ON MAHLER'S TRANSCENDENCE MEASURE FOR e

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Abstract. We present a completely explicit transcendence measure for e. This is a continuation and an improvement over the works of Borel, Mahler and Hata on the topic. Furthermore, we also prove a transcendence measure for an arbitrary positive integer power of e. The results are based on Hermite-Padé approximations, and on careful analysis of common factors in the footsteps of Hata.

1. Introduction

A positive function $\omega(m,H)$ satisfying an estimate of type

$$(1)\qquad \left|\lambda_0+\lambda_1 e+\lambda_1 e^2+\ldots+\lambda_m e^m\right| > \frac{1}{H^{\omega(m,H)}},\quad H=\max_{1\le i\le m}\{1,|\lambda_i|\},$$

for all $\overline{\lambda}=(\lambda_0,\ldots,\lambda_m)^T\in\mathbb{Z}^{m+1}\setminus\{\overline{0}\}$ is called a transcendence measure for e. The quest to obtain good transcendence measures for e dates back to Borel [2]. He proved that $\omega(m,H)$ can be chosen to be $c\log\log H$ for some positive constant c depending only on m. This was considerably improved by Popken [10, 11], who showed that $\omega(m,H)<m+\frac{c}{\log\log H}$ for some positive constant c depending on m. Soon afterwards, Mahler [8] was able to get the dependance on m explicit:

$$\omega(m,H)<m+\frac{cm^2\log(m+1)}{\log\log H}$$

with c an absolute positive constant. The price he had to pay was that he was only able to prove the validity of the result in some subset of the set consisting of $m,H\in\mathbb{Z}_+$ with $H\ge 3$, unlike the results by Borel and Popken. Finally, in 1991, Khassa and Srinivasan [7] proved that the constant can be chosen to be 98 in the set $n,H\in\mathbb{Z}_+$ with $\log\log H\ge d(n+1)^{6n}$ for some absolute constant $d>e^{950}$. Soon after, in 1995, Hata [5] proved that the constant can be chosen to be 1 in the set of m and H with $\log H\ge\max\{(m!)^{3\log m},e^{24}\}$. A broader view about questions concerning transcendence measures can be found for instance in the book of Fel'dman and Nesterenko [3].

In [5] Hata introduced a striking observation about big common factors hiding in the auxiliary numerical approximation forms. These numerical approximation forms are closely related to classical Hermite-Padé approximations (simultaneous approximations of the second type) of exponential function used already by Hermite. The impact of the common factors was utilized in an asymptotic manner resulting in Theorem 1.2 in [5]. This theorem is sharper than Theorem 1.1 in Hata's paper but it was only valid for H in an asymptotic sense: no explicit lower bound was given, instead, the theorem was formulated for a large enough H.

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In this article we present a more extensive result, Theorem 2.1. The improvements compared to Hata are made visible in its corollary, Theorem 1.1 below. Our Theorem 1.1 improves Hata's bound for the function ω in his Theorem 1.1, and extends the set of values of H for which the result is valid whenever $m \geq 5$. In addition, this result makes Hata's Theorem 1.2 completely explicit, mainly due to our rigorous treatment of the common factors, giving rise to a more complicated behaviour visible in the term

$$\kappa_m := \frac{1}{m} \sum_{\substack{p \leq \frac{m+1}{2} \\ p \in \mathbb{P}}} \min_{0 \leq j \leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \frac{\log p}{p-1} \, w_p\big(s(m) e^{s(m)}\big) , \tag{2}$$

where $w_p(x) := 1 - \frac{p}{x} - \frac{p-1}{\log p} \frac{\log x}{x}$. We also give the exact asymptotic impact in (4), as well as approximations for values of κ_m for specific values of m.

Theorem 1.1. *Assume $m \geq 5$ and $\log H \geq s(m) e^{s(m)}$, where the function $s(m)$ is defined to be $s(m) = m(\log m)^2$. Now*

$$\omega(m, H) \leq \begin{cases} m + \left(1 - \frac{2\kappa_m}{(\log m)^2}\right)\left(1 - \frac{\kappa_m}{\log m}\right) \cdot \frac{m^2 \log m}{\log \log H}, & 5 \leq m \leq 14 \\ m + \left(1 - \frac{1+\kappa_m}{(\log m)^2}\right)\left(1 - \frac{\kappa_m}{\log m}\right) \cdot \frac{m^2 \log m}{\log \log H}, & 15 \leq m \end{cases}. \tag{3}$$

Asymptotically, we have

$$\lim_{m \to \infty} \kappa_m = \kappa = \sum_{p \in \mathbb{P}} \frac{\log p}{p(p-1)} = 0.75536661083\ldots. \tag{4}$$

Throughout his work, Hata assumed $\log H \geq \max\left\{e^{s_1(m)}, e^{24}\right\}$ with the choice $s_1(m) \sim 3m(\log m)^2$. The bound e^{24} is considerably larger than our bound $s(m)e^{s(m)}$ at its smallest: $s(5)e^{s(5)} \approx e^{15.51\ldots}$. The choice of the function $s(m)$ was made as an attempt to balance between the amount of technical details, and the improvement of the function ω against the size of the set of the values of H.

In our Main theorem 2.1 we present a completely explicit transcendence measure for e, in terms of m and H. The proof starts with Lemma 3.2, which gives a suitable criterion for studying lower bounds of linear forms in given numbers. Furthermore, we exploit estimates for the exact inverse function $z(y)$ of the function $y(z) = z \log z$, $z \geq 1/e$, in the lines suggested in [4]. As an important consequence of using the function $z(y)$, the functional dependence in H is improved compared to earlier considerations.

The method displayed in this paper is applicable to proving bounds of the type displayed in (1). As an example, we will consider the case where the polynomial is sparse, namely, where several of the coefficients λ_j in (1) are equal to zero. As a corollary of this, we derive a transcendence measure for positive integer powers of e.

It should be noted that all our results are actually valid over an imaginary quadratic field $\mathbb{I}$.

2. Main result

Let $z : \mathbb{R} \to \mathbb{R}$, denote the inverse function of the function $y(z) = z \log z$, $z \geq 1/e$ and denote further

$$e_1 = \left(m + \frac{1}{2}\right) \log m - (1 + \kappa_m) m - 1.0239,$$

$$B = m^2 \log m - (1 + \kappa_m) m^2 + (m+1) \log(m+1) +$$

$$\frac{1}{2} m \log m - (1.0239 + \kappa_m) m + 0.0000525,$$

$$D = m(m+1)\log(m+1) - \kappa_m m^2 + 0.0000525m + m^3 e^{-s(m)}$$
for all $m \geq 5$, where κ_m was given in (2).

Theorem 2.1. *Assume $m \geq 5$. Then, by the above notations, the bound*

$$|\lambda_0 + \lambda_1 e + \ldots + \lambda_m e^m| > \frac{1}{2e^D} (2H)^{-m-\epsilon(H)}, \tag{5}$$

where

$$\epsilon(H) \log 2H = Bz\left(\frac{\log(2H)}{1 - \frac{e_1}{s(m)}}\right) + m^2 \log\left(z\left(\frac{\log(2H)}{1 - \frac{e_1}{s(m)}}\right)\right),$$

holds for all $\overline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$ with $\log H \geq s(m)e^{s(m)}$ and $H := \max_{1 \leq j \leq m}\{1, |\lambda_j|\}$.

Corollary 2.2. *With the assumptions of Theorem 2.1 we have*

$$|\lambda_0 + \lambda_1 e + \ldots + \lambda_m e^m| \geq \frac{1}{2e^D}\left(\frac{s(m)}{s(m) + \log(s(m))} \cdot \frac{\log\log(2H)}{2\log(2H)}\right)^{m^2} \cdot (2H)^{-m-\frac{\hat{B}}{\log\log(2H)}},$$

where

$$\hat{B} := \left(1 + \frac{\log(s(m))}{s(m)}\right)\left(1 - \frac{e_1}{s(m)}\right)^{-1} \cdot B.$$

3. Preliminaries, lemmas and notation

Throughout this work, let $\mathbb{I}$ denote an imaginary quadratic field and $\mathbb{Z}_{\mathbb{I}}$ its ring of integers.

Fix now $\Theta_1, \ldots, \Theta_m \in \mathbb{C}^*$. Assume that we have a sequence of simultaneous linear forms

$$L_{k,j}(n) = B_{k,0}(n)\Theta_j + B_{k,j}(n), \tag{6}$$

$k = 0, 1, \ldots, m,\ j = 1, \ldots, m$, where the coefficients

$$B_{k,j} = B_{k,j}(n) \in \mathbb{Z}_{\mathbb{I}}, \quad k, j = 0, 1, \ldots, m,$$

satisfy the determinant condition

$$\Delta := \begin{vmatrix} B_{0,0} & B_{0,1} & \ldots & B_{0,m} \\ B_{1,0} & B_{1,1} & \ldots & B_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,0} & B_{m,1} & \ldots & B_{m,m} \end{vmatrix} \neq 0. \tag{7}$$

Further, let $a, c, b, e \in \mathbb{R},\ a, c > 0$, and suppose that

$$|B_{k,0}(n)| \leq Q(n) = e^{q(n)},$$

$$\sum_{j=1}^{m} |L_{k,j}(n)| \leq R(n) = e^{-r(n)},$$

where

$$q(n) = an\log n + b_1 n, \tag{8}$$

$$-r(n) = -cn\log n + e_1 n \tag{9}$$

for all $k, j = 0, 1, \ldots, m$. Let the above assumptions be valid for all $n \geq n_0$.

Before presenting a criterion for lower bound, Lemma 3.2, we introduce a function $z : \mathbb{R} \to \mathbb{R}$, the inverse function of the function $y(z) = z\log z,\ z \geq 1/e$, considered in [4].

Lemma 3.1. [4] *The inverse function $z(y)$ of the function $y(z) = z \log z$, $z \geq 1/e$, is strictly increasing. Define $z_0(y) = y$ and $z_n(y) = \frac{y}{\log z_{n-1}}$ for $n \in \mathbb{Z}^+$. Suppose $y > e$, then $z_1 < z_3 < \cdots < z < \cdots < z_2 < z_0$. Thus the inverse function may be given by the infinite nested logarithm fraction*

$$z(y) = \lim_{n \to \infty} z_n(y) = \frac{y}{\log \frac{y}{\log \frac{y}{\log \cdots}}}, \quad y > e.$$

Further, we denote

$$(10) \quad B = b_1 + \frac{ae_1}{c}, \; C = am, \; D = (a + b_1)m + am^2 e^{-s(m)}, \; F^{-1} = 2e^D,$$
$$c_1 := c - \frac{e_1}{s(m)}, \; n_1 := \max\left\{n_0, e, m, e^{s(m)}\right\}.$$

Lemma 3.2. *Let $m \geq 1$. Then, under the above assumptions* (7)-(9), *the bound*

$$|\lambda_0 + \lambda_1 \Theta_1 + \ldots + \lambda_m \Theta_m| > F\,(2H)^{-\frac{a}{c} - \epsilon(H)},$$

$$\epsilon(H) \log 2H = Bz\left(\frac{\log(2H)}{c_1}\right) + C \log\left(z\left(\frac{\log(2H)}{c_1}\right)\right)$$

holds for all $\overline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$ with $\log(2H) \geq c_1 n_1 \log n_1$.

Proof. We use the notation

$$\Lambda := \lambda_0 + \lambda_1 \Theta_1 + \ldots + \lambda_m \Theta_m, \quad \lambda_j \in \mathbb{Z}_{\mathbb{I}},$$

for the linear form to be estimated. Using our simultaneous linear forms

$$L_{k,j}(n) = B_{k,0}(n)\Theta_j + B_{k,j}(n)$$

from (6) we get

$$(11) \qquad B_{k,0}(n)\Lambda = W_k + \lambda_1 L_{k,1}(n) + \ldots + \lambda_m L_{k,m}(n),$$

where

$$(12) \qquad W_k(n) = B_{k,0}(n)\lambda_0 - \lambda_1 B_{k,1}(n) - \ldots - \lambda_m B_{k,m}(n) \in \mathbb{Z}_{\mathbb{I}}.$$

If now $W_k(n) \neq 0$, then by (11) and (12) we get

$$\begin{aligned} 1 \leq |W_k(n)| &= |B_{k,0}(n)\Lambda - (\lambda_1 L_{k,1} + \ldots + \lambda_m L_{k,m})| \\ &\leq |B_{k,0}(n)||\Lambda| + \sum_{j=1}^{m} |\lambda_j||L_{k,j}(n)| \leq Q(n)|\Lambda| + HR(n). \end{aligned}$$

Now we take the largest $\hat{n}$ with

$$(13) \qquad \hat{n} \geq n_1 := \max\left\{n_0, e, m, e^{s(m)}\right\}$$

such that $\frac{1}{2} \leq HR(\hat{n})$ with big enough H (to be determined later). Consequently $HR(\hat{n} + k) < \frac{1}{2}$ for all $k \geq 1$.

According to the non-vanishing of the determinant (7) and the assumption $\overline{\lambda} \neq \overline{0}$ it follows that $W_k(\hat{n} + m) \neq 0 \in \mathbb{Z}_{\mathbb{I}} \setminus \{0\}$ for some integer $k \in [0, m]$. Hence we get the estimate

$$(14) \qquad 1 < 2|\Lambda|Q(\hat{n} + m)$$

for our linear form Λ, where we need to write $Q(\hat{n} + m)$ in terms of $2H$.

Since $\frac{1}{2} \leq HR(\hat{n})$, we have

$$\log(2H) \geq r(\hat{n}) = c\hat{n}\log\hat{n} - e_1\hat{n} = \hat{n}\log\hat{n}\left(c - \frac{e_1}{\log\hat{n}}\right) \tag{15}$$

By (13) we have $\log\hat{n} \geq s(m)$. Thus

$$\log(2H) \geq \left(c - \frac{e_1}{s(m)}\right)\hat{n}\log\hat{n} =: c_1\hat{n}\log\hat{n},$$

or equivalently $\hat{n}\log\hat{n} \leq \frac{\log(2H)}{c_1}$. Now we have $\hat{n} \geq n_1 \geq e^{s(m)}$ by (13), which implies

$$\frac{\log(2H)}{c_1} \geq \hat{n}\log\hat{n} \geq s(m)e^{s(m)}. \tag{16}$$

Then, by the properties of the function $z(y)$ given in Lemma 3.1, we get

$$\hat{n} \leq z\left(\frac{\log(2H)}{c_1}\right). \tag{17}$$

Now we are ready to estimate $Q(\hat{n}+m) = e^{q(\hat{n}+m)}$ as follows:

$$\begin{aligned} q(\hat{n}+m) &= a(\hat{n}+m)\log(\hat{n}+m) + b_1(\hat{n}+m) \\ &< a(\hat{n}+m)\left(\log\hat{n} + \frac{m}{\hat{n}}\right) + b_1(\hat{n}+m) \\ &= a\hat{n}\log\hat{n} + am\log\hat{n} + b_1\hat{n} + (a+b_1)m + \frac{am^2}{\hat{n}}. \end{aligned} \tag{18}$$

The bound above is always valid but not very good if $\hat{n}$ is considerably smaller than m. This is why we take $\hat{n}$ as in (13). By (15) we get

$$\hat{n}\log\hat{n} \leq \frac{1}{c}\left(\log(2H) + e_1\hat{n}\right). \tag{19}$$

Substituting (19) into (18) gives

$$\begin{aligned} q(\hat{n}+m) &\leq \frac{a}{c}\left(\log(2H) + e_1\hat{n}\right) + am\log\hat{n} + b_1\hat{n} + (a+b_1)m + \frac{am^2}{\hat{n}} \\ &\leq \frac{a}{c}\log(2H) + \left(b_1 + \frac{ae_1}{c}\right)\hat{n} + am\log\hat{n} + (a+b_1)m + am^2e^{-s(m)}, \end{aligned}$$

where we applied (13).

Hence

$$\begin{aligned} Q(\hat{n}+m) &\leq \exp\left(\frac{a}{c}\log(2H) + \left(b_1 + \frac{ae_1}{c}\right)\hat{n} + am\log\hat{n} + (a+b_1)m + am^2e^{-s(m)}\right) \\ &= (2H)^{\frac{a}{c}}\, e^{B\hat{n} + C\log\hat{n} + D}, \end{aligned}$$

where B, C and D are precisely as in the formulation of Lemma 3.2. The claim now follows from (17). $\square$

Let us now formulate a lemma that can be used to bound the function z. It is extremely useful while comparing our results with the results of others.

Lemma 3.3. *If $y \geq s(m)e^{s(m)}$, we have $z \geq e^{s(m)}$. When $s(m) \geq e$, for the inverse function of $z(y)$ of the function $y(z) = z\log z$ it holds*

$$z(y) \leq \left(1 + \frac{\log(s(m))}{s(m)}\right)\frac{y}{\log y}.$$

Proof. Denote $z := z(y)$ with $y > s(m)e^{s(m)}$. Then,

$$z = \frac{y}{\log z} = \frac{y}{\log y}\frac{\log y}{\log z} = \frac{y}{\log y}\left(1 + \frac{\log\log z}{\log z}\right) \leq \frac{y}{\log y}\left(1 + \frac{\log(s(m))}{s(m)}\right).$$

□

Corollary 3.4. *If* $c \leq 1 + \frac{e_1}{s(m)}$, $\log(2H) \geq c_1 n_1 \log n_1$ *and* $\hat{e} := 1 + \frac{\log(s(m))}{s(m)}$, *then*

$$|\lambda_0 + \lambda_1\Theta_1 + \ldots + \lambda_m\Theta_m| \geq \frac{c^C}{2^{C+1}e^D\hat{e}^C}\left(\frac{\log\log(2H)}{\log(2H)}\right)^C \cdot (2H)^{-\frac{a}{c} - \frac{B\hat{e}}{c_1 \log\log(2H)}}.$$

Proof. According to (16) and (17) we have $\hat{n} \leq z\left(\frac{\log(2H)}{c_1}\right)$ with the condition

$$\log(2H) \geq c_1 n_1 \log n_1 \geq c_1 s(m) e^{s(m)}.$$

When $c \leq 1 + \frac{e_1}{s(m)}$, Lemma 3.3 implies

$$\hat{n} \leq z\left(\frac{\log(2H)}{c - \frac{e_1}{s(m)}}\right) \leq \left(1 + \frac{\log(s(m))}{s(m)}\right)\frac{\frac{\log(2H)}{c - \frac{e_1}{s(m)}}}{\log\left(\frac{\log(2H)}{c - \frac{e_1}{s(m)}}\right)} \leq \frac{\hat{e}}{c_1}\cdot\frac{\log(2H)}{\log\log(2H)}, \tag{20}$$

when we denote $\hat{e} := 1 + \frac{\log(s(m))}{s(m)}$. Lemma 3.2 now implies

$$\begin{aligned}
|\lambda_0 + \lambda_1\Theta_1 + \ldots + \lambda_m\Theta_m| &> \frac{1}{2e^D}(2H)^{-\frac{a}{c}} e^{-Bz\left(\frac{\log(2H)}{c_1}\right) - C\log\left(z\left(\frac{\log(2H)}{c_1}\right)\right)} \\
&\geq \frac{1}{2e^D}(2H)^{-\frac{a}{c}} e^{-\frac{\hat{e}B}{c_1}\frac{\log(2H)}{\log\log(2H)}}\left(\frac{\hat{e}}{c_1}\cdot\frac{\log(2H)}{\log\log(2H)}\right)^{-C} \\
&= \frac{c^C}{2^{C+1}e^D\hat{e}^C}\left(\frac{\log\log(2H)}{\log(2H)}\right)^C (2H)^{-\frac{a}{c} - \frac{B\hat{e}}{c_1\log\log(2H)}}.
\end{aligned}$$

□

4. Hermite-Padé approximants for the exponential function

Hermite-Padé approximants of the exponential function date back to Hermite's [6] transcendence proof of of e, see also [14].

Lemma 4.1. *Let* $\beta_0 = 0$, $\overline{\beta} = (\beta_0, \beta_1, \ldots, \beta_m)^T \in \mathbb{C}^{m+1}$ *and* $\overline{l} = (l_0, l_1, \ldots, l_m)^T \in \mathbb{Z}_{\geq 1}^{m+1}$ *be given and define* $\sigma_i = \sigma_i(\overline{l}, \overline{\beta})$ *by*

$$\Omega(w, \overline{\beta}) = \prod_{j=0}^{m}(\beta_j - w)^{l_j} = \sum_{i=l_0}^{L}\sigma_i w^i, \quad L = l_0 + \ldots + l_m.$$

Then

$$\sigma_i = \sigma_i(\overline{l}, \overline{\beta}) = (-1)^i \sum_{l_0 + i_1 + \ldots + i_m = i}\binom{l_1}{i_1}\cdots\binom{l_m}{i_m}\cdot\beta_1^{l_1 - i_1}\cdots\beta_m^{l_m - i_m} \tag{21}$$

and

$$\sum_{i=0}^{L}\sigma_i i^{k_j}\beta_j^i = \sum_{i=l_0}^{L}\sigma_i i^{k_j}\beta_j^i = 0$$

for all $j \in \{0, 1, \ldots, m\}$ *and* $k_j \in \{0, \ldots, l_j - 1\}$.

Theorem 4.2. *Let $\alpha_0, \alpha_1, \dots, \alpha_m$ be $m+1$ distinct complex numbers. Denote $\overline{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_m)^T \in \mathbb{C}^{m+1}$ and $\overline{l} = (l_0, l_1, \dots, l_m)^T \in \mathbb{Z}_{\geq 1}^{m+1}$. Put*

$$A_{\overline{l},0}(t,\overline{\alpha}) = \sum_{i=l_0}^{L} t^{L-i} i! \sigma_i\left(\overline{l},\overline{\alpha}\right). \tag{22}$$

Then there exist polynomials $A_{\overline{l},j}(t,\overline{\alpha})$ and remainders $R_{\overline{l},j}(t,\overline{\alpha})$ such that

$$e^{\alpha_j t} A_{\overline{l},0}(t,\overline{\alpha}) - A_{\overline{l},j}(t,\overline{\alpha}) = R_{\overline{l},j}(t,\overline{\alpha}), \tag{23}$$

where

$$\begin{cases} \deg_t A_{\overline{l},0}(t,\overline{\alpha}) = L - l_0, \\ \deg_t A_{\overline{l},j}(t,\overline{\alpha}) = L - l_j, \\ \operatorname*{ord}_{t=0} R_{\overline{l},j}(t,\overline{\alpha}) \geq L+1 \end{cases}$$

for $j = 1, \dots, m$.

Proof. First we have

$$A_{\overline{l},0}(t,\overline{\alpha}) = \sum_{i=l_0}^{L} t^{L-i} i! \sigma_i\left(\overline{l},\overline{\alpha}\right) = \sum_{i=0}^{L} t^{L-i} i! \sigma_i\left(\overline{l},\overline{\alpha}\right) = t^{L+1} \sum_{i=0}^{L} \frac{i! \sigma_i\left(\overline{l},\overline{\alpha}\right)}{t^{i+1}},$$

since $\sigma_i\left(\overline{l},\overline{\alpha}\right) = 0$ for $0 \leq i < l_0$. Using Laplace transform, we can write this as

$$t^{L+1} \sum_{i=0}^{L} \frac{i! \sigma_i\left(\overline{l},\overline{\alpha}\right)}{t^{i+1}} = t^{L+1} \sum_{i=0}^{L} \mathcal{L}\left(\sigma_i\left(\overline{l},\overline{\alpha}\right) x^i\right)(t) = t^{L+1} \int_0^\infty e^{-xt} \Omega(x,\overline{\alpha}) \mathrm{d}x. \tag{24}$$

Then

$$\begin{aligned} e^{\alpha t} A_{\overline{l},0}(t,\overline{\alpha}) &= t^{L+1} \int_0^\infty e^{(\alpha - x)t} \Omega(x,\overline{\alpha}) \mathrm{d}x \\ &= t^{L+1} \int_0^\infty e^{-yt} \Omega(y+\alpha,\overline{\alpha}) \mathrm{d}y + t^{L+1} \int_0^\alpha e^{(\alpha - x)t} \Omega(x,\overline{\alpha}) \mathrm{d}x. \end{aligned}$$

We have $\Omega(x,\overline{\alpha}) = \prod_{j=0}^{m} (\alpha_j - x)^{l_j}$ and consequently

$$\Omega(y+\alpha,\overline{\alpha}) = \prod_{j=0}^{m} (\alpha_j - \alpha - y)^{l_j} = \Omega\left(y, (\alpha_0 - \alpha, \dots, \alpha_m - \alpha)^T\right). \tag{25}$$

By setting $\alpha = \alpha_j$, $j = 1, \dots, m$, we get the approximation formula

$$e^{\alpha_j t} A_{\overline{l},0}(t,\overline{\alpha}) - A_{\overline{l},j}(t,\overline{\alpha}) = R_{\overline{l},j}(t,\overline{\alpha}),$$

where

$$A_{\overline{l},j}(t,\overline{\alpha}) = A_{\overline{l},0}\left(t, (\alpha_0 - \alpha_j, \dots, \alpha_m - \alpha_j)^T\right) = t^{L+1} \int_0^\infty e^{-yt} \Omega(y+\alpha_j,\overline{\alpha}) \mathrm{d}y \tag{26}$$

and

$$R_{\overline{l},j}(t,\overline{\alpha}) = t^{L+1} \int_0^{\alpha_j} e^{(\alpha_j - x)t} \Omega(x,\overline{\alpha}) \mathrm{d}x, \quad j = 1, \dots, m.$$

Going backwards in (24) with (25) in mind we see that

$$\begin{aligned}A_{\bar{l},j}(t,\overline{\alpha}) &= t^{L+1}\int_0^{\infty} e^{-yt}\Omega(y+\alpha_j,\overline{\alpha})\mathrm{d}y \\ &= t^{L+1}\sum_{i=0}^{L}\mathcal{L}\left(\sigma_i\big(\bar{l},(\alpha_0-\alpha_j,\dots,\alpha_m-\alpha_j)^T\big)\,y^i\right)(t) \\ &= \sum_{i=l_j}^{L} t^{L-i} i!\sigma_i\big(\bar{l},(\alpha_0-\alpha_j,\dots,\alpha_m-\alpha_j)^T\big)\,.\end{aligned}$$

Note that the coordinate $\alpha_j-\alpha_j=0$ corresponds to $\beta_0=0$ in Lemma 4.1, and consequently we now have l_j in the place of l_0 in the definition of σ_i (21). Hence $\sigma_i\big(\bar{l},(\alpha_0-\alpha_j,\dots,\alpha_m-\alpha_j)^T\big)=0$ for $0\le i<l_j$, and $\deg_t A_{\bar{l},j}(t,\overline{\alpha})=L-l_j$. In addition, $\operatorname{ord}_{t=0} R_{\bar{l},j}(t,\overline{\alpha})\ge L+1$ for $j=1,\dots,m$, since the function

$$t\mapsto\int_0^{\alpha_j} e^{(\alpha_j-x)t}\Omega(x,\overline{\alpha})\mathrm{d}x$$

is analytic at the origin. □

Lemma 4.3. *We have* $\frac{1}{l_j!}A_{\bar{l},j}(t,\overline{\alpha})\in\mathbb{Z}[t,\alpha_1,\dots,\alpha_m]$ *for all* $j=0,1,\dots,m$.

Proof. In the case $j=0$ we have

$$\frac{1}{l_0!}A_{\bar{l},0}(t,\overline{\alpha})=\sum_{i=l_0}^{L}t^{L-i}\sigma_i\big(\bar{l},\overline{\alpha}\big)\frac{i!}{l_0!}$$

by (22). Since $\sigma_i=0$ when $0\le i\le l_0-1$, the claim clearly holds. Next write

$$A_{\bar{l},0}(t,\overline{\alpha})e^{\alpha_j t}=\sum_{N=0}^{\infty}r_N t^N,$$

where

$$r_N=\sum_{N=h+n}\frac{\sigma_{L-h}\big(\bar{l},\overline{\alpha}\big)(L-h)!}{n!}\alpha_j^n. \tag{27}$$

It follows from (23) that we need to show that $\frac{r_N}{l_j!}\in\mathbb{Z}[\alpha_1,\dots,\alpha_m]$ for $N=0,\dots,L-l_j$. By (27) we have

$$\frac{1}{l_j!}r_N=\sum_{N=h+n}\sigma_{L-h}\big(\bar{l},\overline{\alpha}\big)\frac{(L-h)!}{l_j!n!}\alpha_j^n,$$

where $h+n=N\le L-l_j$ implies $l_j+n\le L-h$, thus giving the result. □

5. Determinant

In order to fulfil the determinant condition (7) we choose

$$\bar{l}^{(k)}=(l,l,\dots,l-1,\dots,l)^T,\quad k=0,1,\dots,m, \tag{28}$$

i.e. $l_i=l$ for $i=0,1,\dots,k-1,k+1,\dots,m$, and $l_k=l-1$. Now $L=(m+1)l-1$. Then we write

$$\begin{cases}A^*_{k,j}(t):=A_{\bar{l}^{(k)},j}(t,\overline{\alpha}), & j=0,1,\dots,m;\\ R^*_{k,j}(t):=R_{\bar{l}^{(k)},j}(t,\overline{\alpha}), & j=1,\dots,m,\end{cases} \tag{29}$$

for all $k=0,1,\dots,m$.

The non-vanishing of the determinant Δ follows from the next well-known lemma (see for example Mahler [9, p. 232] or Waldschmidt [14, p. 53]).

Lemma 5.1. *There exists a constant $c \neq 0$ such that*

$$\Delta(t) = \begin{vmatrix} A_{0,0}^*(t) & A_{0,1}^*(t) & \dots & A_{0,m}^*(t) \\ A_{1,0}^*(t) & A_{1,1}^*(t) & \dots & A_{1,m}^*(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,0}^*(t) & A_{m,1}^*(t) & \dots & A_{m,m}^*(t) \end{vmatrix} = ct^{m(m+1)l}.$$

Proof. According to Theorem 4.2 and equations in (29), the degrees of the entries of the matrix defining Δ are

$$\begin{pmatrix} ml & ml-1 & \dots & ml-1 \\ ml-1 & ml & \dots & ml-1 \\ \vdots & \vdots & \ddots & \vdots \\ ml-1 & ml-1 & \dots & ml \end{pmatrix}_{(m+1)\times(m+1)}.$$

We see that $\deg_t \Delta(t) = (m+1)ml$ and the leading coefficient c is a product of the leading coefficients of $A_{0,0}^*(t), A_{1,1}^*(t), \dots, A_{m,m}^*(t)$, which are non-zero.

On the other hand, column operations yield

$$\Delta(t) = \begin{vmatrix} A_{0,0}^*(t) & -R_{0,1}^*(t) & \dots & -R_{0,m}^*(t) \\ A_{1,0}^*(t) & -R_{1,1}^*(t) & \dots & -R_{1,m}^*(t) \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,0}^*(t) & -R_{m,1}^*(t) & \dots & -R_{m,m}^*(t) \end{vmatrix},$$

as $R_{k,j}^*(t) = e^{\alpha_j t} A_{k,0}^*(t) - A_{k,j}^*(t)$. By Theorem 4.2, the order of each element in columns $1, \dots, m$ is $L+1 = (m+1)l$. Therefore $\underset{t=0}{\operatorname{ord}} \Delta(t) \geq m(m+1)l$. □

6. Common factors

From now on we set $\alpha_j = j$ for $j = 0, 1, \dots, m$ and denote

$$B_{k,j}^*(t) := \frac{1}{(l-1)!} A_{k,j}^*(t)$$

for $j = 0, 1, \dots, m$, $k = 0, 1, \dots, m$ and

$$L_{k,j}^*(t) := \frac{1}{(l-1)!} R_{k,j}^*(t)$$

for $j = 1, \dots, m$, $k = 0, 1, \dots, m$. Then, by Theorem 4.2, we have a system of linear forms

$$B_{k,0}^*(t) e^{\alpha_j t} + B_{k,j}^*(t) = L_{k,j}^*(t), \quad j = 1, \dots, m; \ k = 0, 1, \dots, m, \tag{30}$$

where

$$B_{k,j}^*(t) = \frac{t^{L+1}}{(l-1)!} \int_0^\infty e^{-yt} (0-j-y)^l (1-j-y)^l \cdots (k-j-y)^{l-1} \cdots (m-j-y)^l \mathrm{d}y \tag{31}$$

for $j, k = 0, 1, \dots, m$, and

$$L_{k,j}^*(t) = \frac{t^{L+1}}{(l-1)!} \int_0^j e^{(j-x)t} (0-x)^l (1-x)^l \cdots (k-x)^{l-1} \cdots (m-x)^l \mathrm{d}x \tag{32}$$

for $j = 1, \dots, m$, $k = 0, 1, \dots, m$.

Further, by Lemma 4.3 holds $B^*_{k,j}(t) \in \mathbb{Z}[t]$ for all $j = 0, 1, \ldots, m$, $k = 0, 1, \ldots, m$. Next we try to find a common factor from the integer coefficients of the new polynomials $B^*_{k,j}(t)$.

Let $m \in \mathbb{Z}^+$ in this section. We will also need the p-adic valuation $v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$ and its well-known property

$$\frac{n}{p-1} - \frac{\log n}{\log p} - 1 \leq v_p(n!) \leq \frac{n-1}{p-1}. \tag{33}$$

Theorem 6.1. *For $k = 0, 1, \ldots, m$, we have*

$$\prod_{\substack{p \leq m \\ p \in \mathbb{P}}} p^{\left\lfloor \frac{m}{p} \right\rfloor v_p(l!) - v_p(l)} \Bigg| B^*_{k,0}(t).$$

Proof. Let us start by writing the polynomial $B^*_{k,0}(t)$ from (31) in a different way, using the representation (22):

(34)

$$\begin{aligned} B^*_{k,0}(t) &= \frac{1}{(l-1)!} A_{\bar{l}^{(k)},0}(t, (0,1,\ldots,m)^T) = \sum_{i=l_0}^{L} t^{L-i} \sigma_i \left(\bar{l}^{(k)}, (0,1,\ldots,m)^T \right) \cdot \frac{i!}{(l-1)!} \\ &= \sum_{r=0}^{L-l_0} t^r \sigma_{L-r} \left(\bar{l}^{(k)}, (0,1,\ldots,m)^T \right) \cdot \frac{(L-r)!}{(l-1)!}, \end{aligned}$$

where $L = (m+1)l - 1$ and, by (21),

$$\begin{aligned} \sigma_{L-r} \left(\bar{l}^{(k)}, (0,1,\ldots,m)^T \right) = (-1)^{L-r} \sum_{h_1+\ldots+h_m=r} \frac{l_1!}{(l_1-h_1)!h_1!} \cdot \frac{l_2!}{(l_2-h_2)!h_2!} \cdots \\ \frac{l_m!}{(l_m-h_m)!h_m!} 1^{h_1} 2^{h_2} \cdots m^{h_m}. \end{aligned}$$

So

$$B^*_{k,0}(t) = \sum_{r=0}^{L-l_0} t^r (-1)^{L-r} \sum_{h_1+\ldots+h_m=r} \frac{(L-r)!}{(l-1)!(l_1-h_1)! \cdots (l_m-h_m)!} \cdot \frac{l_1!}{h_1!} \cdots \frac{l_m!}{h_m!} \cdot 1^{h_1} \cdots m^{h_m}.$$

Here $\frac{(L-r)!}{(l-1)!(l_1-h_1)!\cdots(l_m-h_m)!} \in \mathbb{Z}$ because

$$(l-1) + (l_1 - h_1) + \ldots + (l_m - h_m) \leq l_0 + l_1 + \ldots + l_m - (h_1 + \ldots h_m) = L - r.$$

So, we may expect some common factors from the terms $\frac{l_1!}{h_1!} \cdots \frac{l_m!}{h_m!} \cdot 1^{h_1} 2^{h_2} \cdots m^{h_m}$.

Let $p \le m$. Now, using (33),

$$
\begin{aligned}
v_p\left(\frac{l_1!}{h_1!}\cdot\frac{l_2!}{h_2!}\cdots\frac{l_m!}{h_m!}\cdot 1^{h_1}2^{h_2}\cdots m^{h_m}\right) &= \sum_{i=1}^{m}\left(v_p(l_i!)-v_p(h_i!)\right)+\sum_{\substack{i=1\\p|i}}^{m} h_i v_p(i)\\
&\ge \sum_{\substack{i=1\\p|i}}^{m} v_p(l_i!)+h_i\left(v_p(i)-\frac{1}{p-1}\right)\\
&\ge \begin{cases}\left(\left\lfloor\frac{m}{p}\right\rfloor-1\right)v_p(l!)+v_p((l-1)!), & k\in\{1,\dots,m\};\\ \left\lfloor\frac{m}{p}\right\rfloor v_p(l!), & k=0\end{cases}\\
&\ge \left(\left\lfloor\frac{m}{p}\right\rfloor-1\right)v_p(l!)+v_p((l-1)!).
\end{aligned}
\tag{35}
$$

Recall from (28) that $l_k = l-1$ while $l_j = l$ for $j \neq k$. Since $v_p(l!) = v_p(l) + v_p((l-1)!)$, the result (35) can be written as

$$v_p\left(\frac{l_1!}{h_1!}\cdot\frac{l_2!}{h_2!}\cdots\frac{l_m!}{h_m!}\cdot 1^{h_1}2^{h_2}\cdots m^{h_m}\right) \ge \left\lfloor\frac{m}{p}\right\rfloor v_p(l!) - v_p(l).$$

So, there is a factor

$$p^{\left\lfloor\frac{m}{p}\right\rfloor v_p(l!)-v_p(l)}\,\left|\,\frac{l!}{h_1!}\cdots\frac{l!}{h_m!}\cdot 1^{h_1}2^{h_2}\cdots m^{h_m}\right.,$$

which is a common divisor of all the coefficients of $B^*_{k,0}(t)$. The proof is complete. □

Now we need to find a common factor dividing all $B^*_{k,j}(t)$.

Theorem 6.2. *Assume $j \in \{1,\dots,m\}$. Then there exists a positive integer $D_{m,l}$ satisfying*

$$D_{m,l} := \prod_{\substack{p\le\frac{m+1}{2}\\p\in\mathbb{P}}} p^{\nu_p}\,\Big|\,B^*_{k,j}(t)$$

with

$$\nu_p \ge \left(\left\lfloor\frac{j}{p}\right\rfloor + \left\lfloor\frac{m-j}{p}\right\rfloor\right) v_p((l-1)!)$$

for all $k = 0,1,\dots,m$.

Proof. From our assumption $\alpha_i = i$, $i = 1,\dots,m$, and equations (26) and (34) it follows

$$
\begin{aligned}
B^*_{k,j}(t,\overline{\alpha}) &= \frac{1}{(l-1)!}A_{\bar{l}^{(k)},0}\left(t,(0-j,1-j,\dots,m-j)^T\right)\\
&= \sum_{r=0}^{L-l_j} t^r\sigma_{L-r}\left(\bar{l}^{(k)},(0-j,1-j,\dots,m-j)^T\right)\frac{(L-r)!}{(l-1)!},
\end{aligned}
$$

where

$$\sigma_{L-r}\left(\bar{l}^{(k)}, (0-j, 1-j, \ldots, m-j)^T\right)$$
$$= (-1)^{L-r} \sum_{h_0+\ldots+h_{j-1}+h_{j+1}\ldots+h_m=r} \frac{l_0!}{(l_0-h_0)!h_0!} \cdots \frac{l_{j-1}!}{(l_{j-1}-h_{j-1})!h_{j-1}!}$$
$$\cdot \frac{l_{j+1}!}{(l_{j+1}-h_{j+1})!h_{j+1}!} \cdots \frac{l_m!}{(l_m-h_m)!h_m!}$$
$$\cdot (0-j)^{h_0}(1-j)^{h_1} \cdots (-1)^{h_{j-1}} 1^{h_{j+1}} \cdots (m-j)^{h_m}.$$

So

$$B^*_{k,j}(t) = \sum_{r=0}^{L-l_j} t^r (-1)^{L-r} \sum_{h_0+\ldots+h_{j-1}+h_{j+1}\ldots+h_m=r} \frac{(L-r)!}{(l-1)!}$$
$$\cdot \frac{1}{(l_0-h_0)! \cdots (l_{j-1}-h_{j-1})! \cdot (l_{j+1}-h_{j+1})! \cdots (l_m-h_m)!}$$
$$\cdot \frac{l_0!}{h_0!} \cdots \frac{l_{j-1}!}{h_{j-1}!} \cdot \frac{l_{j+1}!}{h_{j+1}!} \cdots \frac{l_m!}{h_m!} (0-j)^{h_0}(1-j)^{h_1} \cdots (-1)^{h_{j-1}} 1^{h_{j+1}} \cdots (m-j)^{h_m}.$$

As before, we may expect some common factors from the terms

$$T_j := \frac{l_0!}{h_0!} \cdots \frac{l_{j-1}!}{h_{j-1}!} \cdot \frac{l_{j+1}!}{h_{j+1}!} \cdots \frac{l_m!}{h_m!} \cdot (0-j)^{h_0}(1-j)^{h_1} \cdots (-1)^{h_{j-1}} 1^{h_{j+1}} \cdots (m-j)^{h_m}.$$

Let $p \leq \frac{m+1}{2}$. With considerations similar to those in (35), we get

$$v_p(T_j) \geq \begin{cases} \left(\left\lfloor \frac{j}{p} \right\rfloor - 1\right) v_p(l!) + v_p((l-1)!) + \left\lfloor \frac{m-j}{p} \right\rfloor v_p(l!), & k \in \{0, 1, \ldots j-1\}; \\ \left\lfloor \frac{j}{p} \right\rfloor v_p(l!) + \left(\left\lfloor \frac{m-j}{p} \right\rfloor - 1\right) v_p(l!) + v_p((l-1)!), & k \in \{j+1, \ldots m-j\}; \\ \left\lfloor \frac{j}{p} \right\rfloor v_p(l!) + \left\lfloor \frac{m-j}{p} \right\rfloor v_p(l!), & k = j \end{cases}$$
$$\geq \left(\left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor\right) v_p(l!) - v_p(l)$$
$$\geq \left(\left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor\right) v_p((l-1)!).$$

□

Corollary 6.3. *For all $k, j = 0, 1, \ldots, m$ we have*

$$D_{m,l} \Big| B^*_{k,j}(t).$$

Theorem 6.4. *Let $l \geq s(m)e^{s(m)}$. Then the common factor $D_{m,l}$ satisfies the bound*

$$D_{m,l} \geq e^{\kappa_m m l}, \tag{36}$$

where

$$\kappa_m := \frac{1}{m} \sum_{\substack{p \leq \frac{m+1}{2} \\ p \in \mathbb{P}}} \min_{0 \leq j \leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \frac{\log p}{p-1} w_p\big(s(m)e^{s(m)}\big), \tag{37}$$

and $w_p(x) := 1 - \frac{p}{x} - \frac{p-1}{\log p}\frac{\log x}{x}$. *Further,*

$$\kappa_m \geq w_{\frac{m+1}{2}}\big(s(m)e^{s(m)}\big)\,\frac{1}{m}\sum_{\substack{p\leq\frac{m+1}{2}\\ p\in\mathbb{P}}}\left(\left\lfloor\frac{m+1}{p}\right\rfloor - 1\right)\frac{\log p}{p-1}, \tag{38}$$

and asymptotically we have

$$\lim_{m\to\infty}\kappa_m = \kappa = \sum_{p\in\mathbb{P}}\frac{\log p}{p(p-1)} = 0.75536661083\ldots. \tag{39}$$

Proof. We begin with the estimate of Theorem 6.2:

$$\nu_p \geq \left(\left\lfloor\frac{j}{p}\right\rfloor + \left\lfloor\frac{m-j}{p}\right\rfloor\right)v_p((l-1)!).$$

Then

$$\begin{aligned}\prod_{p\leq\frac{m+1}{2}}p^{\nu_p} &\geq \prod_{p\leq\frac{m+1}{2}}p^{\left(\left\lfloor\frac{j}{p}\right\rfloor+\left\lfloor\frac{m-j}{p}\right\rfloor\right)v_p((l-1)!)}\\ &= \exp\left(\sum_{p\leq\frac{m+1}{2}}\left(\left\lfloor\frac{j}{p}\right\rfloor + \left\lfloor\frac{m-j}{p}\right\rfloor\right)v_p((l-1)!)\log p\right).\end{aligned}$$

We need

$$\left\lfloor\frac{j}{p}\right\rfloor + \left\lfloor\frac{m-j}{p}\right\rfloor \geq \min_{0\leq j\leq m}\left\{\left\lfloor\frac{j}{p}\right\rfloor + \left\lfloor\frac{m-j}{p}\right\rfloor\right\}$$

since we are estimating the common divisor of all $B^*_{k,j}$. Next we use the property (33) and the assumption $l \geq s(m)e^{s(m)}$ in order to estimate $v_p((l-1)!)\log p$:

$$\begin{aligned}v_p((l-1)!)\log p &\geq \left(\frac{l-1}{p-1} - \frac{\log(l-1)}{\log p} - 1\right)\log p\\ &\geq l\frac{\log p}{p-1}\left(1 - \frac{p}{s(m)e^{s(m)}} - \frac{p-1}{\log p}\frac{\log\big(s(m)e^{s(m)}\big)}{s(m)e^{s(m)}}\right).\end{aligned}$$

Altogether $\prod_{p\leq\frac{m+1}{2}}p^{\nu_p} \geq e^{\kappa_m ml}$, where

$$\kappa_m := \frac{1}{m}\sum_{p\leq\frac{m+1}{2}}\min_{0\leq j\leq m}\left\{\left\lfloor\frac{j}{p}\right\rfloor + \left\lfloor\frac{m-j}{p}\right\rfloor\right\}\frac{\log p}{p-1}\left(1 - \frac{p}{s(m)e^{s(m)}} - \frac{p-1}{\log p}\frac{\log\big(s(m)e^{s(m)}\big)}{s(m)e^{s(m)}}\right),$$

proving the estimate (36).

Next we study the bound (38). Let $x \in \mathbb{R}_{>1}$ be fixed, then

$$w_y(x) > w_z(x) \tag{40}$$

when $2 \leq y < z$. To prove (40) above we differentiate the function:

$$\frac{\partial}{\partial y}w_y(x) = -\frac{1}{x} - \frac{1}{\log y}\frac{\log x}{x} + \frac{y-1}{y(\log y)^2}\frac{\log x}{x} = -\frac{1}{x} - \frac{\log x}{x\log y}\left(1 - \left(1 - \frac{1}{y}\right)\frac{1}{\log y}\right) < 0,$$

since $\log y > 1 - \frac{1}{y}$ when $y \geq 2$. Next write

$$m+1 = hp + \overline{m},\quad j = lp + \overline{j},\quad h, l, \overline{m}, \overline{j}, \in \mathbb{Z}_{\geq 0},\quad 0 \leq \overline{m}, \overline{j} \leq p-1. \tag{41}$$

Then

$$\begin{aligned}
(42)\qquad \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor &= \left\lfloor l + \frac{\bar{j}}{p} \right\rfloor + \left\lfloor \frac{m - lp - \bar{j}}{p} \right\rfloor \\
&= l + \left\lfloor \frac{m+1}{p} - l - 1 + \frac{p-1-\bar{j}}{p} \right\rfloor \geq \left\lfloor \frac{m+1}{p} \right\rfloor - 1 = h - 1.
\end{aligned}$$

Thus, the bound $\min_{0\leq j\leq m}\left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \geq \left\lfloor \frac{m+1}{p} \right\rfloor - 1 \geq \frac{m}{p} - 2$ together with (40) verifies the estimate

$$\begin{aligned}
\kappa_m &\geq w_{\frac{m+1}{2}}\big(s(m)e^{s(m)}\big) \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \left(\left\lfloor \frac{m+1}{p} \right\rfloor - 1 \right) \frac{\log p}{p-1} \\
&\geq w_{\frac{m+1}{2}}\big(s(m)e^{s(m)}\big) \sum_{p \leq \frac{m+1}{2}} \left(1 - \frac{2p}{m} \right) \frac{\log p}{p(p-1)}.
\end{aligned}$$

Hence, by assuming $p \leq \sqrt{m}$, we get

$$\kappa_m \geq w_{\frac{m+1}{2}}\big(s(m)e^{s(m)}\big) \left(1 - \frac{2}{\sqrt{m}} \right) \sum_{p \leq \sqrt{m}} \frac{\log p}{p(p-1)} \overset{m\to\infty}{\to} \sum_{p\in\mathbb{P}} \frac{\log p}{p(p-1)}.$$

On the other hand,

$$\begin{aligned}
\kappa_m &= \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \min_{0\leq j\leq m} \left\{ \left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right\} \frac{\log p}{p-1}\, w_p\big(s(m)e^{s(m)}\big) \\
&\leq \frac{1}{m} \sum_{p \leq \frac{m+1}{2}} \left\lfloor \frac{m}{p} \right\rfloor \frac{\log p}{p-1} \leq \sum_{p \leq \frac{m+1}{2}} \frac{\log p}{p(p-1)} \overset{m\to\infty}{\to} \sum_{p\in\mathbb{P}} \frac{\log p}{p(p-1)}.
\end{aligned}$$

This proves the asymptotic behaviour (39). As for the numerical value in (39), see the sequence A138312 in [13]. □

With $s(m) = m(\log m)^2$, for instance (37) gives

$$\kappa_m \geq \begin{cases} 0.387099, & m = 5 \\ 0.322600, & m = 6 \\ 0.375535, & m = 7 \\ 0.397256, & m = 8 \\ 0.474840, & m = 9 \end{cases} \quad \text{and} \quad \kappa_m \geq \begin{cases} 0.427356, & m = 10 \\ 0.501455, & m = 11 \\ 0.459667, & m = 12 \\ 0.502575, & m = 13 \\ 0.534653, & m = 14. \end{cases}$$

Note that to simplify numerical computations for large m, the estimate (38) is already rather sharp, where in addition the factor $w_{\frac{m+1}{2}}(m)$ is very close to 1.

Lemma 6.5. *With $s(m) = m(\log m)^2$ it holds that $\kappa_m \geq 0.5$ for all $m \geq 13$.*

Proof. By (41) and (42) we get

$$\frac{1}{m}\left(\left\lfloor \frac{j}{p} \right\rfloor + \left\lfloor \frac{m-j}{p} \right\rfloor \right) \geq \frac{h-1}{m} \geq \frac{1}{p}\left(1 - \frac{2p-2}{m} \right).$$

Choosing, for example, $1 - \frac{2p-2}{m} \geq \frac{9}{10}$, implies $p \leq \frac{m}{20} + 1$. Then

$$(43)\qquad \kappa_m \geq \frac{9}{10} w_{\frac{m+1}{2}}\big(s(m)e^{s(m)}\big) \sum_{p \leq \frac{m}{20}+1} \frac{\log p}{p(p-1)}.$$

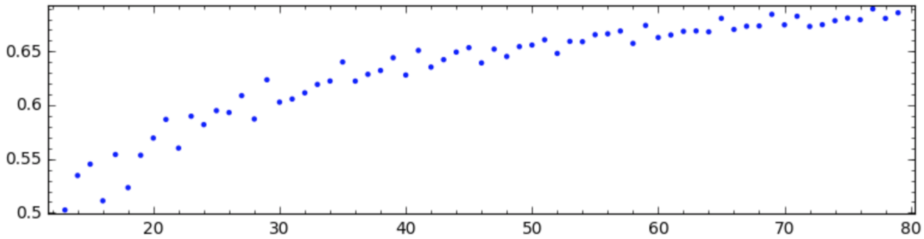
Now

$$w_{\frac{m+1}{2}}\left(s(m)e^{s(m)}\right) = 1 - \frac{\frac{m+1}{2} + \frac{\frac{m+1}{2}-1}{\log\left(\frac{m+1}{2}\right)} \cdot \log\left(m(\log m)^2 e^{m(\log m)^2}\right)}{m(\log m)^2 e^{m(\log m)^2}} > 1 - 10^{-666},$$

when $m \geq 80$. In (43) we have an increasing lower bound for κ_m, and therefore

$$\kappa_m \geq \frac{9}{10} w_{\frac{80+1}{2}}\left(s(80)e^{s(80)}\right) \sum_{p \leq \frac{80}{20}+1} \frac{\log p}{p(p-1)} > \frac{9}{10} \cdot \left(1 - 10^{-666}\right) \cdot \sum_{p \leq 5} \frac{\log p}{p(p-1)} \geq 0.549133,$$

when $m \geq 80$. As for $13 \leq m \leq 79$, the estimate $\kappa_m \geq 0.5$ is quickly verified using Sage [12] and estimate (38).



□

7. Numerical linear forms

By extracting the common factor $D_{m,l}$ from the linear forms (30) we are led to the numerical linear forms

$$B_{k,0}e^j + B_{k,j} = L_{k,j}, \quad j = 1, \ldots, m; \; k = 0, 1, \ldots, m, \tag{44}$$

where

$$B_{k,j} := \frac{1}{D_{m,l}} B^*_{k,j}(1), \quad j = 0, 1, \ldots, m; \; k = 0, 1, \ldots, m,$$

are integers and

$$L_{k,j} := \frac{1}{D_{m,l}} L^*_{k,j}(1), \quad j = 1, \ldots, m; \; k = 0, 1, \ldots, m, .$$

According to (28), now $L = (m+1)l - 1$. We have $s(m) = m(\log m)^2$. Because of the condition (13) we have the assumption $l \geq e^{m(\log m)^2}$.

The following lemma gives necessary estimates for the coefficients $B_{k,0}$ and the remainders $L_{k,j}$ of the linear forms (44). In the subsequent estimates we shall use Stirling's formula (see eg. [1], formula 6.1.38) in the form

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n+\frac{\theta(n)}{12n}}, \quad 0 < \theta(n) < 1.$$

Lemma 7.1. *We have the bounds*

$$|B_{k,0}| \leq \exp\left(ml\log l + l((m+1)\log(m+1) - (1+\kappa_m)m + 0.0000525)\right);$$

$$\sum_{j=1}^{m} |L_{k,j}| \leq \exp\left(-l\log l + l\left(\left(m + \frac{1}{2}\right)\log m - (\kappa_m + 1)m - 0.02394\right)\right)$$

for $j = 1, \ldots, m$, $k = 0, 1, \ldots, m$, *when* $l \geq e^{m(\log m)^2}$.

In other words, $|B_{k,0}(l)| \leq Q(l) = e^{q(l)}$ and $\sum_{j=1}^{m} |L_{k,j}| \leq R(l) = e^{-r(l)}$, where

$$q(l) = ml \log l + l((m+1)\log(m+1) - (1+\kappa_m)m + 0.0000525), \tag{45}$$

$$-r(l) = -l \log l + \left(\left(m + \frac{1}{2}\right)\log m - (1+\kappa_m)m - 0.02394\right) l, \tag{46}$$

for all $k, j = 0, 1, \ldots, m$, $m \geq 5$.

Proof. The structure of the proof is the following: First we treat the term $B^*_{k,0}(1)$ by first using the formulas given for it to obtain a bound. Then we factor out the common factor $D_{m,l}$ of $B^*_{k,0}(1)$ yielding to $B_{k,0}$. Finally, the bound $Q(l)$ is then the bound for $B_{k,0}$. With $L^*_{k,j}(1)$ we proceed in a similar way: we first bound them, then sum them, and finally factor out the common divisor to obtain the bound $R(l)$.

By (31) we have

$$B^*_{k,0}(1) = \frac{1}{(l-1)!}\int_0^\infty e^{-x}\frac{\prod_{j=0}^{m}(j-x)^l}{k-x}\mathrm{d}x.$$

Let us split the integral into following pieces:

$$\int_0^\infty e^{-x}\frac{\prod_{j=0}^{m}(j-x)^l}{k-x}\mathrm{d}x = \left(\int_0^m + \int_m^{2(m+1)l} + \int_{2(m+1)l}^\infty\right) e^{-x}\frac{\prod_{j=0}^{m}(j-x)^l}{k-x}\mathrm{d}x.$$

When $x \geq m$,

$$\left|\frac{\prod_{j=0}^{m}(j-x)^l}{k-x}\right| = x^l(x-1)^l \cdots (x-k)^{l-1} \cdots (x-m)^l \leq x^{(m+1)l-1} \leq x^{(m+1)l}.$$

Hence, we may estimate

$$\left|\left(\int_m^{2(m+1)l} + \int_{2(m+1)l}^\infty\right) e^{-x}\frac{\prod_{j=0}^{m}(j-x)^l}{k-x}\mathrm{d}x\right| \leq \left(\int_m^{2(m+1)l} + \int_{2(m+1)l}^\infty\right) e^{-x}x^{(m+1)l}\mathrm{d}x.$$

Write $f(x) = e^{-x}x^{(m+1)l}$. Now $f'(x) = (-x+(m+1)l)e^{-x}x^{(m+1)l-1}$, which has a unique zero at $x = (m+1)l$. The function $f(x)$ is increasing for $x \leq (m+1)l$ and decreasing for $x \geq (m+1)l$. Let us now estimate the integrals. The function $f(x)$ obtains its maximum at $x = (m+1)l$, and we may thus estimate

$$\int_m^{2(m+1)l} e^{-x}x^{(m+1)l}dx \leq 2(m+1)le^{-(m+1)l}((m+1)l)^{(m+1)l}.$$

On the interval $x \geq 2(m+1)l$, the function $f(x)$ is decreasing. Our aim is to find an upper bound for the integral using a geometric sum. Let us first write

$$\int_{2(m+1)l}^\infty e^{-x}x^{(m+1)l} \leq \sum_{h=0}^{\infty} e^{-2(m+1)l-h}(2(m+1)l+h)^{(m+1)l}.$$

Notice that $\frac{f(x+1)}{f(x)} = e^{-1}\left(1+\frac{1}{x}\right)^{(m+1)l} \leq e^{-1/2}$, when $x \geq 2(m+1)l$. It follows that $f(x+h) \leq \left(e^{-\frac{1}{2}}\right)^h f(x)$. Hence

$$\begin{aligned}\sum_{h=0}^{\infty} e^{-2(m+1)l-h}(2(m+1)l+h)^{(m+1)l} &\leq \frac{e^{-2(m+1)l}(2(m+1)l)^{(m+1)l}}{(1-e^{-1/2})} \\ &< 2.55e^{-2(m+1)l}(2(m+1)l)^{(m+1)l}.\end{aligned}$$

Finally, we have to estimate the first integral. We have

$$\max_{n\le x\le n+1}\prod_{j=0}^{m}|j-x|^{l-1}\le \max_{0\le x\le 1}\prod_{j=0}^{m}|j-x|^{l-1}$$

for $0\le n\le m-1$. Now

$$\left|\frac{\prod_{j=0}^{m}(j-x)^l}{k-x}\right|\le m!\max_{0\le x\le 1}\prod_{j=0}^{m}|j-x|^{l-1}.$$

Hence

$$\left|\int_0^m e^{-x}\frac{\prod_{j=0}^{m}(j-x)^l}{k-x}\mathrm{d}x\right|\le \frac{m!(m!)^{l-1}}{(5!)^{l-1}}\int_0^m e^{-x}\max_{0\le x\le 1}\prod_{j=0}^{5}|j-x|^{l-1}\mathrm{d}x\le \frac{(m!)^l 16.91^{l-1}}{120^{l-1}}$$

when $m\ge 5$. We may conclude that

$$\begin{aligned}|B^*_{k,0}(1)|&\le \frac{(m!)^l16.91^{l-1}}{(l-1)!120^{l-1}}+\frac{2.55e^{-2(m+1)l}(2(m+1)l)^{(m+1)l}}{(l-1)!}+\frac{2(m+1)l((m+1)l)^{(m+1)l}}{(l-1)!e^{(m+1)l}}\\&\le \frac{6(m+1)l}{(l-1)!}e^{-(m+1)l}((m+1)l)^{(m+1)l}\\&\le \exp\big(ml\log l+l\left((m+1)\log(m+1)-m+\log l-\log(l-1)\right)\\&\qquad+\log l+\frac{1}{2}\log(l-1)+\log(m+1)+\log 6-1-\log\sqrt{2\pi}\big).\end{aligned}$$

Next we take into account the common factor $D_{m,l}$ estimated by $e^{\kappa_m ml}$. Remember that $B_{k,0}$ will be the expression that is obtained when $B^*_{k,0}(1)$ is divided by the common factor. Now

$$\begin{aligned}|B_{k,0}|\le \exp\Bigg(&ml\log l+l\left((m+1)\log(m+1)-(1+\kappa_m)m+\log l-\log(l-1)\right)\\&+\log l+\frac{1}{2}\log(l-1)+\log(m+1)+\log 6-1-\log\sqrt{2\pi}\Bigg).\end{aligned}\tag{47}$$

Since $m\ge 5$ and $l\ge e^{m(\log m)^2}\ge e^{5(\log 5)^2}$, we have

$$\log l-\log(l-1)\le 0.000002372774\tag{48}$$

and

$$\frac{\log l}{l}+\frac{\log(l-1)}{2l}+\frac{\log(m+1)}{l}+\frac{\log 6-1-\log\sqrt{2\pi}}{l}\le 0.00005004591565.\tag{49}$$

At last, estimate (47) with (48) and (49) yields

$$|B_{k,0}|\le \exp\left(ml\log l+l((m+1)\log(m+1)-(1+\kappa_m)m+0.0000525)\right).$$

We may now estimate the remainder. According to equation (32) we have the representation

$$L^*_{k,j}(t)=\frac{t^{L+1}}{(l-1)!}\int_0^j e^{(j-x)t}(0-x)^l(1-x)^l\cdots(k-x)^{l-1}\cdots(m-x)^l\mathrm{d}x.$$

The expression $|x(1-x)\cdots(m-x)|$ attains its maximum in the interval $]0,m[$ for the first time when $0<x<1$, so

$$\max_{0<x<m}|x(1-x)\cdots(m-x)|\le\frac{m!}{5!}\max_{0<x<1}|x(1-x)(2-x)(3-x)(4-x)(5-x)|=\frac{m!16.91}{120}.$$

Thus we may estimate

$$|L^*_{k,j}(1)| \leq \frac{e^j}{(l-1)!}\int_0^j e^{-x}\frac{\prod_{r=0}^m |r-x|^l}{|k-x|}\mathrm{d}x \leq \frac{(m!)^l 16.91^{l-1}}{120^{l-1}(l-1)!}\left(e^j-1\right)$$

when $m \geq 5$. Using the estimate $\sum_{j=1}^m (e^j-1) < e^m\frac{e}{e-1}$, and summing together the terms $L^*_{j,k}(1)$, we get

$$\sum_{j=1}^m |L^*_{j,k}(1)| < \frac{1.582e^m (m!)^l 16.91^{l-1}}{120^{l-1}(l-1)!}.$$

Again we divide by the common factor $D_{m,l}$. Thus the new values $L_{k,j}$ satisfy:

$$\sum_{j=1}^m |L_{k,j}| < \frac{1.582(m!)^l 16.91^{l-1}}{120^{l-1}(l-1)!}\cdot e^{m-\kappa_m ml} < \exp\left(-\left(l-\frac{1}{2}\right)\log(l-1)+l-\log\sqrt{2\pi}\right)$$
$$\times\exp\left(l\left(\left(m+\frac{1}{2}\right)\log m - m + \log\sqrt{2\pi}+\frac{1}{12m}+\frac{m}{l}-\kappa_m m+\log 16.91-\log 120\right)\right)$$
$$\times\exp\left(\log 120-\log 16.91+\log 1.582\right).$$

Now

$$\frac{m}{l}+\log(l)-\log(l-1)+\frac{\log(l-1)}{2l}+1-\frac{\log(\sqrt{2\pi})}{l} \leq 1.000027388,$$
$$\log(\sqrt{2\pi})+\frac{1}{12m}+\log 16.91-\log 120 \leq -1.023981380000218$$

and

$$\frac{\log 120-\log 16.91+\log 1.582}{l} \leq 5.738016903383273\cdot 10^{-6}$$

because $m \geq 5$ and $l \geq e^{5(\log 5)^2}$. Together these estimates yield

$$\sum_{j=1}^m |L_{k,j}| \leq \exp\left(-l\log l + l\left(\left(m+\frac{1}{2}\right)\log m-(\kappa_m+1)m-0.0239481\right)\right). \tag{50}$$

□

8. Measure

We will apply Lemma 3.2. The determinat condition 7 is certainly satisfied by Lemma 5.1 and 44. Comparing formulas (8) and (9) with (45) and (46), we have

$$\begin{cases} a &= m, \\ b_1 &= (m+1)\log(m+1)-(1+\kappa_m)m+\delta, \quad \delta = 0.0000525; \\ c &= 1, \\ e_1 &= \left(m+\frac{1}{2}\right)\log m-(1+\kappa_m)m-0.0239. \end{cases}$$

Now, with $s(m) = m(\log m)^2$, the formulas in (10) give

$$\begin{cases} B = b_1+\frac{ae_1}{c} \\ \quad = m^2\log m-(1+\kappa_m)m^2+(m+1)\log(m+1)+\frac{1}{2}m\log m-(1.0239+\kappa_m)m+\delta, \\ C = am = m^2, \\ D = (a+b_1)m+am^2e^{-s(m)} = m(m+1)\log(m+1)-\kappa_m m^2+\delta m+\frac{m^3}{e^{m(\log m)^2}} \end{cases}$$

for all $m \geq 5$, represented using our shorthand notations $c_1 = 1-\frac{e_1}{s(m)}$ and $\hat{e} = 1+\frac{\log(s(m))}{s(m)}$.

We have thus finally accomplished our Main result 2.1:

Proofs of Theorem 2.1 and Corollary 2.2. The values above have been achieved with the choice $\Theta_j = e^j$. Combining them with Lemma 3.2 leads straight to the result (5). Corollary 2.2 follows likewise by plugging these values into Corollary 3.4. □

Estimate (3) still requires a bit more work.

Proof of Theorem 1.1. Starting directly from (14) we have

$$1 < 2|\Lambda| Q(\hat{n}+m) \leq |\Lambda|\, 2\, (2H)^{\frac{a}{c}}\, e^{B\hat{n}+C\log\hat{n}+D} = |\Lambda|\, H^{\frac{a}{c}+Y} = |\Lambda|\, H^{m+Y},$$

where

$$\begin{aligned} Y &= \frac{1}{\log H}\left(B\hat{n} + m^2\log\hat{n} + D + (m+1)\log 2\right) \\ &\leq \frac{1}{\log H}\left(\frac{\hat{e}B}{c_1}\frac{\log(2H)}{\log\log(2H)} + m^2\log\left(\frac{\hat{e}}{c_1}\frac{\log(2H)}{\log\log(2H)}\right) + D + (m+1)\log 2\right) \end{aligned} \tag{51}$$

by the estimate (20) for $\hat{n}$. By recalling our assumption $\log H \geq s(m)e^{s(m)}$ it is obvious from the expression (51), the terms corresponding to parameters C and D contribute much less than the term corresponding to parameter B. The first task is to bound them in such a way that they only slightly increase the constant term in the expression for the parameter B. Let us start with the terms D and $(m+1)\log 2$. We have

$$\begin{aligned} D + (m+1)\log 2 &= m(m+1)\log(m+1) - \kappa_m m^2 + \delta m + \frac{m^3}{e^{m(\log m)^2}} + (m+1)\log 2 \\ &= m(m+1)\log(m+1) + m\left(\delta + \frac{m^2}{e^{m(\log m)^2}} + \log 2 + \frac{\log 2}{m} - \kappa_m m\right) \\ &\leq m(m+1)\log(m+1). \end{aligned}$$

Since $c_1\log\log(2H) \geq 1$, we may estimate

$$m^2\log\left(\frac{\hat{e}\log(2H)}{c_1\log\log(2H)}\right) \leq m^2\log(\hat{e}\log(2H)).$$

Hence, the estimate becomes

$$\begin{aligned} Y &\leq \frac{1}{\log H}\left(\frac{\hat{e}B}{c_1}\frac{\log(2H)}{\log\log(2H)} + m^2\log\left(\hat{e}\log(2H)\right) + m(m+1)\log(m+1)\right) \\ &\leq \frac{1}{\log H}\left(\frac{\hat{e}B}{c_1}\frac{\log(2H)}{\log\log(2H)} + m^2\log\left(2\log H\right) + m(m+1)\log(m+1)\right). \end{aligned}$$

Let us now show that $m(m+1)\log(m+1) \leq m^2\log(2\log H)$. This is equivalent to showing that $\left(1+\frac{1}{m}\right)\log(m+1) \leq \log(2\log H)$. Notice that $1+\frac{1}{m} \leq 2$, so it suffices to show that $(m+1)^2 \leq 2\log(H)$, which is clearly true. We have now derived

$$\begin{aligned} Y &\leq \frac{1}{\log H}\left(\frac{\hat{e}B}{c_1}\frac{\log(2H)}{\log\log(2H)} + 2m^2\log\left(2\log H\right)\right) \\ &\leq \frac{1}{\log\log H}\left(\frac{\log(2H)}{\log H}\frac{\hat{e}B}{c_1} + \frac{4m^2(\log\log H)^2}{\log H}\right) \\ &= \frac{\hat{e}}{c_1\log\log H}\left(B + \frac{1}{\log H}\left(\log(2)B + \frac{c_1 4m^2(\log\log H)^2}{\hat{e}}\right)\right). \end{aligned}$$

Notice now that roughly estimating we have $B \le m^2 \log m$ because $\log(m+1) - (1.0239 + \kappa_m)m + \delta \le 0$ and $-(1+\kappa_m)m^2 + m\log(m+1) + \frac{1}{2}m\log m \le 0$. Since $0 < c_1 \le 1 \le \hat{e}$, we have now derived the inequality

$$Y \le \frac{\hat{e}}{c_1 \log\log H}\left(B + \frac{m^2}{\log H}\left(\log(2)\log m + 4(\log\log H)^2\right)\right) \le \frac{\hat{e}}{c_1 \log\log H}(B+0.005).$$

Let us take a closer look at

$$f(m) := \frac{\hat{e}(B+0.005)}{c_1 m^2 \log m}$$
$$= \frac{\left(1 + \frac{1}{m\log m} + \frac{2\log\log m}{m(\log m)^2}\right)\left(1 - \frac{1+\kappa_m}{\log m} + \frac{(m+1)\log(m+1)}{m^2\log m} + \frac{1}{2m} - \frac{1.0239+\kappa_m}{m\log m} + \frac{0.0000525+0.005}{m^2\log m}\right)}{1 - \frac{1}{\log m} - \frac{1}{2m\log m} + \frac{1+\kappa_m}{(\log m)^2}}.$$

Before moving any further, notice that the value of the expression $f(m)$ can be estimated, and compared against the value of $\left(1 - \frac{\kappa_m}{\log m}\right)\left(1 - \frac{2\kappa_m}{(\log m)^2}\right)$ when $5 \le m \le 14$. The calculations are performed by Sage [12]. The values of both functions are presented in the following table:

m	$f(m)$	$\left(1 - \frac{\kappa_m}{\log m}\right)\left(1 - \frac{2\kappa_m}{(\log m)^2}\right)$
5	0.464069782428510	0.532483829657739
6	0.616080767646427	0.655166009179873
7	0.603297796976212	0.646940824358811
8	0.615927378914005	0.660320196448654
9	0.576858338943839	0.629691071273270
10	0.636648247235239	0.683112999357727
11	0.599534862167035	0.652930086080100
12	0.644422623292065	0.693671559327988
13	0.628657937147328	0.681213991293286
14	0.620367090378663	0.674978350619014

It is evident from these values that $f(m) \le \left(1 - \frac{\kappa_m}{\log m}\right)\left(1 - \frac{2\kappa_m}{(\log m)^2}\right)$ when $5 \le m \le 14$.

We have thus shown $f(m) \le \left(1 - \frac{2\kappa_m}{(\log m)^2}\right)\left(1 - \frac{\kappa_m}{\log m}\right)$ when $5 \le m \le 14$, and the proof is ready for $5 \le m \le 14$. For the rest of the proof we assume that $m \ge 15$ meaning also that $0.5 \le \kappa_m \le 0.756$. Let us continue by writing

$$f(m) = g(m)h(m),$$

where

$$g(m) := \frac{1 - \frac{1}{\log m}}{1 - \frac{1}{\log m} - \frac{1}{2m\log m} + \frac{1+\kappa_m}{(\log m)^2}} = 1 - \frac{1+\kappa_m}{(\log m)^2}\frac{1 - \frac{\log m}{2m(1+\kappa_m)}}{1 - \frac{1}{\log m} - \frac{1}{2m\log m} + \frac{1+\kappa_m}{(\log m)^2}}$$

and

$$h(m) := \frac{\left(1 + \frac{1}{m\log m} + \frac{2\log\log m}{m(\log m)^2}\right)\left(1 - \frac{1+\kappa_m}{\log m} + \frac{(m+1)\log(m+1)}{m^2\log m} + \frac{1}{2m} - \frac{1.0239+\kappa_m}{m\log m} + \frac{0.0050525}{m^2\log m}\right)}{1 - \frac{1}{\log m}}.$$

First we show that $g(m) \le 1 - \frac{1+\kappa_m}{(\log m)^2}$. This claim is equivalent to

$$2 + \frac{1}{m} - \frac{2(1+\kappa_m)}{\log m} - \frac{(\log m)^2}{(1+\kappa_m)m} \ge 0,$$

which is true when $m \geq 15$ because $\frac{2(1+\kappa_m)}{\log m} < \frac{4}{\log 15} < 1.48$ and $\frac{(\log m)^2}{(1+\kappa_m)m} < \frac{(\log m)^2}{m} < 0.49$. We still need to prove that

$$h(m) \leq 1 - \frac{\kappa_m}{\log m}. \tag{52}$$

Let us now look at the second term in the numerator of $h(m)$. First take a look at the ratio

$$\frac{(m+1)\log(m+1)}{m^2 \log m} \leq \frac{1}{m} + \frac{1}{m^2 \log m} + \frac{1}{m^2} + \frac{1}{m^3 \log m}.$$

We have

$$\begin{aligned}
&1 - \frac{1+\kappa_m}{\log m} + \frac{(m+1)\log(m+1)}{m^2 \log m} + \frac{1}{2m} - \frac{1.0239+\kappa_m}{m \log m} + \frac{0.0050525}{m^2 \log m} \\
&\leq 1 - \frac{1+\kappa_m}{\log m} + \frac{1}{m} + \frac{1}{m^2 \log m} + \frac{1}{m^2} + \frac{1}{m^3 \log m} + \frac{1}{2m} - \frac{1.0239+\kappa_m}{m \log m} + \frac{0.0050525}{m^2 \log m} \\
&\qquad < 1 - \frac{1+\kappa_m}{\log m} + \frac{3}{2m},
\end{aligned}$$

since

$$\frac{1}{m^2} + \frac{1.0050525}{m^2 \log m} + \frac{1}{m^3 \log m} - \frac{1.0239+\kappa_m}{m \log m} < 0.$$

Thus, we have

$$\begin{aligned}
&\frac{\left(1 + \frac{1}{m \log m} + \frac{2 \log\log m}{m(\log m)^2}\right)\left(1 - \frac{1+\kappa_m}{\log m} + \frac{(m+1)\log(m+1)}{m^2 \log m} + \frac{1}{2m} - \frac{1.0239+\kappa_m}{m \log m} + \frac{0.0050525}{m^2 \log m}\right)}{1 - \frac{1}{\log m} - \frac{1}{2m \log m} + \frac{1+\kappa_m}{(\log m)^2}} \\
&\qquad < \left(1 + \frac{1}{m \log m} + \frac{2 \log\log m}{m(\log m)^2}\right)\left(1 - \frac{\frac{\kappa_m}{\log m} - \frac{3}{2m}}{1 - \frac{1}{\log m}}\right).
\end{aligned}$$

Let us now prove that

$$\frac{\left(1 + \frac{1}{m \log m} + \frac{2 \log\log m}{m(\log m)^2}\right)\left(1 - \frac{1+\kappa_m}{\log m} + \frac{3}{2m}\right)}{1 - \frac{1}{\log m}} < 1 - \frac{\kappa_m}{\log m}.$$

This is done by showing that

$$1 - \frac{1+\kappa_m}{\log m} + \frac{3}{2m} < \left(1 - \frac{\kappa_m}{\log m}\right)\left(1 - \frac{1}{\log m}\right)\left(1 - \frac{1}{m \log m} - \frac{2 \log\log m}{m(\log m)^2}\right),$$

because then

$$\begin{aligned}
&\frac{\left(1 + \frac{1}{m \log m} + \frac{2 \log\log m}{m(\log m)^2}\right)\left(1 - \frac{1+\kappa_m}{\log m} + \frac{3}{2m}\right)}{1 - \frac{1}{\log m}} < \\
&\frac{\left(1 + \frac{1}{m \log m} + \frac{2 \log\log m}{m(\log m)^2}\right)\left(1 - \frac{\kappa_m}{\log m}\right)\left(1 - \frac{1}{\log m}\right)\left(1 - \frac{1}{m \log m} - \frac{2 \log\log m}{m(\log m)^2}\right)}{1 - \frac{1}{\log m}} < 1 - \frac{\kappa_m}{\log m}.
\end{aligned}$$

Notice first that

$$\left(1-\frac{\kappa_m}{\log m}\right)\left(1-\frac{1}{\log m}\right)\left(1-\frac{1}{m\log m}-\frac{2\log\log m}{m(\log m)^2}\right)$$
$$>1-\frac{1+\kappa_m}{\log m}+\frac{\kappa_m}{(\log m)^2}-\frac{1}{m\log m}+\frac{1+\kappa_m}{m(\log m)^2}-\frac{2\log\log m}{m(\log m)^2}$$
$$>1-\frac{1+\kappa_m}{\log m}+\frac{\kappa_m}{(\log m)^2}-\frac{2}{m\log m}+\frac{1+\kappa_m}{m(\log m)^2},$$

so we have to show that

$$\frac{3}{2m}<\frac{\kappa_m}{(\log m)^2}-\frac{2}{m\log m}+\frac{1+\kappa_m}{m(\log m)^2}.$$

This is equivalent to $3(\log m)^2+4\log m<2\kappa_m m+2+2\kappa_m$. When $m\geq 15$, the right hand side of the inequality is at least $2m+3$, since $\kappa_m\geq 0.5$. The inequality

$$3(\log m)^2+4\log m<2m+3$$

is true when $m\geq 14.74$, and hence for all integer values $m\geq 15$. The proof is complete. □

9. Sparse polynomials

The method presented in this paper suits very well for obtaining bounds for sparse polynomials of e, namely, polynomials which have a considerable number of coefficients equal to zero. Let the pairwise different non-negative integers $\beta_0=0,\beta_1,\ldots,\beta_{m_1}$ be the exponents of the sparse polynomial $P(x)=\lambda_0+\lambda_1x^{\beta_1}+\ldots+\lambda_{m_1}x^{\beta_{m_1}}\in\mathbb{Z}_{\mathbb{I}}[x]$.

Theorem 9.1. *Let $P(x)=\lambda_0+\lambda_1x^{\beta_1}+\ldots+\lambda_{m_1}x^{\beta_{m_1}}$ be a polynomial with at most $m_1+1\geq 2$ non-zero coefficients, and of degree $m_2\geq 4$, where $m_2\geq m_1+1$. Then*

$$|P(e)|>H^{-m_1-\frac{\rho(m_1^2+3m_1+2)\log m_2}{\log\log H}}$$

holds for all $\overline{\lambda}=(\lambda_0,\lambda_1,\ldots,\lambda_m)^T\in\mathbb{Z}_{\mathbb{I}}^{m+1}\setminus\{\overline{0}\}$ with $\log H\geq m_2(\log m_2)^2e^{m_2(\log m_2)^2}$, $H=\max_{1\leq i\leq m}\{1,|\lambda_i|\}$, where the constant $\rho\leq 13.79$ for all $m_2\geq 4$, and $\rho\leq 2$ when $m_2\geq 11$.

Proof. This boils down to estimating the size of the terms $Q(n)$ and $R(n)$. We use the polynomial expression $\Omega(w,\overline{\beta})$. Now the polynomial in question is $\prod_{j=0}^{m_1}(\beta_j-w)^{l_j}$, where β_j are the exponents of the polynomial, so $0\leq\beta_j\leq m_2$ for all j. Furthermore, we know that $l_j=l$ with the exception of one index, in which case it is $l-1$. We may assume that the index in question is k, namely, that the terms $B_{k,0}$, $B_{k,j}$ and $L_{k,j}$ correspond to the polynomials with $l_k=l-1$. Furthermore, we assume $l\geq s(m_2)e^{s(m_2)}$.

Let us now estimate the size of the polynomial using the same method as earlier. We have

$$B_{k,0}^*(t)=\frac{1}{(l-1)!}\int_0^\infty e^{-xt}\frac{\prod_{j=0}^{m_1}(\beta_j-x)^l}{\beta_k-x}dx,$$

and we need the value at $t=1$. First the integral needs to be split into integrals over the intervals $[0,m_2]$, $[m_2,2m_2l]$ and $[2m_2l,\infty)$. Let us start by looking at the first integral. We have

$$\frac{1}{(l-1)!}\int_0^{m_2}e^{-xt}\frac{\prod_{j=0}^{m_1}|\beta_j-x|^l}{|\beta_k-x|}dx\leq\frac{1}{(l-1)!}\int_0^{m_2}m_2^{l(m_1+1)-1}=\frac{m_2^{(m_1+1)l}}{(l-1)!}.$$

Next we estimate the integral on the interval $[m_2, 2m_2l]$. Now

$$\frac{\prod_{j=0}^{m_1} |\beta_j - x|^l}{|\beta_k - x|} dx \leq x^{(m_1+1)l-1}.$$

Let us now look at the function $f(x) = e^{-x}x^{(m_1+1)l-1}$. We have

$$f'(x) = -e^{-x}x^{(m_1+1)l-1} + ((m_1+1)l-1)e^{-x}x^{(m_1+1)l-2} = 0,$$

when $x_0 = (m_1+1)l - 1$. Hence, the integral can be estimated to be

$$\frac{1}{(l-1)!}\int_{m_2}^{2m_2l} e^{-x}\frac{\prod_{j=0}^{m_1} |\beta_j - x|^l}{|\beta_k - x|} dx \leq \frac{2m_2le^{-(m_1+1)l+1}((m_1+1)l-1)^{(m_1+1)l-1}}{(l-1)!}.$$

Finally, let us estimate the third integral

$$\frac{1}{(l-1)!}\int_{2m_2l}^{\infty} e^{-x}\frac{\prod_{j=0}^{m_1} |\beta_j - x|^l}{|\beta_k - x|} dx \leq \frac{1}{(l-1)!}\int_{2m_2l}^{\infty} e^{-x}x^{(m_1+1)l-1}dx.$$

Again, we use the function $f(x) = e^{-x}x^{(m_1+1)l-1}$. Sinve this function obtains its maximum at $x_0 = (m_1+1)l-1$, it is decreasing when $x > x_0$. We also have $2m_2l \geq (m_1+1)l \geq x_0$. Hence, we may estimate

$$\frac{1}{(l-1)!}\int_{2m_2l}^{\infty} e^{-x}x^{(m_1+1)l-1}dx \leq \frac{1}{(l-1)!}\sum_{h=0}^{\infty} e^{-2m_2l-h}(2m_2l+h)^{(m_1+1)l-1}.$$

Let us estimate the ratio between consecutive terms:

$$\frac{e^{-2m_2l-h-1}(2m_2l+h+1)^{(m_1+1)l-1}}{e^{-2m_2l-h}(2m_2l+h)^{(m_1+1)l-1}} = e^{-1}\left(1 + \frac{1}{2m_2l+h}\right)^{(m_1+1)l-1} \leq e^{-1/2}.$$

The third integral can thus be estimated as a geometric sum:

$$\frac{1}{(l-1)!}\int_{2m_2l}^{\infty} e^{-x}x^{(m_1+1)l-1}dx \leq \frac{e^{-2m_2l}(2m_2l)^{(m_1+1)l-1}}{(l-1)!(1-e^{-1/2})}.$$

Hence,

$$\frac{1}{(l-1)!}\int_{0}^{\infty} e^{-xt}\frac{\prod_{j=0}^{m_1} |\beta_j - x|^l}{|\beta_k - x|} dx$$
$$\leq \frac{m_2^{(m_1+1)l}}{(l-1)!} + \frac{1}{(l-1)!}2m_2le^{-(m_1+1)l+1}((m_1+1)l-1)^{(m_1+1)l-1} + \frac{e^{-2m_2l}(2m_2l)^{(m_1+1)l-1}}{(l-1)!(1-e^{-1/2})}.$$

Since $m_2 \leq \frac{l(m_1+1)}{e}$, we have

$$\frac{m_2^{(m_1+1)l}}{(l-1)!} < \frac{1}{(l-1)!}2m_2le^{-(m_1+1)l+1}((m_1+1)l-1)^{(m_1+1)l-1},$$

and since the function $f(x)$ peaks at $(m_1+1)l-1$, we have

$$\frac{1}{(l-1)!}2m_2le^{-(m_1+1)l+1}((m_1+1)l-1)^{(m_1+1)l-1} > \frac{e^{-2m_2l}(2m_2l)^{(m_1+1)l-1}}{(l-1)!(1-e^{-1/2})}.$$

Therefore,

$$\frac{1}{(l-1)!}\int_{0}^{\infty} e^{-xt}\frac{\prod_{j=0}^{m_1} |\beta_j - x|^l}{|\beta_k - x|} dx \leq 3\frac{1}{(l-1)!}2m_2le^{-(m_1+1)l+1}((m_1+1)l-1)^{(m_1+1)l-1}$$
$$\leq \frac{2m_2le^{-(m_1+1)l}((m_1+1)l)^{(m_1+1)l}}{(l-1)!}.$$

We need to write the estimate as an exponential function:

$$\frac{2m_2 l e^{-(m_1+1)l}((m_1+1)l)^{(m_1+1)l}}{(l-1)!}$$
$$\leq \exp\left(m_1 l \log l + l\log\frac{l}{l-1} + l(m_1+1)\log(m_1+1) - m_1 l + \log l\right)$$
$$\times \exp\left(\frac{1}{2}\log(l-1) + \log m_2 - 1 + \log 2 - \frac{1}{2}\log(2\pi)\right).$$

Since $l\log\frac{l}{l-1} \leq 1$ and

$$\frac{1}{l}\left(\log l + \frac{1}{2}\log(l-1) + \log m_2 + \log 2 - \frac{1}{2}\log(2\pi)\right) \leq 0.006 < \log(m_1+1),$$

we have

$$\frac{1}{(l-1)!}\int_0^\infty e^{-xt}\frac{\prod_{j=0}^{m_1}|\beta_j - x|^l}{|\beta_k - x|}dx \leq \exp\left(m_1 l \log l + l((m_1+2)\log(m_1+1) - m_1)\right).$$

Let us now estimate the terms $L_{k,j}$. They have the following integral representations:

$$|L_{k,j}| = \left|\frac{1}{(l-1)!}\int_0^{\beta_j} e^{\beta_j - x}\frac{\prod_{i=0}^{m_1}(\beta_i - x)^l}{\beta_j - x}dx\right| \leq \frac{e^{\beta_j}m_2^{l(m_1+1)-1}}{(l-1)!}\int_0^{\beta_j} e^{-x}dx \leq \frac{e^{\beta_j}m_2^{l(m_1+1)-1}}{(l-1)!}.$$

We obtain

$$\sum_{j=1}^{m_1}|L_{k,j}| \leq \sum_{j=1}^{m_1}\frac{e^{\beta_j}m_2^{l(m_1+1)-1}}{(l-1)!} \leq \frac{e^{m_2+1}m_2^{l(m_1+1)-1}}{(l-1)!}.$$

Now we need to write this as an exponential function:

$$\sum_{j=1}^{m_1}|L_{k,j}| \leq \sum_{j=1}^{m_1}\frac{e^{\beta_j}m_2^{l(m_1+1)-1}}{(l-1)!} \leq \frac{e^{m_2+1}m_2^{l(m_1+1)-1}}{(l-1)!}$$
$$\leq \exp\left(m_2 + 1 + (l(m_1+1)-1)\log m_2 - \left(l - \frac{1}{2}\right)\log(l-1) + l - 1 - \frac{1}{2}\log(2\pi)\right)$$
$$\leq \exp\left(-l\log l + l(m_1+2)\log m_2\right).$$

Now

$$\begin{cases} e^{q(l)} \leq \exp\left(m_1 l\log l + l((m_1+2)\log(m_1+1) - m_1)\right) \\ e^{-r(l)} \leq \exp\left(-l\log l + l(m_1+2)\log m_2\right). \end{cases}$$

Comparing the above to (8) and (9), we get

$$a = m_1, \quad b_1 = (m_1+2)\log(m_1+1) - m_1, \quad c = 1, \quad \text{and} \quad e_1 = (m_1+2)\log m_2,$$

and by (10),

$$\begin{cases} B = b_1 + \frac{ae_1}{c} \leq (m_1^2 + 3m_1 + 2)\log m_2 - m_1, \\ C = am_1 = m_1^2, \\ D = (a+b_1)m_1 + am_1^2 e^{-m_1(\log m_1)^2} \\ \quad = m_1(m_1 + (m_1+2)\log(m_1+1) - m_1) + m_1^3 e^{-m_1(\log m_1)^2} \\ \quad \leq 2m_1(m_1+2)\log(m_1+1). \end{cases}$$

Next we sum together the terms arising from the terms C and D. Since

$$4\frac{\log(2H)}{\log\log(2H)} \geq 2e^{(m_1+1)(\log(m_1+1))^2} \geq 2(m_1+1)^2,$$

we may estimate (see (51))

$$
\begin{aligned}
&C\log\hat{n}+D+(m_1+1)\log 2\\
&\qquad\le m_1^2\log\left(\frac{\hat{e}}{c_1}\cdot\frac{\log(2H)}{\log\log(2H)}\right)+2m_1(m_1+2)\log(m_1+1)+(m_1+1)\log 2\\
&\qquad\qquad\le 2(m_1+1)^2\log\left(4\frac{\log(2H)}{\log\log(2H)}\right).
\end{aligned}
$$

Now we can combine this term with the term coming from the term B:

$$
\begin{aligned}
&\frac{1}{\log H}\left(B\hat{n}+m_1^2\log\hat{n}+D+(m_1+1)\log 2\right)\\
&\le\frac{1}{\log H}\left(\frac{\hat{e}((m_1^2+3m_1+2)\log m_2-m_1)\log(2H)}{c_1\log\log(2H)}+2(m_1+1)^2\log\left(4\frac{\log(2H)}{\log\log(2H)}\right)\right).
\end{aligned}
$$

Let us start by eliminating the last term with the term $-m_1\frac{\hat{e}\log(2H)}{c_1\log\log(2H)}$. Notice that

$$2(m_1+1)^2\log\left(4\frac{\log(2H)}{\log\log(2H)}\right)<2(m_1+1)^2\log\log(2H).$$

Hence, it suffices to show that $\frac{\log(2H)}{(\log\log(2H))^2}\ge 2\frac{(m_1+1)^2}{m_1}$. Notice first that the function $f(x)=\frac{x}{(\log x)^2}$ is increasing when $x\ge e^2$, so we may estimate

$$\frac{\log(2H)}{(\log\log(2H))^2}\ge\frac{\log H}{(\log\log H)^2}\ge\frac{e^{m_2(\log m_2)^2}}{4m_2(\log m_2)^2}\ge\frac{e^{m_2(\log m_2)^2}}{4m_2(\log m_2)^4}.$$

Observe that the function $f(x)=\frac{e^x}{4x^2}$ is increasing when $x\ge 2$. Since $4(\log 4)^2>2$, we obtain

$$\frac{e^{m_2(\log m_2)^2}}{4m_2^2(\log m_2)^4}\ge\frac{e^{4(\log 4)^2}}{4(4(\log 4)^2)^2}\ge 9.2,$$

and hence

$$\frac{e^{m_2(\log m_2)^2}}{4m_2(\log m_2)^4}>9.2m_2\ge 2\frac{(m_1+1)^2}{m_1},$$

and thus,

$$
\begin{aligned}
&\frac{1}{\log H}\left(\frac{\hat{e}((m_1^2+3m_1+2)\log m_2-m_1)\log(2H)}{c_1\log\log(2H)}+2(m_1+1)^2\log\left(4\frac{\log(2H)}{\log\log(2H)}\right)\right)\\
&\qquad\le\frac{1}{\log\log H}\left(1+\frac{\log 2}{\log H}\right)\frac{\hat{e}}{c_1}(m_1^2+3m_1+2)\log m_2.
\end{aligned}
$$

Finally, $\left(1+\frac{\log 2}{\log H}\right)\frac{\hat{e}}{c_1}$ is always at most 13.79 (the biggest value for $m_2=4$) and it is decreasing. When $m_2\ge 11$, the value of this expression is at most 2. Computations are performed by Sage [12]. □

As a corollary of the bound obtained for sparse polynomials, we get the following transcendence measure for an arbitrary integer power of e:

Corollary 9.2. *Assume $d\ge 2$ is an integer. Then the bound*

$$\left|\lambda_0+\lambda_1e^d+\lambda_2e^{2d}+\ldots+\lambda_me^{md}\right|>\frac{1}{H^{\omega(m,H)}},$$

where
$$\omega(m,H) < m + \frac{\rho(m^2+3m+2)\log(dm)}{\log\log H},$$
holds for all $\overline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)^T \in \mathbb{Z}_{\mathbb{I}}^{m+1} \setminus \{\overline{0}\}$ *with* $\log H \geq dm(\log(dm))^2 e^{dm(\log(dm))^2}$, $H = \max_{1\leq i\leq m}\{1, |\lambda_i|\}$, *and* ρ *as in the previous theorem.*

Proof. Notice that now $m_1 = m$ and $m_2 = dm$. Substituting these values into the previous theorem immediately yields the result. □

References

[1] M. Abramowitz, I. A. Stegun: Handbook of mathematical functions with formulas, graphs, and mathematical tables, National Bureau Of Standards Applied Mathematics Series, 55, Washington, D.C, 1964.

[2] É. Borel: Sur la nature arithmétique du nombre e, *C. R. Acad. Sci. Paris*, 128 (1899), 596–599.

[3] N. I. Fel'dman, Yu. V. Nesterenko: Transcendental Numbers, Number Theory IV, 1–345, Encyclopaedia Math. Sci., 44, Springer, Berlin, 1998.

[4] J. Hančl, M. Leinonen, K. Leppälä and T. Matala-aho: Explicit irrationality measures for continued fractions, *J. Number Theory* 132 (2012), 1758–1769.

[5] M. Hata: Remarks on Mahler's Transcendence Measure for e, *J. Number Theory* 54 (1995), 81–92.

[6] Ch. Hermite: Sur la fonction exponentielle, C. R. Acad. Sci. 77 (1873), 18–24, 74–79, 226–233, 285–293.

[7] D. S. Khassa, S. Srinivasan: A transcendence measure for e, *J. Indian Math. Soc.* 56 (1991), 145–152.

[8] K. Mahler: Zur Approximation der Exponentialfunktion und des Logarithmus. Teil I, *J. Reine Angew. Math.* 166 (1931), 118–136.

[9] K. Mahler: Lectures on Transcendental numbers, Springer-Verlag, 1976.

[10] J. Popken: Sur la nature arithmetique du nombre e, *C. R. Acad. Sci. Paris*, 186 (1928), 1505–1507.

[11] J. Popken: Zur Transzendenz von e, *Math. Z.* 29 (1929), 525–541.

[12] *SageMath, the Sage Mathematics Software System (Version 6.5)*, The Sage Developers, 2015, `http://www.sagemath.org`.

[13] N. J. A. Sloane: The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/`.

[14] M. Waldschmidt: Introduction To Diophantine methods: irrationality and transcendence, 2014, `https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/IntroductionDiophantineMethods.pdf`.

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