roles of the numbers in $\underline{6}$.

One can consider the symbols p_I as generators of a quadratic commutative algebra, the quadratic quotient $\mathcal{O}_{k,n}$ of the polynomial algebra $\mathbb{C}[p_I \mid I \subset \underline{n}]$ by the relations (2), and skew-symmetry with respect to commuting indices. It is well-known that $\mathcal{O}_{k,n}$ is a Koszul ring since the relations give a quadratic Gröbner basis. This was proved in [DRS74], and also follows from [Kem90], using results of [DCEP82]. See [MS05, Theorem 14.6] for a textbook exposition.

The Hilbert series of $\mathcal{O}_{k,n}$ can be computed combinatorially using methods from [Stu08]. In the above example of $G_{2,4}$, one obtains the closed formula (see [GW11, Section 7])

$$(6) \quad H(\mathcal{O}_{2,4}, t) = \frac{(1 + 6t + 6t^2 + t^3)}{(1 - t)^9} = 1 + 15t + 105t^2 + 490t^3 + O(t^4).$$

The Plücker coordinates $p_I(A)$ define an embedding of the Grassmannian $G_{k,n}$ into projective space of dimension $\binom{n}{k} - 1$. The coordinate ring of $G_{k,n}$ via the Plücker embedding is the quadratic algebra $\mathcal{O}_{k,n}$ considered above.

1.2. Non-commutative Plücker Coordinates. Analogues of Plücker coordinates for a $k \times n$ -matrix with non-commuting entries are obtained using the theory of quasi-determinants [GR97, GGRW05] as ratios of two quasi-minors. More precisely, given a choice of two indices $i, j \in \underline{n}$, and a subset $i \notin I \subset \underline{n}$ of size $k - 1$ and a matrix A with coefficients in a division ring, the *quasi-Plücker coordinate* q_{ij}^I is defined as the following ratio of non-commutative analogues of maximal minors:

$$(7) \quad q_{ij}^I = q_{ij}^I(A) = \frac{\begin{vmatrix} a_{1i} & a_{1i_1} & \cdots & a_{1i_{k-1}} \\ \vdots & & & \vdots \\ a_{ki} & a_{ki_1} & \cdots & a_{ki_{k-1}} \end{vmatrix}_{si}^{-1}}{\begin{vmatrix} a_{1j} & a_{1i_1} & \cdots & a_{1i_{k-1}} \\ \vdots & & & \vdots \\ a_{kj} & a_{ki_1} & \cdots & a_{ki_{k-1}} \end{vmatrix}_{sj}},$$

$p_{i|I}^{(s)}(A)^{-1}$ $p_{j|I}^{(s)}(A)$

which is independent of choice in s , undefined if $i \in I$, and zero if $j \in I$. The following analogue of the Plücker relations holds for these non-commutative analogues of Plücker coordinates:

$$(8) \quad \sum_{j \in L} q_{ij}^M q_{ji}^{L \setminus j} = 1.$$

In the case where the entries of A commute, (8) recovers the classical relation (2). Moreover, symmetry in changing the order of elements of I holds,

replacing skew-symmetry for these ratios, and q_{ji}^I is inverse to q_{ij}^I if non-zero. By considering the ratios q_{ij} , an additional relation appears:

$$(9) \quad q_{ij}^{N \setminus \{i,j\}} q_{jm}^{N \setminus \{j,m\}} = -q_{im}^{N \setminus \{i,m\}}.$$

See Section 2 for the list of relations among quasi-Plücker coordinates.

For example, in the case $k = 2$ and $n = 4$, Equation (8) gives the formula

$$(10) \quad q_{13}^2 q_{31}^4 + q_{14}^2 q_{41}^3 = 1.$$

This translates to

$$(11) \quad p_{12}^{-1} p_{32} p_{34}^{-1} p_{14} + p_{12}^{-1} p_{42} p_{43}^{-1} p_{13} = 1.$$

If the elements p_{ij} commute, this equality reduces to the classical formula (3).

As a second example, consider the case $k = 2$ and $n = 6$. Let $M = \{1, 2\}$ and $L = \{3, 4, 5\}$. Then, with $i = 6$, we obtain the equation

$$(12) \quad \begin{aligned} & q_{63}^{12} q_{36}^{45} + q_{64}^{12} q_{46}^{35} + q_{65}^{12} q_{56}^{34} = 1 \\ \iff & p_{612}^{-1} p_{312} p_{345}^{-1} p_{645} + p_{612}^{-1} p_{412} p_{435}^{-1} p_{635} + p_{612}^{-1} p_{512} p_{543}^{-1} p_{643} = 1. \end{aligned}$$

Assuming that the variables commute, this recovers relation (4). Similarly, Equation (5) can be recovered using $M = \{1, 2\}$, $L = \{2, 3, 4\}$, $i = 5$.

1.3. Quantum Plücker Coordinates. A quantum analogue of Equation (2) was considered in [TT91, Eq. (3.2c)] in order to construct a quantum analogue of the coordinate algebra $\mathcal{O}_{G_{k,n}}$ of the Grassmannian $G_{k,n}$. For this, more general exchange relations appear, called *Young symmetry relations*:

$$(13) \quad \sum_{\Lambda \subseteq I, |\Lambda|=n} (-q)^{-l(I \setminus \Lambda | \Lambda)} f_{I \setminus \Lambda} f_{\Lambda | J},$$

for $1 \leq r \leq d$, and I, J index sets of size $d + r$ and $d - r$, respectively. Here, we use notation adapted from [Lau06, Eq. (9)]. The classical Plücker relations (2) can be recovered as the case $r = 1$, $q = 1$. It was further shown in [Lau06] that the relations (13) can be reduced successively to relations with $r = 1$.

In fact, the Young symmetry relations are consequences of the quasi-Plücker relations (8) [Lau06, Theorem 28].

1.4. Sagbi and Gröbner Basis for Coordinates of Grassmannians.

In the commutative setting, maximal minors form a *sagbi* (canonical subalgebra basis) according to [Stu08, 3.2.9]. (The relations among these maximal minors give a quadratic Gröbner basis as mentioned in Section 1.1.)

This result was generalized to another approach to quantum Grassmannians, which emerges from geometry and quantum cohomology and is a commutative construction (rather than using q -commutators). The coordinate ring, denoted by $k[\mathcal{C}_{k,n}^q]$ of the quantum Grassmannian consists of maximal graded minors (of degree up to $q \in \mathbb{N}$) of $k \times n$ -matrices with graded

entries. In [SS01, Theorem 1] it is proved that these maximal graded minors give a sagbi basis for the coordinate ring $k[\mathcal{C}_{k,n}^q]$ within the polynomial ring of graded entries of the matrix. Further, the relations among these maximal graded minors have a quadratic Gröbner basis [SS01, Theorem 2].

1.5. This Paper's Approach. This paper takes the approach to start with a quadratic algebra of quasi-Plücker Coordinates (introduced in Section 2). This algebra is quadratic-linear, and a theory for Koszulity of such algebras has been developed in [PP05].

Our main result is that the associated quadratic algebra of this algebra has a quadratic Gröbner basis. Hence the algebra of non-commutative Plücker coordinates is a non-homogeneous Koszul algebra (Theorem 4). In Section 4 we consider colimits of these algebras, varying $k \geq 2$. We further study a second version of algebras of quasi-Plücker coordinates which is not quadratic-linear, but also non-homogeneous Koszul in Section 5.

In Section 6 we study the Koszul dual dg algebras explicitly in the case $k = 2$, and we finish the exposition by considering an algebra of non-commutative *flag coordinates* which is also non-homogeneous Koszul in Section 7.

There are different approaches to non-commutative Grassmannian coordinate rings, see e.g. [Kap98, KR00], which are not discussed here.

2. DEFINITION OF THE ALGEBRA $R_n^{(k)}$

We want to define a quadratic algebra of quasi-Plücker coordinates. As outlined in Section 1.2, quasi-Plücker coordinates were constructed using quasi-determinants in [GR97, Section II], cf. [GGRW05, 4.3]. For fixed integers $n, k \geq 2$ we define the algebra $Q_n^{(k)}$ as having generators q_{ij}^I , where $I \subset \underline{n}$ has size $k-1$ and $i \notin I$, which satisfy the following relations, obtained from [GGRW05, 4.3]:

- (i) q_{ij}^I does not depend on the ordering of the elements of I ;
- (ii) $q_{ij}^I = 0$ whenever $j \in I$ and $i \neq j$;
- (iii) $q_{ii}^I = 1$, and $q_{ij}^I q_{jl}^I = q_{il}^I$;
- (iv) $q_{ij}^{N \setminus \{i,j\}} q_{jm}^{N \setminus \{j,m\}} = -q_{im}^{N \setminus \{i,m\}}$.
- (v) If $i \notin M$, then $\sum_{j \in L} q_{ij}^M q_{ji}^{L \setminus \{j\}} = 1$.

Relation (iv) is called *non-commutative skew-symmetry*, and (v) is a non-commutative analogue of the Plücker relations.

The algebra $Q_n^{(2)}$ is studied in [BR15], as the algebra of *non-commutative sectors*, where it is denoted by \mathcal{Q}_n . Given an $k \times n$ -matrix A with entries in a division ring, we note that the description of q_{ij}^I in terms of quasi-determinants in Equation (7) provides a morphism of algebras from $Q_n^{(k)}$ to the skew-field generated by the non-commutative entries a_{ij} of the matrix A .

We also consider the subalgebra $R_n^{(k)}$ of $Q_n^{(k)}$ generated by those of the q_{ij}^I for which $i < j$. The restriction to $R_n^{(k)}$ can be justified by noting that the skew-fields generated by the images of $R_n^{(k)}$ and $Q_n^{(k)}$ in the skew-field generated by the matrix entries a_{ij} coincide. One advantage of considering $R_n^{(k)}$ is that it admits a presentation as a quadratic-linear algebra:

Proposition 1. *The subalgebra $R_n^{(k)}$ can be described by the relations*

$$(14) \quad q_{ij}^I q_{jl}^I = q_{il}^I, \quad \forall i < j < l, \quad i, j \notin I,$$

$$(15) \quad \sum_{j=1}^{k-1} q_{l_0 l_j}^M q_{l_j l_k}^{L \setminus \{l_j, l_k\}} + q_{l_0 l_k}^{L \setminus \{l_0, l_k\}} = q_{l_0 l_k}^M,$$

where $L = \{l_0 < l_1 < \dots < l_k\}$, $l_0 \notin M$, which are read in the way that $q_{ij}^M = 0$ if $j \in M$. Hence, $R_n^{(k)}$ is a quadratic-linear algebra.

Proof. Starting with formula (v), we distinguish three cases, depending on the value of the index i . In the case when $l_j < i < l_{j+1}$ for some $j = 1, \dots, k-1$ we obtain Equation (15) by multiplying with $q_{l_1, i}^M$ on the left, and $q_{i, l_k}^{L \setminus l_k}$ on the right. If $i < l_1$, it suffices to multiply by $q_{i, l_k}^{L \setminus l_k}$ on the right; and if $l_k < i$, it is enough to multiply by $q_{l_1, i}^M$ on the left. In all three cases, we obtain the same relation after relabelling so that the index-set L contains i in the correct order. These are all possible relations between generators q_{ij}^I with $i < j$ as in this case Equation (iv) is a special case of Equation (15), with $M = L \setminus \{l_0, l_j\}$. \square

Example 2. Let us consider the case $k = 2$. In this case, the skew-symmetry relation (iv) and the Plücker relation (v) in $Q_n^{(2)}$ become

$$(16) \quad q_{ij}^l q_{jl}^i = -q_{il}^j$$

$$(17) \quad q_{ij}^m q_{ji}^l + q_{il}^m q_{li}^j = 1,$$

where all indices are distinct. In this case, the algebra $R_n^{(2)}$ has $(n-2) \binom{n}{2}$ generators q_{ij}^l with $i < j$ and $l \neq i, j$. The relations governing this algebra are

$$(18) \quad q_{ij}^m q_{jl}^i + q_{il}^j = q_{il}^m,$$

$$(19) \quad q_{ij}^m q_{jl}^m - q_{il}^m = 0.$$

for all $i < j < l$ and m a distinct element from i, j, l , and if $m = l$ we have the relation

$$(20) \quad q_{ij}^l q_{jl}^i + q_{il}^j = 0.$$

3. KOSZULNESS OF THE ALGEBRA $R_n^{(k)}$

The Koszul property for quadratic algebras can, more generally, be studied for non-homogeneous quadratic algebras [PP05, Chapter 5]. A non-homogeneous quadratic algebra is *Koszul* if the corresponding quadratic algebra $A^{(0)}$ obtained by taking the homogeneous parts of the quadratic relations is Koszul. In this case, $A^{(0)}$ is isomorphic to the associated graded algebra $\text{gr } A$.

The associated quadratic algebra $(R_n^{(k)})^{(0)}$ is generated by the relations

$$(21) \quad q_{ij}^I q_{jl}^I = 0, \quad \forall i < j < l, \quad i, j \notin I,$$

$$(22) \quad \sum_{j=1}^{k-1} q_{i_0 l_j}^M q_{l_j l_k}^{L \setminus \{l_j, l_k\}} = 0,$$

where $L = \{l_0 < l_1 < \dots < l_t\}, l_0 \notin M$, which are read in the way that $q_{ij}^M = 0$ if $j \in M$.

We consider the quadratic dual of $(R_n^{(k)})^{(0)}$ which is denoted by $B_n^{(k)}$. It consists of generators r_{ij}^I for $i < j$ and $i \notin I$, and $r_{ij}^I = 0$ if $j \in I$. Again, we regard I as a strictly ordered set of indices.

We will write $i < K < j$ to denote that $i < l < j$ for every element $l \in K$.

Lemma 3. *The algebra $B_n^{(k)}$ is given by the relations*

$$(23) \quad r_{ij}^I = 0, \quad \text{if } j \in I,$$

$$(24) \quad r_{ij}^I r_{ab}^J = 0, \quad \text{unless } j = a \text{ and } \begin{cases} i \in J \text{ and } i < J \setminus i < b \\ \text{or } I = J \end{cases},$$

$$(25) \quad r_{ij}^I r_{jl}^J = r_{ij'}^I r_{j'k}^{J \setminus j' \cup j}, \quad \forall j' \in J \setminus (J \cap I), \text{ provided } i < J \setminus i < l, \\ \text{provided that } i \notin I, a \notin J \text{ for (24), and } i \notin I, j \notin J \text{ for (25)}.$$

Theorem 4. *The algebra $B_n^{(k)}$ has a quadratic non-commutative Gröbner basis, for $k, n \geq 2$, given by the relations (24)–(25) on generators $r_{ij}^I, i, j \notin I$.*

Note that for $k \geq n$, $B_n^{(k)} = 0$ is trivial as no subset of size $k + 1$ of \underline{n} can be chosen.

Proof. In order to prove the theorem, we have to show that there are no obstructions of degree larger than two (see e.g. [Ani86, CPU99] for the terminology). We chose the following ordering on the generators q_{ij}^I . We first order by size of (i, j) lexicographically. Given same subscripts, we order according to the lexicographic order on the superscripts I . The monomials are then ordered graded reverse lexicographically (**degrevlex** order). There are two different types of normal words of degree two:

$$\begin{aligned} r_{ij}^I r_{jl}^J, & \quad \text{for } i < j < J \setminus \{J \cap I \cup i\} < l, \\ r_{ij}^I r_{jl}^I, & \quad j \notin I, \end{aligned}$$

with $i \notin I$.

We claim that a basis for $B_n^{(k)}$ is given by monomials of the form

$$(26) \quad r_{i_0, i_1}^{I_0} r_{i_1, i_2}^{I_1} \cdots r_{i_{t-1}, i_t}^{I_{t-1}},$$

with $i_0 < i_1 < \dots < i_t$, where for each $j = 1, \dots, t-1$ we have for $r_{i_{j-1}, i_j}^{I_{j-1}} r_{i_j, i_{j+1}}^{I_j}$ either

$$(27) \quad I_{j-1} = I_j, \quad \text{or}$$

$$(28) \quad i_{j-1} \in I_j \quad \text{and} \quad i_j < I_j \setminus \{I_j \cap I_{j-1} \cup i_{j-1}\} < i_{j+1}.$$

To prove this, we note in an arbitrary non-zero monomial M we might have degree two sub-word of the form $r_{i_{j-1}, i_j}^{I_{j-1}} r_{i_j, i_{j+1}}^{I_j}$ where i_j is not necessarily smaller than all element in $I_j \setminus \{I_j \cap I_{j-1} \cup i_{j-1}\}$. In this case, we can replace the sub-word by $r_{i_{j-1}, i'_j}^{I_{j-1}} r_{i'_j, i_{j+1}}^{I_j}$, where i'_j is smaller than all elements in $I_j \setminus \{I_j \cap I_{j-1} \cup i_{j-1}\}$ using relation (25). Assume that j corresponds to right-most occurrence of such a degree two sub-word. If now $r_{i_{j-2}, i_{j-1}}^{I_{j-2}} r_{i_{j-1}, i_j}^{I_{j-1}}$ is of the same form, but there exists an element of I_{j-1} which is larger than i'_j , the monomial M was zero by relation (24). Hence, such a situation cannot occur and by replacing all the non-normal degree two sub-words of M we obtain that M equals a monomial of the form (26).

It is now clear by the description of the monomial basis in (26) that the quadratic relations given in Lemma 3 give a non-commutative Gröbner basis (non-commutative PBW basis) for the algebra $B_n^{(k)}$. \square

We note, in particular, that $B_n^{(k)}$ is a monomial algebra if and only if $k = 2$.

Corollary 5. *The algebra $B_n^{(k)}$ is Koszul, and hence the algebra $R_n^{(k)}$ is non-homogeneous Koszul for all $k, n \geq 2$.*

Example 6.

- (i) Consider $B_n^{(2)}$ for small values of n . The algebra $B_3^{(2)}$ has a basis given by

$$1 < r_{12}^3 < r_{13}^2 < r_{23}^1 < r_{12}^3 r_{23}^1,$$

so the Hilbert series are

$$(29) \quad H(B_3^{(2)}, t) = 1 + 3t + t^2,$$

$$(30) \quad \begin{aligned} H(R_3^{(2)}, t) &= (1 - 3t + t^2)^{-1} \\ &= 1 + 3t + 8t^2 + 21t^3 + 55t^4 + 144t^5 + O(t^6). \end{aligned}$$

According to [Ani85, Ani86] (see [CPU99, Theorem 7.1]), this implies that the global dimension of $R_3^{(2)}$ equals two. The n -th coefficient of $H(R_3^{(2)}, t)$ is the $2(n-1)$ -th Fibonacci number.¹

¹According to the *On-Line Encyclopaedia of Integer Sequence*®, <https://oeis.org/A001906>.

(ii) The Hilbert series $k = 2$ and $n = 4$ is

$$(31) \quad \begin{aligned} H(R_4^{(2)}, t) &= (1 - 12t + 12t^2 - 5t^3)^{-1} \\ &= 1 + 12t + 132t^2 + 1,445t^3 + O(t^4). \end{aligned}$$

(iii) The Hilbert series $k = 2$ and $n = 5$ is

$$(32) \quad \begin{aligned} H(R_5^{(2)}, t) &= (1 - 30t + 50t^2 - 45t^3 + 17t^4)^{-1} \\ &= 1 + 30t + 850t^2 + 24,045t^3 + 680,183t^4 + O(t^5). \end{aligned}$$

In general, the top degree of $H(B_n^{(2)}, t)$ is $n - 1$. The leading coefficient h_{n-1} is given by

$$(33) \quad h_{n-1} = \sum_{i=3}^n (n - i + 1)2^{n-i} = \sum_{i=0}^{n-3} (i + 1)2^i,$$

while the coefficient $h_1 = \frac{1}{2}n(n - 1)(n - 2)$. The other coefficients can be computed as

$$(34) \quad h_{n-l} = \binom{n}{l-1} \left(l - 1 + \sum_{i=0}^{n-l-2} (i + l)2^i \right),$$

for $1 \leq l \leq n - 1$.

Example 7. If $k = 3$ and $n = 4$, then $r_{12}^{\{3,4\}} r_{24}^{\{1,3\}}$ is the only non-zero quadratic monomial in $B_4^{(3)}$, and hence

$$(35) \quad \begin{aligned} H(R_4^{(3)}, t) &= (1 - 6t + 1t^2)^{-1} \\ &= 1 + 6t + 35t^2 + 204t^3 + 1,189t^4 + O(t^5), \end{aligned}$$

for which the coefficients satisfy the recursion $a_n = 6a_{n-1} - a_{n-2}$, with $a_0 = 1$.

In general, we find that $H(R_n^{(n-1)}, t) = (1 - \frac{1}{2}n(n - 1)t + t^2)^{-1}$ since the monomial $r_{12}^{\underline{n} \setminus \{1,2\}} r_{2n}^{\underline{n} \setminus \{1,n\}}$ is the only non-zero quadratic monomial in $B_4^{(3)}$, and hence the coefficients of this Hilbert series satisfy the recursion $a_n = \frac{1}{2}n(n - 1)a_{n-1} - a_{n-2}$.

4. KOSZULNESS OF THE COLIMIT ALGEBRA R_n

The relations in [GGRW05, 4.8.1] link the quasi-Plücker coordinates for $k \times n$ -matrices with those of $(k - 1) \times n$ -matrices. In our algebraic setting, this gives the non-homogeneous relations

$$(36) \quad q_{ij}^J = q_{ij}^{J \cup m} + q_{im}^J \cdot q_{mj}^{J \cup i},$$

where $i, m \notin J$. This relation links $Q_n^{(k+1)}$ and $Q_n^{(k)}$, where J is a set of size $k - 1$. We can inductively define the quadratic algebra $Q_n^{(\leq k)}$ as the coproduct

of the algebras $Q_n^{(k')}$, for $k' \leq k$, with the additional relations of the form (36). Accordingly, we define the *algebra of quasi-Plücker coordinates* Q_n

$$Q_n := Q_n^{(\leq n)}.$$

Note that $Q_n^{(k)} = 0$ for $k \geq n$ as then it is not possible to choose an index set of size $k + 1$ in \underline{n} . The colimit algebra Q_n is again a quadratic-linear algebra with finitely many generators.

Lemma 8. *The subalgebra $R_n^{(\leq k)}$ of $Q_n^{(\leq k)}$ generated by q_{ij}^J with $i < j$ can be described as the quotient of the colimit over the subalgebras $R_n^{(k)}$ together with the relations (36) for $i < m < j$.*

Proof. In the larger algebra $Q_n^{(\leq k)}$, all relations of the form (36) can be transformed into relations of the same form where the lower indices are in strictly increasing order. This can be checked distinguishing cases depending on the order of $\{i, m, j\}$ according to size, and multiplying by the correct inverse. Hence all the relations in the subalgebra $R_n^{(\leq k)}$ are of the same form. \square

Therefore, we define the colimit algebra

$$R_n := R_n^{(\leq n)}.$$

Theorem 9. *The algebras $R_n^{(\leq k)}$ are quadratic-linear Koszul algebras, and hence the quadratic-linear algebra R_n is Koszul.*

Proof. The quadratic part of the relation (36) gives that

$$(37) \quad q_{ij}^J q_{jm}^{J \cup i} = 0, \quad \forall i < j < m,$$

where $i, j \notin J$. Consider the quadratic dual $B_n^{(\leq k)}$ of $(R_n^{(\leq k)})^{(0)}$. In this algebras, all products of generators $r_{ij}^J r_{ab}^K$ with different sizes of the index sets J, K are zero unless $j = a$ and $K = J \cup i$. We extend the linear ordering on generators by requiring that $q_{ij}^I < q_{kl}^K$ if $|K| < |L|$, again using the **degrevlex** ordering on monomials. Then a non-commutative PBW basis is given by products $M_1 M_2 \dots M_s$ of monomials of the form from (26) which only give a non-zero product if the last generator in M_t is of the form r_{ij}^K , and the first generator or M_{t+1} has the form $r_{ji}^{K \cup i}$. This shows that a quadratic non-commutative Gröbner basis exists for $B_n^{(\leq k)}$. In particular, $(R_n^{(\leq k)})^{(0)}$ is Koszul, and so $R_n^{(\leq k)}$ is non-homogeneous Koszul. \square

Example 10. Consider the algebra Q_4 . The quadratic dual has the PBW basis

$$\begin{aligned}
& r_{12}^3 < r_{12}^4 < r_{13}^2 < r_{13}^4 < r_{14}^2 < r_{14}^3 < r_{23}^1 < r_{23}^4 < r_{24}^1 < r_{24}^3 < r_{34}^1 < r_{34}^2 \\
& \quad < r_{12}^{34} < r_{13}^{24} < r_{14}^{23} < r_{23}^{14} < r_{24}^{13} < r_{34}^{12} \\
& < r_{12}^3 r_{23}^1 < r_{12}^3 r_{24}^1 < r_{12}^3 r_{24}^3 < r_{12}^3 r_{24}^{13} < r_{12}^4 r_{23}^1 < r_{12}^4 r_{23}^4 < r_{12}^4 r_{24}^1 < r_{12}^4 r_{23}^{14} \\
& < r_{13}^2 r_{34}^1 < r_{13}^2 r_{34}^2 < r_{13}^2 r_{34}^{12} < r_{13}^4 r_{34}^1 < r_{23}^1 r_{34}^1 < r_{23}^1 r_{34}^2 < r_{23}^1 r_{34}^{12} < r_{23}^4 r_{34}^2 \\
& \quad < r_{12}^3 r_{23}^1 r_{34}^1 < r_{12}^3 r_{23}^1 r_{34}^2 < r_{12}^3 r_{23}^1 r_{34}^{12} < r_{12}^4 r_{23}^1 r_{34}^1 \\
& \quad < r_{12}^4 r_{23}^1 r_{34}^2 < r_{12}^4 r_{23}^1 r_{34}^{12} < r_{12}^4 r_{23}^4 r_{34}^2.
\end{aligned}$$

Hence the Hilbert series for Q_4 is given by

$$\begin{aligned}
(38) \quad H(Q_4, t) &= (1 - 18t + 16t^2 - 7t^3)^{-1} \\
&= 1 + 18t + 308t^2 + 5,263t^3 + 89,932t^4 + O(t^5)
\end{aligned}$$

5. THE ALGEBRAS $Q_n^{(k)}$ AND Q_n ARE ALSO KOSZUL

The non-homogeneous quadratic algebras $Q_n^{(k)}$ can also be shown to be Koszul. However, it is not quadratic-linear, as constant terms appear in the relations (cf. [PP05, Chapter 5]). We change the presentation from [GGRW05, 4.3] slightly:

Lemma 11. *The algebra $Q_n^{(k)}$ has generators q_{ij}^I , where $|I| = k - 1$ and $i \notin I$, subject to the relations*

- (i) q_{ij}^I does not depend on the ordering of the elements of I ;
- (ii) $q_{ij}^I = 0$ whenever $j \in I$;
- (iii) $q_{ii}^I = 1$, and $q_{ij}^I q_{jl}^I = q_{il}^I$, $i, j \notin I$;
- (v') If $i \notin M$, $i \in L$, then $\sum_{j \in L \setminus \{i\}} q_{ij}^M q_{jl}^{L \setminus \{j, l\}} + q_{il}^{L \setminus \{i, l\}} = 0$.

Theorem 12. *The non-homogeneous quadratic algebras $Q_n^{(k)}$ and $Q_n^{(\leq k)}$ (and hence, in particular, Q_n) are non-homogeneous Koszul.*

Proof. The proof is similar to that for $R_n^{(k)}$ in Theorem 4, but there are less restrictions of the order of indices. We consider $C_n^{(k)}$, which is the quadratic dual of the associated quadratic algebra $(Q_n^{(k)})^{(0)}$. Denote generators for this algebra by r_{ij}^K (dual to q_{ij}^K). Then $r_{ij}^K r_{ab}^L = 0$ if $j \neq a$ or, if $j = a$, $r_{ij}^K r_{jb}^L = 0$ if $K \neq L$ and $i \notin L$. The relations in $C_n^{(k)}$ are fully described by

$$(39) \quad r_{ij}^M r_{jl}^{L \setminus \{j, l\}} = r_{ij'}^M r_{j'l}^{L \setminus \{j', l\}}, \quad \forall j, j' \in L \setminus \{L \cap M \cup i\},$$

requiring distinct sub-indices and $i \in L$. This means we have a rewriting rule

$$r_{ij}^M r_{jl}^L \mapsto r_{il_{L \setminus M}}^M r_{l_{L \setminus M} l}^L,$$

where $l_{L \setminus M} = \min L \setminus (L \cap M \cup i)$, and $L' = (L \setminus l_{L \setminus M}) \cup j$. One checks that for a critical triple $r_{ij}^I, r_{jl}^J, r_{lm}^L$, applying the rewriting rule to the first two generators and then to the last two generators gives a reduce monomial, and the same reduced monomial emerges if we apply the rewriting rules in opposite order. Hence, the process of applying rewriting rules stabilizes after two steps. This means every critical pair is confluent, and hence the algebra is Koszul (cf. e.g. [LV12, Section 4.1] for these general results and terminology). This implies that $Q_n^{(k)}$ is non-homogeneous Koszul. Further, after adding relation (36), the same is true. Moreover, the process of passing to the quadratic part of relations still commutes with taking the coproduct, and hence the algebras $Q_n^{(\leq k)}$ are also non-homogeneous Koszul. \square

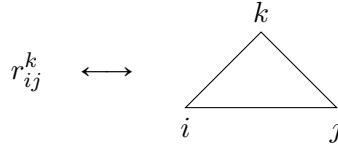
6. DIFFERENTIAL GRADINGS ON THE QUADRATIC DUALS

Using the non-homogeneous quadratic duality of [PP05, 5.4], it follows that the algebras $B_n^{(k)}$ are *differentially graded (dg) algebras*. That is, for each of these algebras, there exist a graded map d of degree one such that

$$(40) \quad d^2 = 0, \quad d(xy) = d(x)y + (-1)^{\deg x} x d(y),$$

for homogeneous elements x, y , which is referred to as the *differential*. We study the case $k = 2$ in more detail and relate it to certain refinements of triangles with labelled corners.

To a generator r_{ij}^k with $i < j$, $k \neq i, j$ associate the triangle with labelled corners



Consider three types of ways to add a corner to the triangles:

$$(41) \quad \begin{array}{c} k \\ \triangle \\ i \quad j \end{array} \longrightarrow - \begin{array}{c} k \\ \triangle \\ i \quad l \quad j \end{array}, \quad i < l < j, \text{ and } l \neq k,$$

$$(42) \quad \begin{array}{c} k \\ \triangle \\ i \quad j \end{array} \longrightarrow - \begin{array}{c} k \\ \triangle \\ i \quad j \end{array} \begin{array}{c} l \end{array}, \quad i < l < j, \text{ and } l \neq k,$$

$$(43) \quad \begin{array}{c} k \\ \triangle \\ i \quad j \end{array} \longrightarrow \begin{array}{c} l \\ \triangle \\ i \quad j \end{array} \quad , \quad i < k < j, \text{ and } l \neq i, k.$$

The triangulations on the right hand side correspond to the products

$$-r_{il}^k r_{lj}^k \longleftrightarrow (41), \quad -r_{il}^k r_{lj}^i \longleftrightarrow (42), \quad r_{ik}^l r_{kj}^i \longleftrightarrow (43).$$

We can recover a quadratic monomial from a triangulation by reading from left to right, reflecting the second triangle in the cases (42) and (43) so that the left corner becomes the top corner.

Now the map $d(r_{ij}^k)$ is the sum over all ways to triangulate $\begin{array}{c} k \\ \triangle \\ i \quad j \end{array}$ by adding one corner in any of the three ways described in (41)–(43).

Corollary 13. *The map $d_B: B_n^{(2)} \rightarrow B_n^{(2)}$ given by*

$$(44) \quad d_B(r_{ij}^k) = \begin{cases} -\sum_{l=i+1}^{j-1} (r_{il}^k r_{lj}^k + r_{il}^k r_{lj}^i) + \sum_{l \neq i} r_{ik}^l r_{kj}^i, & i < k < j, \\ -\sum_{l=i+1}^{j-1} (r_{il}^k r_{lj}^k + r_{il}^k r_{lj}^i), & \text{otherwise.} \end{cases}$$

is a differential for $B_n^{(2)}$.

We can explicitly compute the homology of these dg algebras in small examples: $B_3^{(2)}$ has homology $1 + 2t$, and $B_4^{(2)}$ has homology $1 + 7t + 2t^2$.

The algebras $C_n^{(k)}$ will not give dg algebras, but rather examples of non-trivial *curved* dg algebras [PP05, 5.4, Definition 1].

7. QUASI-FLAG COORDINATES

As an alternative to Plücker coordinates, one can consider *flag coordinates*. These also generalize to non-commutative entries, using quasi-determinants [GGRW05, Section 4.10]. Given a $k \times n$ -matrix, $n \geq k$, choose distinct indices i, j_1, \dots, j_{k-1} in \underline{n} and denote

$$(45) \quad f_{i, \{j_1, \dots, j_k\}}(A) = \begin{vmatrix} a_{1i} & a_{1j_1} & \cdots & a_{1j_{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{a_{ki}} & a_{kj_1} & \cdots & a_{kj_{k-1}} \end{vmatrix}_{ki},$$

which is independent of the order of $\{j_1, \dots, j_k\}$. These functions are referred to as *flag coordinates* and were introduced in [GR92].

For a set I of smaller size, one can consider $f_{i,I}(A)$ by restricting to the first $|I| + 1$ rows of A . Then the following relations hold [GGRW05, Section

4.10.2]:

$$(46) \quad f_{i,I} f_{i,I \setminus k}^{-1} = -f_{k,I \setminus k \cup i} f_{k,I \setminus k}^{-1} \quad \forall k \in I,$$

$$(47) \quad \sum_{i=1}^k f_{j_i, J \setminus j_i} f_{j_i, J \setminus \{j_i, j_{i-1}\}} = 0,$$

where $J = \{j_1, \dots, j_k\}$ and we also denote $j_0 = j_k$.

Definition 14. For $n \geq 2$, we denote by F_n the non-homogeneous quadratic algebra with generators $f_{i,I}$, where I is a subset of \underline{n} and $i \notin I$, and relations given by (46)–(47) as well as

$$(48) \quad f_{i,I} f_{i,I}^{-1} = f_{i,I}^{-1} f_{i,I} = 1.$$

Note that quasi-Plücker and flag coordinates are closely related as by [GGRW05, Section 4.10],

$$(49) \quad q_{i,j}^I(A) = f_{i,I}(A)^{-1} f_{j,I}(A).$$

Theorem 15. *The algebras F_n are non-homogeneous Koszul.*

Proof. Consider the quadratic dual $G_n := (F_n^{(0)})^!$ of the homogeneous part of the relations (46)–(48). In this algebra, denoting the dual generator for $f_{i,I}$ by $g_{i,I}$, we have

$$(50) \quad g_{i,I} g_{i,I}^{-1} + g_{i,I}^{-1} g_{i,I} = 0,$$

$$(51) \quad g_{i,I} g_{i,I \setminus k}^{-1} = g_{k,I \setminus k \cup i} g_{k,I \setminus k}^{-1} \quad \forall k \in I,$$

$$(52) \quad g_{j,J} g_{j,J \setminus \{j,k\}}^{-1} = g_{l,J \setminus l} g_{l,J \setminus \{l,j\}}^{-1}, \quad l, j \in J,$$

plus all quadratic monomials not appearing in these relations are zero. We order the generators $g_{i,I}^{\pm 1}$ lexicographically according to the triple $(|I|, i, I)$ and $g < g^{-1}$, and use the **degrevlex** ordering on monomials. Then the normal words of degree two are

$$g_{i,I} g_{i,I}^{-1}, \quad g_{j,J} g_{j,J \setminus k}^{-1}$$

where $i \notin I$, and $j < J$ (in particular $j < k$, otherwise no restriction on $k \in J$). The rewriting rule is given by

$$g_{i,I} g_{i,I \setminus k}^{-1} \mapsto g_{m_I, I \setminus m_I \cup i} g_{m_I, I \setminus m_I}^{-1}$$

where $m_I = \min I$.

Monomials in which every two neighboring generators are one of these normal words give a basis for G_n , which is thus a non-commutative PBW basis, and hence F_n is non-homogeneous Koszul. \square

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