The genesis of involutions (polarizations and lattice paths)

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Abstract

The number of Borel orbits in polarizations (the symmetric variety $SL(n)/S(GL(p)\times GL(q)))$ is analyzed, various (bivariate) generating functions are found. Relations to lattice path combinatorics are explored.

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1 Introduction

Let n be a positive integer and denote by $SL(n)$ the special linear group of $n\times n$ matrices with determinant 1. We know from classification of involutions on algebraic groups that there are essentially four different types of involutory automorphisms associated with $SL(n)$. The first of these is the automorphism $\theta_1 : SL(n) \to SL(n)$ defined by $\theta_1(g) = (g^{-1})^\top$ whose fixed point set is $SO(n)$. The second one is $\theta_2 : SL(2n) \to SL(2n)$ defined by $\theta_2(g) = J(g^{-1})^\top J$, where J denotes the skew form

$$J = \begin{pmatrix} 0 & id_n \\ -id_n & 0 \end{pmatrix}, \tag{1.1}$$

and id_n is the $n \times n$ identity matrix. The third one is actually a family; let $0 < q < p$ be two positive integers such that $n = q + p$ and define

$$J_{p,q} := \begin{pmatrix} 0 & 0 & s_q \\ 0 & id_{p-q} & 0 \\ s_q & 0 & 0 \end{pmatrix}, \tag{1.2}$$

where s_q is the $q \times q$ matrix with 1's on the anti-diagonal and 0's elsewhere. Then $\theta_3 = \theta_3(p,q)$ is the automorphism $\theta_3 : SL(n) \to SL(n)$ defined by $\theta_3(g) = J_{p,q}(g^{-1})^\top J_{p,q}$. The final automorphism is the identity $\theta_4(g) = g$. (Actually, $SL(n)$ can be viewed as the diagonal in $SL(n) \times SL(n)$, θ_4 is induced from the nontrivial involutory automorphism $\widetilde{\theta}_4 : SL(n) \times SL(n) \to SL(n) \times SL(n)$ defined by $\widetilde{\theta}_4((g,h)) = (h,g)$.) Note that for $i = 1, \dots, 4$, the fixed point subgroups $SL(n)^{\theta_i} = \{g \in SL(n) : \theta_i(g) = g\}$, are equal to the special orthogonal group $SO(n)$, the symplectic group $Sp(2n)$, the Levi subgroup $S(GL(p) \times GL(q))$, and $SL(n)$, respectively.

The symmetric group S_n of permutations on $[n] := \{1, \dots, n\}$ plays a fundamental role for the homogenous spaces of $SL(n)$. Let us denote by B_n the standard Borel subgroup of $SL(n)$, that is the subgroup consisting of upper triangular matrices from $SL(n)$. A Schubert variety, by definition, is the Zariski closure of a B_n orbit in the quotient $\mathscr{B}_n := SL(n)/B_n$, which is commonly known as the "full flag variety" of $SL(n)$. On one hand, S_n gives a parametrization of all Schubert varieties in $\mathscr{B}_n$. On the other hand, each involution θ_i, $i = 1, 2, 3$ induces an involutory automorphism on S_n with the corresponding fixed point sets:

1. $S_n^{\theta_1} = I_n = \{\pi \in S_n : \pi^2 = id\}$;
2. $S_n^{\theta_2} = FI_n$, the set of involutions with no fixed points;
3. $S_n^{\theta_3} = I^{\pm}_{p,q}$, the set of "signed (p,q)-involutions" in I_n, which we define below;
4. $S_n^{\theta_4} = S_n$.

For $i = 1, 2, 3$, these sets of involutions parametrize the orbits of fixed point subgroup $SL(n)^{\theta_i}$ in $\mathscr{B}_n$. Similarly, S_n parametrizes the $SL(n)^{\theta_i}$ orbits in $\mathscr{B}_n \times \mathscr{B}_n$.

Definition 1.3. A signed (p,q)-involution $\pi \in S_n$ is an involution with an assignment of $+$ and $-$ signs to the fixed points of π such that there are $p - q$ more $+$'s than $-$'s if $q < p$. (If $p < q$, there are $q - p$ more $-$ signs than $+$ signs.) The set of all signed (p,q)-involutions in S_n is denoted by $I^{\pm}_{p,q}$. The cardinality of $I^{\pm}_{p,q}$ is denoted by $\alpha_{p,q}$.

For example, $\pi = (1\ 6)(2\ 3)(4^+)(5^-)(7^+)(8^+)$ is an element of $I^{\pm}_{5,3}$. Observe here that p is equal to the number of fixed points in π with a $+$ sign attached plus the number of two-cycles in π, while q is equal to the number of fixed points in π with a $-$ sign attached plus the number of two-cycles in π.

Our goal in this manuscript is to show that the numbers $\alpha_{p,q}$, $p, q \geq$ have a great deal of interesting combinatorics pertaining to them. Let us give a brief summary of our results and explain the structure of our paper.

Let $\gamma_{k,p,q}$ denote the number of signed involutions from $I^{\pm}_{p,q}$ with k 2-cycles. Clearly, $\alpha_{p,q} = \gamma_{0,p,q} + \gamma_{1,p,q} + \cdots$. First by finding a 3-term recurrence for $\gamma_{k,p,q}$'s we prove in Section 3 that

$$\alpha_{p,q} = \alpha_{p-1,q} + \alpha_{p,q-1} + (p+q-1)\alpha_{p-1,q-1}$$

with obvious initial conditions $\alpha_{p,0} = \alpha_{0,q} = 1$ for all $p, q \geq 1$. The generating function $u_p(x) = \sum_{q\geq 0} \alpha_{p,q} x^q$ and its bivariate extensions are worked out in Section 4. (Somewhat surprisingly, the modified Bessel functions appear in our calculations.)

One of the most exciting main results of our paper is on the relationship between Borel orbits and the lattice path combinatorics. We show in Section 5 that $\alpha_{p,q}$'s count the number of weighted lattice paths in $(p+1)\times(q+1)$-grid starting at the origin and ending at (p, q) with $(0, 1), (1, 0)$, and $(1, 1)$-steps only. In particular, by constructing an explicit bijection between $I^{\pm}_{p,q}$ and the paths we make note of the fact that now there is a way to reinterpret and study the action of the "Richardson-Springer monoid of symmetric group" in terms of weighted lattice paths. This connection is particularly exciting because intersection theory on Borel orbit closures in $SL(n)/S(GL(p)\times GL(q))$ is determined by the action of Richardson-Springer monoid (see [11]) and there is an interplay between tableaux and lattice paths.

Let us pointed out the other similar result, that we know of, relating Schubert varieties to lattice paths. Let $Y = Gr(k, n)$ denote the Grassmann variety of k dimensional subspaces of the n dimensional vector space $\mathbb{C}^n$. It is well known that this variety is a homogeneous space of the form $SL(n)/P$, where P is the subgroup

$$P := \left\{ \begin{pmatrix} * & * \\ \mathbf{0} & * \end{pmatrix} \in SL(n) : \ \mathbf{0} \text{ is the } k \times n - k \text{ matrix whose entries are all } 0. \right\}.$$

The orbits of Borel subgroup B_n, via its left translation action on $SL(n)/P$ give a cellular decomposition of Y; the cells are in bijection with the lattice paths in the $k \times (n - k)$-grid starting at the origin ending at $k\times(n-k)$ and moving with $(0, 1)$- and $(1, 0)$-steps only. The inclusion order on Borel orbit has a natural interpretation in terms of these lattice paths and the length generating function of this poset is given by the t-analog of binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$, where $[m] := 1 + \cdots + t^{m-1}$ and $[m]! = [m]\cdots[2][1]$. It obeys the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + t^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

See [9, Chapter I].

In Section 6, we study three generating functions pertaining $\alpha_{p,q}$'s. Among them is the length generating function $E_{p,q}(t) = \sum_{\pi\in I^{\pm}_{p,q}} t^{L(\pi)}$ of the Bruhat order on $I^{\pm}_{p,q}$. We prove that $E_{p,q}(t)$'s satisfy the 3-term recurrence

$$E_{p,q}(t) = E_{p-1,q}(t) + E_{p,q-1}(t) + ([p + q] - 1)E_{p-1,q-1}(t).$$

We finish our paper with a section on the plan of a future work.

2 Preliminaries (remarks and notation)

For $i = 1, 2, 3$, let θ_i be one of the involutions mentioned in the introduction and set $K_i := SL(n)^{\theta_i}$ to denote the group of θ_i-fixed elements in $SL(n)$. As we mentioned it earlier, the K_i orbits in $\mathscr{B}_n$ are parametrized by the corresponding sets of involutions in

S_n. Equivalently, these sets of involutions parametrize the B_n orbits in the symmetric varieties $SL(n)/K_i$, respectively. Borel orbit closures form a graded poset with respect to (set-theoretic) inclusion. These posets are first considered by Richardson and Springer in their seminal paper [8]. We briefly review some well known results pertaining to these posets, which are commonly referred to as Bruhat posets.

We start with setting up our notation. In combinatorics, it is customary to denote by $[n]$ the set $\{1, \dots, n\}$. We write the elements of the symmetric group S_n in cycle notation using parentheses, as well as in one-line notation using brackets. We omit brackets in one-line notation if there is no danger of confusion. For example, $w = 4213 = [4, 2, 1, 3] = (1, 4, 3)$ is the permutation that maps 1 to 4, 2 to 2, 3 to 1, and 4 to 3.

An involution is an element of S_n of order ≤ 2 and the set of involutions in S_n is denoted by I_n. The Bruhat order on I_n has a minimal element $\alpha_n := \mathrm{id}$, and a maximal element $\beta_n := w_0$, where $w_0 = [n, n-1, \cdots, 2, 1]$, the longest permutation. We drop the subscript n when it is clear from context.

Let $\pi \in I_n$ be an involution. The standard way of writing π is as a product of 2-cycles. Since we often need the data of fixed points (1-cycles) of π, we are always going to include them in our notation. Thus, our *standard form* for π is

$$\pi = (a_1, b_1)(a_2, b_2) \cdots (a_k, b_k) c_1 \dots c_{n-2k},$$

where $a_i < b_i$ for all $1 \leq i \leq k$, $a_1 < a_2 < \cdots < a_k$, and $c_1 < \cdots < c_{n-2k}$.

The Bruhat-Chevalley order on S_n is a ranked poset and its grading is given by $\ell : S_n \to \mathbb{Z}$, $\ell(\pi) =$ the number of inversions in π. The Bruhat order on I_n is also a ranked poset; for $\pi = (a_1, b_1) \cdots (a_k, b_k) c_1 \dots c_{n-2k} \in I_n$, the length $L(\pi)$ is defined by

$$L(\pi) := \frac{\ell(\pi) + k}{2},$$

where $\ell(\pi)$ is the length of π in S_n, and k is the number 2-cycles that appear in the standard form of π.

For more on the combinatorics of Bruhat order on (fixed point free) involutions we recommend works [5] and [1]. The PhD thesis [11] of Ben Wyser has a concrete description of the Bruhat order on signed involutions, although it is described in a different terminology ((p, q)-clans and degeneracy loci).

There is another "geometric" partial ordering on involutions, namely the weak order, that is extremely useful for studying Bruhat ordering. In particular, the weak order is a ranked poset and its length function agrees with that of the Bruhat order. In general, a convenient way of defining the weak order is via the so called "Richardson-Springer monoid" action. Since our goal in this paper is *not* studying the poset structure, and since the descriptions of both Bruhat and weak orders are lengthy we skip their definitions but mention a relevant fact. (See Figure 2.1, where we illustrate the weak order on $I^{\pm}_{2,2}$.)

It is observed in [2] that the length function of the weak order on signed involutions $I^{\pm}_{p,q}$ agrees with that of the weak order on I_n. In other words, the length of $\pi \in I^{\pm}_{p,q}$ is equal to $L(\pi)$, where π is identified with its supporting involution. We occasionally speak of the

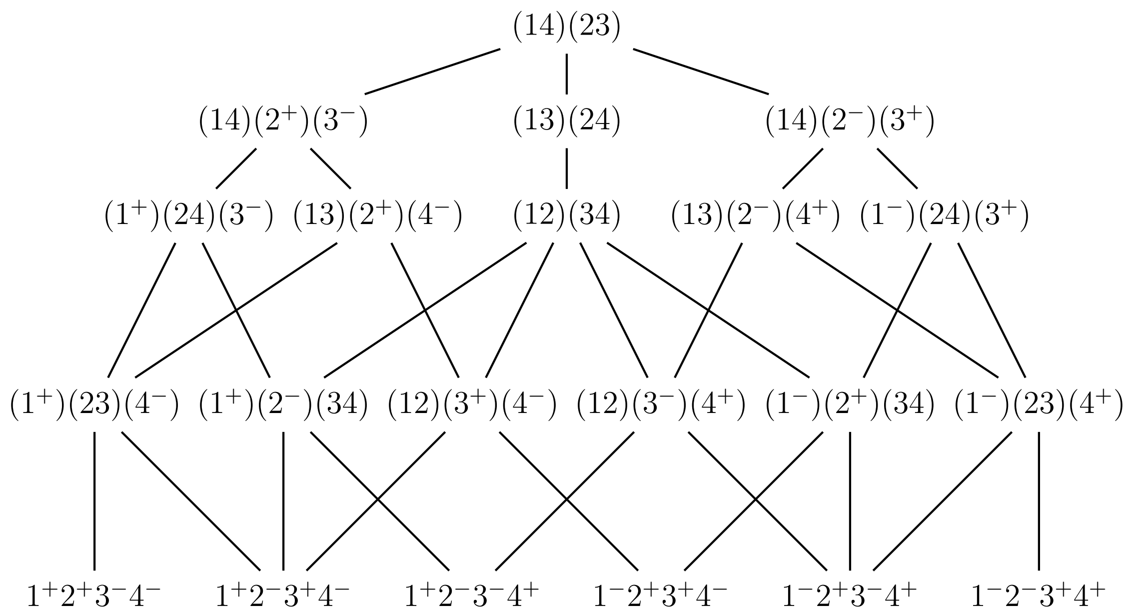


Figure 2.1: Weak order on $I^{\pm}_{2,2}$

support of a signed involution to mean the underlying involution without reference to its signs. Thus, from now on, by abusing notation we are going to use $L(\cdot)$ for denoting the length function of Bruhat as well as the weak order on $I^{\pm}_{p,q}$. Our final remark on the length function $L(\cdot)$ is that if $\pi \in I^{\pm}_{p,q}$ is the signed involution corresponding to the Borel orbit $\mathcal{O}$, then the dimension of $\mathcal{O}$ is equal to $L(\pi) + c$, where c is the dimension of (any) closed Borel orbit in $SL(p+q)/S(GL(p) \times GL(q))$. Thus, studying $L(\pi)$, and its generating function, is equivalent to studying dimensions of orbits.

2.1 Signed involutions with k 2-cycles.

Let n and k be two integers such that $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and let $\pi = (i_1 j_1) \dots (i_k j_k) l_1 \dots l_{n-2k}$ be an involution with fixed points $l_1 < \cdots < l_{n-2k}$. Let us assume that p and q are two numbers such that $0 \leq q \leq p$ and $p + q = n$.

We denote by $I^{\pm}_{k,p,q}$ the set of all signed involutions π from $I^{\pm}_{p,q}$ such that π has $n-2k$ fixed points, $p - q \leq n - 2k$, and there are $p-q$ more $+$'s than $-$'s. We denote the cardinality of $I^{\pm}_{k,p,q}$ by $\gamma_{k,p,q}$. Our aim in this section is to give a practical formula for $\gamma_{k,p,q}$.

Remark 2.1. Since $q + p - 2k = n - 2k \geq p - q$, it holds true that $0 \leq k \leq q$.

Now, if $\pi \in I^{\pm}_{k,p,q}$, then its support is one of the $\binom{n}{2k}\frac{(2k)!}{2^k k!}$ involutions $\pi \in I_n$ with $n-2k$ fixed points. This is easy to see but let us justify it for completeness: Once the entries to appear in the transpositions of π are chosen the fixed points, which are ordered in an increasing manner, are uniquely determined. So, the question is equivalent to choosing k

transposition from $[n]$ and ordering them to give π. It is not difficult to see that this is indeed given by $\binom{n}{2k}\frac{(2k)!}{2^k k!}$.

Next, we look into ways to place a +'s and b -'s on the string $l_1 \dots l_{n-2k}$ so that there are exactly $p-q = a-b$ +'s more than -'s. Clearly, this number is equivalent to $\binom{n-2k}{a}$. Since $a+b = n-2k = q+p-2k$ and $a-b = p-q$, we have $a = p-k$. Therefore,

$$\gamma_{k,p,q} = \binom{q+p}{2k}\frac{(2k)!}{2^k k!}\binom{q+p-2k}{q-k}, \tag{2.2}$$

or more symmetrically expressed as

$$\gamma_{k,p,q} = \frac{(q+p)!}{(q-k)!(p-k)!}\frac{1}{2^k k!}. \tag{2.3}$$

Observe that the formula (2.3) is defined independently of the inequality $q < p$. From now on, for our combinatorial purposes, we skip mentioning this comparison between p and q and use the equality $\gamma_{k,p,q} = \gamma_{k,q,p}$ whenever it is needed. Also, we record the following obvious recurrences for future reference:

$$\gamma_{k,p,q} = \frac{1}{2(p-k)(q-k)(k-1)}\gamma_{k-1,p,q}, \quad \gamma_{k,p,q} = \frac{p+q}{p-k}\gamma_{k,p-1,q}, \quad \gamma_{k,p,q} = \frac{p+q}{q-k}\gamma_{k,p,q-1}. \tag{2.4}$$

which hold true (whenever they are defined) for all $p, q, k \geq 1$.

It is well known that the (exponential) generating function for the number of involutions I_n, which we denote by c_n, is given by $e^{t+\frac{t^2}{2}}$. In fact, define polynomials $K_n(x)$ by the formula

$$\sum_{n\geq 0} K_n(x)\frac{t^n}{n!} = e^{xt+\frac{t^2}{2}}.$$

It is well known that $K_n(x) = \sum_{\pi\in I_n} x^{a_1(\pi)}$, where $a_1(\pi)$ denotes the number of 1-cycles (fixed points) of π. See [9, Exercise 5.19]. It easily follows that if $c_{n,r}$ denotes the number of elements of I_n with exactly r 1-cycles, then

$$\sum_{n,r\geq 0} c_{n,r}\frac{t^n}{n!}x^r = e^{xt+\frac{t^2}{2}}. \tag{2.5}$$

Remark 2.6. The numbers $c_{n,r}$ appear in our context rather naturally. Suppose we have $q = k \leq p$. Then $\gamma_{k,p,k} = \gamma_{k,k,p}$ is the number of signed involutions on $[p+k]$ such that there are $p-k$ +'s more than -'s on the fixed points. It is not difficult to see in this case that the number of $-$'s is 0. Therefore, $\gamma_{k,p,k}$ is the number of involutions on $[p+k]$ with $p-k$ 1-cycles and whose fixed points have + signs only. In other words, $\gamma_{k,p,k} = c_{p+k,p-k}$.

The polynomial $K_n(x)$ is the sum of the entries of the nth row of Table 1. In the sequel, we are going to need the following finite *diagonal* sums of the same table:

$$G_m(x) = \sum_{k=1}^{m} c_{m+k,m-k}x^k. \tag{2.7}$$

$$\begin{array}{cccccc}
c_{0,0}x^0 & 0 & 0 & 0 & 0 & \cdots \\
c_{1,0}x^0 & c_{1,1}x^1 & 0 & 0 & 0 & \cdots \\
c_{2,0}x^0 & c_{2,1}x^1 & c_{2,2}x^2 & 0 & 0 & \cdots \\
c_{3,0}x^0 & c_{3,1}x^1 & c_{3,2}x^2 & c_{3,3}x^3 & 0 & \cdots
\end{array}$$

Table 1: Dissection of $K_n(x)$'s.

3 Recurrences

We are going to show that $\gamma_{k,p,q}$'s obey a 3-term recurrence and exploit its consequences.

Theorem 3.1. *Let p and q be two positive integers. If $k \geq 1$, then the following recurrence relation and its initial condition holds true:*

$$\gamma_{k,p,q} = \gamma_{k,p-1,q} + \gamma_{k,p,q-1} + (q+p-1)\gamma_{k-1,p-1,q-1} \quad \textit{and} \quad \gamma_{0,p,q} = \binom{p+q}{q}. \tag{3.2}$$

Proof. The proof of the second equality (initial condition) is straightforward. We are going to construct our proof of the recurrence by analyzing what happens to an involution $\pi \in I^{\pm}_{k,p,q}$ when we remove its largest entry n. Clearly, n appears in π either as a fixed point, or in one of the 2-cycles. Thus we partition $I^{\pm}_{k,p,q}$ into $n+1$ disjoint subsets;

$$I^{\pm}_{k,p,q} = I^{\pm}_{k,p,q}(+) \cup I^{\pm}_{k,p,q}(-) \cup \bigcup_{i=1}^{n-1} I^{\pm}_{k,p,q}(i),$$

where

1. $I^{\pm}_{k,p,q}(+) := \{\pi \in I^{\pm}_{k,p,q} : \text{ n is a fixed point with a + sign}\}$,
2. $I^{\pm}_{k,p,q}(-) := \{\pi \in I^{\pm}_{k,p,q} : \text{ n is a fixed point with a - sign}\}$,
3. $I^{\pm}_{k,p,q}(i) := \{\pi \in I^{\pm}_{k,p,q} : \text{ n appears in the 2-cycle } (i,n)\}$ for $i = 1, \ldots, n-1$.

First, we assume that $\pi \in I^{\pm}_{k,p,q}(+)$, so

$$\pi = (i_1\ j_1)\ldots(i_k\ j_k)l_1 \ldots l_{n-2k-1}n^+.$$

It follows that by removing n we reduce the total number of + signs by 1. Note that this makes sense because $p-q$ is fixed. Thus, the number of such signed (p,q)-involutions on $[n]$ is counted by $\gamma_{k,p-1,q}$. By using a similar argument for the case $\pi \in I^{\pm}_{k,p,q}(-)$, we conclude that there are $\gamma_{k,p,q-1}$ such signed involutions on $[n]$.

Next, and finally, we consider the case where n appears in a 2-cycle (i,j) of π. Then $j = n$ and therefore $\pi \in I^{\pm}_{k,p,q}(i)$. It is obvious that there are $(n-1)$ possibilities for i from n numbers. Removing this 2-cycle from π leaves us with $(n-2)$ elements and $(k-1)$ 2-cycles but it does not change the signs on the fixed points. Of course, once this 2-cycle is removed, we decrease the numbers that are greater than i_r by 1 so that we have valid signed involution whose support lies in I_{n-2}. In particular, since the difference of $+$ and $-$ signs is preserved, we see that the number of such signed (p,q)-involutions on $[n]$ is given by $(n-1)\gamma_{k-1,p-1,q-1}$.

Notice that in each of these cases we get an injective map into a set of signed involutions of a smaller size. Indeed, by removing n, in the cases 1. and 2. we get injections into $I^{\pm}_{k,p-1,q}$ and $I^{\pm}_{k,p,q-1}$, respectively. In the case of 3. we get an injection into $I^{\pm}_{k-1,p-1,q-1}$. Conversely, if π' is a signed involution from $I^{\pm}_{k,p-1,q}$ (or, from $I_{k,p,q-1}$), then we append n^+ (resp. n^-) to get an element $\psi_k(+)(\pi') \in I^{\pm}_{k,p,q}$ (resp. $\psi_k(-)(\pi') \in I^{\pm}_{k,p,q}$). If $\pi' \in I^{\pm}_{k-1,p-1,q-1}$, then we pick a number, say $i \in [n-1]$ in $n-1$ different ways; we add 1 to every number j such that $i < j$ and j appears in the standard form of π'; and insert the 2-cycle (i,n) into π'. Let us denote the resulting map by $\psi_k(i)(\pi') \in I^{\pm}_{k,p,q}$ Obviously, these maps, $\psi(\pm)$ and $\psi(i)$'s, are well defined inverses to the procedures that are described in the previous paragraph. Thus, it is now clear that we have built a bijection between $I^{\pm}_{k,p,q}$ and the disjoin union $I^{\pm}_{k,p-1,q} \cup I^{\pm}_{k,p,q-1} \cup \bigsqcup_{i=1}^{n-1} I^{\pm}_{k-1,p-1,q-1}$, proving our claimed recurrence. (We use square-union to indicate that it is a disjoint union of $n-1$ copies of the same set.)

□

We list two important corollaries.

Corollary 3.3. *The number of involutions $\pi \in I_n$ with exactly r 1-cycles, $c_{n,r}$, is a special case of $\gamma_{k,p,q}$'s.*

Proof. We already know from Remark 2.6 that $\gamma_{k,p,k} = c_{p+k,p-k}$. The result follows from the fact that the equations $n = p+k$, $r = p-k$ have a unique solution. □

Corollary 3.4. *Set $\alpha_{p,0} = \alpha_{0,q} = 1$ for all nonnegative integers p and q. Then the numbers $\alpha_{p,q}$ satisfy the following recurrence relation*

$$\alpha_{p,q} = \alpha_{p-1,q} + \alpha_{p,q-1} + (p+q-1)\alpha_{p-1,q-1}, \qquad p,q \geq 1. \tag{3.5}$$

Proof. Taking the sum of both sides of the equation (3.2) over k, where $1 \leq k \leq p-1$, gives us

$$\begin{aligned}\alpha_{p,q} - \alpha_{p-1,q} - \alpha_{p,q-1} - (p+q-1)\alpha_{p-1,q-1} &= \gamma_{0,p,q} - \gamma_{0,p-1,q} - \gamma_{0,p,q-1} \\ &+ \gamma_{p,p,q} - \gamma_{p,p,q-1} - (p+q-1)\gamma_{p-1,p-1,q-1} \\ &= 0.\end{aligned}$$

By using the recurrence relation in (3.2) once more, we see that the latter becomes 0. □

4 Generating functions

In this section we are going to describe the generating function $u_p(x) := \sum_{q\geq 0} \alpha_{p,q}x^q$. We first assume that $p \geq 1$. By tabulating a few terms of $u_p(x)$ as in

$$\begin{aligned}
\alpha_{p,0} &= \gamma_{0,p,0} \\
\alpha_{p,1}x &= \gamma_{0,p,1}x + \gamma_{1,p,1}x \\
\alpha_{p,2}x^2 &= \gamma_{0,p,2}x^2 + \gamma_{1,p,2}x^2 + \gamma_{2,p,2}x^2 \\
\alpha_{p,3}x^3 &= \gamma_{0,p,3}x^3 + \gamma_{1,p,3}x^3 + \gamma_{2,p,3}x^3 + \gamma_{3,p,3}x^3
\end{aligned}$$

we find that it is going to be useful to determine the following infinite sums first:

$$F_k(x) := \sum_{q\geq k} \gamma_{k,p,q}x^q \qquad (k \geq 1).$$

Indeed, it is clear from the above table that

$$u_p(x) = \frac{1}{1-x} + \sum_{k\geq 1} F_k(x). \tag{4.1}$$

By using recurrences in (2.4) we rewrite $F_k(x)$'s:

$$\begin{aligned}
F_k(x) = \sum_{q\geq k} \gamma_{k,p,q}x^q &= \gamma_{k,p,k}x^k + \sum_{q\geq k+1} \gamma_{k,p,q}x^q \\
&= \gamma_{k,p,k}x^k + \sum_{q\geq k+1} \gamma_{k,p,q-1}\frac{p+q}{q-k}x^q \\
&= \gamma_{k,p,k}x^k + \sum_{q\geq k+1} \left(\gamma_{k,p,q-1}x^q + \gamma_{k,p,q-1}\frac{p+k}{q-k}x^q\right) \\
&= \gamma_{k,p,k}x^k + x\sum_{q\geq k+1} \gamma_{k,p,q-1}x^{q-1} + \sum_{q\geq k+1} \gamma_{k,p,q-1}\frac{p+k}{q-k}x^q \\
&= \gamma_{k,p,k}x^k + xF_k(x) + (p+k)\sum_{q\geq k+1} \gamma_{k,p,q-1}\frac{1}{q-k}x^q \\
&= \gamma_{k,p,k}x^k + xF_k(x) + (p+k)x^k\sum_{q\geq k+1} \gamma_{k,p,q-1}\frac{1}{q-k}x^{q-k} \\
&= \gamma_{k,p,k}x^k + xF_k(x) + (p+k)x^k\int\sum_{q\geq k+1} \gamma_{k,p,q-1}x^{q-k-1}dx.
\end{aligned}$$

We re-organize the last equation as follows:

$$\frac{F_k(x) - xF_k(x)}{x^k} = \gamma_{k,p,k} + (p+k) \int x^{-k} F_k(x) dx.$$

Equivalently,

$$F_k(x)(x^{-k} - x^{-k+1}) = \gamma_{k,p,k} + (p+k) \int F_k(x) x^{-k} dx. \tag{4.2}$$

Taking the derivative of both sides gives us a first order differential equation with variable coefficients:

$$F_k'(x^{-k} - x^{-k+1}) - F_k(kx^{-k-1} + (-k+1)x^{-k} + (p+k)x^{-k}) = 0$$

or

$$F_k' + \frac{k + x(1+p)}{x^2 - x} F_k = 0, \tag{4.3}$$

which is a first order linear separable homogeneous ODE with the initial condition

$$\left. \frac{F_k(x)}{x^k} \right|_{x=0} = \gamma_{k,p,k}.$$

Therefore,

$$F_k(x) = x^k((1-x)^{-(k+p+1)} + \gamma_{k,p,k} - 1).$$

Recall our assumption that $p \geq 1$. Now, taking the sum of both sides over all $k \geq 1$ gives us

$$\begin{aligned}
u_p(x) &= \frac{1}{1-x} + \sum_{k \geq 1} F_k(x) \\
&= \frac{1}{1-x} + \sum_{k \geq 1} x^k (1-x)^{-(k+p+1)} + \sum_{k=1}^{p} \gamma_{k,p,k} x^k - \sum_{k \geq 1} x^k \\
&= \frac{1}{1-x} + \sum_{k \geq 1} x^k (1-x)^{-(k+p+1)} + \sum_{k=1}^{p} \gamma_{k,p,k} x^k - \frac{x}{1-x} \\
&= 1 + \frac{1}{(1-x)^{1+p}} \sum_{k \geq 1} \frac{x^k}{(1-x)^k} + \sum_{k=1}^{p} \gamma_{k,p,k} x^k \\
&= 1 + \frac{1}{(1-x)^{1+p}} \left(\frac{1}{1 - \frac{x}{(1-x)}} - 1 \right) + \sum_{k=1}^{p} \gamma_{k,p,k} x^k \\
&= 1 + \frac{1}{(1-x)^{1+p}} \left(\frac{x}{1-2x} \right) + \sum_{k=1}^{p} \gamma_{k,p,k} x^k. \qquad (4.4)
\end{aligned}$$

Recall also that $c_{m,r}$ stands for the number of involutions on $[m]$ with exactly r 1-cycles.

Theorem 4.5. *Let p be a nonnegative integer. The generating function $u_p = \sum_{q\geq 0} \alpha_{p,q} x^q$ is equal to*

$$u_p(x) = \begin{cases} \frac{1}{1-x}; & \text{if } p = 0; \\ 1 + \frac{1}{(1-x)^{1+p}}\left(\frac{x}{1-2x}\right) + \sum_{k=1}^{p} c_{p+k,p-k} x^k; & \text{if } p \geq 1. \end{cases}$$

Proof. The proof follows from (4.4) and Remark 2.6. $\square$

One of the many options for a bivariate generating function for $\alpha_{p,q}$'s is

$$v(x,y) := \sum_{p,q\geq 0} \alpha_{p,q} x^q \frac{y^p}{p!} = \sum_{p\geq 0} u_p(x) \frac{y^p}{p!}, \tag{4.6}$$

which is easily seen (by Theorem 4.5) to reduce to the calculation of

$$\sum_{p\geq 0} G_p(x) \frac{y^p}{p!}, \tag{4.7}$$

where $G_p(x) = \sum_{k=1}^{p} c_{p+k,p-k} x^k$. Substituting $\gamma_{k,p,k} = c_{p+k,p-k} = \frac{(p+k)!}{(p-k)!2^k k!}$ into (4.7), we have

$$\begin{aligned}
\sum_{p\geq 0} G_p(x) \frac{y^p}{p!} &= \sum_{p\geq 0} \sum_{k=1}^{p} \frac{\gamma_{k,p,k}}{p!} x^k y^p \\
&= \sum_{p\geq 0} \sum_{k=1}^{p} \frac{(p+k)!}{(p-k)!k!p!} \left(\frac{x}{2}\right)^k y^p \\
&= \sum_{p\geq 0} \frac{\sqrt{\frac{2}{\pi}} e^{\frac{1}{x}} \sqrt{\frac{1}{x}} \widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x}) - 1}{p!} y^p \\
&= e^{\frac{1}{x}} \sqrt{\frac{2}{\pi x}} \sum_{p\geq 0} \frac{\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!} y^p - \sum_{p\geq 0} \frac{y^p}{p!} \\
&= -e^y + e^{\frac{1}{x}} \sqrt{\frac{2}{\pi x}} \sum_{p\geq 0} \frac{\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!} y^p,
\end{aligned}$$

where $\widetilde{K}_n(x)$ denotes the modified Bessel function of the second kind, which is one of the solutions to the modified Bessel differential equation. Now the following consequence is immediate from Theorem 4.5.

Corollary 4.8. *The bivariate generating function $\sum_{p,q\geq 0} \alpha_{p,q} x^q \frac{y^p}{p!}$ is given by*

$$\frac{e^y}{1-x} + \frac{x e^{\frac{y}{1-x}}}{(1-2x)(1-x)} + e^{\frac{1}{x}} \sqrt{\frac{2}{\pi x}} \sum_{p\geq 0} \frac{\widetilde{K}_{p+\frac{1}{2}}(\frac{1}{x})}{p!} y^p.$$

Another approach for deriving the bivariate generating function (4.6) is by transforming the recurrence relation (3.5) into a partial differential equation as follows. Multiplying both side of the recurrence relation by $\frac{x^q y^p}{p!}$ and taking the sum over all $p, q \geq 1$ gives us

$$\sum_{p,q\geq 1} \frac{\alpha_{p,q}}{p!} x^q y^p = \sum_{p,q\geq 1} \frac{\alpha_{p-1,q}}{p!} x^q y^p + \sum_{p,q\geq 1} \frac{\alpha_{p,q-1}}{p!} x^q y^q + \sum_{p,q\geq 1} (p+q-1) \frac{\alpha_{p-1,q-1}}{p!} x^q y^p. \tag{4.9}$$

Since

$$\begin{aligned} v(x,y) = \sum_{p,q\geq 0} \frac{\alpha_{p,q}}{p!} x^q y^p &= \alpha_{0,0} + \alpha_{0,1} x + \cdots + \alpha_{0,q} x^q + \dots \\ &+ \frac{\alpha_{1,0}}{1!} y + \cdots + \frac{\alpha_{p,0}}{p!} y^p + \dots \\ &+ \frac{\alpha_{1,1}}{1!} xy + \cdots + \frac{\alpha_{p,1}}{p!} xy^p + \dots \\ &+ \frac{\alpha_{1,2}}{1!} x^2 y + \cdots + \frac{\alpha_{p,2}}{p!} x^2 y^p + \dots \end{aligned}$$

the equation (4.9) combined with the initial conditions $\alpha_{p,0} = \alpha_{0,q} = 1$ gives

$$\begin{aligned} v(x,y) - \frac{1}{1-x} - e^y + 1 &= \int \sum_{p\geq 1, q\geq 0} \frac{\alpha_{p-1,q}}{(p-1)!} x^q y^{p-1} dy - e^y + x \Bigg(\sum_{p,q\geq 0} \frac{\alpha_{p,q}}{p!} x^q y^p - \frac{1}{1-x} \Bigg) \\ &+ \sum_{p,q\geq 1} p \frac{\alpha_{p-1,q-1}}{p!} x^q y^p + \sum_{p,q\geq 1} q \frac{\alpha_{p-1,q-1}}{p!} x^q y^p - \sum_{p,q\geq 1} \frac{\alpha_{p-1,q-1}}{p!} x^q y^p \\ &= \int \sum_{p\geq 1, q\geq 0} \frac{\alpha_{p-1,q}}{(p-1)!} x^q y^{p-1} dy - e^y + x \Bigg(\sum_{p,q\geq 0} \frac{\alpha_{p,q}}{p!} x^q y^p - \frac{1}{1-x} \Bigg) \\ &+ xy \sum_{p,q\geq 0} \frac{\alpha_{p-1,q-1}}{(p-1)!} x^{q-1} y^{q-1} + \int \sum_{p,q\geq 1} \frac{q\alpha_{p-1,q-1}}{(p-1)!} x^{q-1} y^{p-1} dy \\ &- x \int \sum_{p,q\geq 1} \frac{\alpha_{p-1,q-1}}{(p-1)!} x^{q-1} y^p dy. \end{aligned}$$

Thus, we have

$$\begin{aligned} v(x,y) - \frac{1}{1-x} - e^y + 1 &= \int v(x,y) dy - e^y + xv(x,y) - \frac{x}{1-x} + xyv(x,y) \\ &+ x \int \left(\frac{\partial}{\partial x} (xv(x,y)) \right) dy - x \int v(x,y) dy, \end{aligned}$$

or equivalently,

$$(1 - x - xy) v(x,y) = (1-x) \int v(x,y) dy + x \int \left(\frac{\partial}{\partial x} (xv(x,y)) \right) dy.$$

Taking the integral of both sides with respect to y yields the following PDE

$$-xv(x,y)+(1-x-xy)\frac{\partial v(x,y)}{\partial y}=(1-x)v(x,y)+x\bigg(v(x,y)+x\frac{\partial v(x,y)}{\partial x}\bigg),$$

or

$$(-x^2)\frac{\partial v(x,y)}{\partial x}+(1-x-xy)\frac{\partial v(x,y)}{\partial y}=(1+x)v(x,y),$$

with the initial conditions $v(0,y)=e^y$ and $v(x,0)=\frac{1}{1-x}$.

Solutions of such PDE's are easily obtained by applying the method of "characteristic curves." Our characteristic curves are $x(r,s),y(r,s)$, and $v(r,s)$. Their tangents are equal to

$$\frac{\partial x}{\partial r}=-x^2 \qquad \frac{\partial y}{\partial r}=1-x-xy, \qquad \frac{\partial v}{\partial r}=(1+x)v, \tag{4.10}$$

with the initial conditions

$$x(0,s)=s,\ \ y(0,s)=1,\ \ \text{and}\ \ v(0,s)=e^s.$$

From the first equation given in (4.10) and its initial condition below, we have

$$x(r,s)=\frac{s}{rs+1}. \tag{4.11}$$

Plugging this into the second equation gives us $\frac{\partial y}{\partial r}=1-\frac{s}{rs+1}(1+y)$, which is a first order linear ODE. The general solution for this ODE is

$$y(r,s)=\frac{r^2s-2rs+2r+2}{2(rs+1)}. \tag{4.12}$$

Finally, from the last equation in (4.10) together with its initial condition we conclude that

$$v(r,s)=e^{r+s}(rs+1).$$

In summary we outlined the proof of our next result.

Corollary 4.13. *Let $v(x,y)$ denote the function that is represented by the series $\sum_{p,q\geq 0}\alpha_{p,q}x^q\frac{y^p}{p!}$ around the origin. If r and s are the variables related to x and y as in equations (4.12) and (4.11), then*

$$v(r,s)=e^{r+s}(rs+1) \tag{4.14}$$

around $(r,s)=(-1,0)$.

Unfortunately, the beautiful form in (4.14) of $v(r,s)$ diminishes once the variables r and s are solved in terms of x and y. In fact, this seems to be a nontrivial task due to complicated nature of (4.12). However, one can still recover some information by computing (by brute force of long division) the inverses of power series. We anticipate that this approach will be helpful for understanding special values (at small numbers) of the modified Bessel functions. Such information is useful in number theory.

5 A combinatorial interpretation

The *Delannoy numbers*, denoted by $D(p, q)$ $(p, q \in \mathbb{N})$ are defined by the recurrence relation

$$D(p, q) = D(p-1, q) + D(p, q-1) + D(p-1, q-1) \tag{5.1}$$

and the initial conditions $D(p, 0) = D(0, q) = D(0, 0) = 0$. Their generating series is

$$\sum_{\substack{a+b\geq 0\\ a,b\in\mathbb{N}}} D(p, q)x^i y^j = \frac{1}{1-x-y-xy}$$

Our goal in this section is to show that our involution numbers $\alpha_{p,q}$ have a beautiful interpretation in terms of lattice paths and they are related to the generalized Delannoy numbers.

We first set up our terminology. For us, a *step* in $\mathbb{R}^2$ is a pair $((a, b), (a', b'))$ of points from $\mathbb{Z}^2$ such that $(a-a')^2+(b-b')^2 \leq 2$. We always assume that $|a'|+|b'| > |a|+|b|$. A step is called *diagonal* if $a' = a+1$ and $b' = b+1$. We are going to need two other types of steps; a horizontal step is a step from (a, b) to $(a+1, b)$ and a vertical step is a step from (a, b) to $(a, b+1)$. Occasionally, if there is no need to be specific about coordinates, we denote a horizontal step by N, a vertical step by V, and a diagonal step by NE.

A lattice path π is a sequence of steps $(\pi_1, \dots, \pi_r)$, where $\pi_i \in \{N, E, NE\}$ for $i = 1, \dots, r$, and moreover, the second entry of π_i is the first entry of π_{i+1} for $i = 1, \dots, r-1$. The set of all lattice paths from $(0, 0)$ to (p, q) is denoted by $L(p, q)$. The subset of $L(p, q)$ which consists of lattice paths with no diagonal step is denoted by $L^0(p, q)$. More generally, of $a \leq c, b \leq d$ are four nonnegative integers, then we are going to denote by $L((a, b), (c, d))$ the set of paths that starts at (a, b) and ends at (c, d).

Clearly, the number of elements of $L(p, q)$ is the Delannoy number $D(p, q)$. Let us denote the number of elements of $L^0(p, q)$ by $b_{p,q}$. Then

$$b_{p,q} := \#L^0(p, q) = \binom{p+q}{q}.$$

Now suppose we have 3 sequences of complex numbers $\mathrm{h} = (h_1, h_2, \dots)$, $\mathrm{v} = (v_1, v_2, \dots)$, and $\mathrm{d} = (d_1, d_2, \dots)$, and suppose ζ is a complex number. Let $\pi = (\pi_1, \dots, \pi_r)$ be a lattice path. Put $\mathcal{L} := (\mathrm{h}, \mathrm{v}, \mathrm{d}, \zeta)$. The $\mathcal{L}$-weight of π is defined as the product of weights of the steps of π. More precisely, if $\pi_i = ((a, b), (c, e))$ is a step in π, then its weight is defined as

$$\omega^{\mathcal{L}}(\pi_i) = \begin{cases} v_c & \text{if } \pi_i \text{ is a vertical step,} \\ h_c\zeta^e & \text{if } \pi_i \text{ is a horizontal step,} \\ d_c\zeta^{e-1} & \text{if } \pi_i \text{ is a diagonal step.} \end{cases}$$

The weight of π is

$$\omega^{\mathcal{L}}(\pi) := \omega^{\mathcal{L}}(\pi_1)\cdots\omega^{\mathcal{L}}(\pi_r). \tag{5.2}$$

In his seminal article [4], Dziemianczuk describes a generalization of Delannoy numbers as follows. The (p, q) $\mathcal{L}$-Delannoy number, denoted by $\left\langle {p \atop q} \right\rangle_{\mathcal{L}}$, is the sum

$$\left\langle {p \atop q} \right\rangle_{\mathcal{L}} = \sum_{\pi \in P(p,q)} \omega^{\mathcal{L}}(\pi). \tag{5.3}$$

If the sequences $\mathrm{h}, \mathrm{v}, \mathrm{d}$ are all the same and equal to $(1, 1, \dots)$, and if $\zeta = 1$, then it is not difficult to see that

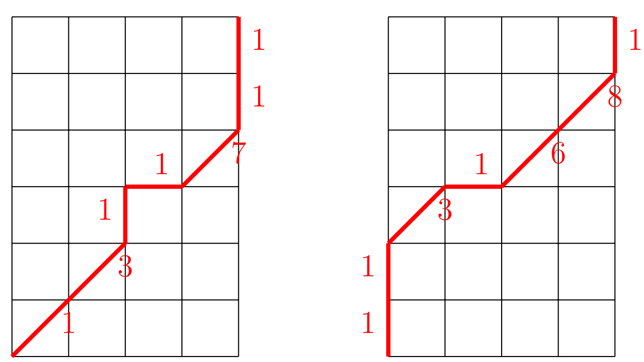
$$\left\langle {p \atop q} \right\rangle_{\mathcal{L}} = D(p, q) \quad \text{for all } p, q \geq 0.$$

Thus, $\left\langle {p \atop q} \right\rangle_{\mathcal{L}}$ is indeed a generalization of the Delannoy number $D(p, q)$.

Now we re-interpret our $\alpha_{p,q}$'s in terms of weighted lattice paths.

Definition 5.4. For us, different than Dziemianczuk's weights (see part 1 of Remark 5.8), the weights of horizontal and vertical steps are 1, and the weight of the step $\pi_i = ((a, b), (c, e))$ is $c + e - 1$. If $\pi = (\pi_1, \dots, \pi_r)$ is a lattice path from $(0, 0)$ to (p, q), then we define our weight, similar to the above definition, as the product of weights of the steps π_i, $i = 1, \dots, r$, and denote it simply by $\omega(\pi)$ with no decoration.

Example 5.5. Consider the following paths given in $(4, 6)$-grid.



The weight of the first path π is $\omega(\pi) = 3 \cdot 7 = 21$ and the weight of the second path $\pi^{'}$ is $\omega(\pi^{'}) = 3 \cdot 6 \cdot 8 = 144$.

It is now immediate from our definition of weight and Corollary 3.4 that

Proposition 5.6. *If p and q are two nonnegative integers, then*

$$\alpha_{p,q} = \sum_{\pi \in L(p,q)} \omega(\pi). \tag{5.7}$$

We have a remark in order.

Remark 5.8. 1. The weights that we use in equation (5.7) are constant along each antidiagonal in the plane. Indeed, the weight of a NE-step which crosses the mth antidiagonal $x+y=m$ is $m-1$. (In a sense, this gives a "force field" in $\mathbb{R}^2$, and $\alpha_{p,q}$ is the count of paths in $L(p,q)$ which are weighted against this force field.) It is easy to check that the weight systems (that is to say $\omega^{\mathcal{L}}$'s) that are introduced by Dziemianczuk do not satisfy this property, therefore, our results are not included in [4].

2. The proof of Proposition 5.7 does not depend on an explicit bijection but rather relies on the equality of the numbers arising from the recurrence in Corollary 3.4 and the obvious recurrence that is satisfied by our weights.

It is now desirable to produce the an explicit bijection between the set of signed involutions $I^{\pm}_{p,q}$ and the paths in $L(p,q)$ with certain labels.

5.1 Lattice paths and signed involutions.

Let $\pi = \pi^{(0)} = (i_1, j_1)\dots(i_k, j_k)l_1 \dots l_{n-2k}$ be a signed involution from $I^{\pm}_{p,q}$. To construct the corresponding weighted path we proceed algorithmically as in the proof of Theorem 3.1.

First we look at where $n = p+q$ appears in π. If it appears as a fixed point with a $+$ sign, then we draw a E-step between (p,q) and $(p-1,q)$. If it appears as a fixed point with a $-$ sign, then we draw a N-step between (p,q) and $(p,q-1)$. In the these cases, removing n from π results in an involution, that we denote by $\pi^{(1)}$, in either $I^{\pm}_{k,p-1,q}$ or $I^{\pm}_{k,p,q-1}$. If n appears as the second entry of one of the 2-cycles, say $(i_r, j_r) = (i,n)$ (for some r and i), then we draw a NE-step between (p,q) and $(p-1,q-1)$, label it with i, and then we remove the two cycle (i,n) from π and reduce every number that is bigger than i in π by -1. Hence we obtain an element $\pi^{(1)}$ of $I^{\pm}_{k-1,p-1,q-1}$. By Theorem 3.1 we know that this algorithm results in a bijection.

By abusing notation we denote the map that we obtain by ϕ, without giving any reference to the indices p and q. Let us demonstrate this bijection step by step in an example.

Example 5.9. Let $\pi = (1,4)(3,8)2^+5^+6^+7^-$. Then $p+q=8$ and $p-q=2$, hence $p=5, q=3$. Then $\phi(\pi)$ is the path that is in the last picture of Figure 5.1.

Let us denote by $P(p,q)$ the set of labeled paths that lie in the image of the bijection ϕ. In other words, $P(p,q) = \phi(I^{\pm}_{p,q})$. It is interesting, though we postpone its investigation to an upcoming article that the weak order on $I^{\pm}_{p,q}$ is easy to describe using $P(p,q)$. (The covering relations of the weak order, equivalently the action of the Richardson-Springer monoid on $I^{\pm}_{p,q}$, as described in [2, Figure 2.5] are easy to express in terms of certain simple operations on the paths in $P(p,q)$.) In this notation, for example, the unique maximal element of $I^{\pm}_{p,q}$ is the path depicted in Figure 5.2. It corresponds to the maximal dimensional Borel orbit in $X = SL(p+q)/S(GL(p)\times GL(q))$. Closed Borel orbits in X correspond to the elements of $L^0(p,q)$ in $P(p,q)$, that is the set of paths with no diagonal step.

$$\pi = (1,4)(3,8)2^+5^+6^+7^- \;\longrightarrow\; \pi^{(1)} = (1,3)2^+4^+5^+6^-$$

$$\pi^{(1)} = (1,3)2^+4^+5^+6^- \;\longrightarrow\; \pi^{(2)} = (1,3)2^+4^+5^+$$

$$\pi^{(2)} = (1,3)2^+4^+5^+ \;\longrightarrow\; \pi^{(3)} = (1,3)2^+4^+$$

$$\pi^{(3)} = (1,3)2^+4^+ \;\longrightarrow\; \pi^{(4)} = (1,3)2^+$$

$$\pi^{(4)} = (1,3)2^+ \;\longrightarrow\; \pi^{(5)} = 1^+$$

$$\pi^{(5)} = 1^+ \;\longrightarrow\; \pi^{(6)} = \cdot$$

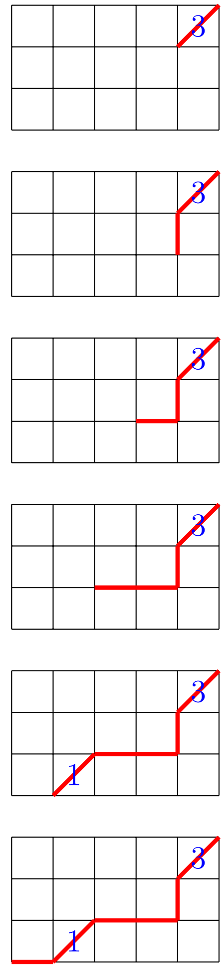


Figure 5.1: Algorithmic construction of ϕ.

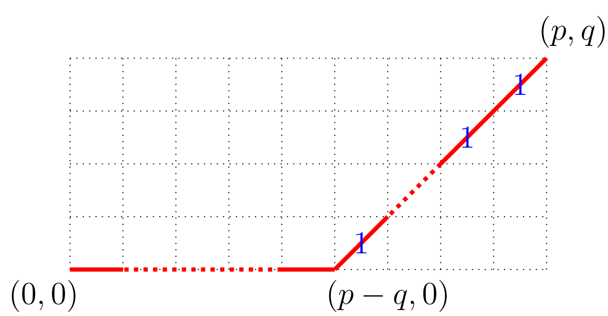


Figure 5.2: The unique maximal element of the weak order on $I^{\pm}_{p,q}$.

6 Polynomial analogs of $\alpha_{p,q}$'s

In this section we consider 3-term t-analogs of the numbers $\alpha_{p,q}$, $p, q = 0, 1, \dots$

6.1 The weight generating function.

Our first version is as follows:

$$D_{p,q}(t) := \frac{1}{t} \sum_{\pi \in L(p,q)} t^{\omega(\pi)}. \tag{6.1}$$

This is the generating function, up to a factor of t, for the weight function ω as in Definition 5.4. It follows from definitions that $D_{p,q}(t)$ obeys the recurrence

$$D_{p,q}(t) = D_{p-1,q}(t) + D_{p,q-1}(t) + t^{p+q-2} D_{p-1,q-1}(t^{p+q-1}).$$

Obviously,

$$\frac{\partial}{\partial t}(t D_{p,q}(t))|_{t=1} = \alpha_{p,q}.$$

It is also obvious that $D_{p,q}(1)$ is nothing but the cardinality of the set $L(p,q)$, the Delannoy number $D(p,q)$. The value at $t = 0$ of $D_{p,q}(t)$ is also easy to find and described below.

Evaluating $D_{p,q}(t)$'s at other roots of unities also gives Delannoy numbers.

Proposition 6.2. *The value of the difference $D_{p,q}(t) - D_{p-1,q}(t) - D_{p,q-1}(t)$ at a $(p+q-1)$'th root of unity ζ is equal to $D(p-1, q-1)\zeta^{-1}$.*

Proof. It follows immediately from the definition (6.1) that

$$t(D_{p,q}(t) - D_{p-1,q}(t) - D_{p,q-1}(t)) = \sum_{P \in L(p-1,q-1)} t^{(p+q-1)\omega(P)}.$$

Therefore, evaluating both sides at ζ and then dividing by ζ gives

$$\frac{1}{\zeta} \sum_{P \in L(p-1,q-1)} \zeta^{(p+q-1)\omega(P)} = \frac{1}{\zeta} \sum_{P \in L(p-1,q-1)} 1^{\omega(P)} = D(p-1, q-1)\zeta^{-1}.$$

□

Next, we are going to have a careful look at the coefficients of $D_{p,q}(t)$. It turns out they are always sums of products of binomial coefficients. Let $n \geq 1$ denote the degree of $D_{p,q}(t)$ and set

$$D_{p,q}(t) = a_0 + a_1 t + \cdots + a_n t^n \qquad (a_i \in \mathbb{N}).$$

We start with the constant term. It is clear from our definition of a path $\pi \in L(p,q)$ that $\omega(\pi) = 1$ if and only if π has at most one diagonal step, which occurs as an initial diagonal step (otherwise the weight would be greater than 1). Consequently,

$$a_0 = b_{p,q} + b_{p-1,q-1} = \binom{p+q}{q} + \binom{p+q-2}{q-1}. \tag{6.3}$$

Let $0 \le a_1 < \cdots < a_r \le p$ and $0 \le b_1 < \cdots < b_r \le q$ be two sequences. Next, we are going to focus on the set of paths $L_{\mathrm{a,b}}(p,q)$ consisting of lattice paths $\pi \in L(p,q)$ with diagonal steps at $\pi_i = ((a_i, b_i), (a_i + 1, b_i + 1))$ for $i = 1, \ldots, r$. Clearly each element $\pi \in L_{\mathrm{a,b}}(p,q)$ is a concatenation of $r+1$ lattice paths $\pi^{(1)}, \ldots, \pi^{(r)}$ each having no diagonal steps. More precisely, $\pi^{(i)} \in L((a_i + 1, b_i + 1), (a_{i+1}, b_{i+1}))$ for $i = 0, \ldots, r$. Here $(a_0, b_0) = (0,0)$ and $(a_{r+1}, b_{r+1}) = (p,q)$. Clearly, the number of lattice paths in $L_{\mathrm{a,b}}(p,q)$ is then

$$\prod_{i=0}^{r} |L((a_i+1, b_i+1), (a_{i+1}, b_{i+1}))| = \prod_{i=0}^{r} \binom{a_{i+1} + b_{i+1} - a_i - b_i - 2}{a_{i+1} - a_i - 1}.$$

Note that the weight of any element $\pi \in L_{\mathrm{a,b}}(p,q)$ is equal to

$$\omega(\pi) = \prod_{i=0}^{r} (a_i + b_i - 1).$$

Thus, by varying the number and choice of diagonal entries we obtain a formula for $D_{p,q}(t)$:

$$D_{p,q}(t) = \sum_{r=0}^{\min\{q,p\}} \sum_{\substack{0 \le a_1 < \cdots < a_r \le p \\ 0 \le b_1 < \cdots < b_r \le q}} \left(\prod_{i=0}^{r} \binom{a_{i+1} + b_{i+1} - a_i - b_i - 2}{a_{i+1} - a_i - 1} \right) t^{\prod_{i=0}^{r}(a_i + b_i - 1)}. \tag{6.4}$$

Remark 6.5. The polynomials $D_{p,q}(t)$ are in general not unimodal.

6.2 Rank generating function.

Our second t-analog has algebro-geometric significance. Recall that the inclusion poset on Borel orbit closures in $X = SL(n)/S(GL(p) \times GL(q))$ is the Bruhat order on signed involutions $I^{\pm}_{p,q}$. This is a graded poset and its rank is equal to the length of its maximal element, which is $(1,n) \cdots (q, n+1-q)(q+1)^{+} \cdots (n-q)^{+}$. Thus,

$$\begin{aligned}
\mathrm{rank}(I^{\pm}_{p,q}) &= \frac{((n-1) + \cdots + (n-q-1)) + (1 + \cdots + q)) + q}{2} \\
&= \frac{\frac{(q+1)(2n-q-2)}{2} + \frac{q(q+1)}{2} + q}{2} \\
&= \frac{pq + +q^2 + p + q - 1}{2} = nq + n - 1
\end{aligned}$$

Note that the dimension of X is $(n^2-1)-(p^2+q^2-1) = n^2-p^2-q^2 = 2pq$. Therefore, the smallest possible dimension for a Borel orbit in X is

$$\begin{aligned} f_{min}(p,q) &:= \dim X - \operatorname{rank}(I^{\pm}_{p,q}) \\ &= 2pq - \frac{pq+q^2+p+q-1}{2} \\ &= \frac{3pq-q^2-p-q+1}{2}. \end{aligned}$$

If we denote the dimension of the Borel orbit in X attached to $\pi \in I^{\pm}_{p,q}$ by $\dim \pi$, then

$$\dim \pi = L(\pi) + f_{min}(p,q).$$

Since, $f_{min}(p,q)$ is constant (relative to p and q), the study of the function $\pi \mapsto \dim \pi$ is equivalent to studying the length function $L(\cdot)$ on $I^{\pm}_{p,q}$, so we consider the following (length) generating function

$$E_{p,q}(t) := \sum_{\pi \in I^{\pm}_{p,q}} t^{L(\pi)}.$$

Our goal is to find a recurrence for $E_{p,q}(t)$. To this end we go back to our ideas in Section 3. Indeed, there is an important consequence of the proof of Theorem 3.1, where we essentially constructed a bijection $\psi_k = (\psi_k(+), \psi_k(-), \psi_k(1), \dots, \psi_k(n-1))$ from

$$I^{\pm}_{k,p-1,q} \times I^{\pm}_{k,p,q-1} \times \underbrace{I^{\pm}_{k-1,p-1,q-1} \times \cdots \times I^{\pm}_{k-1,p-1,q-1}}_{(n-1\text{-copies})}$$

to $I^{\pm}_{k,p,q}$.

In the light of Corollary 3.4, we obtain the bijection $\psi = (\psi(+), \psi(-), \psi(1), \dots, \psi(n-1))$

$$\psi : I^{\pm}_{p,q-1} \times I^{\pm}_{p-1,q} \times \underbrace{I^{\pm}_{p-1,q-1} \times \cdots \times I^{\pm}_{p-1,q-1}}_{(n-1\text{-copies})} \longrightarrow I^{\pm}_{p,q}. \tag{6.6}$$

Next, we analyze the effect of maps $\psi(\pm)$ and $\psi(i)$, $i = 1, \dots, n-1$ on the length of $\pi \in I^{\pm}_{p,q}$.

We know from Section 2 that $L(\pi)$ is equal to $(\ell(\pi)+k)/2$, where k is the number of 2-cycles in π and $\ell(\pi)$ is the number of inversions in π viewed as a permutation. Thus, if n is a fixed point of π, then removing it from π has no effect on the length:

$$L(\psi(\pm)^{-1}(\pi)) = L(\pi). \tag{6.7}$$

For $\psi(i)$'s, it is more interesting. Suppose π has the standard form $\pi = (i_1, j_1) \cdots (i_k, j_k) c_1 \dots c_{n-k}$. If n appears in the 2-cycle $(i_r, j_r) = (i, n)$, then by removing (i_r, j_r) from π we loose $n-i$ inversions of the form $n > j$ and we loose $n-i-1$ inversions of the form $j > i$. Moreover, we loose one 2-cycle. Therefore,

$$L(\psi(i)^{-1}(\pi)) = \frac{\ell(\pi)+k-(2n-2i-1)-1}{2} = \frac{\ell(\pi)+k}{2} - (n-i) = L(\pi) - (n-i). \tag{6.8}$$

First we partition $I^{\pm}_{p,q}$ into three disjoint sets $\psi(+)(I^{\pm}_{p-1,q})$, $\psi(-)(I^{\pm}_{p,q-1})$, and $\psi(i)(I^{\pm}_{p-1,q-1})$ $(i = 1, \ldots, n-1)$, then re-organize the sums by using our observations (6.7) and (6.8):

$$\begin{aligned}
E_{p,q}(t) &= \sum_{\pi\in\psi(+)(I^{\pm}_{p-1,q})} t^{L(\pi)} + \sum_{\pi\in\psi(-)(I^{\pm}_{p,q-1})} t^{L(\pi)} + \sum_{i=1}^{n-1} \sum_{\pi\in\psi(i)(I^{\pm}_{p-1,q-1})} t^{L(\pi)} \\
&= \sum_{\pi\in I^{\pm}_{p-1,q}} t^{L(\pi)} + \sum_{\pi\in I^{\pm}_{p,q-1}} t^{L(\pi)} + \sum_{i=1}^{n-1} \sum_{\pi\in I^{\pm}_{p-1,q-1}} t^{L(\pi)+(n-i)} \\
&= E_{p-1,q}(t) + E_{p,q-1}(t) + (t + t^2 + \cdots + t^{n-1}) E_{p-1,q-1}(t).
\end{aligned}$$

The coefficient of the last term is equal to $[n]_t - 1$, where $[n]_t$ stands for the t-analog of the natural number n:

$$[n]_t := \frac{t^n - 1}{t - 1}.$$

Thus we obtained the proof of the following.

Proposition 6.9. *The family of polynomials $E_{p,q}(t)$, $p, q \geq 0$ satisfies the following recurrence:*

$$E_{p,q} = E_{p-1,q}(t) + E_{p,q-1}(t) + ([q+p]_t - 1) E_{p-1,q-1}(t).$$

Remark 6.10.

1. The polynomials $E_{p,q}(t)$ are unimodal.
2. It appears that the sequence $(E_{n,n}(-1))_{n\geq 1}$ is the sequence of number of "grand Motzkin paths" of length n. The sequence $(E_{n,n}(0))_{n\geq 1}$ is the sequence of number of "central binomial coeffients" $\binom{2n}{n}$, $n \geq 1$. (For other interpretations of these sequences, see The On-Line Encyclopedia of Integer Sequences, https://oeis.org.)
3. There is another closely related t-analogue. We define $\widetilde{E}_{p,q}(t)$'s by the recurrence

   $$\widetilde{E}_{p,q}(t) = \widetilde{E}_{p-1,q}(t) + \widetilde{E}_{p,q-1}(t) + [q+p-1]_t \widetilde{E}_{p-1,q-1}(t).$$

   Similarly to $E_{p,q}(t)$, $\widetilde{E}_{p,q}(t)$ is a unimodal polynomial as well. Both families have interesting specializations.

6.3 Generating functions of $\gamma_{k,p,q}$'s.

The power series $F_k(x) := \sum_{q\geq k} \gamma_{k,p,q} x^q$ played a significant role in our determination of the generating series of $\alpha_{p,q}$'s. We consider a closely related polynomial generating function. Perhaps, as an initial choice, the most natural choice for a t-analog of $\alpha_{p,q}$'s is the polynomial

$$A_{p,q}(t) = \sum_{k=0}^{q} \gamma_{k,p,q} t^k,$$

where $\gamma_{k,p,q} = \frac{(p+q)!}{(p-k)!(q-k)!} \frac{1}{2^k k!}$. (Recall our convention that $\gamma_{k,p,q} = 0$ whenever $k > \min\{p, q\}$.)

Proposition 6.11. *Let p and q be two nonnegative integers. Then*

$$A_{p,q}(t) = \sum_{\pi \in P(p,q)} t^{\#\text{of diagonal steps in } \pi}.$$

Proof. By definition, the coefficient of t^k in $A_{p,q}(t)$ is the number of signed involutions from $I^{\pm}_{p,q}$ with k fixed points. Equivalently, $A_{p,q}(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} |I^{\pm}_{k,p,q}| t^k$. The proof follows from the fact that the bijection ϕ maps an element of $I^{\pm}_{k,q,p}$ to a weighted path in $L(p,q)$ with k diagonal steps. $\square$

Proposition 6.12. *The family $\{A_{p,q}(t)\}_{p,q\geq 0}$ satisfies the following recurrence relation:*

$$A_{p,q}(t) = A_{p,q-1}(t) + A_{p-1,q}(t) + (p+q-1)tA_{p-1,q-1}(t)$$

with initial conditions $A_{p,0}(t) = A_{0,q}(t) = 1$ for all $p, q \geq 0$.

Proof. The proof is a straightforward application of the recurrences in Theorem 3.1. $\square$

Remark 6.13. The polynomials $A_{p,q}(t)$ are unimodal also.

7 Final comments.

As it is mentioned earlier, it would be very interesting to transfer the description of the (weak) Bruhat order [2, 13] on $I^{\pm}_{p,q}$ to the (more combinatorial) setting of lattice paths. One of the many good reasons to make such an effort is that the lattice path combinatorics is well established and it has relations to many other branches of mathematics including representation theory of classical groups and "Schubert calculus". In this regard, the weak order and tableaux corresponding to maximal chains on symmetric group play a crucial role. We also have tableaux from our lattice path interpretation of the weak order on $I^{\pm}_{p,q}$. In fact there are several ways to attach a tableau to a signed involutions, the obvious one is obtained from a "signed version" of the Richardson-Schensted-Knuth algorithm. We are going to report this and related developments in a future publication, where we are going to focus on the poset theoretic properties of $I^{\pm}_{p,q}$.

There is a well established theory of symmetric varieties. If G denotes a simple simply connected classical linear algebraic group, then according to Cartan's classification, there are 10 types of symmetric varieties 4 of which are already mentioned in the introduction. See [3, Table 26.3]. The polarizations namely, symmetric varieties of the form $SL(n)/S(GL(p) \times GL(q))$ exist in other types as well. In other words, there are analogs of the signed involutions (called "signed shuffles") for types B and D. See [6, 7, 12, 14]. Our enumeration and combinatorial reinterpretation of $I^{\pm}_{p,q}$ has extensions to these types as well.

As it is mentioned in the introduction, the Borel orbit enumeration (and the Bruhat order on the corresponding orbit representatives) in a grassmannian amounts to studying $L(p,q)$. Now, there is a liaison between (partial) flag varieties and symmetric varieties via the well established theory of equivariant embeddings [3]. In particular, for special values of p and q, the wonderful compactification of the corresponding symmetric variety has a unique closed G-orbit, which is isomorphic to a grassmann variety. Therefore, our "path combinatorics" approach to Borel orbit counting is geometrically connected to that of grassmannians. Also related to the wonderful compactifications are the "systems" of lattice paths. Borel orbits in $\overline{X}$, the wonderful compactification of $X = SL(n)/S(GL(p) \times GL(q))$ are parametrized by combinatorial objects called "μ-involutions" where μ stands for a composition of a fixed number (depending on p and q). In particular, when μ has a single part, there the corresponding μ-involutions are just the elements of $I^{\pm}_{p,q}$. As μ runs over all possible compositions, we get different sets of lattice paths. Each of these sets of paths have their own enumeration problem. These connections will be explored in a future work.

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