A NECESSARY AND SUFFICIENT CRITERION FOR THE EXISTENCE OF RATIO LIMITS OF SEQUENCES GENERATED BY LINEAR RECURRENCES

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ABSTRACT. We introduce a necessary and sufficient criterion for determining the existence and the values of ratio limits of complex sequences generated by arbitrary linear recurrences.

1. INTRODUCTION

Sloane's Online Encyclopedia of Integer Sequences [22] and Khovanova's website [16] catalog thousands of integer sequences generated by linear recurrences that are associated with problems in various branches of mathematics and other sciences, such as number theory, abstract algebra, linear algebra, combinatorics, complex numbers, group theory, probability, statistics, affine geometry, electrical networks, infectious diseases, etc., cf. [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 17, 18, 20, 21, 23, 24, 25].

The asymptotic behavior of sequences generated by linear recurrences is characterized by the ratio limit of the sequence's consecutive terms. The knowledge on whether a ratio limit exists is necessary if a problem requires considering the sequence's terms with higher and higher indices. The existence of a ratio limit and its value depend on the choice of the sequence's initial conditions.

In 1997 Dubeau et al. [9] studied linear recurrences $F \in L(\mathbb{C}^n, \mathbb{C}^n)$ with asymptotically simple characteristic polynomials

$$P = \lambda^n - b_1\lambda^{n-1} - \cdots - b_n, \quad b_n \neq 0. \tag{1.1}$$

A polynomial is asymptotically simple iff among its zeros of maximal modulus there is a dominant zero λ_0 of maximal multiplicity.

Dubeau et al. derived *a sufficient* criterion for the existence of ratio limits of sequences $\left(F_k^{\mathbf{a}}\right)_{k=-n+1}^{\infty}$ generated by F from complex initial conditions $\mathbf{a} = (a_{-n+1}, \ldots, a_0)$. Specifically, the authors showed that if

$$a_0\lambda_0^{n-1} + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j}\lambda_0^{n-j-1} \neq 0, \tag{1.2}$$

$$\text{then} \quad \lim_{k_0<k\to\infty} \frac{F_{k+1}^{\mathbf{a}}}{F_k^{\mathbf{a}}} = \lambda_0, \;\; F_k^{\mathbf{a}} \neq 0 \;\text{ for }\; k > k_0, \tag{1.3}$$

$$\text{where } F_k^{\mathbf{a}} = a_k \text{ if } -n+1 \leq k \leq 0 \text{ and } F_k^{\mathbf{a}} = b_1F_{k-1}^{\mathbf{a}} + \cdots + b_nF_{k-n}^{\mathbf{a}} \text{ if } k > 0. \tag{1.4}$$

Condition (1.2) is satisfied, in particular, by all sequences generated by linear recurrences with asymptotically simple characteristic polynomials from initial conditions $(0, \ldots, 0, a_0)$.

An example of a sequence that does not satisfy condition (1.2), but has the ratio limit, is the constant sequence $(1_k)_{k=-n+1}^{\infty}$ generated by the linear recurrence with the signature $(2, -1)$ from the initial conditions $(1, 1)$. The corresponding asymptotically simple characteristic polynomial $P = (\lambda - 1)^2$.

We generalize results obtained by Dubeau et al. by introducing *a necessary and sufficient* criterion for the existence of ratio limits of complex sequences generated by linear recurrences with *arbitrary* characteristic polynomials P. We also prove that if the ratio limit exists, it must be equal to one of the zeros of P.

2. Main results

Given a linear recurrence $F \in L(\mathbb{C}^n, \mathbb{C}^n)$ of an order n with the signature $(b_1, \dots, b_n)$, where $b_n \neq 0$. A sequence $(F_k^{\mathbf{a}})_{k=-n+1}^{\infty} = F^{\mathbf{a}}$ generated by formulas (1.4) is called *a solution of* F.

Theorem 2.1. *If a solution $F^{\mathbf{a}}$ of F generated from initial conditions $\mathbf{a} \in \mathbb{C}^n$ has a ratio limit*

$$\lim_{k_0 < k \to \infty} \frac{F_{k+1}^{\mathbf{a}}}{F_k^{\mathbf{a}}} = \Psi, \quad \text{where} \quad F_k^{\mathbf{a}} \neq 0 \quad \text{for} \quad k > k_0, \tag{2.1}$$

then Ψ is equal to one of the zeros of the characteristic polynomial P of F.

Proof. If $n = 1$, b_1 is the zero of the characteristic monomial, and we have $(F_k^{\mathbf{a}})_{k=0}^{\infty} = a_1 \cdot b_1^k$.

If $n > 1$, we introduce a continuous mapping $f : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ defined as

$$f(z_1, \dots, z_n) = \left(z_2, \dots, z_n, b_1 + \frac{b_2}{z_n} + \frac{b_3}{z_{n-1} z_n} + \dots + \frac{b_n}{z_2 \dots z_n} \right). \tag{2.2}$$

$$f\left(\frac{F_{k_0+2}}{F_{k_0+1}}, \dots, \frac{F_{k_0+n+1}}{F_{k_0+n}} \right) = \left(\frac{F_{k_0+3}}{F_{k_0+2}}, \dots, \frac{F_{k_0+n+1}}{F_{k_0+n}}, \frac{b_1 F_{k_0+n+1} + b_2 F_{k_0+n} + \dots + b_n F_{k_0+2}}{F_{k_0+n+1}} \right). \tag{2.3}$$

It follows from formula (1.4) with $k = k_0 + n + 2$, and equation (2.3) that

$$f\left(\frac{F_{k_0+2}}{F_{k_0+1}}, \dots, \frac{F_{k_0+n+1}}{F_{k_0+n}} \right) = \left(\frac{F_{k_0+3}}{F_{k_0+2}}, \dots, \frac{F_{k_0+n+2}}{F_{k_0+n+1}} \right). \tag{2.4}$$

Our assumption (2.1) and equation (2.4) imply that iterations of f create a sequence convergent to the vector $(\Psi, \dots, \Psi)$. Since f is continuous, it means that the vector $(\Psi, \dots, \Psi)$ is a fixed point of f, [19, p.227], i.e.,

$$f(\Psi, \dots, \Psi) = (\Psi, \dots, \Psi). \tag{2.5}$$

On the other hand, from the continuity of f, equation (2.3), and the fact that due to (2.1)

$$\lim_{k_0 \to \infty} \frac{F_{k_0+i+1}}{F_{k_0+n+1}} = \Psi^{-n+i}, \quad i = 1, \dots, n, \tag{2.6}$$

we obtain that

$$f(\Psi, \dots, \Psi) = \left(\Psi, \dots, \Psi, b_1 + b_2 \Psi^{-1} + \dots + b_n \Psi^{1-n} \right). \tag{2.7}$$

Equations (2.5) and (2.7) imply that $\Psi^n - b_1 \Psi^{n-1} - \dots - b_n = 0$.

□

Let the characteristic polynomial P of a linear recurrence F have ν distinct zeros. For simplicity of the notation, we label them as λ_i, $i = 1, \dots, \nu$. Let μ_i denote the multiplicity of the zero λ_i, i.e., $\sum_{i=1}^{\nu} \mu_i = n$.

Any solution $F^{\mathbf{a}}$ of F is a linear combination of the following n *basic solutions* of F [14, 15]:

$$(k^j \lambda_i^k)_{k=-n+1}^{\infty}, \quad i = 1, \dots, \nu, \quad j = 0, \dots, \mu_i - 1, \tag{2.8}$$

So, for $k \geq -n + 1$, we have

$$F_k^{\mathbf{a}} = \sum_{i=1}^{\nu} \sum_{j=0}^{\mu_i - 1} c_{ij}^{\mathbf{a}} k^j \lambda_i^k. \tag{2.9}$$

The coefficients $(c^{\mathbf{a}}_{1,0}, \dots, c^{\mathbf{a}}_{\nu,\mu_\nu-1}) = \mathbf{c}^{\mathbf{a}}$ are solutions of the system of linear equations

$$\mathbf{c}^{\mathbf{a}} = C^{-1}\mathbf{a}, \tag{2.10}$$

where columns of matrix C consists of linearly independent vectors built from the initial conditions of basic solutions (2.8), i.e.,

$$C = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \lambda_1^{-1} & \cdots & (-1)^{\mu_1-1}\lambda_1^{-1} & \cdots & \lambda_\nu^{-1} & \cdots & (-1)^{\mu_\nu-1}\lambda_\nu^{-1} \\ \lambda_1^{-2} & \cdots & (-2)^{\mu_1-1}\lambda_1^{-2} & \cdots & \lambda_\nu^{-2} & \cdots & (-2)^{\mu_\nu-1}\lambda_\nu^{-2} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \lambda_1^{-n+1} & \cdots & (-n+1)^{\mu_1-1}\lambda_1^{-n+1} & \cdots & \lambda_\nu^{-n+1} & \cdots & (-n+1)^{\mu_\nu-1}\lambda_\nu^{-n+1} \end{bmatrix}.$$

We define *the characteristic polynomial* $P^{\mathbf{a}}$ *of a solution* $F^{\mathbf{a}}$ as follows:

- $P^{\mathbf{a}}$ has as its zeros all those zeros λ_i of P for which there exists j such that $c^{\mathbf{a}}_{ij} \neq 0$;
- The multiplicity of a zero λ_i in $P^{\mathbf{a}}$ is equal to the largest index j for which $c^{\mathbf{a}}_{ij} \neq 0$.

In what follows, we say that a solution $F^{\mathbf{a}}$ of a linear recurrence F *is associated* with the characteristic polynomial $P^{\mathbf{a}}$.

Theorem 2.2. *Given a solution $F^{\mathbf{a}}$ of a linear recurrence F. The ratio limit*

$$\lim_{k_0<k\to\infty} \frac{F^{\mathbf{a}}_{k+1}}{F^{\mathbf{a}}_k}, \quad \text{where} \quad F^{\mathbf{a}}_k \neq 0 \quad \textit{for} \quad k > k_0, \tag{2.11}$$

exists iff the characteristic polynomial $P^{\mathbf{a}}$ of the solution $F^{\mathbf{a}}$ is asymptotically simple.

If the latter is true, then

$$\lim_{k_0<k\to\infty} \frac{F^{\mathbf{a}}_{k+1}}{F^{\mathbf{a}}_k} = \lambda_{i_0}, \tag{2.12}$$

where λ_{i_0} is the dominant zero of $P^{\mathbf{a}}$.[1]

Proof. $\Longleftarrow$ Let λ_{i_0} be the dominant zero with the multiplicity j_0 of the asymptotically simple polynomial $P^{\mathbf{a}}$. Then, we have

$$\lim_{k\to\infty} \frac{F^{\mathbf{a}}_k}{k^{j_0}\lambda^k_{i_0}} = c_{i_0 j_0}. \tag{2.13}$$

Formula (2.13) implies that

$$\lim_{k_0<k\to\infty} \frac{F^{\mathbf{a}}_{k+1}}{F^{\mathbf{a}}_k} = \lambda_{i_0} \cdot \lim_{k_0<k\to\infty} \left(\frac{F^{\mathbf{a}}_{k+1}}{(k+1)^{j_0}\lambda^{k+1}_{i_0}} \cdot \frac{k^{j_0}\lambda^k_{i_0}}{F^{\mathbf{a}}_k} \right) = \lambda_{i_0}. \tag{2.14}$$

$\Longrightarrow$ Let us assume that the ratio limit exists and that the characteristic polynomial $P^{\mathbf{a}}$ of a solution $F^{\mathbf{a}}$ is not asymptotically simple. Then $P^{\mathbf{a}}$ has $\eta > 1$ *distinct* zeros, say $\lambda_1, \dots, \lambda_\eta$, such that

(i) the modulus $R = |\lambda_1| = \cdots = |\lambda_\eta|$ is greater than or equal to the moduli of other zeros of the polynomial $P^{\mathbf{a}}$; and
(ii) there exist nonzero coefficients $c^{\mathbf{a}}_{1j_0}, \dots, c^{\mathbf{a}}_{\eta j_0}$ with the index j_0 greater than all indices j corresponding to zeros of $P^{\mathbf{a}}$ with the same modulus as R.

[1]Condition (1.2) ensures that the dominant zero λ_{i_0} of $P^{\mathbf{a}} \neq P$ coincides with the dominant zero λ_0 of P.

We decompose each sequence element $F_k^{\mathbf{a}}$ given by formula (2.9) into a part $D_k^{\mathbf{a}}$ containing linear combinations of the basic solutions (2.8) with the dominant moduli equal to $|k^{j_0}R^k|$, and a part $E_k^{\mathbf{a}}$ containing linear combinations of the basic solutions with moduli smaller than $|k^{j_0}R^k|$. Thus, for any $k \geq -n+1$, we have $F_k^{\mathbf{a}} = D_k^{\mathbf{a}} + E_k^{\mathbf{a}}$, where

$$D_k^{\mathbf{a}} = k^{j_0} \sum_{l=1}^{\eta} c_{lj_0}^{\mathbf{a}} \lambda_l^k. \tag{2.15}$$

According to Theorem 2.1, if limit (2.11) exists, it is equal to a zero of the characteristic polynomial P, say $\tilde{\lambda}$. So, we obtain that

$$\tilde{\lambda} = \lim_{k_0<k\to\infty} \frac{F_{k+1}^{\mathbf{a}}}{F_k^{\mathbf{a}}} = \lim_{k_0<k\to\infty} \frac{D_{k+1}^{\mathbf{a}} + E_{k+1}^{\mathbf{a}}}{D_k^{\mathbf{a}} + E_k^{\mathbf{a}}} = \lim_{k_0<k\to\infty} \frac{D_{k+1}^{\mathbf{a}}}{D_k^{\mathbf{a}} + E_k^{\mathbf{a}}}. \tag{2.16}$$

Formula (2.16) implies that

$$\tilde{\lambda}^{-1} = \lim_{k_0<k\to\infty} \frac{D_k^{\mathbf{a}} + E_k^{\mathbf{a}}}{D_{k+1}^{\mathbf{a}}} = \lim_{k_0<k\to\infty} \frac{D_k^{\mathbf{a}}}{D_{k+1}^{\mathbf{a}}}. \tag{2.17}$$

It follows from (2.15) and (2.17) that

$$\lim_{k_0<k\to\infty} \left(\frac{\sum_{l=1}^{\eta} c_{lj_0}^{\mathbf{a}} (\lambda_l/R)^{k+1}}{\sum_{l=1}^{\eta} c_{lj_0}^{\mathbf{a}} (\lambda_l/R)^{k}} \right) = \tilde{\lambda}/R. \tag{2.18}$$

To simplify the notation, let us set $c_l^{\mathbf{a}} = c_{lj_0}^{\mathbf{a}}$, and let us introduce normalized zeros $\gamma_l = \lambda_l/R$, i.e., $|\gamma_l| = 1$, $l = 1, \dots, \eta$. Since limit (2.18) exists, the sequence

$$\left(\frac{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+1}}{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k}} \right)_{k=k_0+1}^{\infty} \tag{2.19}$$

is a Cauchy sequence. Thus, for any $\epsilon > 0$, there exist k_ϵ such that for $k > k_\epsilon$

$$\left| \frac{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+2}}{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+1}} - \frac{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+1}}{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k}} \right| < \epsilon. \tag{2.20}$$

We transform inequality (2.20) into

$$\left| \frac{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+2} \sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k} - \left(\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+1} \right)^2}{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+1} \sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k}} \right| = \tag{2.21}$$

$$= \left| \frac{\sum_{l=1}^{\eta} \sum_{m=l+1}^{\eta} c_l^{\mathbf{a}} c_m^{\mathbf{a}} \gamma_l^k \gamma_m^k (\gamma_l - \gamma_m)^2}{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+1} \sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k}} \right| < \epsilon. \tag{2.22}$$

It follows from inequality (2.22) that the sequence

$$\left(\frac{\sum_{l=1}^{\eta} \sum_{m=l+1}^{\eta} c_l^{\mathbf{a}} c_m^{\mathbf{a}} \gamma_l^k \gamma_m^k (\gamma_l - \gamma_m)^2}{\sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k+1} \sum_{l=1}^{\eta} c_l^{\mathbf{a}} \gamma_l^{k}} \right)_{k=k_0+1}^{\infty} \tag{2.23}$$

must converge to 0.

The denominators in sequence (2.23) are bounded from above due to the fact that the moduli $|\gamma_l| = 1$, $l = 1, \dots, \eta$. Thus, the numerators of this sequence must form a sequence converging to 0. However, the sequences $(\gamma_l^k \gamma_m^k)_{k=k_0+1}^{\infty}$ oscillate for each pair of indices (l, m), and therefore their linear combination does not converge to 0. Consequently, sequence (2.23) converges to 0 only when it is the constant sequence $(0)_{k=k_0+1}^{\infty}$, i.e., all normalized zeros γ_l are equal one to another.

So, if the ratio limit (2.11) exists, there can be only one zero that satisfies conditions (i) and (ii) listed above. The latter contradicts our assumption that the characteristic polynomial $P^{\mathbf{a}}$ of the solution $F^{\mathbf{a}}$ is not asymptotically simple.

□

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MSC2010: 40A05, 40A30

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