Binomial Polynomials mimicking Riemann's Zeta Function

Mark W. Coffey and Matthew C. Lettington

March 29, 2017

Abstract

The (generalised) Mellin transforms of certain Chebyshev and Gegenbauer functions based upon the Chebyshev and Gegenbauer polynomials, have polynomial factors $p_n(s)$, whose zeros lie all on the 'critical line' $\Re s = 1/2$ or on the real axis (called critical polynomials). The transforms are identified in terms of combinatorial sums related to H. W. Gould's S:4/3, S:4/2 and S:3/1 binomial coefficient forms. Their 'critical polynomial' factors are then identified as variants of the S:4/1 form, and more compactly in terms of certain ${}_3F_2(1)$ hypergeometric functions. Furthermore, we extend these results to a 1-parameter family of polynomials with zeros only on the critical line. These polynomials possess the functional equation $p_n(s;\beta) = \pm p_n(1-s;\beta)$, similar to that for the Riemann xi function.

It is shown that via manipulation of the binomial factors, these 'critical polynomials' can be simplified to an S:3/2 form, which after normalisation yields the rational function $q_n(s)$. The denominator of the rational form has singularities on the negative real axis, and so $q_n(s)$ has the same 'critical zeros' as the 'critical polynomial' $p_n(s)$. Moreover as $s \to \infty$ along the real axis, $q_n(s) \to 1$ from below, mimicking $1/\zeta(s)$ on the real axis.

In the case of the Chebyshev parameters we deduce the simpler S:2/1 binomial form, and with $\mathcal{C}_n$ the nth Catalan number, s an integer, we show that polynomials $4\mathcal{C}_{n-1}p_{2n}(s)$ and $\mathcal{C}_n p_{2n+1}(s)$ yield integers with only odd prime factors. The results touch on analytic number theory, special function theory, and combinatorics.

1 Introduction

As stated by K. Dilcher and K. B. Stolarsky, [15],

"Two of the most ubiquitous objects in mathematics are the sequence of prime numbers and the binomial coefficients (and thus Pascal's triangle). A connection between the two is given by a well-known characterisation of the prime numbers: Consider the entries in the kth row of Pascal's triangle, without the initial and final entries. They are all divisible by k if and only if k is a prime".

By considering a modified form of Pascal's triangle, whose kth row consists of the integers

$$a(k,j) := \frac{(2k-1)(2k+1)}{2j+3}\binom{k+j}{2j+1}, \qquad k \in \mathbb{N}, \qquad 0 \le j \le k-1, \tag{1.1}$$

The authors would like to thank Dr J. L. Hindmarsh and Prof. M N Huxley for their helpful comments and suggestions.

2010 *Mathematics Subject Classification*: 11B65 (primary), 05A10, 33C20, 33C45, 42C05, 44A20, 30D05 (secondary).

Key words and phrases: Critical polynomials, binomials coefficients, Gould combinatorial summations, Mellin transforms, hypergeometric functions.

Dilcher and Stolarsky obtained an analogous characterisation of pairs of twin prime numbers $(2k-1, 2k+1)$. This says that the entries in the kth row of the $a(k,s)$ number triangle are divisible by $2k-1$ with exactly one exception, and are divisible by $2k+1$ with exactly one exception, if and only if $(2k-1, 2k+1)$ is a pair of twin prime numbers.

Similarly by considering the number triangle whose kth row consists of the integers

$$b(k,j) := \frac{2k+1}{2j+1}\binom{k+j}{2j}, \qquad k \in \mathbb{N}_0, \qquad 0 \le j \le k, \tag{1.2}$$

they showed that the entries in the kth row are divisible by $2k+1$ with exactly one exception, if and only if $2k+1$ is a prime number.

The above are two instances of the numerous connections that exist between the prime numbers and the binomial coefficients [16, 19, 29]. As discovered by Euler, the prime numbers are closely connected to the Riemann zeta function $\zeta(s)$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}} = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s},$$

with the latter series (globally convergent on $s \in \mathbb{C}\backslash\{1\}$), utilising powers of 2 and binomial coefficients (e.g. [22] p206).

Riemann demonstrated that the zeta function $\zeta(s)$, can be obtained via a Mellin transform of a shifted theta function $\theta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}$, deducing the analytic continuation of $\zeta(s)$ to the whole complex plane (except for the simple pole at $s=1$), and the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.3}$$

His original objective was to verify the observation made by the young Gauss (aged 15), that the number $\pi(x)$ of primes $< x$ is closely approximated by the logarithmic integral

$$\pi(x) \approx \mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}u}{\log u}, \tag{1.4}$$

which he surpassed, obtaining the more accurate expression for $\pi(x)$ given by

$$\pi(x) \sim \mathrm{Li}(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/n}).$$

Here $\mu(n)$ is the möbius function, which returns 0 if n is divisible by a prime squared and $(-1)^k$ if n is the product of k distinct primes.

Riemann's Hypothesis (1859) states that all of the non-trivial zeros of $\zeta(s)$ (the trivial zeros lie at the negative even integers) lie on the critical line $\Re s = 1/2$. In 1901 Koch demonstrated that the Riemann Hypothesis is equivalent to the statement that the error term for $\pi(x)$ is of order of magnitude $\sqrt{x}\log(x)$ [24], so that $\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\log(x)\right)$.

The Báez-Duarte equivalence to the Riemann Hypothesis [4, 25] links the binomial coefficients, the Riemann zeta function, and the Riemann Hypothesis (and so the prime numbers) through the coefficients c_t, defined such that

$$c_t := \sum_{s=0}^{t} (-1)^s \binom{t}{s} \frac{1}{\zeta(2s+2)},$$

with the assertion that the Riemann hypothesis is true if and only if $c_t = O(t^{-3/4+\epsilon})$, for integers $t \geq 0$, and for all $\epsilon > 0$.

From the above, one might conclude therefore, that it is important to understand the triangle of connections that exist between the three objects consisting of the prime numbers, the binomial coefficients, and functions which only have critical zeros (those on the line $\Re s = 1/2$ or zeros on the real line), and henceforth referred to as *critical polynomials*). Of these three connections, it is those between the binomial coefficients and the 'critical polynomials' that appears to be the least studied, thus motivating the results contained in this paper.

We recall that the Gegenbauer polynomials with $\lambda > -1/2$, $\lambda \neq 0$ (e.g., [2]) can be written explicitly in terms of binomial coefficients and powers of 2 such that

$$C_n^\lambda(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} \binom{n-r-1+\lambda}{n-r} (2x)^{n-2r}. \tag{1.5}$$

In this article, by addressing the Mellin transforms of the Gegenbauer (and so Chebyshev) functions $C_n^\lambda(x)$, we obtain families of 'critical polynomials' $p_n^\lambda(s)$, $n = 0, 1, 2, \ldots$, of degree $\lfloor n/2 \rfloor$, satisfying the functional equation $p_n^\lambda(s) = (-1)^{\lfloor n/2 \rfloor} p_n^\lambda(1-s)$. Additionally we find that (up to multiplication by a constant) these polynomials can be written explicitly as variants of Gould S:4/1 and S:3/2 binomial sums (see [20]), the latter form being

$$p_{2n}^\lambda(s) = n!(2n)! \binom{n+\lambda-1}{n} \binom{n+\frac{1}{2}(s+\lambda)-\frac{3}{4}}{n} \sum_{r=0}^{n} \frac{(-1)^{n-r} 2^{2r-1} \binom{n+r+\lambda-1}{r} \binom{n+r}{2r} \binom{\frac{1}{2}(s-2)+r}{r}}{\binom{n+r}{r} \binom{\frac{1}{2}(s+\lambda)-\frac{3}{4}+r}{r}}, \tag{1.6}$$

$$p_{2n+1}^\lambda(s) = n!(2n+1)! \binom{n+\lambda}{n+1} \binom{n+\frac{1}{2}(s+\lambda)-\frac{1}{4}}{n} \sum_{r=0}^{n} \frac{(-1)^{n-r} 2^{2r} \binom{n+r+\lambda}{r} \binom{n+r+1}{2r+1} \binom{\frac{1}{2}(s-1)+r}{r}}{\binom{n+r+1}{r} \binom{\frac{1}{2}(s+\lambda)-\frac{1}{4}+r}{r}}. \tag{1.7}$$

In the case of the Chebyshev polynomials ($\lambda = 1$), this simplifies to the $S:2/1$ form, due to cancellation of binomial factors, and with $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, s an integer, we show that polynomials $4\mathcal{C}_{n-1} p_{2n}(s)$ and $\mathcal{C}_n p_{2n+1}(s)$ yield integers with only odd prime factors. The first few transformed Chebyshev polynomials $p_n(s)$, are given by $p_0(s) = 1/2$, $p_1(s) = 1$, $p_2(s) = \frac{3s}{2} - \frac{3}{4}$, $p_3(s) = 6s - 3$, $p_4(s) = 15s^2 - 15s + \frac{63}{4}$.

Before elaborating further, we mention some standard notation in which ${}_2F_1$ denotes the Gauss hypergeometric function, ${}_pF_q$ the generalized hypergeometric function,

$$(a)_n = \Gamma(a+n)/\Gamma(a) = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}$$

the Pochhammer symbol, with Γ the Gamma function [2, 5, 18]. We also define $\varepsilon = 0$ for n even and $\varepsilon = 1$ for n odd.

The "perfect-reflection" functional equation $p_n^\lambda(s) = (-1)^{\lfloor n/2 \rfloor} p_n^\lambda(1-s)$, is similar to that for Riemann's xi function $\xi(s)$, defined by $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$, and which satisfies $\xi(s) = \xi(1-s)$, so that for $t \in \mathbb{R}$, the zeros of $\xi(1/2+it)$ and $\zeta(1/2+it)$ are identical. Drawing upon this analogy, one interpretation is that the polynomials $p_n^\lambda(s)$, are normalised (from a functional equation perspective) polynomial forms of the rational functions $q_n^\lambda(s)$, defined for $n \in \mathbb{N}$ by

$$q_{2n}^\lambda(s) = \frac{2\, p_{2n}^\lambda(s)}{\lambda(n-1)!(2n)!\binom{2n+2\lambda-1}{2n-1}\binom{n+\frac{1}{2}(s+\lambda)-\frac{3}{4}}{n}} = \frac{2^{2n+1} p_{2n}^\lambda(s)}{(2\lambda)_{2n} \prod_{j=1}^{n}(2s+2\lambda+4j-3)}, \tag{1.8}$$

$$q_{2n+1}^{\lambda}(s) = \frac{p_{2n+1}^{\lambda}(s)}{\lambda(n)!(2n)!\binom{2n+2\lambda}{2n}\binom{n+\frac{1}{2}(s+\lambda)-\frac{1}{4}}{n}} = \frac{2^{2n+1}p_{2n}^{\lambda}(s)}{(2\lambda)_{2n+1}\prod_{j=1}^{n}(2s+2\lambda+4j-1)}, \tag{1.9}$$

where both numerator and denominator polynomials of $q_n^{\lambda}(s)$ are of degree $\lfloor n/2 \rfloor$.

For $\lambda > -1/2$, $\lambda \neq 0$, and $\Re s > 0$, the $\lfloor n/2 \rfloor$ linear factors of the denominator polynomials of $q_n^{\lambda}(s)$, are each non-zero, so that for these values of λ, we have $q_n^{\lambda}(s)$ has no singularities with $\Re s > 0$. Hence the rational function $q_n^{\lambda}(s)$ has the same 'critical zeros' as the polynomial $p_n^{\lambda}(s)$, and for $t \in \mathbb{R}$, the roots of $p_n^{\lambda}(1/2+it)$ and $q_n^{\lambda}(1/2+it)$ are identical.

Moreover (Theorem 1.6), for $s \in (1,\infty)$, $q_n^{\lambda}(s)$ takes values on $(0,1)$, with $\lim_{s\to\infty} q_n^{\lambda}(s) = 1$ (from below), and obeys the functional equation

$$q_n^{\lambda}(s) = (-1)^{\lfloor n/2 \rfloor}\binom{\lfloor n/2 \rfloor + \frac{1-s+\lambda+\epsilon}{2} - \frac{3}{4}}{\lfloor n/2 \rfloor}\binom{\lfloor n/2 \rfloor + \frac{s+\lambda+\epsilon}{2} - \frac{3}{4}}{\lfloor n/2 \rfloor}^{-1} q_n^{\lambda}(1-s). \tag{1.10}$$

It follows that on $\mathbb{R}_{>1}$, the behaviour of $q_n^{\lambda}(s)$ has similarities to that of $1/\zeta(s)$, albeit with a rate of convergence to the limit point 1, considerably slower than for $1/\zeta(s)$.

The 'critical polynomials' under consideration here, in a sense motivate the Riemann hypothesis, and have many important applications to analytic number theory. For example, using the Mellin transforms of Hermite functions, Hermite polynomials multiplied by a Gaussian factor, Bump and Ng [7] were able to generalise Riemann's second proof of the functional equation (1.3) of the zeta function $\zeta(s)$, and to obtain a new representation for it.

The polynomial factors of the Mellin transforms of Bump and Ng are realized as certain ${}_2F_1(2)$ Gauss hypergeometric functions [10]. In a different setting, the polynomials $p_n(x) = {}_2F_1(-n,-x;1;2) = (-1)^n\ {}_2F_1(-n,x+1;1;2)$ and $q_n(x) = i^n n! p_n(-1/2-ix/2)$ were studied [23], and they directly correspond to the Bump and Ng polynomials with $s=-x$. Kirschenhofer, Pethö, and Tichy considered combinatorial properties of p_n, and developed Diophantine properties of them. Their analytic results for p_n include univariate and bivariate generating functions, and that its zeros are simple, lie on the line $x=-1/2+it$, $t\in\mathbb{R}$, and that its zeros interlace with those of p_{n+1} on this line. We may observe that these polynomials may as well be written as $p_n(x) = \binom{n+x}{n}\ {}_2F_1(-n,-x;-n-x;-1)$, or

$$p_n(x) = \frac{(-1)^n 2^n \Gamma(n-x)}{n!\Gamma(-x)}\ {}_2F_1\left(-n,-n;x+1-n;\frac{1}{2}\right).$$

Previous results obtained by the authors related to this area of research are discussed in [12, 13, 9], where in the former paper families of 'critical polynomials' are obtained from generalised Mellin transforms of classical orthogonal Legendre polynomials. In [13] the first author has addressed the Mellin transform of certain generalized Hermite functions, where the resulting critical polynomials possess a reciprocity relation. In the latter paper it is demonstrated that sequences of 'critical polynomials' can also be obtained by generalised Mellin transforms of families of orthogonal polynomials whose coefficients are the weighted binomial coefficients given in (1.2), defined by

$$B_k(x) = \sum_{j=0}^{k} b(k,j)x^j = \sum_{j=0}^{k}\frac{2k+1}{2j+1}\binom{k+j}{2j}x^j.$$

Here it is shown that for $\Re s > -1/4$, the Mellin transforms

$$M_n^B(s) = \int_{-4}^{0}\frac{B_n(x)x^{s-3/4}}{(4+x)^{3/4}}dx = (-1)^{s+5/4}4^s 4^{-n-1}\Gamma(1/4)p_n(s)\frac{\Gamma\left(s+\frac{1}{4}\right)}{\Gamma\left(s+\frac{2n+1}{2}\right)},$$

yield critical polynomial factors $p_n(s)$, which obey the 'perfect reflection' functional equation $p_n(s) = \pm p_n(1 - s)$.

The analogous sequence of polynomials $A_k(x)$ obtained from the kth row of the number triangle generated by the integers $a(k,j)$ is similarly given by

$$A_k(x) = \sum_{j=0}^{k-1} a(k,j)x^j = \sum_{j=0}^{k-1} \frac{(2k-1)(2k+1)}{2j+3} \binom{k+j}{2j+1} x^j.$$

It was shown in [15] that this polynomial family satisfies the four-term recurrence relation

$$A_{k+4}(x) = (2x+4)\left(A_{k+3}(x) + A_{k+1}(x)\right) - (4x^2+4x+6)A_{k+2}(x) - A_k(x),$$

as opposed to a three-term recurrence relation required for orthogonality, and so they do not constitute an orthogonal polynomial system. The orthogonality condition is a key ingredient in obtaining critical polynomials via generalised Mellin transforms, as discussed further in the short section entitled *Connection with continuous Hahn polynomials*, following on from the proofs of the main results. However it is also shown in [15] that the polynomials $A_k(x)$ are closely linked to the Gegenbauer polynomials $C_k^{\lambda}(x)$ (defined earlier in (1.5)), with $\lambda = 2$, by the relation

$$A_k(x) = C_{k-1}^2\left(\frac{x+2}{2}\right) + (x+6)C_{k-2}^2\left(\frac{x+2}{2}\right) + C_{k-3}^2\left(\frac{x+2}{2}\right).$$

The Gegenbauer polynomials with $\lambda > -1/2$ have the hypergeometric series representation [1] (p. 773-802) given by

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(2\lambda+n, -n; \lambda+\frac{1}{2}; \frac{1-x}{2}\right), \tag{1.11}$$

immediately showing that $C_n^{\lambda}(1) = (2\lambda)_n/n!$.

The Chebyshev polynomials of the second kind $U_n(x)$ (e.g.,[28]), are the $\lambda = 1$ case of the Gegenbauer polynomials $C_n^{\lambda}(x)$. The latter polynomials are orthogonal on $[-1,1]$ with weight function $(1-x^2)^{\lambda-1/2}$. The Gegenbauer polynomials, and thus the Chebyshev polynomials, satisfy ordinary differential equations. The Gegenbauer polynomials are also referred to as "ultra spherical polynomials", a name acquired from their application to spherical harmonics.

To give an overview, the present work is split into six sections, with the main results concerning the critical polynomials arising from Mellin transforms of Gegenbauer polynomials concluding this introduction. In the second section we prove these results, referring to the *Lemmas* section (Section 4), in which connections to Chebyshev polynomials [26] of both varieties, respectively T_n and U_n, themselves both special cases of the Gegenbauer polynomials, are examined. This follows a short third section concerning continuous Hahn polynomials where another method of proof for the 'critical polynomial' property is outlined. In Section 5 we briefly consider certain Mellin transforms of Chebyshev functions of the first kind T_n, whose polynomial factors have zeros at either the even or the odd integers. We note in passing that the polynomials $B_k(x)$ are related to the Chebyshev polynomials of the second kind such that

$$B_k(x) = U_{2k}\left(\frac{\sqrt{x+4}}{2}\right).$$

In [7, 10, 11], Mellin transforms were used on $[0,\infty)$. Here we consider Mellin transformations for functions supported on $[0,1]$,

$$(\mathcal{M}_0 f)(s) = \int_0^1 f(x)x^s \frac{dx}{x}.$$

For properties of the Mellin transform, we mention [8].

For $\lambda > -1/2$, we put

$$M_n^\lambda(s) = \int_0^1 \frac{C_n^\lambda(x)x^{s-1}}{(1-x^2)^{3/4-\lambda/2}}dx = \int_0^{\pi/2} \cos^{s-1}\theta\ C_n^\lambda(\cos\theta)\sin^{\lambda-1/2}\theta\ d\theta, \tag{1.12}$$

wherein we assume $\Re s > 0$ for n even and $\Re s > -1$ for n odd, denoting by $p_n^\lambda(s)$ the polynomial factor of $M_n^\lambda(s)$. Then for $\lambda = 1$ we have the Mellin transform of the Chebyshev functions of the second kind given by

$$M_n(s) \equiv \int_0^1 x^{s-1}U_n(x)\frac{dx}{(1-x^2)^{1/4}}. \tag{1.13}$$

In fact, we could equally well employ

$$(\mathcal{M}_1 f)(s) = \int_1^\infty f(x)x^s\frac{dx}{x},$$

and $(\mathcal{M}f)(s) = (\mathcal{M}_0 f)(s) + (\mathcal{M}_1 f)(s)$, where the latter representation is to be taken as the analytic continuation of each term. For what we present, it is specifically the analytic continuation of the Gamma function to the whole complex plane that permits $(\mathcal{M}f)(s)$ to exist also through out $\mathbb{C}$. Indeed, here the contributions $(\mathcal{M}_0 f)(s)$ and $(\mathcal{M}_1 f)(s)$ are 'companions'–the analytic continuations of one another.

Theorem 1.1. *(a) When $\lambda = 1$, the polynomials p_n, corresponding to the Chebyshev polynomials of the second kind, satisfy the simplified recursion relation, with $p_0 = \Gamma(3/4)/2$ and $p_1 = \Gamma(3/4)$. for n even,*

$$p_n(s) = sp_{n-1}(s+1) - \frac{1}{2}\left(s+n-\frac{1}{2}\right)p_{n-2}(s), \tag{1.14}$$

and for n odd,

$$p_n(s) = 2p_{n-1}(s+1) - \frac{1}{2}\left(s+n-\frac{1}{2}\right)p_{n-2}(s). \tag{1.15}$$

(b) The polynomials $p_n(s)$, of degree $\lfloor n/2\rfloor$, satisfy the functional equation

$$p_n(s) = (-1)^{\lfloor n/2\rfloor}p_n(1-s).$$

These polynomials have zeros only on the critical line. Further, all zeros $\neq 1/2$ occur in complex conjugate pairs.

Theorem 1.2. *Let $M_n^\lambda(s)$ be defined as in (1.12). Then we have*
(a) (the mixed recurrence relation)

$$nM_n^\lambda(s) = 2(\lambda+n-1)M_{n-1}^\lambda(s+1) - (2\lambda+n-2)M_{n-2}^\lambda(s),$$

(b) the generating function

$$G^\lambda(s,t) \equiv \sum_{k=0}^\infty M_k^\lambda(s)t^k = \int_0^1 \frac{1}{(1-x^2)^{3/4-\lambda/2}}\frac{x^{s-1}}{(1-2tx+t^2)^\lambda}dx$$

$$= \frac{1}{(1+t^2)^\lambda}\frac{\Gamma\left(\frac{1}{4}+\frac{\lambda}{2}\right)}{2}\left[\frac{\Gamma(\lambda)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+\lambda}{2}+\frac{1}{4}\right)}\ {}_3F_2\left(\frac{\lambda+1}{2},\frac{\lambda}{2},\frac{s}{2};\frac{1}{2},\frac{s+\lambda}{2}+\frac{1}{4};\frac{4t^2}{(1+t^2)^2}\right)\right.$$

$$+\frac{2t\Gamma(\lambda+1)}{(1+t^2)}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+\lambda}{2}+\frac{3}{4}\right)}\ {}_3F_2\left(\frac{\lambda+1}{2},1+\frac{\lambda}{2},\frac{s+1}{2};\frac{3}{2},\frac{s+\lambda}{2}+\frac{3}{4};\frac{4t^2}{(1+t^2)^2}\right)\Bigg],$$

(c) the polynomial factors satisfy the functional equation $p_n^\lambda(s)=(-1)^{\lfloor n/2\rfloor}p_n^\lambda(1-s)$,
(d) the recurrence relation in s

$$\begin{gathered}[6-4(\lambda+2\lambda n+n^2)-16s+8s(s+1)]M_n^\lambda(s)\\ +[-9+4(n+\lambda)^2+16(s+2)-4(s+2)(s+3)]M_n^\lambda(s+2)\\ -4(s-1)(s-2)M_n^\lambda(s-2)=0,\end{gathered}$$

with

$$M_0^\lambda(s)=\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+\lambda}{2}+\frac{1}{4}\right)},\qquad \text{and}\qquad M_1^\lambda(s)=2\lambda M_0^\lambda(s+1),$$

and (e) (location of zeros) the polynomial factors $p_n^\lambda(s)$ *of* $M_n^\lambda(s)$*, have zeros only on the critical line.*

Theorem 1.3. *The Mellin transforms (1.12) may be written as a constant multiplied by a variant on Gould's combinatorial S:4/2 and S:3/1 functions, as well a* ${}_3F_2(1)$ *hypergeometric functions, such that*

$$\begin{aligned}M_{2n}^\lambda(s)&=M_0^\lambda(s)\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda-1}{n+r}\binom{n+r}{2r}\binom{\frac{s-2}{2}+r}{r}\binom{n+\frac{s+\lambda}{2}-\frac{3}{4}}{n-r}}{\binom{n}{r}\binom{n+\frac{s+\lambda}{2}-\frac{3}{4}}{n}}\\ &=M_0^\lambda(s)\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda-1}{n+r}\binom{n+r}{2r}\binom{\frac{s-2}{2}+r}{r}}{\binom{\frac{s+\lambda}{2}-\frac{3}{4}+r}{r}}\\ &=(-1)^nM_0^\lambda(s)\binom{\lambda+n-1}{n}\ {}_3F_2\left(-n,\lambda+n,\frac{s}{2};\frac{1}{2},\frac{\lambda}{2}+\frac{s}{2}+\frac{1}{4};1\right),\end{aligned}$$

and

$$\begin{aligned}M_{2n+1}^\lambda(s)&=M_0^\lambda(s+1)\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r+1}\binom{n+r+\lambda}{n+r+1}\binom{n+r+1}{2r+1}\binom{\frac{s-1}{2}+r}{r}\binom{n+\frac{s+\lambda}{2}-\frac{1}{4}}{n-r}}{\binom{n}{r}\binom{n+\frac{s+\lambda}{2}-\frac{1}{4}}{n}}\\ &=M_0^\lambda(s+1)\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r+1}\binom{n+r+\lambda}{n+r+1}\binom{n+r+1}{2r+1}\binom{\frac{s-1}{2}+r}{r}}{\binom{\frac{s+\lambda}{2}-\frac{1}{4}+r}{r}}\\ &=(-1)^n2M_0^\lambda(s+1)(n+1)\binom{\lambda+n}{n+1}\ {}_3F_2\left(-n,\lambda+n+1,\frac{s}{2}+\frac{1}{2};\frac{3}{2},\frac{\lambda+s}{2}+\frac{3}{4};1\right),\end{aligned}$$

where the latter identities for odd and even cases give the equivalent hypergeometric form which can be written for general n *as*

$$M_n^\lambda(s)=\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+n}{2}\right)}{2(n!)\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)}\ {}_3F_2\left(\frac{\lambda}{2}+\frac{1}{4},\frac{1-n}{2},-\frac{n}{2};\frac{1}{2}+\lambda,1-\frac{(n+s)}{2};1\right).$$

Corollary 1.4. *When $s = 2s_1$ is even we have*

$$M_0^\lambda(s) = \frac{1}{2s}\binom{\frac{\lambda}{2}+\frac{1}{4}+s_1-1}{s_1}^{-1},$$

and when $s = 2s_1+1$ is odd

$$M_1^\lambda(s) = 2\lambda M_0^\lambda(s+1) = 2\lambda M_0^\lambda(2s_1+2) = \frac{\lambda}{s+1}\binom{\frac{\lambda}{2}+\frac{1}{4}+s_1}{s_1+1}^{-1}.$$

Hence, when s is a non-negative integer then we have the Gould's S:4/3 summation variants

$$M_{2n}^\lambda(2s) = \frac{1}{2s}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda-1}{n+r}\binom{n+r}{2r}\binom{s-1+r}{r}\binom{n+s+\frac{\lambda}{2}-\frac{3}{4}}{n-r}}{\binom{n}{n-r}\binom{s+\frac{\lambda}{2}-\frac{3}{4}}{s}\binom{n+s+\frac{\lambda}{2}-\frac{3}{4}}{n}},$$

and

$$M_{2n+1}^\lambda(2s+1) = \frac{\lambda}{s+1}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r+1}\binom{n+r+\lambda}{n+r+1}\binom{n+r+1}{2r+1}\binom{s+r}{r}\binom{n+s+\frac{\lambda}{2}+\frac{1}{4}}{n-r}}{\binom{n}{r}\binom{s+\frac{\lambda}{2}+\frac{1}{4}}{s+1}\binom{n+s+\frac{\lambda}{2}+\frac{1}{4}}{n}}.$$

Theorem 1.5. *Let $\varepsilon = (1-(-1)^n)/2$. Then the Mellin transforms are of the form*

$$M_n^\lambda(s) = \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+\epsilon}{2}\right)}{2(n!)\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)}\,p_n^\lambda(s),$$

and the polynomial factors $p_n^\lambda(s)$ satisfy the difference equations

$$(s+2)(2\lambda+4n-2s-5)p_{2n}^\lambda(s+4) - \left(8\lambda n+2\lambda+8n^2-4s^2-12s-11\right)p_{2n}^\lambda(s+2)$$
$$-(s+1)(2\lambda+4n+2s+1)p_{2n}^\lambda(s) = 0,$$

and

$$(s+3)(2\lambda+4n-2s-3)p_{2n+1}^\lambda(s+4) - \left(8\lambda n+6\lambda+8n^2+8n-4s^2-12s-9\right)p_{2n+1}^\lambda(s+2)$$
$$-s(2\lambda+4n+2s+3)p_{2n+1}^\lambda(s) = 0.$$

.

Theorem 1.6 (Binomial 'critical polynomial' theorem)**.** *(a) The polynomials $p_n^\lambda(s)$, can be written (up to multiplication by a constant) in terms of binomial coefficients and powers of 2, as a variant of a Gould S:4/1 combinatorial function, such that*

$$p_{2n}^\lambda(s) = n!(2n)!\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{n+r}\binom{n+r}{2r}\binom{\frac{1}{2}(s-2)+r}{r}\binom{n+\frac{1}{2}(s+\lambda)-\frac{3}{4}}{n-r}}{\binom{n}{r}},$$

$$p_{2n+1}^\lambda(s) = n!(2n+1)!\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{n+r+1}\binom{n+r+1}{2r+1}\binom{\frac{1}{2}(s-1)+r}{r}\binom{n+\frac{1}{2}(s+1+\lambda)-\frac{3}{4}}{n-r}}{\binom{n}{r}},$$

or as a Gould S:3/2 combinatorial function variant, such that

$$p_{2n}^\lambda(s) = n!(2n)!\binom{n+\lambda-1}{n}\binom{n+\frac{1}{2}(s+\lambda)-\frac{3}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{r}\binom{n+r}{2r}\binom{\frac{1}{2}(s-2)+r}{r}}{\binom{n+r}{r}\binom{\frac{1}{2}(s+\lambda)-\frac{3}{4}+r}{r}},$$

$$p^{\lambda}_{2n+1}(s) = n!(2n+1)!\binom{n+\lambda}{n+1}\binom{n+\frac{1}{2}(s+\lambda)-\frac{1}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{r}\binom{n+r+1}{2r+1}\binom{\frac{1}{2}(s-1)+r}{r}}{\binom{n+r+1}{r}\binom{\frac{1}{2}(s+\lambda)-\frac{1}{4}+r}{r}},$$

thus establishing the binomial 'critical polynomial' relationships (1.6) and (1.7).

Setting $\lambda = 1$ we obtain the polynomial factors arising from the Mellin transform of the Chebyshev polynomials, which have the simpler form as a variant of a Gould S:2/1 combinatorial function, such that

$$p_{2n}(s) = n!(2n)!\binom{n+\frac{s}{2}-\frac{1}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r}{2r}\binom{\frac{1}{2}(s-2)+r}{r}}{\binom{\frac{s}{2}-\frac{1}{4}+r}{r}},$$

$$p_{2n+1}(s) = n!(2n+1)!\binom{n+\frac{s}{2}+\frac{1}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+1}{2r+1}\binom{\frac{1}{2}(s-1)+r}{r}}{\binom{\frac{s}{2}+\frac{1}{4}+r}{r}}.$$

(b) Let $q^{\lambda}_{n}(s)$ denote the rational function in s derived from the S:3/2 form of the 'critical polynomial' $p^{\lambda}_{n}(s)$ such that

$$q^{\lambda}_{2n}(s) = \frac{2n\binom{n+\lambda-1}{n}}{\lambda\binom{2n+2\lambda-1}{2n-1}}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{r}\binom{n+r}{2r}\binom{\frac{1}{2}(s-2)+r}{r}}{\binom{n+r}{r}\binom{\frac{1}{2}(s+\lambda)-\frac{3}{4}+r}{r}},$$

$$q^{\lambda}_{2n+1}(s) = \frac{(2n+1)\binom{n+\lambda}{n+1}}{\lambda\binom{2n+2\lambda}{2n}}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{r}\binom{n+r+1}{2r+1}\binom{\frac{1}{2}(s-1)+r}{r}}{\binom{n+r+1}{r}\binom{\frac{1}{2}(s+\lambda)-\frac{1}{4}+r}{r}}.$$

Then we have

$$q^{\lambda}_{2n}(s) = \frac{2\,p^{\lambda}_{2n}(s)}{\lambda(n-1)!(2n)!\binom{2n+2\lambda-1}{2n-1}\binom{n+\frac{1}{2}(s+\lambda)-\frac{3}{4}}{n}} = \frac{2^{2n+1}p^{\lambda}_{2n}(s)}{(2\lambda)_{2n}\prod_{j=1}^{n}(2s+2\lambda+4j-3)},$$

$$q^{\lambda}_{2n+1}(s) = \frac{p^{\lambda}_{2n+1}(s)}{\lambda(n)!(2n)!\binom{2n+2\lambda}{2n}\binom{n+\frac{1}{2}(s+\lambda)-\frac{1}{4}}{n}} = \frac{2^{2n+1}p^{\lambda}_{2n}(s)}{(2\lambda)_{2n+1}\prod_{j=1}^{n}(2s+2\lambda+4j-1)},$$

where both numerator and denominator polynomials of $q^{\lambda}_{n}(s)$ are of degree $\lfloor n/2\rfloor$.
(c) For $\lambda > -1/2$, $\lambda \neq 0$, and $\Re s > 0$, the rational function $q^{\lambda}_{n}(s)$ has no singularities, and has the same 'critical zeros' as the polynomial $p^{\lambda}_{n}(s)$, so that for $t \in \mathbb{R}$, the roots of $p^{\lambda}_{n}(1/2+it)$ and $q^{\lambda}_{n}(1/2+it)$ are identical.

When $s \in \mathbb{R}_{>1}$, $q^{\lambda}_{n}(s)$ takes values on $(0,1)$, with $\lim_{s\to\infty} q^{\lambda}_{n}(s) = 1$ (from below), and with $\varepsilon = 0$ for n even and $\varepsilon = 1$ for n odd, obeys the functional equation

$$q^{\lambda}_{n}(s) = (-1)^{\lfloor n/2\rfloor}\binom{\lfloor n/2\rfloor+\frac{1-s+\lambda+\epsilon}{2}-\frac{3}{4}}{\lfloor n/2\rfloor}\binom{\lfloor n/2\rfloor+\frac{s+\lambda+\epsilon}{2}-\frac{3}{4}}{\lfloor n/2\rfloor}^{-1} q^{\lambda}_{n}(1-s). \tag{1.16}$$

As $s \to \infty$ along the positive real axis is follows that $q^{\lambda}_{n}(s) \to 1$, from below, as does $1/\zeta(s)$.

Corollary 1.7. *The polynomial factors arising from the Mellin transform of the Chebyshev polynomials have the simpler form as a variant of a Gould S:2/1 combinatorial function, such that*

$$p_{2n}(s) = n!(2n)!\binom{n+\frac{s}{2}-\frac{1}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r}{2r}\binom{\frac{1}{2}(s-2)+r}{r}}{\binom{\frac{s}{2}-\frac{1}{4}+r}{r}},$$

$$p_{2n+1}(s) = n!(2n+1)!\binom{n+\frac{s}{2}+\frac{1}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+1}{2r+1}\binom{\frac{1}{2}(s-1)+r}{r}}{\binom{\frac{s}{2}+\frac{1}{4}+r}{r}}.$$

For s a positive integer and and with $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, the polynomials $4\mathcal{C}_{n-1}p_{2n}(s)$ and $\mathcal{C}_n p_{2n+1}(s)$ yield integers with only odd prime factors. Moreover the polynomials

$$\frac{2^{2n+1}}{(2n)!}p_{2n}(s), \quad \text{and} \quad \frac{2^{2n+1}\mathcal{T}_{n+1}}{(2n+2)!}p_{2n+1}(s),$$

with $\mathcal{T}_{n+1}$ the largest odd factor of $n+1$, yield odd integers with fewer prime factors.

Theorem 1.8 (Perfect reflection property theorem)**.** *We say that "$f(s)$ has the* perfect reflection property*" to mean $f(\overline{s}) = \overline{f(s)}$, $f(s) = \chi f(1-s)$, with $\chi = \pm 1$, $f(s) = 0$, only when $\Re s = 1/2$.*

Then the polynomials

$$p_n(s;\beta) = \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)}\, {}_3F_2\left(1-\beta, \frac{1-n}{2}, -\frac{n}{2}; 2(1-\beta), 1-\frac{(n+s)}{2}; 1\right),$$

have the perfect reflection property *with $\chi(n) = (-1)^{\lfloor n/2 \rfloor}$, wherein $\varepsilon = 0$ for n even and $= 1$ for n odd, $\beta < 1$, of degree $\lfloor n/2 \rfloor$, satisfy the functional equation $p_n(s;\beta) = (-1)^{\lfloor n/2 \rfloor}p_n(1-s;\beta)$. These polynomials have zeros only on the critical line, and all zeros $\neq 1/2$ occur in complex conjugate pairs.*

Corollary 1.9. *(a) The properties of Theorem1.8 are satisfied by the polynomials*

$$p_n(s;0) = \frac{2(n+s)}{(n+1)(n+2)}\frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)}\left[1 - {}_2F_1\left(-\frac{(n+1)}{2}, -\frac{n}{2}-1; -\frac{(n+s)}{2}; 1\right)\right]$$

$$= \frac{2(n+s)}{(n+1)(n+2)}\frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)}\left[1 - \frac{\Gamma\left(-\frac{(n+s)}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\frac{\Gamma\left(\frac{n+3-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\right].$$

(b) More generally, for β a negative integer, the properties of Theorem1.8 are satisfied by the polynomials $p_n(s;-m)$, and these polynomials may be written in terms of elementary factors and the Gamma function.

2 Proof of Theorems

Proof of Theorem 1.1. (a) According to Lemma 4.6 or 4.8, the Mellin transforms are of the form

$$M_n(s) = \Gamma\left(\frac{3}{4}\right)\frac{p_n(s)\Gamma\left(\frac{s+\varepsilon}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{2n+3}{4}\right)}, \tag{2.1}$$

where $\varepsilon = 0$ for n even and $= 1$ for n odd. The recursions (1.14) and (1.15) follow by inserting the form (2.1) into (4.1), and using the functional equation of the Gamma function, $\Gamma(z+1) = z\Gamma(z)$.

(b) From Lemma 4.8(a), up to factors not involving s, the polynomials p_n may be taken as

$$p_n(s) = (n+1)\frac{\Gamma\left(\frac{3}{4}\right)}{2}\frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)}\, {}_3F_2\left(\frac{3}{4}, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, 1-\frac{(n+s)}{2}; 1\right), \tag{2.2}$$

wherein $\varepsilon = 0$ for n even and $= 1$ for n odd. From the form of the numerator parameters, the degree of p_n is evident.

By a 'Beta transformation' [18] (p. 850) we have the following integral representation:

$$
{}_3F_2\left(\frac{3}{4}, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, 1-\frac{(n+s)}{2}; 1\right)
$$

$$
= \frac{\sqrt{\pi}}{2\Gamma^2(3/4)} \int_0^1 (1-x)^{-1/4} x^{-1/4}\ {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\frac{(n+s)}{2}; x\right) dx.
$$

We use (2.2), together with an $x \to 1-x$ transformation of the ${}_2F_1$ function [18] (p. 1043). Owing to the poles of the Γ function, and that n is a nonnegative integer, the ${}_2F_1(x)$ function then transforms to a single ${}_2F_1(1-x)$ function, and there results

$$
p_n(s) = \frac{(n+1)}{4} \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)} \frac{\sqrt{\pi}}{\Gamma(3/4)} \frac{\Gamma\left(1-\frac{(n+s)}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} \frac{\Gamma\left(\frac{1+n-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}
$$

$$
\times \int_0^1 (1-x)^{-1/4} x^{-1/4}\ {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{s-n+1}{2}; 1-x\right) dx
$$

$$
= \frac{(n+1)}{4} \frac{\pi}{\Gamma\left(\frac{s+\varepsilon}{2}\right)} \frac{\sqrt{\pi}}{\sin \pi \left(\frac{n+s}{2}\right) \Gamma(3/4)} \frac{1}{\Gamma\left(1-\frac{s}{2}\right)} \frac{\Gamma\left(\frac{1+n-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}
$$

$$
\times \int_0^1 (1-x)^{-1/4} x^{-1/4}\ {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{s-n+1}{2}; x\right) dx.
$$

The following observations lead to verification of the functional equation. When n is even, $\varepsilon = 0$,

$$
\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right) = \frac{\pi}{\sin \pi(s/2)},
$$

leaving the denominator factor $\Gamma\left(\frac{1-s}{2}\right)$. When n is odd, $\varepsilon = 1$,

$$
\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos \pi(s/2)},
$$

leaving the denominator factor $\Gamma\left(1-\frac{s}{2}\right)$.

Hence the factor $(-1)^{\lfloor n/2 \rfloor}$ emerges as $\sin(\pi s/2)/\sin[\pi(n+s)/2] = (-1)^{n/2}$ when n is even and as $\cos(\pi s/2)/\sin[\pi(n+s)/2] = (-1)^{(n-1)/2}$ when n is odd, and the functional equation of $p_n(s)$ follows.

Let u be the shifted polynomial $u(s) = p_n(s+1/2)$, so that $p_n(s) = u(s-1/2)$. Furthermore, let r_i be the roots of u, $u(s) = c\prod_i(s-r_i)$, where c is a constant. As a special case (of $\lambda = 1$) in the difference equation (2.3), it follows that for s being a root r_k that

$$
-(r_k - n)\left(r_k + \frac{1}{2}\right) c \prod_i (r_k + 2 - r_i) = (r_k + n)\left(r_k - \frac{1}{2}\right) c \prod_i (r_k - 2 - r_i).
$$

The equality of the absolute value of both sides of this equation provides a necessary condition that $\Re\, r_i = 0$. With all zeros of u being pure imaginary, all of those of $p_n(s)$ lie on the critical line. Since the polynomial coefficients are real, in fact, rational numbers, the zeros of p_n aside from 1/2 occur as complex conjugates. $\square$

Proof of Theorem 1.2. (a) follows from ([2], p. 303 or [18], p. 1030)

$$(n+2)C_{n+2}^{\lambda}(x) = 2(\lambda+n+1)xC_{n+1}^{\lambda}(x) - (2\lambda+n)C_n^{\lambda}(x),$$

$C_0^{\lambda}(x) = 1$, and $C_1^{\lambda}(x) = 2\lambda x$. (b) follows from ([2], p. 302 or [18], p. 1029)

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n.$$

(c) The Mellin transforms may be computed explicitly by several means. In particular, from

$$C_n^{\lambda}(x) = \frac{(\lambda)_n}{n!}(2x)^n \ {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; 1-n-\lambda; \frac{1}{x^2}\right),$$

transformation of the ${}_2F_1$ function to argument x^2 and use of [18] (p. 850), we find that

$$M_n^{\lambda}(s) = \frac{(\lambda)_n}{n!}2^{n-1}\sqrt{\pi}(-i)^n\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma(1-\lambda-n)$$

$$\times\left\{\frac{-2i\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{1}{2}-\lambda-\frac{n}{2}\right)}\frac{{}_3F_2\left(\frac{1-n}{2}, \lambda+\frac{n+1}{2}, \frac{s+1}{2}; \frac{3}{2}, \frac{3}{4}+\frac{\lambda+s}{2}; 1\right)}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{\lambda+s}{2}+\frac{3}{4}\right)}\right.$$

$$\left.+\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-n}{2}\right)}\frac{{}_3F_2\left(-\frac{n}{2}, \lambda+\frac{n}{2}, \frac{s}{2}; \frac{1}{2}, \frac{1}{4}+\frac{\lambda+s}{2}; 1\right)}{\Gamma\left(1-\lambda-\frac{n}{2}\right)\Gamma\left(\frac{\lambda+s}{2}+\frac{1}{4}\right)}\right\}.$$

The second line of the right member provides the transform for n odd and the third line for n even. Then transformation of the ${}_3F_2$ functions and arguments similar to the proof of Theorem 1.1 may be used to show that the degree of the polynomial factors of the transforms is again $\lfloor n/2 \rfloor$, and that they satisfy the functional equation $p_n^{\lambda}(s) = (-1)^{\lfloor n/2 \rfloor}p_n^{\lambda}(1-s)$.

(e) To show that the resulting zeros occur only on $\Re\, s = 1/2$ we first demonstrate the difference equation

$$[6-4(\lambda+2\lambda n+n^2)-16s+8s(s+1)]\left(\frac{s+\epsilon}{2}-1\right)\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)p_n^{\lambda}(s)$$

$$+[-9+4(n+\lambda)^2-4(s-1)(s+2)]\left(\frac{s+\epsilon}{2}\right)\left(\frac{s+\epsilon}{2}-1\right)p_n^{\lambda}(s+2) \tag{2.3}$$

$$-4(s-1)(s-2)\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)\left(\frac{s+n+\lambda}{2}-\frac{3}{4}\right)p_n^{\lambda}(s-2) = 0,$$

wherein $\epsilon = 0$ for n even and $= 1$ for n odd. We apply the ordinary differential equation satisfied by Gegenbauer polynomials (e.g., [18], p. 1031)

$$(x^2-1)y''(x) + (2\lambda+1)xy'(x) - n(2\lambda+n)y(x) = 0.$$

If $f(x) \equiv C_n^{\lambda}(x)/(1-x^2)^{3/4-\lambda/2}$, we then substitute $C_n^{\lambda}(x) = (1-x^2)^{3/4-\lambda/2}f(x)$ into this differential equation. We then find that

$$\frac{1}{4}(1-x^2)^{-1/4-\lambda/2}\left[(6-4(\lambda+2\lambda n+n^2)+(-9+4(\lambda+n)^2)x^2)f(x)\right.$$

$$\left.+4(x^2-1)(-4xf'(x)+(1-x^2)f''(x))\right] = 0.$$

It follows that the quantity in square brackets is zero. We multiply it by x^{s-1} and integrate from $x=0$ to 1, integrating the f' term once by parts, and the f'' term twice by parts. We determine that the Mellin transforms satisfy the following difference equation:

$$[6-4(\lambda+2\lambda n+n^2)-16s+8s(s+1)]M_n^\lambda(s)$$
$$+[-9+4(n+\lambda)^2+16(s+2)-4(s+2)(s+3)]M_n^\lambda(s+2)$$
$$-4(s-1)(s-2)M_n^\lambda(s-2)=0,$$

and hence (d).

As follows from either part (c) or Theorem 1.3, the Mellin transforms are of the form

$$M_n^\lambda(s)=\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+\epsilon}{2}\right)}{2(n!)\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)}p_n^\lambda(s).$$

Noting that the factor $\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)/(2n!)$ is independent of s, and repeatedly applying the functional equation $\Gamma(z+1)=z\Gamma(z)$ leads to (2.3).

Using the shifted polynomial $u(s)=p_n^\lambda(s+1/2)$, so that $p_n^\lambda(s)=u(s-1/2)$ is convenient. Then with the translation $s\to s+1/2$, (2.3) gives

$$\left[6-4(\lambda+2\lambda n+n^2)-16\left(s+\frac{1}{2}\right)+8\left(s+\frac{1}{2}\right)\left(s+\frac{3}{2}\right)\right]\left(\frac{s+\epsilon}{2}-\frac{3}{4}\right)\left(\frac{s+n+\lambda}{2}+\frac{1}{2}\right)u(s)$$
$$+\left[-9+4(n+\lambda)^2-4\left(s-\frac{1}{2}\right)\left(s+\frac{5}{2}\right)\right]\left(\frac{s+\epsilon}{2}+\frac{1}{4}\right)\left(\frac{s+\epsilon}{2}-\frac{3}{4}\right)u(s+2)$$
$$-4\left(s-\frac{1}{2}\right)\left(s-\frac{3}{2}\right)\left(\frac{s+n+\lambda}{2}+\frac{1}{2}\right)\left(\frac{s+n+\lambda}{2}-\frac{1}{2}\right)u(s-2)=0.$$

After some simplification, in particular cancelling a factor of $s+n+\lambda+1$ on both sides, it follows that if r_k is a root of u, $u(r_k)=0$, that

$$-(r_k-n-\lambda+1)\left(r_k+\epsilon+\frac{1}{2}\right)\left(r_k+\epsilon-\frac{3}{2}\right)u(r_k+2)$$
$$=(r_k+n+\lambda-1)\left(r_k-\frac{1}{2}\right)\left(r_k-\frac{3}{2}\right)u(r_k-2).$$

It may be noted that when n is even, a factor of $r_k-3/2$ cancels on both sides, and when n is odd, a factor of $r_k-1/2$ cancels on both sides. In either case, equality of the absolute value of both sides provides a necessary condition that $\Re\, r_i=0$ for all the zeros of u. Hence the zeros of $p_n^\lambda(s)$ lie on the critical line. □

Proof of Theorem 1.3. Here we determine the hypergeometric form of $M_n^\lambda(s)$, from which the binomial forms then follow by algebraic manipulation.

We use the series representation ([27], p. 278 (6))

$$C_n^\lambda(x)=\sum_{k=0}^{[n/2]}\frac{(2\lambda)_n x^{n-2k}(x^2-1)^k}{4^k k!(\lambda+1/2)_k(n-2k)!},$$

and recall a form of the Beta integral,

$$\int_0^1 x^{a-1}(1-x^2)^{b-1}dx=\frac{1}{2}B\left(\frac{a}{2},b\right),\qquad \Re\, a>0,\quad \Re\, b>0.$$

Then

$$M_n^\lambda(s) = \sum_{k=0}^{[n/2]} \frac{(2\lambda)_n(-1)^k}{4^k k!(\lambda+1/2)_k(n-2k)!} \int_0^1 x^{s+n-2k-1}(1-x^2)^{k+\lambda/2-3/4}dx$$

$$= \sum_{k=0}^{[n/2]} \frac{(2\lambda)_n(-1)^k}{4^k k!(\lambda+1/2)_k(n-2k)!} \frac{1}{2} B\left(\frac{s+n}{2}-k, k+\frac{\lambda}{2}+\frac{1}{4}\right)$$

$$= \frac{(2\lambda)_n}{2\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)} \sum_{k=0}^{[n/2]} \frac{(-1)^k}{4^k k!} \frac{\Gamma\left(\frac{s+n}{2}-k\right)\Gamma\left(k+\frac{\lambda}{2}+\frac{1}{4}\right)}{(\lambda+1/2)_k(n-2k)!}$$

$$= \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+n}{2}\right)}{2n!\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)} {}_3F_2\left(\frac{\lambda}{2}+\frac{1}{4}, \frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2}+\lambda, 1-\frac{(n+s)}{2}; 1\right),$$

as required.

The above has used the duplication formula for the Gamma function so that

$$\frac{1}{(n-2k)!} = \frac{1}{\Gamma(n-2k+1)} = \frac{4^k}{n!}\left(-\frac{n}{2}\right)_k \left(\frac{1-n}{2}\right)_k.$$

The latter form of $M_n^\lambda(s)$ exhibits the key feature of a denominator parameter twice a numerator parameter and we identify $\beta = 3/4-\lambda/2$ in Theorem 1.8. The stated properties follow in accord with that result.

The binomial forms of $M_n^\lambda(s)$ are obtained by rewriting the hypergeometric form in terms of Pochhammer symbols, and then (in a variety of orders) applying the five identities

$$\binom{a+s-1}{s} = \frac{a(a+1)\dots(a+s-1)}{s!} = \frac{(a)_s}{s!},$$

$$\binom{n+m}{k}\binom{n}{k}^{-1} = \frac{(n+m)_m}{(n-k+1)_m}, \qquad \binom{n}{k+m}\binom{n}{k}^{-1} = \frac{(n-m-k+1)_m}{(k+1)_m},$$

$$\binom{n}{k-m}\binom{n}{k}^{-1} = \frac{(n-k+1)_m}{(k-m+1)}, \qquad \binom{c}{b}\binom{a+b}{b}^{-1} = \binom{a+c}{c}^{-1}\binom{a+c}{c-b}.$$

Corollary 1.4 then follows immediately by by replacing $M_0^\lambda(s)$ and $M_0^\lambda(s+1)$ with their equivalent binomial coefficients forms when s is respectively an even or an odd integer. □

Remark. *With a simple change of variable, the Mellin transforms of* (1.12) *may be obtained from [18] (p. 830). Alternative forms of these transforms and their generating functions may be realised by using various expressions from [27] (pp. 279–280).*

Proof of Theorem 1.5. It follows from either part (c) of Theorem 1.2 or the hypergeometric form in Theorem 1.3, that the Mellin transforms are of the form

$$M_n^\lambda(s) = \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+\epsilon}{2}\right)}{2(n!)\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)} p_n^\lambda(s).$$

The recurrence relations for $p_n^\lambda(s)$ are obtained by considering the general recurrence relation for $M_n^\lambda(s)$ and $p_n^\lambda(s)$, given in the proof of Theorem 1.2 (d), separately for the two cases of n odd or n even. □

Proof of Theorem 1.6. (a) The S:4/1 type combinatorial expressions for the polynomial factors $p_n^\lambda(s)$ are obtained from the S:4/2 type expressions for $M_n^\lambda(s)$ in Theorem 1.3, by respectively multiplying through by the factors

$$\frac{n!(2n)!}{2M_0^\lambda(s)}\binom{n+\frac{s+\lambda}{2}-\frac{3}{4}}{n}, \qquad \text{or} \qquad \frac{n!(2n+1)!}{2M_0^\lambda(s+1)}\binom{n+\frac{s+1+\lambda}{2}-\frac{3}{4}}{n},$$

depending on whether n is odd or even.
(b) The two expressions for $q_n^\lambda(s)$ given can be verified by inserting the explicit expressions for p_n^λ given in part (a) into the latter expression for $q_n^\lambda(s)$ given in part (b) and rearranging. The degree of both numerator and denominator polynomials of $q_n^\lambda(s)$ being $\lfloor n/2\rfloor$, then follows from the degree of the polynomials $p_n^\lambda(s)$ given in Theorem 1.2, and the number of s-linear factors appearing in the denominator product of $q_n^\lambda(s)$.
(c) The zeros of the denominator polynomials (and so poles of q_n^λ), correspond to the zeros of the linear factors $2s+2\lambda+4j-3$, or $2s+2\lambda+4j-1$, with $1\le j\le\lfloor n/2\rfloor$. For $\lambda>-1/2$, $\lambda\neq 0$ and $\Re s>0$, each linear factor is non-zero, ensuring that the rational function $q_n^\lambda(s)$ has no singularities. Hence the 'critical zeros' of the polynomials $p_n^\lambda(s)$, are the same as for $q_n^\lambda(s)$, and so for $t\in\mathbb{R}$, the roots of $p_n^\lambda(1/2+it)$ and $q_n^\lambda(1/2+it)$ are identical.

To see that the rational functions $q_n^\lambda(s)$ are normalised with limit 1 as $s\to\infty$, we consider the limit term by term as $s\to\infty$ in the S:3/2 sums of (1.6) and (1.7), giving

$$\lim_{s\to\infty}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{r}\binom{n+r}{2r}\binom{\frac{1}{2}(s-2)+r}{r}}{\binom{n+r}{r}\binom{\frac{1}{2}(s+\lambda)-\frac{3}{4}+r}{r}}=\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{r}\binom{n+r}{2r}}{\binom{n+r}{r}},$$

$$\lim_{s\to\infty}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{r}\binom{n+r+1}{2r+1}\binom{\frac{1}{2}(s-1)+r}{r}}{\binom{n+r+1}{r}\binom{\frac{1}{2}(s+\lambda)-\frac{1}{4}+r}{r}}=\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{r}\binom{n+r+1}{2r+1}}{\binom{n+r+1}{r}}.$$

Applying the combinatorial identities

$$\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{r}\binom{n+r}{2r}}{\binom{n+r}{r}}=\frac{1}{2}\binom{2n+2\lambda-1}{2n-1}\binom{n+\lambda-1}{n-1}^{-1},$$

$$\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{r}\binom{n+r+1}{2r+1}}{\binom{n+r+1}{r}}=\frac{n+1}{2n+1}\binom{2n+2\lambda}{2n}\binom{n+\lambda}{n}^{-1},$$

we then have the upper bounds for the combinatorial sums, so that $\lim_{s\to\infty}q_n^\lambda(s)=1$ from below, as required. The functional equation follows from that for $p_n^\lambda(s)$, by considering the third and fourth displays in part (b) of the theorem.

To see Corollary 1.7, substituting $\lambda=1$ in the S/3:2 forms for $p_n^\lambda(s)$, simplifies the sums to the Gould S/2:1 combinatorial functions stated. Term-by-term analysis of the $n+1$ terms in each sum then reveals that for s an integer, each term is an even integer apart form the $r=0$ term, given by

$$\frac{n!(2n)!}{2}\binom{n+\frac{s}{2}-\frac{1}{4}}{n}, \quad \text{and} \quad \frac{n!(2n+2)!}{2}\binom{n+\frac{s}{2}+\frac{1}{4}}{n},$$

depending of whether n is respectively even or odd. The binomial coefficients contribute the power of two 2^{-2n}, so that the power of 2 in the $r=0$ term is determined by $(2n)!/2^{2n+1}$ when n is even, and $(2n{+}2)!/2^{2n+1}$, when n is odd. Noting that the $n!$ terms cancel between numerator and denominator, we see that multiplying through by the reciprocal of these respective powers of 2 will produce odd integer values for the $r = 0$ term, whilst leaving the others terms $r = 1, 2, \ldots, n$ even. Hence the summation results in integers having only odd prime factors, being the sum of n even numbers and one odd number.

Analysis of the nth Catalan number

$$\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n},$$

shows that the power of 2 in the $\mathcal{C}_n$ is determined by $2^{2n+1}/(2n+2)!$ (A048881 in the OEIS) so that $4\mathcal{C}_{n-1}$ and $\mathcal{C}_n$ have the respective reciprocal powers of 2 to p_{2n} and p_{2n+1}. It follows that for $s \in \mathbb{Z}$ we have $4\mathcal{C}_{n-1}p_{2n}$ and $\mathcal{C}_n p_{2n+1}$ are odd integers. A slight modification of this argument also removes the odd factors arising in the $(2n)!$ and $(2n+1)!$ polynomial factors such that generated by

$$\frac{2^{2n+1}}{(2n)!}p_{2n}(s), \quad \text{and} \quad \frac{2^{2n+1}\mathcal{T}_{n+1}}{(2n+2)!}p_{2n+1}(s),$$

where $\mathcal{T}_{n+1}$ is the largest odd factor of $n+1$. Therefore the above two expressions yield odd integers with fewer prime factors than $4\mathcal{C}_{n-1}p_{2n}$ and $\mathcal{C}_n p_{2n+1}$, as required. □

Proof of Theorem 1.8. The proof follows that of Theorem 1.1, noting the integral representation

$$\int_0^1 (1-x)^{-\beta}x^{-\beta}\ {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\frac{(n+s)}{2}; x\right) dx$$

$$= 2^{2\beta-1}\frac{\sqrt{\pi}\Gamma(1-\beta)}{\Gamma(3/2-\beta)}\ {}_3F_2\left(1-\beta, \frac{1-n}{2}, -\frac{n}{2}; 2(1-\beta), 1-\frac{(n+s)}{2}; 1\right), \tag{2.4}$$

with $\beta < 1$. The ${}_2F_1(x)$ function is again transformed to a ${}_2F_1(1-x)$ function and the other steps are very similar to before.

The location of the zeros follows from Theorem 1.2, setting $\lambda = 3/2 - 2\beta$. □

To see Corollary 1.9 (a) The initial $\beta = 0$ reduction of Theorem 1.8 to ${}_2F_1$ form follows from the series definition of the ${}_3F_2$ function with a shift of summation index and the relations $(1)_j/(2)_j = 1/(j+1)$ and $(\kappa)_{j-1} = (\kappa-1)_j/(\kappa-1)$. The second reduction is a consequence of Gauss summation. (b) Similarly, with m a positive integer, $(m+1)_j/(2(m+1))_j$ may be reduced and partial fractions applied to this ratio. Then with shifts of summation index, the ${}_3F_2$ function may be reduced to a series of ${}_2F_1(1)$ functions. These in turn may be written in terms of ratios of Gamma functions from Gauss summation.

3 Connection with continuous Hahn polynomials

These polynomials are given by (e.g., [2] p. 331, [3])

$$p_n(x; a, b, c, d) = i^n\frac{(a+c)_n(a+d)_n}{n!}\ {}_3F_2\left(-n, n+a+b+c+d-1, a+ix; a+c, a+d; 1\right).$$

By using the second transformation of a terminating ${}_3F_2(1)$ series given in the Appendix, we have

$$p_n(x; a, b, c, d) = \frac{i^n}{n!}(a+b+c+d+n-1)_n(1-b-n-ix)_n$$
$$\times\ {}_3F_2(1-b-c-n, 1-b-d-n, -n; 2-a-b-c-d-2n, 1-b-n-ix; 1).$$

Then comparing with the ${}_3F_2(1)$ function of Theorem 1.3, we see that our polynomial factors are proportional to the continuous Hahn polynomial

$$p_n\left(-\frac{is}{2}; \frac{1}{4}-\frac{\lambda}{2}-n, 0, \frac{3}{4}-\frac{\lambda}{2}-n, \frac{1}{2}\right).$$

The continuous Hahn polynomials are orthogonal on the line with respect to the measure

$$\frac{1}{2\pi}\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)dx.$$

Due to the Parseval relation for the Mellin transform,

$$\int_0^\infty f(x)g^*(x)dx = \frac{1}{2\pi i}\int_{(0)} (\mathcal{M}f)(s)(\mathcal{M}g)^*(s)ds,$$

the polynomial factors $p_n(1/2+it)$ form an orthogonal family with respect to a suitable measure with Γ factors. Since orthogonal polynomials have real zeros, this approach provides another way of showing that $p_n(s)$ has zeros only on the critical line.

4 Lemmas

Lemma 4.1. *With $T_n(x)$ and $U_n(x)$ respectively the Chebyshev polynomials of the first and second kinds, we have that (a)*

$$M_{mn-1}(s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{1/4}} U_{m-1}[T_n(x)]U_{n-1}(x)dx$$
$$= \int_0^1 \frac{x^{s-1}}{(1-x^2)^{1/4}} U_{n-1}[T_m(x)]U_{m-1}(x)dx,$$

and (b)

$$\frac{1}{2}[M_{m+n-1}(s)+M_{m-n-1}(s)] = \int_0^1 x^{s-1}T_n(x)U_{m-1}(x)\frac{dx}{(1-x^2)^{1/4}}.$$

For polynomials of negative index, we have $U_{-n}(x) = -U_{n-2}(x)$.

Proof. Part (a) follows from the composition-product property

$$U_{mn-1}(x) = U_{m-1}[T_n(x)]U_{n-1}(x) = U_{n-1}[T_m(x)]U_{m-1}(x),$$

and (b) from $(1/2)[U_{m+n-1}(s) + U_{m-n-1}(s)] = T_n(x)U_{m-1}(x)$. □

Lemma 4.2. *We have the mixed recursion relation*

$$M_n(s) = 2M_{n-1}(s+1) - M_{n-2}(s), \tag{4.1}$$

and

$$M_0(s) = \frac{\Gamma\left(\frac{3}{4}\right)}{2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)}, \qquad M_1(s) = \Gamma\left(\frac{3}{4}\right)\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)}. \tag{4.2}$$

Here $M_1(s) = 2M_0(s+1)$ and $M_0(s) = B(s/2, 3/4)/2$, with B the Beta function.

Proof. The recursion (4.1) follows from $U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0$. With $U_0 = 1$, and applying a simple change of variable we have

$$M_0(s) = \frac{1}{2}\int_0^1 u^{s/2-1}(1-u)^{-1/4}du = \frac{1}{2}B\left(\frac{s}{2}, \frac{3}{4}\right),$$

giving (4.2). Since $U_1(x) = 2x$, $M_1(s) = 2M_0(s+1)$ immediately follows. □

Lemma 4.3. *We have (a)*

$$M_n(s+m) = \frac{1}{2^m}\sum_{r=0}^{m}\binom{m}{r}M_{m+n-2r}(s),$$

and (b)

$$M_n(s) = \sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}2^r M_{n-2k+r}(s+r).$$

Proof. (a) For $m < n$, we have

$$x^m U_n(x) = \frac{1}{2^m}\sum_{r=0}^{m}\binom{m}{r}U_{m+n-2r}(x),$$

provable by induction, using the property $xU_n(x) = \frac{1}{2}[U_{n+1}(x)+U_{n-1}(x)]$. Then by the definition (1.13) the result follows. For the inductive step, we easily have

$$x^{m+1}U_n(x) = \frac{1}{2^m}\sum_{r=0}^{m}\binom{m}{r}xU_{m+n-2r}(x)$$

$$= \frac{1}{2^{m+1}}\sum_{r=0}^{m}\binom{m}{r}\left[U_{m+n-2r+1}(x) + U_{m+n-2r-1}(x)\right]$$

$$= \frac{1}{2^{m+1}}\sum_{r=0}^{m+1}\left[\binom{m}{r} + \binom{m}{r-1}\right]U_{m+n-2r+1}$$

$$= \frac{1}{2^{m+1}}\sum_{r=0}^{m+1}\binom{m+1}{r}U_{m+n-2r+1}.$$

(b) follows from the binomial transform

$$f_n = \sum_{i=0}^{n}\binom{n}{i}g_i \iff g_n = \sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}f_i,$$

which is based upon the orthogonality property of binomial coefficients

$$\sum_{k=m}^{n}(-1)^{k+m}\binom{n}{k}\binom{k}{m} = \delta_{nm},$$

where δ_{nm} is the Kronecker symbol.

For the Chebyshev polynomials of the second kind we have the generating function

$$\sum_{k=0}^{\infty}U_k(x)t^k = \frac{1}{1-2tx+t^2}. \tag{4.3}$$

□

Lemma 4.4. *For $\Re\, s > 0$, the generating function of the Mellin transforms is given by*

$$G(t,s) \equiv \sum_{k=0}^{\infty} M_k(s)t^k = \int_0^1 \frac{1}{(1-x^2)^{1/4}} \frac{x^{s-1}}{(1-2tx+t^2)} dx$$

$$= \frac{1}{(1+t^2)} \frac{\Gamma\left(\frac{3}{4}\right)}{2} \left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)} \, {}_2F_1\left(1, \frac{s}{2}; \frac{2s+3}{4}; \frac{4t^2}{(1+t^2)^2}\right) \right.$$

$$\left. + \frac{2t}{(1+t^2)} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)} \, {}_2F_1\left(1, \frac{s+1}{2}; \frac{2s+5}{4}; \frac{4t^2}{(1+t^2)^2}\right) \right]. \tag{4.4}$$

As given below, the first line of the right member of (4.4) furnishes $M_{2k}(s)$ and the second line, $M_{2k+1}(s)$.

Proof. Using (4.3), we first binomially expand so that

$$\int_0^1 \frac{1}{(1-x^2)^{1/4}} \frac{x^{s-1}}{(1-2tx+t^2)} dx = \sum_{\ell=0}^{\infty} \binom{-\frac{1}{4}}{\ell} (-1)^\ell \int_0^1 \frac{x^{2\ell+s-1}}{1-2tx+t^2} dx$$

$$= \frac{1}{1+t^2} \sum_{\ell=0}^{\infty} \binom{-\frac{1}{4}}{\ell} \frac{(-1)^\ell}{(2\ell+s)} \, {}_2F_1\left(1, 2\ell+s; 1+2\ell+s; \frac{2t}{1+t^2}\right).$$

Then we interchange sums, and separate terms of even and odd summation index:

$$\sum_{k=0}^{\infty} M_k(s)t^k = \frac{1}{1+t^2} \sum_{\ell=0}^{\infty} \binom{-\frac{1}{4}}{\ell} \frac{(-1)^\ell}{(2\ell+s)} \sum_{j=0}^{\infty} \frac{(2\ell+s)_j}{(2\ell+s+1)_j} \left(\frac{2t}{1+t^2}\right)^j$$

$$= \frac{\Gamma(3/4)}{2(1+t^2)} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+s}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{j+s}{2}\right)} \left(\frac{2t}{1+t^2}\right)^j$$

$$= \frac{\Gamma(3/4)}{2(1+t^2)} \left[\sum_{m=0}^{\infty} \frac{\Gamma\left(m+\frac{s}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{s}{2}+m\right)} \left(\frac{2t}{1+t^2}\right)^{2m} + \sum_{m=0}^{\infty} \frac{\Gamma\left(m+1+\frac{s}{2}\right)}{\Gamma\left(\frac{5}{4}+\frac{s}{2}+m\right)} \left(\frac{2t}{1+t^2}\right)^{2m+1} \right]$$

$$= \frac{\Gamma(3/4)}{2(1+t^2)} \left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)} \sum_{m=0}^{\infty} \frac{(1)_m}{m!} \frac{\left(\frac{s}{2}\right)_m}{\left(\frac{3}{4}+\frac{s}{2}\right)_m} \left(\frac{2t}{1+t^2}\right)^{2m} \right.$$

$$\left. + \frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)} \sum_{m=0}^{\infty} \frac{(1)_m}{m!} \frac{\left(1+\frac{s}{2}\right)_m}{\left(\frac{5}{4}+\frac{s}{2}\right)_m} \left(\frac{2t}{1+t^2}\right)^{2m+1} \right].$$

□

We mention a second method of proof of Lemma 4.4. With the use of partial fractions, we have

$$\sum_{k=0}^{\infty} M_k(s)t^k = \int_0^1 \frac{(1-x^2)^{3/4}}{(1-x^2)} \frac{x^{s-1}}{(1-2tx+t^2)} dx$$

$$= \int_0^1 x^{s-1}(1-x^2)^{3/4} \left\{ \frac{1}{2} \left[\frac{1}{(1+t)^2} \frac{1}{(1+x)} + \frac{1}{(1-t)^2} \frac{1}{(1-x)} \right] - \frac{4t^2}{(1-t^2)^2(1-2tx+t^2)} \right\} dx.$$

The Beta function suffices to evaluate the first two integrals on the right side, while the ${}_2F_1$ function is required for the third,

$$\sum_{k=0}^{\infty} M_k(s)t^k = \frac{\Gamma\left(\frac{3}{4}\right)}{2}\frac{1}{(1-t^2)^2}\left[(1+t^2)\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{s}{2}\right)} + 2t\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{5}{4}+\frac{s}{2}\right)}\right]$$
$$-\frac{4t^2}{(1-t^2)^2}\frac{\Gamma\left(\frac{7}{4}\right)}{2(1+t^2)}\left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{7}{4}+\frac{s}{2}\right)}\ {}_2F_1\left(1,\frac{s}{2};\frac{7}{4}+\frac{s}{2};\frac{4t^2}{(1+t^2)^2}\right)\right.$$
$$\left.+\frac{2t}{(1+t^2)}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{9}{4}+\frac{s}{2}\right)}\ {}_2F_1\left(1,\frac{s+1}{2};\frac{9}{4}+\frac{s}{2};\frac{4t^2}{(1+t^2)^2}\right)\right].$$

Then contiguous relations for the ${}_2F_1$ function (e.g., [18], pp. 1044-1045) may be used to show the equivalence with (4.4).

Corollary 4.5. *For t sufficiently small we have the summation identity*

$$\frac{1}{(1+t^2)}\frac{\Gamma\left(\frac{3}{4}\right)}{2}\left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)}\ {}_2F_1\left(1,\frac{s}{2};\frac{2s+3}{4};\frac{4t^2}{(1+t^2)^2}\right)\right.$$
$$\left.+\frac{2t}{(1+t^2)}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)}\ {}_2F_1\left(1,\frac{s+1}{2};\frac{2s+5}{4};\frac{4t^2}{(1+t^2)^2}\right)\right]$$
$$=\frac{\Gamma\left(\frac{3}{4}\right)}{2}\sum_{k=0}^{\infty}(-1)^k t^{2k}\left[\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)}\ {}_3F_2\left(\frac{1-k}{2},\frac{s}{2},-\frac{k}{2};\frac{1}{2},\frac{2s+3}{4};\frac{4}{t^2}\right)\right.$$
$$\left.-\frac{2k\Gamma\left(\frac{s+1}{2}\right)}{t\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)}\ {}_3F_2\left(\frac{1-k}{2},1-\frac{k}{2},\frac{s+1}{2};\frac{3}{2},\frac{2s+5}{4};\frac{4}{t^2}\right)\right].$$

Proof. This follows by equating the result for $G(t,s)$ of Lemma 4.4 with the geometric series expansion and integration

$$G(t,s) = \int_0^1 \frac{x^{s-1}}{(1-x^2)^{1/4}}\sum_{k=0}^{\infty}(-1)^k t^k (t-2x)^k dx.$$

□

Remark. *Transformations of the ${}_2F_1$ functions in Lemma 4.4 include the following.*

$${}_2F_1\left(1,\frac{s}{2};\frac{2s+3}{4};\frac{4t^2}{(1+t^2)^2}\right) = \left(\frac{1+t^2}{1-t^2}\right)^2\ {}_2F_1\left(1,\frac{3}{4};\frac{2s+3}{4};-\frac{4t^2}{(1-t^2)^2}\right),$$

and

$${}_2F_1\left(1,\frac{s+1}{2};\frac{2s+5}{4};\frac{4t^2}{(1+t^2)^2}\right) = \left(\frac{1+t^2}{1-t^2}\right)^2\ {}_2F_1\left(1,\frac{3}{4};\frac{2s+5}{4};-\frac{4t^2}{(1-t^2)^2}\right).$$

Letting $'$ denote differentiation with respect to t, (4.4) is consistent with the differential equation

$$G'(t,s) + \frac{1}{2t}G(t,s) = \frac{(t^2-1)}{2t}\int_0^1 \frac{x^{s-1}}{(1-x^2)^{1/4}}\frac{1}{(1-2xt+t^2)^2}dx,$$

obtained by using partial fractions.

The following comes from the application of Corollary 1.9 to the generating function (4.4).

Lemma 4.6. *Let $k \geq 0$. Then*

$$M_{2k}(s) = \frac{\Gamma\left(\frac{3}{4}\right)}{2} \frac{4^k \Gamma\left(k + \frac{s}{2}\right)}{\Gamma\left(k + \frac{2s+3}{4}\right)} \,{}_3F_2\left(\frac{1}{2} - k, -k - \frac{s}{2} + \frac{1}{4}, -k; 1 - \frac{s}{2} - k, -2k; 1\right), \tag{4.5}$$

and

$$M_{2k+1}(s) = \Gamma\left(\frac{3}{4}\right) \frac{4^k \Gamma\left(k + \frac{s+1}{2}\right)}{\Gamma\left(k + \frac{2s+5}{4}\right)} \,{}_3F_2\left(-\frac{1}{2} - k, -k - \frac{s}{2} - \frac{1}{4}, -k; \frac{1-s}{2} - k, -1 - 2k; 1\right). \tag{4.6}$$

Remark. *From (4.5) and (4.6) expressions for the special values $M_{2k+1}(0)$ and $M_{2k+1}(1)$ and $M_{2k}(1)$ may be written.*

These transforms may be re-expressed by using Thomae's identity [2] (p. 143) such that

$${}_3F_2(a, b, c; d, e; 1) = \frac{\Gamma(d)\Gamma(e)\Gamma(w)}{\Gamma(a)\Gamma(w+b)\Gamma(w+c)} \,{}_3F_2(d-a, e-a, w; w+b, w+c; 1),$$

where $w = d + e - a - b - c$.

The following allows us to obtain other equivalent hypergeometric forms for the Mellin transforms, and hence for the polynomial factors.

Lemma 4.7. *We have (a)*

$$U_n(x) = \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} (x^2-1)^k x^{n-2k} = (n+1)x^n \,{}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}; 1 - \frac{1}{x^2}\right),$$

(b)

$$U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k} = (2x)^{2n} \,{}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; -n; \frac{1}{x^2}\right),$$

and (c)

$$T_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (x^2-1)^k x^{n-2k} = x^n \,{}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2}; 1 - \frac{1}{x^2}\right).$$

Proof. These expressions follow by applying quadratic transformations (e.g., [18], p. 1043) to known ${}_2F_1$ forms of U_n and T_n,

$$U_n(x) = (n+1) \,{}_2F_1\left(n+2, -n; \frac{3}{2}; \frac{1-x}{2}\right),$$

and

$$T_n(x) = \,{}_2F_1\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right).$$

□

Lemma 4.8. *With $M_n(s)$ the Mellin transform of the Chebyshev function given in (1.13), we have that*

(a)

$$M_n(s) = (n+1)\frac{\Gamma\left(\frac{3}{4}\right)}{2}\frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{n+s}{2}+\frac{3}{4}\right)}\ {}_3F_2\left(\frac{3}{4}, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, 1-\frac{n+s}{2}; 1\right),$$

and (b)

$$M_n(s) = 2^{n-1}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{n+s}{2}+\frac{3}{4}\right)}\ {}_3F_2\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1}{4}-\frac{(n+s)}{2}; -n, 1-\frac{n+s}{2}; 1\right).$$

Proof. The interchange of finite summation and the integration of (1.13) is used in both instances. □

Lemma 4.9. *Further relations concerning the Chebyshev, Gegenbauer and Legendre polynomials $P_n(x)$ are given by (a)*

$$U_m(x) = \sum_{k=0}^{m} P_k(x)P_{m-k}(x),$$

(b)

$$\frac{1}{2}U'_{m+1}(x) = \sum_{k=0}^{m} U_k(x)U_{m-k}(x) = \frac{1}{2(x^2-1)}[(m+1)xU_{m+1}(x) - (m+2)U_m(x)],$$

and (c)

$$C_m^{\lambda_1+\lambda_2}(x) = \sum_{k=0}^{m} C_k^{\lambda_1}(x)C_{m-k}^{\lambda_2}(x).$$

Proof. (a) The generating function (4.3) is used, along with

$$\sum_{k=0}^{\infty} P_k(x)t^k = \frac{1}{\sqrt{1-2tx+t^2}}.$$

Since

$$\sum_{k=0}^{\infty} U_k(x)t^k = \frac{1}{1-2tx+t^2} = \left(\sum_{k=0}^{\infty} P_k(x)t^k\right)^2$$

holds over a common range of t, the result follows.

(b) The generating function

$$\sum_{k=0}^{\infty} C_k^{\lambda}(x)t^k = \frac{1}{(1-2tx+t^2)^{\lambda}}$$

is used, together with (4.3) and the expression for $C_n^2(x)$ given in (6.2). (c) Again uses the generating function given just above. □

Lemma 4.10 (Limit relation lemma)**.** *The Gegenbauer polynomials $C_n^{\lambda}(x)$ satisfy*

$$\lim_{\lambda\to\infty} \frac{C_n^{\lambda}(x)}{C_n^{\lambda}(1)} = x^n.$$

Proof. We have

$$\lim_{\lambda\to\infty}\frac{C_n^\lambda(x)}{C_n^\lambda(1)}=\lim_{\lambda\to\infty}\ {}_2F_1\left(2\lambda+n,-n;\lambda+\frac{1}{2};\frac{1-x}{2}\right),$$

and, by Stirling's formula

$$\frac{(2\lambda+n)_k}{\left(\lambda+\frac{1}{2}\right)_k}=2^k\left[1+O\left(\frac{1}{\lambda}\right)\right].$$

Then

$$\lim_{\lambda\to\infty}\frac{C_n^\lambda(x)}{C_n^\lambda(1)}=\sum_{k=0}^n\frac{(-n)_k}{k!}(1-x)^k=\sum_{k=0}^n(-1)^k\binom{n}{k}(1-x)^k=x^n.$$

□

Lemma 4.11. *The Gegenbauer polynomials $C_n^\lambda(x)$ are related to the Chebyshev polynomials of the first kind such that*

$$\left.\frac{C_n^\lambda(x)}{C_n^\lambda(1)}\right|_{\lambda=0}=T_n(x).$$

Proof. We employ the ${}_2F_1$ form of $C_n^\lambda(x)/C_n^\lambda(1)$ given in (1.11).

Then

$$\sum_{k=0}^\infty\frac{(n)_k(-n)_k}{(1/2)_k}\frac{1}{k!}\left(\frac{1-x}{2}\right)^k=\cos\left(2n\sin^{-1}\left(\frac{1-x}{2}\right)\right)=\cos(n\cos^{-1}x)=T_n(x)=\left.\frac{C_n^\lambda(x)}{C_n^\lambda(1)}\right|_{\lambda=0}.$$

□

Remark. *We show elsewhere [12] that the $\beta=1/2$ special case of Theorem 1.8 corresponds to a particular Mellin transform of Legendre functions.*

Owing to the form of $p_n(s;\beta)$ in terms of $s(s-1)+b$, where b is a rational number, the polynomials $p_n(s;\beta_1)\pm p_n(s;\beta_2)$ also have zeros only on the critical line, when this combination does not degenerate to a constant.

5 Mellin transforms of Chebyshev polynomials T_n

The Chebyshev polynomials of the first kind begin with $T_0=1$ and $T_1(x)=x=xT_0$, and satisfy the same recursion as $U_n(x)$. We now put, for $\Re\, s>0$,

$$M_n^T(s)\equiv\int_0^1 x^{s-1}T_n(x)(1-x^2)^{1/2}dx, \tag{5.1}$$

where for n odd these also hold for $\Re\ s>-1$. $M_n^T(s)$ satisfies the mixed recursion relation (4.1), with

$$M_0(s)=\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{2}\right)},\qquad M_1(s)=M_0(s+1)=\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+2\right)}. \tag{5.2}$$

For $T_n(x)$ we have the generating function

$$T_0(x)+2\sum_{k=1}^\infty T_k(x)t^k=\frac{1-t^2}{1-2tx+t^2}. \tag{5.3}$$

Lemma 5.1. *(Generating function lemma.) For $\Re\ s > 0$,*

$$G^T(t,s) \equiv M_0(s) + 2\sum_{k=1}^{\infty} M_k^T(s)t^k = (1-t^2)\int_0^1 \frac{x^{s-1}}{(1-2tx+t^2)}(1-x^2)^{1/2}dx$$

$$= \frac{\sqrt{\pi}}{4}(1-t^2)\left[\frac{\Gamma\left(\frac{s}{2}\right)}{(1+t^2)\Gamma\left(\frac{s}{2}+\frac{3}{2}\right)}\ {}_2F_1\left(1,\frac{s}{2};\frac{s+3}{2};\frac{4t^2}{(1+t^2)^2}\right)\right.$$

$$\left.+\frac{2t}{(1+t^2)^2}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+2\right)}\ {}_2F_1\left(1,\frac{s+1}{2};\frac{s+4}{2};\frac{4t^2}{(1+t^2)^2}\right)\right]. \tag{5.4}$$

Proof. The proof is similar to that for Lemma 4.4. □

The generating function $G^T(t,s)$ may be expanded similarly as in Lemma 4.6.

Theorem 5.2. *The polynomial factors of $M_n^T(s)$ have their zeros at the even or odd integers up to $n-3$, and at n^2-1.*

Proof. This follows since when $n \geq 3$ is odd,

$$M_n^T(s) = \frac{\sqrt{\pi}}{4\cdot 2^n}(s-2)(s-4)\cdots(s-(n-3))(s-(n^2-1))\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+n+3}{2}\right)},$$

and when $n \geq 2$ is even,

$$M_n^T(s) = \frac{\sqrt{\pi}}{4\cdot 2^n}(s-1)(s-3)\cdots(s-(n-3))(s-(n^2-1))\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+n+3}{2}\right)}.$$

For $n=3$ and $n=2$ only the $s-(n^2-1)$ linear factor is present. It is then easily verified that these expressions satisfy the recursion (4.1) with the starting functions $M_0(s)$ and $M_1(s)$. □

6 Discussion

Given the Gould variant combinatorial expressions obtained for $p_n^\lambda(s)$ and $q_n^\lambda(s)$, our results invite several other research questions, such as: is there a combinatorial interpretation of $p_n^\lambda(s)$ or $q_n^\lambda(s)$, and more generally, of $p_n(s;\beta)$? Relatedly, is there a reciprocity relation for $p_n(s)$ and $p_n(s;\beta)$?

Two instances when the combinatorial sums produce "nice" combinatorial expressions are

$$q_{2n}^\lambda(1) = \sum_{r=0}^{n} \frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{r}\binom{n+r}{2r}\binom{r-\frac{1}{2}}{r}}{\binom{n+r}{r}\binom{\frac{\lambda}{2}-\frac{1}{4}+r}{r}} = \frac{1}{2}\binom{n+\frac{2\lambda-3}{4}}{n}\binom{n+\frac{2\lambda-1}{4}}{n}^{-1},$$

$$q_{2n+1}^\lambda(2) = \sum_{r=0}^{n} \frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{r}\binom{n+r+1}{2r+1}\binom{r+\frac{1}{2}}{r}}{\binom{n+r+1}{r}\binom{\frac{\lambda}{2}+\frac{3}{4}+r}{r}} = (n+1)\binom{n+\frac{2\lambda-3}{4}}{n}\binom{n+\frac{2\lambda+3}{4}}{n}^{-1}.$$

In fact, polynomials with only real zeros commonly arise in combinatorics and elsewhere. Two examples are Bell polynomials [21] and Eulerian polynomials [14] (p. 292). This suggests that there may be other relations of our results to discrete mathematics and other areas. In this regard, we mention matching polynomials with only real zeros. The matching polynomial $M(G,x) = \sum_k (-1)^k p(G,k)x^k$ counts the matchings in a graph G. Here, $p(G,k)$ is the number of matchings of size k, i.e., the number of sets of k edges of G, no two edges having a common vertex. $M(G,x)$ satisfies recurrence relations and has only real zeros and $M(G-v,x)$ interlaces $M(G,x)$ for any v in the vertex set of G (e.g., [17]). As regards [6, 10], we note that the classical orthogonal polynomials are closely related to the matching polynomials. For example, the Chebyshev polynomials of the first two kinds are the matching polynomials of paths and cycles respectively, and the Hermite polynomials and the Laguerre polynomials are the matching polynomials of complete graphs and complete bipartite graphs, respectively.

There are several open topics surrounding the recursions (1.14) and (1.15). These include: is it possible to reduce this three-term recursion to two terms, can a pure recursion be obtained, and, can some form of it be used to demonstrate the occurrence of the zeros only on the critical line?

The Gegenbauer polynomials have the integral representation

$$C_n^\lambda(x) = \frac{1}{\sqrt{\pi}} \frac{(2\lambda)_n}{n!} \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} \int_0^\pi (x+\sqrt{x^2-1}\cos\theta)^n \sin^{2\lambda-1}\theta \, d\theta. \tag{6.1}$$

Then binomial expansion of part of the integrand of $M_n^\lambda(s)$ is another way to obtain this Mellin transform explicitly. The representation (6.1) is also convenient for showing further special cases that reduce in terms of Chebyshev polynomials U_n or Legendre or associated Legendre polynomials P_n^m. We mention as examples

$$C_n^2(x) = \frac{1}{2(x^2-1)}[(n+1)xU_{n+1}(x) - (n+2)U_n(x)], \tag{6.2}$$

and

$$C_n^{3/2}(x) = \frac{(n+1)}{(x^2-1)}[xP_{n+1}(x) - P_n(x)] = -\frac{P_{n+1}^1(x)}{\sqrt{1-x^2}}.$$

The Gegenbauer polynomials are a special case of the two-parameter Jacobi polynomials $P_n^{\alpha,\beta}(x)$ (e.g., [2]) as follows:

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{\lambda-1/2,\lambda-1/2}(x).$$

The Jacobi polynomials are orthogonal on $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. Therefore, it is also of interest to consider Mellin transforms such as

$$M_n^{\alpha,\beta}(s) = \int_0^1 x^{s-1} P_n^{\alpha,\beta}(x)(1-x)^{\alpha/2-1/2}(1+x)^{\beta/2-1/2}dx,$$

especially as the Jacobi polynomials can be written in the binomial form

$$P_n^{\alpha,\beta}(x) = \sum_{j=0}^n \binom{n+\alpha}{k}\binom{n+\beta}{n-s}\left(\frac{x-1}{2}\right)^{n-s}\left(\frac{x+1}{2}\right)^s.$$

In fact this line of enquiry may provide a far more general approach to investigate 'critical polynomials' arising from combinatorial sums.

Appendix

Below are collected various transformations of terminating ${}_3F_2(1)$ series [5], where again $(a)_n$ denotes the Pochhammer symbol.

$$ {}_3F_2(-n,a,b;c,d;1) = \frac{(c-a)_n(d-a)_n}{(c)_n(d)_n}\,{}_3F_2(-n,a,a+b-c-d-n+1;a-c-n+1,a-d-n+1;1) $$

$$ = \frac{(a)_n(c+d-a-b)_n}{(c)_n(d)_n}\,{}_3F_2(-n,c-a,d-a;1-a-n,c+d-a-b;1) $$

$$ = \frac{(c+d-a-b)_n}{(c)_n}\,{}_3F_2(-n,d-a,d-b;d,c+d-a-b;1) $$

$$ = (-1)^n\frac{(a)_n(b)_n}{(c)_n(d)_n}\,{}_3F_2(-n,1-c-n,1-d-n;1-a-n,1-b-n;1) $$

$$ = (-1)^n\frac{(d-a)_n(d-b)_n}{(c)_n(d)_n}\,{}_3F_2(-n,1-d-n,a+b-c-d-n+1;a-d-n+1,b-d-n+1;1) $$

$$ = \frac{(c-a)_n}{(c)_n}\,{}_3F_2(-n,a,d-b;d,a-c-n+1;1) $$

$$ = \frac{(c-a)_n(b)_n}{(c)_n(d)_n}\,{}_3F_2(-n,d-b;1-c-n;1-b-n,a-c-n+1;1). $$

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Washington, National Bureau of Standards (1964).

[2] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press (1999).

[3] R. Askey, Continuous Hahn polynomials, J. Phys. A **18**, L1017-L1019 (1985).

[4] L. Báez-Duarte, A sequential Riesz-like criterion for the Riemann hypothesis, Int. J. Math. Sci., **21**, 2005, 3527-3537.

[5] W. N. Bailey, Generalized hypergeometric series, Cambridge University Press (1935).

[6] D. Bump, K.-K. Choi, P. Kurlberg, and J. Vaaler, A local Riemann hypothesis, I, Math. Z. **233**, 1-19 (2000).

[7] D. Bump and E. K.-S. Ng, On Riemann's zeta function, Math. Z. **192**, 195-204 (1986).

[8] P. Butzer and S. Jansche, A direct approach to the Mellin transform, J. Fourier Analysis Appls. **3**, 325-376 (1997).

[9] M. W Coffey, J. L Hindmarsh, M. C. Lettington and J. Pryce, *On Higher Dimensional Interlacing Fibonacci Sequences, Continued Fractions and Chebyshev Polynomials*, J. Theor. Nom. Bordx. (to appear).

[10] M. W. Coffey, Special functions and the Mellin transforms of Laguerre and Hermite functions, Analysis **27**, 95-108 (2007).

[11] M. W. Coffey, Theta and Riemann xi function representations from harmonic oscillator eigenfunctions, Phys. Lett. A **362**, 352-356 (2007).

[12] M. W. Coffey and M. C. Lettington, Mellin transforms with only critical zeros: Legendre functions, J. Number Th., **148**, 507-536 (2015).

[13] M. W. Coffey, Mellin transforms with only critical zeros: generalized Hermite functions, arXiv:1308.6821 (2013).

[14] L. Comtet, Advanced combinatorics, Reidel, Dordrecht (1974).

[15] K. Dilcher and K. B. Stolarsky, A Pascal-Type Triangle Characterizing Twin Primes, Amer. Math. Monthly **112**, 673-681 (2005).

[16] R.D. Fray, Congruence Properties of Ordinary and q-Binomial Coefficients, Duke Math. J., **34**, 467-480 (1967).

[17] C. D. Godsil and I. Gutman, On the theory of the matching polynomial, J. Graph Theory **5**, 137-144 (1981).

[18] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York (1980).

[19] A. Granville, Arithmetic Properties of Binomial Coefficients I: Binomial coefficients modulo prime powers Canadian Mathematical Society Conference Proceedings **20**, 253-275 (1997).

[20] H. W. Gould, *Combinatorial Identities*, Morgantown, W. Va. 1972.

[21] L. H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Statist. **38**, 410-414 (1967).

[22] J. Havil, Gamma: Exploring Euler's Constant, Princeton University Press, (2003).

[23] P. Kirschenhofer, A. Pethö, and R. F. Tichy, On analytical and diophantine properties of a family of counting polynomials, Acta Sci. Math. (Szeged) **65**, 47-59 (1999).

[24] H. von Koch, Sur lat distribution des nombres premiers, Acta Math., **24**, 159-182, (1901).

[25] K. Maślanka, Hypergeometric-like representation of the zeta-function of Riemann., arXiv:math-ph/0105007v1, 2001.

[26] J. C. Mason and D. C. Handscomb, Chebyshev polynomials, Chapman & Hall (2003).

[27] E. D. Rainville, Special functions, Macmillan (1960).

[28] T. J. Rivlin, Chebyshev polynomials: From approximation theory to algebra and number theory, John Wiley (1990).

[29] Z. W. Sun and R. Tauroso, Congruences for sums of binomial coefficients, J. Number Theory **126**, 287-296 (2007).