Spiral determinants

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Abstract. We evaluate determinants of "spiral" matrices, which are matrices in which entries are spiralling from the centre of the matrices towards the outside, with prescribed increments from one entry to the next depending on whether one moves right, up, left, or down along the spiral.

1. Preamble. The evaluation of determinants is a rich topic, and fascinating for many people, in particular so for the authors of this note. A large body of such evaluations has been collected in [1, 2]. Although it may seem to some that everything which is known is to be found there, this is definitely not the case. After all, the task of assembling "everything which is known" is an impossible one. (However, more is contained in these two surveys as one might commonly think since frequently simple transformations or reindexing turns a seemingly "unknown" determinant into a known one.) The purpose of this note is to discuss some attractive determinant evaluations not contained (in any form) in [1, 2].

2. Spiral matrices. Rather than jumping directly to our results in Theorems 1–3 below (and the generalisation of Theorem 3 discussed in Section 6) without any further motivation, we believe that it is of interest to describe the path that led to them.

During a visit of the second author (CK) at the Institut Henri Poincaré in Paris in February 2017, Alexander R. Miller told CK that he had been looking at "spiral" matrices of the form

$$\begin{pmatrix} q^{16} & q^{15} & q^{14} & q^{13} \\ q^{5} & q^{4} & q^{3} & q^{12} \\ q^{6} & q & q^{2} & q^{11} \\ q^{7} & q^{8} & q^{9} & q^{10} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^{17} & q^{16} & q^{15} & q^{14} & q^{13} \\ q^{18} & q^{5} & q^{4} & q^{3} & q^{12} \\ q^{19} & q^{6} & q & q^{2} & q^{11} \\ q^{20} & q^{7} & q^{8} & q^{9} & q^{10} \\ q^{21} & q^{22} & q^{23} & q^{24} & q^{25} \end{pmatrix}, \quad \text{etc.} \tag{2.1}$$

(The pattern should be clear: we start with q in the centre, and then starting from there form a spiral of entries $q, q^2, q^3, \dots, q^{n^2}$, the matrices above being the cases for $n = 4$ and

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$n = 5$.) He had observed that the determinants of these matrices all factor nicely, and he had also an explanation why this happened. He asked CK whether he had already seen such determinant evaluations before.

The reaction of CK was that, while he had not seen these determinants before, he seems to recall determinants of spiral matrices to have appeared in a Monthly Problem several years ago. It did not take long to retrace this problem (namely [5], which, upon further search, turned out to have appeared even earlier as a problem in the Mathematics Magazine [7]). As it turned out, it concerned the spiral matrices

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 12 & 13 & 14 & 5 \\ 11 & 16 & 15 & 6 \\ 10 & 9 & 8 & 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 16 & 17 & 18 & 19 & 6 \\ 15 & 24 & 25 & 20 & 7 \\ 14 & 23 & 22 & 21 & 8 \\ 13 & 12 & 11 & 10 & 9 \end{pmatrix}, \quad \text{etc.,} \tag{2.2}$$

where $1, 2, \dots, n^2$ is winding from the outside into the centre. Again, as the solution [4] showed, the determinants of all these matrices are given by nice product formulae.

Miller then went on to ask himself what happens if, instead of spiralling from outside to inside, we spiral from inside to outside. In other words, what are the determinants of the matrices

$$\begin{pmatrix} 16 & 15 & 14 & 13 \\ 5 & 4 & 3 & 12 \\ 6 & 1 & 2 & 11 \\ 7 & 8 & 9 & 10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 17 & 16 & 15 & 14 & 13 \\ 18 & 5 & 4 & 3 & 12 \\ 19 & 6 & 1 & 2 & 11 \\ 20 & 7 & 8 & 9 & 10 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix}, \quad \text{etc.?} \tag{2.3}$$

The On-Line Encyclopedia of Integer Sequences [6] told him that the determinants of these matrices were given by sequence `A079340`, and a conjectured formula could be found there. (The determinants of the matrices in (2.2) form the sequence `A023999` in [6].) He reported this to CK and asked him if he would know how to evaluate the determinants in (2.2) and (2.3).

After a few moments' thought, CK's reply was that, say for the case of odd n (n being the number of rows and columns of the matrix), one should subtract the next-to-last row from the last; the result is that almost all entries in the last row are the same, except for the one in the first column. Now use the first column to eliminate all the other entries in the last row and expand the determinant with respect to the (new) last row. As a result, one is again left with the determinant of a spiral matrix. Doing this, and something similar in the case of even n, should result into an inductive proof.

When CK thought this argument through more carefully, he discovered that, while it is true that the above manipulations allow one to reduce the evaluation of the determinants of the spiral matrices in (2.2) and (2.3) to the evaluation of a determinant of a smaller spiral matrix, this smaller spiral matrix does however not belong to the same family anymore. Neither is the central entry still 1 respectively n^2, nor are the increments between entries along the spiral anymore always 1. Thus, if an inductive argument in this spirit should work at all, then more parameters would have to be introduced into the game.

3. More parameters. Indeed, to introduce more parameters is one of the important points "preached" in [1, 2]: one should always try to add parameters to the determinants one considers, and hope that even with these new parameters the determinants still evaluate nicely. If this happens, it will (usually) be much easier to evaluate the more general determinants since parameters allow a much bigger flexibility in the arguments.

In our case, what should we do? First of all, instead of 1 (or n^2), we should allow ourselves to start with an arbitrary number in the centre — say a —, and we should allow more general increments from one entry to the next along the spiral. Clearly, if one allows arbitrary increments then one cannot expect anything (since one then faces a completely generic matrix), but maybe we should allow an (additive) increment of x, say, when moving to the right, an increment of b, say, when moving up, an increment of y, say, when moving to the left, and an increment of c, say, when moving down along the spiral. We denote the corresponding $n \times n$ spiral matrix by $M_n(a, b, c, x, y)$. When it is clear from the context what the parameters are, we shall abbreviate this to M_n. For example, we have

$$M_4 = \begin{pmatrix} a+4b+2c+4x+5y & a+4b+2c+4x+4y & a+4b+2c+4x+3y & a+4b+2c+4x+2y \\ a+b+x+2y & a+b+x+y & a+b+x & a+3b+2c+4x+2y \\ a+b+c+x+2y & a & a+x & a+2b+2c+4x+2y \\ a+b+2c+x+2y & a+b+2c+2x+2y & a+b+2c+3x+2y & a+b+2c+4x+2y \end{pmatrix},$$

and

$$M_5 = \begin{pmatrix} a+4b+2c+4x+6y & a+4b+2c+4x+5y & a+4b+2c+4x+4y & a+4b+2c+4x+3y & a+4b+2c+4x+2y \\ a+4b+3c+4x+6y & a+b+x+2y & a+b+x+y & a+b+x & a+3b+2c+4x+2y \\ a+4b+4c+4x+6y & a+b+c+x+2y & a & a+x & a+2b+2c+4x+2y \\ a+4b+5c+4x+6y & a+b+2c+x+2y & a+b+2c+2x+2y & a+b+2c+3x+2y & a+b+2c+4x+2y \\ a+4b+6c+4x+6y & a+4b+6c+5x+6y & a+4b+6c+6x+6y & a+4b+6c+7x+6y & a+4b+6c+8x+6y \end{pmatrix}.$$

Computer experiments suggest that the determinants of these matrices factorise nicely. In fact, it is not difficult to come up with a guess for the result, and the inductive argument sketched above leads to an almost effortless proof.

Theorem 1. *For all non-negative integers n, we have*

$$\det M_{2n}(a,b,c,x,y) = (-1)^{n+1}\big(ax + n^2bx + n(n-1)cx + n^2x^2 + ay + (n-1)^2by \\ + n(n-1)cy + n(n-1)y^2 + n(2n-1)xy\big) \prod_{i=1}^{2n-2} \big(i(b+c) + (i+1)(x+y)\big)$$

and

$$\det M_{2n+1}(a,b,c,x,y) = (-1)^{n}\big(ax + n^2bx + n(n-1)cx + n^2x^2 + ay + n^2by \\ + n(n+1)cy + n(n+1)y^2 + n(2n+1)xy\big) \prod_{i=1}^{2n-1} \big(i(b+c) + (i+1)(x+y)\big).$$

PROOF. One proceeds by induction on n. Let us consider the matrix $M_{2n+1}(a,b,c,x,y)$. It has the form

$$M_{2n+1} = \begin{pmatrix} \dots\dots\dots\dots\dots\dots & & & \\ \vdots & & & \vdots \\ E_1 & \dots & a & \dots. \\ \dots\dots\dots\dots\dots\dots & & & \\ E_2 & E_3 & \dots & E_4 \\ E_5 & E_6 & \dots & E_7 \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned} E_1 &= a + n^2 b + n^2 c + n^2 x + n(n+1)y, \\ E_2 &= a + n^2 b + (n^2+n-1)c + n^2 x + n(n+1)y, \\ E_3 &= a + (n-1)^2 b + n(n-1)c + (n-1)^2 x + n(n-1)y, \\ E_4 &= a + (n-1)^2 b + n(n-1)c + n^2 x + n(n-1)y, \\ E_5 &= a + n^2 b + n(n+1)c + n^2 x + n(n+1)y, \\ E_6 &= a + n^2 b + n(n+1)c + (n^2+1)x + n(n+1)y, \\ E_7 &= a + n^2 b + n(n+1)c + n(n+2)x + n(n+1)y. \end{aligned}$$

Now we subtract the next-to-last row from the last row. This operation does not change the determinant. The matrix we obtain thereby is

$$\begin{pmatrix} \dots\dots\dots\dots\dots\dots & & & \\ \vdots & & & \vdots \\ E_1 & \dots & a & \dots.. \\ \dots\dots\dots\dots\dots\dots & & & \\ E_2 & E_3 & \dots & E_4 \\ c & D_1 & \dots & D_1 \end{pmatrix},$$

where

$$D_1 = (2n-1)b + 2nc + 2nx + 2ny.$$

We eliminate the D_1's by subtracting D_1/c times the first column from the other columns. Again, this does not change the determinant. The resulting matrix has the form

$$\begin{pmatrix} \dots\dots\dots\dots\dots\dots & & & \\ \vdots & & & \vdots \\ E_1 & \dots & A_1 & \dots \\ \dots\dots\dots\dots\dots\dots & & & \\ * & * & \dots. & * \\ c & 0 & \dots. & 0 \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} A_1 &= a - \frac{1}{c} D_1 E_1 \\ &= a - \frac{1}{c}\big((2n-1)b + 2nc + 2nx + 2ny\big)\big(a + n^2 b + n^2 c + n^2 x + n(n+1)y\big). \end{aligned} \tag{3.3}$$

The point here is that, apart from the first column and the last row, we have obtained a spiral matrix, of dimensions $(2n)\times(2n)$, with central entry A_1, with right increment x, up increment B_1, left increment y, and down increment C_1, where

$$B_1 = b + D_1 = 2n(b+c+x+y), \tag{3.4}$$
$$C_1 = c - D_1 = -(2n-1)(b+c) - 2n(x+y). \tag{3.5}$$

Hence, by taking the determinants of (3.1) and (3.2), we infer the relation

$$\det M_{2n+1}(a,b,c,x,y) = c \cdot \det M_{2n}(A_1,B_1,C_1,x,y),$$

where A_1, B_1, C_1 are given in (3.3)–(3.5). If we assume by induction that we already know the formula for M_{2n}, then by the above relation we obtain the claimed formula for M_{2n+1}.

For the proof of the claim on the determinant of $M_{2n}(a,b,c,x,y)$, one proceeds similarly. Here, one subtracts the second row from the first row, and subsequently D_2/b times the last column from the other columns, where

$$D_2 = (2n-1)b + (2n-2)c + (2n-1)x + (2n-1)y.$$

Here, the relation which results by taking determinants is

$$\det M_{2n}(a,b,c,x,y) = -b \cdot \det M_{2n-1}(A_2,B_2,C_2,x,y),$$

where

$$\begin{aligned} A_2 &= a - \frac{1}{b}\big((2n-1)b + (2n-2)c + (2n-1)x + (2n-1)y\big)\\ &\qquad\qquad \times \big(a + n(n-1)b + n(n-1)c + n^2x + n(n-1)y\big),\\ B_2 &= -(2n-2)(b+c) - (2n-1)(x+y),\\ C_2 &= (2n-1)(b+c+x+y). \end{aligned}$$

Again, if we assume by induction that we already know the formula for M_{2n-1}, then the claimed formula for M_{2n} results straightforwardly from the above relation. □

REMARK. It goes without saying that the evaluation of the determinants of the matrices in (2.2) results by specialising Theorem 1 to $a = n^2$, $x = y = b = c = -1$, while the evaluation of the determinants of the matrices in (2.3) is given by the special case $a = x = y = b = c = 1$ of Theorem 1. In particular, this provides a proof of the conjectured formula in [6, `A079340`]. The proof [4] of the formula for the determinants in (2.2) actually yielded more generally the evaluation of the determinants of the matrices $M_n(a,-1,-1,-1,-1)$.

4. A "q-analogue". The "q-analogue" of the above theorem in which one replaces each entry X of $M_n(a,b,c,x,y)$ by q^X is actually simpler to derive. Let $Q_n(a,b,c,x,y)$ denote the matrix which results from this replacement. Clearly, the matrices in (2.1) are the matrices $Q_n(1,1,1,1,1)$.

Theorem 2. *For all non-negative integers n, we have*

$$\det Q_{2n}(a,b,c,x,y) = (-1)^n q^{2na+\frac{1}{3}n(2n^2+1)(b+x)+\frac{2}{3}(n-1)n(n+1)(c+y)} \times \prod_{i=0}^{2n-2}\left(1-q^{i(b+c)+(i+1)(x+y)}\right)$$

and

$$\det Q_{2n+1}(a,b,c,x,y) = (-1)^n q^{(2n+1)a+\frac{1}{3}n(n+1)(2n+1)(b+c+x+y)} \times \prod_{i=0}^{2n-1}\left(1-q^{i(b+c)+(i+1)(x+y)}\right).$$

PROOF. One proceeds analogously to before. When one does the subtraction of the rows (where one multiplies by a suitable power of q), then all entries but one in the last/first row already become zero. So, no column operations are needed. Everything else is just the same. □

REMARK. The evaluation of the determinants of the matrices in (2.1) results by specialising Theorem 2 to $a = x = y = b = c = 1$.

5. A real q-analogue. Alexander Miller pointed out that the "q-analogue" given in Theorem 2 is not a *real* q-analogue: no matter how one specialises Theorem 2 or which limit one takes, it will never be possible to derive Theorem 1.

Without doubt, this is a valid point. How can one obtain a "real" q-analogue? The standard way (in combinatorics) to pass from the "ordinary $q = 1$-world" to the "q-world" is by replacing entries X by $q^X - 1$. If one tries this with the matrix $M_n(a,b,c,x,y)$ in Theorem 1, then the determinant of the resulting matrix turns out to be a huge polynomial in q^a, q^b, q^c, q^x, q^y which does not factor at all.

A second way to arrive at q-analogues is to replace entries X by the symmetric $q^X - q^{-X}$. (This is very common in the theory of quantum algebras.) Again the determinants of the resulting matrices are huge expressions that do not factor at all. However, on setting $x = y$, a miracle happens, and we obtain very nice factorisations. (For the record: nothing of this kind happens if one sets $b = c$, or for any other similar specialisation.) Again, it is not difficult to guess the result. See Theorem 3 below.

The recursive approach which proved Theorems 1 and 2 does not work anymore. However, by calculating a few small cases — this time by hand rather than by a computer — an elimination technique was found which led to a proof.

In order to formulate the result and its proof in a convenient and succinct way, we "get rid" of q: instead of writing $q^X - q^{-X}$, we write $X - \frac{1}{X}$. To be precise, let $Z_n(a,b,c,x,y)$ be the $n \times n$ spiral matrix, where, instead of adding increments of x, b, y, c at each "step" when moving to the right, up, left, down in the spiral, we *multiply* by x, b, y, c, and in the end, if the result is α, we put $\alpha - \alpha^{-1}$ as the corresponding entry. For the sake of simplicity of notation, we write $[\alpha] := \alpha - \alpha^{-1}$ and $\langle \alpha \rangle := \alpha + \alpha^{-1}$. For example, the 4×4 matrix $Z_4(a, b, c, x, y)$ is

$$Z_4 = \begin{pmatrix} [ab^4c^2x^4y^5] & [ab^4c^2x^4y^4] & [ab^4c^2x^4y^3] & [ab^4c^2x^4y^2] \\ [abxy^2] & [abxy] & [abx] & [ab^3c^2x^4y^2] \\ [abcxy^2] & [a] & [ax] & [ab^2c^2x^4y^2] \\ [abc^2xy^2] & [abc^2x^2y^2] & [abc^2x^3y^2] & [abc^2x^4y^2] \end{pmatrix}.$$

In order to see that this is a q-analogue of $M_n(a, b, c, x, y)$, one has to replace each variable α by q^α, divide each entry by $q - q^{-1}$, and then let $q \to 1$.

The announced determinant evaluation is the following.

Theorem 3. *For all non-negative integers n, we have*

$$\det Z_{2n}(a, b, c, x, x) = (-1)^{n+1} \left[a^2 b^{2n^2-2n+1} c^{2n^2-2n} x^{2n(2n-1)}\right] \prod_{k=0}^{2n-2} \left[(bc)^{k/2} x^{k+1}\right] \\ \times \prod_{k=1}^{n-1} \left\langle ab^{k(k+1)} c^{k^2} x^{k(2k+1)} \right\rangle \left\langle ab^{(2k^2-2k+1)/2} c^{(2k^2-1)/2} x^{k(2k-1)} \right\rangle \quad (5.1)$$

and

$$\det Z_{2n+1}(a, b, c, x, x) = (-1)^n \left[a^2 (bc)^{2n^2} x^{2n(2n+1)}\right] \prod_{k=0}^{2n-1} \left[(bc)^{k/2} x^{k+1}\right] \\ \times \prod_{k=1}^{n-1} \left\langle ab^{k(k+1)} c^{k^2} x^{k(2k+1)} \right\rangle \prod_{k=1}^{n} \left\langle ab^{(2k^2-2k+1)/2} c^{(2k^2-1)/2} x^{k(2k-1)} \right\rangle. \quad (5.2)$$

Proof. Our proof will be based on the two relations

$$[a] \langle x \rangle = [ax] + [a/x], \quad (5.3)$$
$$[a] [x] = \langle ax \rangle - \langle a/x \rangle, \quad (5.4)$$

valid for all non-zero a and x. From (5.3), we obtain

$$\left[Ax^2\right] - \langle x \rangle [Ax] + [A] = 0, \quad (5.5)$$

for all non-zero A and x, which is the key identity in the following arguments.

We start with the even case, that is, with the proof of (5.1). In the following, we shall abbreviate $Z_{2n}(a, b, c, x, x)$ by Z_{2n}. Furthermore, we denote the matrix entries by $z_{i,j}$, so that $Z_{2n} = (z_{i,j})_{1 \le i,j \le 2n}$.

Let C_j denote the j-th column of Z_{2n}. We replace C_j by

$$C_j - \langle x \rangle C_{j-1} + C_{j-2}, \quad \text{for } j = 3, 4, \ldots, 2n. \quad (5.6)$$

Let Z'_{2n} be the matrix which is obtained in this way. Clearly, these column operations do not change the determinant, so that

$$\det Z_{2n} = \det Z'_{2n}. \tag{5.7}$$

By (5.5), the effect of these operations is that, starting from the third column, all entries of Z'_{2n} in the upper wedge bordered by the diagonal $z'_{i,i}$, $i = 1, 2, \dots, n$ and the antidiagonal $z'_{i,2n-i+2}$, $i = 2, 3, \dots, n$ vanish, and, similarly, that all entries of Z'_{2n} in the lower wedge bordered by the diagonal $z'_{i,i+1}$, $i = n+1, n+2, \dots, 2n-1$ and the antidiagonal $z'_{i,2n-i+2}$, $i = n+1, n+2, \dots, 2n$ vanish. More precisely, the matrix Z'_{2n} has the form

$$\begin{pmatrix}
z_{1,1} & z_{1,2} & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\
* & * & 0 & 0 & \cdots\cdots & 0 & 0 & z'_{2,2n} \\
* & * & z'_{3,3} & 0 & \cdots\cdots & 0 & z'_{3,2n-1} & * \\
* & * & * & z'_{4,4} \; 0 & \cdots\cdots & 0 \; z'_{4,2n-2} & * & * \\
\vdots & \vdots & & \ddots & & \iddots & & \vdots \\
\vdots & \vdots & & & z'_{n,n} \quad 0 \quad z'_{n,n+2} & & & \vdots \\
\vdots & \vdots & & & z'_{n+1,n+1} \; z'_{n+1,n+2} & & & \vdots \\
\vdots & \vdots & & \iddots & & \ddots & & \vdots \\
\vdots & \vdots & \iddots & & & & \ddots & \vdots \\
* & * & * & z'_{2n-2,4} \; 0 & \cdots\cdots & 0 & z'_{2n-2,2n-1} & * \\
* & * & z'_{2n-1,3} & 0 & \cdots\cdots & 0 & 0 & z'_{2n-1,2n} \\
z_{2n,1} & z_{2n,2} & 0 & 0 & \cdots\cdots & 0 & 0 & 0
\end{pmatrix}. \tag{5.8}$$

We may now do a Laplace expansion of the determinant of this matrix with respect to the first and the last row. The result is that

$$\det Z'_{2n} = \det \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2n,1} & z_{2n,2} \end{pmatrix}$$
$$\times \det \begin{pmatrix}
0 & 0 & \cdots\cdots & 0 & 0 & z'_{2,2n} \\
z'_{3,3} & 0 & \cdots\cdots & 0 & z'_{3,2n-1} & * \\
* & z'_{4,4} \; 0 & \cdots\cdots & 0 \; z'_{4,2n-2} & * & * \\
\vdots & \ddots & & \iddots & & \vdots \\
\vdots & & z'_{n,n} \quad 0 \quad z'_{n,n+2} & & & \vdots \\
\vdots & & z'_{n+1,n+1} \; z'_{n+1,n+2} & & & \vdots \\
\vdots & \iddots & & \ddots & & \vdots \\
\vdots & \iddots & & & \ddots & \vdots \\
* & z'_{2n-2,4} \; 0 & \cdots\cdots & 0 & z'_{2n-2,2n-1} & * \\
z'_{2n-1,3} & 0 & \cdots\cdots & 0 & 0 & z'_{2n-1,2n}
\end{pmatrix}.$$

By expanding the last determinant alternatingly along the first row, along the last row, along the second row, along the next-to-last row, etc., it is straightforward to see that — up to a sign that is easily determined — it equals the product of the entries along the antidiagonal $z'_{i,2n-i+2}$, $i = 2, 3, \dots, 2n-1$. Thus, we obtain

$$\det Z'_{2n} = (-1)^{n-1}(z_{1,1}z_{2n,2} - z_{1,2}z_{2n,1}) \prod_{i=2}^{2n-1} z'_{i,2n-i+2}. \tag{5.9}$$

Hence, the only remaining task is to compute the first term in parentheses on the right-hand side and the entries $z'_{i,2n-i+2}$.

We begin with the first term on the right-hand side of (5.9):

$$\begin{aligned} z_{1,1}z_{2n,2} - z_{1,2}z_{2n,1} &= \left[ab^{n^2}c^{n(n-1)}x^{n(2n-1)+(2n-1)}\right]\left[ab^{(n-1)^2}c^{n(n-1)}x^{n(2n-1)-(2n-2)}\right] \\ &\quad - \left[ab^{n^2}c^{n(n-1)}x^{n(2n-1)+(2n-2)}\right]\left[ab^{(n-1)^2}c^{n(n-1)}x^{n(2n-1)-(2n-1)}\right] \\ &= \left\langle a^2b^{2n^2-2n+1}c^{2n^2-2n}x^{2n(2n-1)+1}\right\rangle \\ &\qquad - \left\langle a^2b^{2n^2-2n+1}c^{2n^2-2n}x^{2n(2n-1)-1}\right\rangle \\ &= \left[a^2b^{2n^2-2n+1}c^{2n^2-2n}x^{2n(2n-1)}\right][x]. \end{aligned} \tag{5.10}$$

Here, we first used (5.4) to rewrite the products of the form $[A][B]$, and subsequently we used (5.4) in the "reverse" direction in order to obtain the last line.

To complete the proof in the current case, we need a formula for the entries $z'_{i,2n-i+2}$ in (5.9). To do that, we need formulae for the entries $z_{i,2n-i}, z_{i,2n-i+1}, z_{i,2n-i+2}$ of the original matrix Z_{2n}. Writing $k = n - i + 1$ for convenience, we find the formulae

$$\begin{aligned} z_{i,2n-i} &= \left[ab^{k^2}c^{k(k-1)}x^{k(2k-1)+1}\right], && \text{for } i = 1, 2, \dots, n; \\ z_{2n-i+1,i-1} &= \left[ab^{k^2}c^{k(k+1)-1}x^{k(2k+1)}\right], && \text{for } i = 2, 3, \dots, n; \\ z_{i,2n-i+1} &= \left[ab^{k^2}c^{k(k-1)}x^{k(2k-1)}\right], && \text{for } i = 1, 2, \dots, n; \\ z_{2n-i+1,i} &= \left[ab^{(k-1)^2}c^{k(k-1)}x^{(k-1)(2k-1)}\right], && \text{for } i = 1, 2, \dots, n; \\ z_{i,2n-i+2} &= \left[ab^{k^2+2k}c^{k(k+1)}x^{k(2k+1)}\right], && \text{for } i = 2, 3, \dots, n; \\ z_{2n-i+1,i+1} &= \left[ab^{(k-1)^2}c^{k(k-1)}x^{(k-1)(2k-1)+1}\right], && \text{for } i = 1, 2, \dots, n. \end{aligned}$$

Thus, by using (5.3) several times, for $i = 2, 3, \dots, n$ we have

$$\begin{aligned} z'_{i,2n-i+2} &= z_{i,2n-i+2} - \langle x\rangle z_{i,2n-i+1} + z_{i,2n-i} \\ &= \left[ab^{k^2+2k}c^{k(k+1)}x^{(k+1)(2k+1)}\right] - \langle x\rangle \left[ab^{k^2}c^{k(k-1)}x^{k(2k-1)}\right] \\ &\qquad + \left[ab^{k^2}c^{k(k-1)}x^{k(2k-1)+1}\right] \\ &= \left[ab^{k^2+2k}c^{k(k+1)}x^{(k+1)(2k+1)}\right] - \left[ab^{k^2}c^{k(k-1)}x^{k(2k-1)-1}\right] \\ &= \left[(bc)^k x^{2k+1}\right] \left\langle ab^{k(k+1)}c^{k^2}x^{k(2k+1)}\right\rangle \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} z'_{2n-i+1,i+1} &= z_{2n-i+1,i+1} - \langle x\rangle z_{2n-i+1,i} + z_{2n-i+1,i-1} \\ &= \left[ab^{(k-1)^2}c^{k(k-1)}x^{(k-1)(2k-1)+1}\right] - \langle x\rangle \left[ab^{(k-1)^2}c^{k(k-1)}x^{(k-1)(2k-1)}\right] \\ &\qquad + \left[ab^{k^2}c^{k(k+1)-1}x^{k(2k+1)}\right] \\ &= -\left[ab^{(k-1)^2}c^{k(k-1)}x^{(k-1)(2k-1)}\right] + \left[ab^{k^2}c^{k(k+1)-1}x^{k(2k+1)}\right] \\ &= \left[(bc)^{(2k-1)/2}x^{2k}\right] \left\langle ab^{(2k^2-2k+1)/2}c^{(2k^2-1)/2}x^{k(2k-1)}\right\rangle. \end{aligned} \tag{5.12}$$

Substitution of (5.10), (5.11), and (5.12) in (5.9) in combination with (5.7) leads to (5.1).

The odd case, that is, our proof of (5.2), is similar. The only difference is that, here, we replace the j-th column of $Z_{2n+1}(a,b,c,x,x)$, C_j say, by

$$C_j - \langle x\rangle C_{j+1} + C_{j+2}, \quad \text{for } j = 1, 2, \dots, 2n-1.$$

(This should be compared with (5.6).) Everything else is completely analogous. We leave the details to the reader.

This completes the proof of the theorem. □

6. Discussion. We may ask ourselves whether Theorem 3 is the most general result existing or whether further generalisations are possible. In particular, Theorem 3 is not a full generalisation of Theorem 1 but only of its special case in which $x = y$. Clearly, our proof of Theorem 3 heavily depends on x being equal to y: the column operations that we perform there create zeroes in the upper *and* lower wedges only in this special case. We must leave it as an open problem whether there is a *full* "q-analogue" of Theorem 1 (in the sense discussed at the very beginning of this section).

Is it possible to generalise Theorem 3 in other directions? An attentive reader will have noticed that our proof only hinged upon the two relations (5.3) and (5.4) satisfied by the brackets $[\,.\,]$ (also involving the functions $\langle\,.\,\rangle$). An even closer perusal of the proof shows that actually relation (5.3) is sufficient for a factorisation of the determinant to happen. Namely, it implies (5.5), which in turn makes the column operations (5.6) create the wedges of zeroes in the matrix Z'_{2n} (cf. (5.8)). Relation (5.4) is only used in the simplification of the 2×2-determinant in (5.10). It then becomes clear that we could have even chosen *arbitrary* increments while moving up or down one step along the spiral, and we would still obtain a closed form product formula for the determinant of the corresponding matrix. Moreover, entries which are in the left and the right wedges do not influence the result at all.

The final question which we address is whether there are more general functions (instead of the brackets $[\,.\,]$ and angle brackets $\langle\,.\,\rangle$)? So, are our functions $[\,.\,]$ and $\langle\,.\,\rangle$ the most general functions possible which satisfy (5.3)?

Our final proposition — which is based on an auxiliary result that we state and prove separately as Lemma 5 below — says that, while there are in fact slightly more general functions satisfying (5.3), they do not allow for an "essential" generalisation of Theorem 3 along this line. See the more detailed discussion of the significance of Proposition 4 in the remark after the statement of the proposition.

Proposition 4. *Let f and g be continuous functions from $\mathbb{R}_{>0}$ to $\mathbb{R}$ satisfying the relation*

$$f(a)g(x) = f(ax) + f(a/x) \tag{6.1}$$

for all positive real numbers a and x. Then there are three possibilities:

(1) $f(x) = 0,\quad$ *for* $x \in \mathbb{R}_{>0}$;
(2) $f(x) = c_1 + c_2 \log x\quad$ *and* $\quad g(x) = 2,\quad$ *for* $x \in \mathbb{R}_{>0}$,
for suitable real numbers c_1 *and* c_2;
(3) $f(x) = c_1 x^{\alpha} + c_2 x^{-\alpha}\quad$ *and* $\quad g(x) = x^{\alpha} + x^{-\alpha},\quad$ *for* $x \in \mathbb{R}_{>0}$,
for a suitable real or (purely) imaginary number $\alpha \neq 0$, *and for suitable real numbers* c_1 *and* c_2.

Remark. We recall that this proposition in combination with the proof of Theorem 3 says that, if we form the spiral matrix which arises from $Z_n(a, b, c, x, x)$ by replacing the bracket $[\,.\,]$ by one of the functions $f(\,.\,)$ given in the proposition, then its determinant will factor in closed product form.

Clearly, the first possibility is without interest, since it produces the zero matrix. As is not difficult to see, the second possibility leads — after reparametrisation — to the earlier spiral matrices $M_n(a, b, c, x, y)$ with $x = y$. By Theorem 1, we know that their determinants factor nicely, but this happens even for the spiral matrices $M_n(a, b, c, x, y)$ with *arbitrary* x and y.

The third possibility does indeed provide functions that are more general than the brackets $[\,.\,]$, which define spiral matrices whose determinants still factor nicely. For the sake of brevity, we omit to work out the explicit statement of the corresponding result, which we leave to the reader. Suffice it to say that it still does not allow for a full q-analogue of Theorem 1, in the sense that a specialisation or limit would be equivalent to Theorem 1, with x and y *different*.

Proof of Proposition 4. Clearly, $f \equiv 0$ solves (6.1), which corresponds to possibility (1). Therefore, from now on, we assume that f is not identically zero.

We start with the observation that under the replacement $x \to 1/x$ nothing changes on the right-hand side of (6.1), and therefore our assumption on f implies

$$g(y) = g(1/y), \quad \text{for } y \in \mathbb{R}_{>0}. \tag{6.2}$$

Next we use (6.1) with $x = y$, with $x = y^2$, and with $x = y^3$ to produce the system of equations

$$f(ya)g(y) = f(y^2a) + f(a), \tag{6.3}$$
$$f(y^2a)g(y) = f(y^3a) + f(ya), \tag{6.4}$$
$$f(y^3a)g(y) = f(y^4a) + f(y^2a), \tag{6.5}$$
$$f(y^4a)g(y) = f(y^5a) + f(y^3a), \tag{6.6}$$
$$f(y^5a)g(y) = f(y^6a) + f(y^4a), \tag{6.7}$$
$$f(y^2a)g(y^2) = f(y^4a) + f(a). \tag{6.8}$$
$$f(y^3a)g(y^3) = f(y^6a) + f(a). \tag{6.9}$$

By adding both sides of (6.3) and (6.5), we obtain

$$g(y)\big(f(ya) + f(y^3a)\big) = f(a) + f(y^4a) + 2f(y^2a).$$

If we now use (6.4) on the left-hand side and (6.8) on the right-hand side, then we arrive at the equation

$$g^2(y)f(y^2a) = \big(g(y^2) + 2\big)f(y^2a).$$

By our assumption on f, this implies the equation

$$g(y^2) = g^2(y) - 2, \quad \text{for } y \in \mathbb{R}_{>0}. \tag{6.10}$$

Similarly, if we add both sides of (6.3) and (6.7), respectively both sides of (6.4) and (6.6), then we obtain

$$g(y)\big(f(ya) + f(y^5a)\big) = f(a) + f(y^6a) + f(y^2a) + f(y^4a),$$
$$g(y)\big(f(y^2a) + f(y^4a)\big) = f(ya) + f(y^5a) + 2f(y^3a).$$

We use the second equation to rewrite the left-hand side of the first equation, and (6.9) to rewrite the right-hand side. The result is

$$g(y)\Big(g(y)\big(f(y^2a) + f(y^4a)\big) - 2f(y^3a)\Big) = g(y^3)f(y^3a) + f(y^2a) + f(y^4a).$$

Finally, we use (6.5) on both sides and arrive at

$$g(y)\big(g^2(y)f(y^3a) - 2f(y^3a)\big) = g(y^3)f(y^3a) + g(y)f(y^3a).$$

By our assumption on f, this implies the equation

$$g(y^3) = g^3(y) - 3g(y), \quad \text{for } y \in \mathbb{R}_{>0}. \tag{6.11}$$

By Lemma 5, we see that (6.2), (6.10), and (6.11) together imply already the claim on the form of the function g, namely that $g(x) = x^\alpha + x^{-\alpha}$ for a suitable real or (purely) imaginary number α. (This includes possibility (2), in which $g(x) \equiv 2$, by choosing $\alpha = 0$.)

It remains to solve the functional equation

$$\left(x^\alpha + x^{-\alpha}\right) f(a) = f(ax) + f(a/x), \quad \text{for } a, x \in \mathbb{R}_{>0}. \tag{6.12}$$

We treat the case where $\alpha \neq 0$ first. It is trivial to check that $f(a) = a^\alpha$ and $f(a) = a^{-\alpha}$ are solutions of (6.12).

Let f be an arbitrary solution of (6.12). We determine c_1 and c_2 as solutions of the system of linear equations

$$\begin{aligned} f(1) &= c_1 + c_2, \\ f(2) &= c_1 2^\alpha + c_2 2^{-\alpha}. \end{aligned}$$

It should be noted that this system can indeed always be (uniquely) solved since $\alpha \neq 0$. Using (6.12) with $x = 2$, it may then inductively be inferred that

$$f(2^n) = c_1 (2^n)^\alpha + c_2 (2^n)^{-\alpha}, \quad \text{for } n \in \mathbb{Z}. \tag{6.13}$$

Furthermore, given y and z with

$$f(y) = c_1 y^\alpha + c_2 y^{-\alpha} \quad \text{and} \quad f(z) = c_1 z^\alpha + c_2 z^{-\alpha},$$

Equation (6.12) with $a = \sqrt{yz}$ and $x = \sqrt{y/z}$ implies that

$$f(\sqrt{yz}) = c_1 \left(\sqrt{yz}\right)^\alpha + c_2 \left(\sqrt{yz}\right)^{-\alpha}.$$

Iteration of this argument starting from the values in (6.13) yields a dense set of values x for which $f(x) = c_1 x^\alpha + c_2 x^{-\alpha}$. By continuity of f, the same equation must hold for *all* $x \in \mathbb{R}_{>0}$. We thus arrived at possibility (3).

If $\alpha = 0$, then (6.12) simplifies to

$$2f(a) = f(ax) + f(a/x), \quad \text{for } a, x \in \mathbb{R}_{>0}.$$

Here, it is straightforward to see that $f(a) = 1$ and $f(a) = \log a$ are solutions of this equation. By an argument that is completely analogous to the one before that led to possibility (3), we conclude that, here, we arrive at possibility (2).

This completes the proof of the proposition. □

Lemma 5. *Let $g : \mathbb{R}^{>0} \to \mathbb{R}$ be a continuous function satisfying the relations*

$$g(x) = g(1/x), \quad \text{for } x \in \mathbb{R}_{>0}, \tag{6.14}$$

$$g(x^2) = g^2(x) - 2, \quad \text{for } x \in \mathbb{R}_{>0}, \tag{6.15}$$

$$g(x^3) = g^3(x) - 3g(x), \quad \text{for } x \in \mathbb{R}_{>0}. \tag{6.16}$$

Then

$$g(x) = x^\alpha + x^{-\alpha} = 2\cos\big(i\alpha \log x\big), \quad \text{for } x \in \mathbb{R}_{>0},$$

for a suitable real or (purely) imaginary number α.

Remark. A corollary is that g satisfies

$$g(a)g(x) = g(ax) + g(a/x), \quad \text{for } a, x \in \mathbb{R}_{>0}. \tag{6.17}$$

In fact, one can show that there holds a(n almost) reverse statement: if g satisfies (6.17), then either g vanishes identically or g satisfies (6.14)–(6.16).

Proof of Lemma 5. From (6.15) and (6.16) we see that $g(1) = 2$. Because of (6.14), we may restrict our attention to the values $g(x)$ with $x \geq 1$, which we shall do from now on.

Our strategy is to find an α and a set $\{x_{n,m} : n, m = 0, 1, 2, \ldots\}$ that is dense in $\mathbb{R}_{>0}$ such that

$$g(x_{n,m}) = x_{n,m}^\alpha + x_{n,m}^{-\alpha}, \quad \text{for } n, m = 0, 1, 2, \ldots. \tag{6.18}$$

We distinguish between two cases:

(1) there exists $x_0 > 1$ such that $g(x_0) \geq 2$;
(2) there exists $x_0 > 1$ such that $g(x_0) < 2$.

It may seem that the two cases may overlap. We shall see however that they are actually distinct.

Case (1). We fix $x_0 > 1$ with $g(x_0) \geq 2$, and we choose a real α such that

$$g(x_0) = x_0^\alpha + x_0^{-\alpha}. \tag{6.19}$$

This is indeed possible since $g(x_0) \geq 2$.

Starting from (6.19), the two relations (6.15) and (6.16) allow us to compute $g(x)$ for all x of the form $x_{n,m} := x_0^{2^n 3^{-m}}$, $n, m \in \mathbb{N}_{\geq 0}$. More precisely, assuming that we already know $g(x_{n,m})$ and that $g(x_{n,m}) \geq 2$ (which is certainly the case for $n = m = 0$), by (6.15) and (6.16) we get

$$\begin{aligned} g(x_{n+1,m}) &= g^2(x_{n,m}) - 2, \\ g(x_{n,m}) &= g^3(x_{n,m+1}) - 3g(x_{n,m+1}). \end{aligned}$$

In particular, we see that $g(x_{n+1,m}) \geq 2$. Moreover, since $g(x_{n,m}) \geq 2$, the second equation determines the value $g(x_{n,m+1})$ *uniquely*, and it satisfies $g(x_{n,m+1}) \geq 2$.

A simple inductive argument — the start of the induction being given by (6.19) — then shows that, in fact, the precise value of $g(x_{n,m})$ is given by (6.18).

We claim that the values $x_{n,m} = x_0^{2^n 3^{-m}}$ are dense in the positive real numbers. Clearly, it suffices to show that the rational numbers $2^n 3^{-m}$ are dense in the positive real numbers. To see this, one takes the logarithm,

$$\log_2\left(2^n 3^{-m}\right) = n - m\log_2(3).$$

It is a well-known fact that the sequence $(n\omega)_{n\ge 0}$ is uniformly distributed modulo 1 if ω is irrational (cf. [3, Ex. 2.1]). Since $\log_2(3)$ is irrational, this implies our claim.

The function g being continuous by assumption, this proves the claim for Case (1).

Case (2). We fix $x_0 > 1$ such that $g(x_0) < 2$. We claim that $|g(x)| \le 2$ for all $x > 1$. Namely, it is certainly not possible to find an $x_1 > 1$ with $g(x_1) \ge 2$ because we would then be in Case (1), where we found that $g(x) \ge 2$ for *all* $x \in \mathbb{R}_{>0}$. Moreover, it is also not possible to find an $x_1 > 1$ with $g(x_1) \le -2$ because we would then obtain $g(x_1^2) \ge 2$ by (6.15), which would again put us in Case (1).

Now, using (6.15) iteratively, we will eventually find a positive integer k such that $g(x_0^{2^k}) < 0$. We recall that $g(1) = 2$. By the continuity of g it then follows that there is a $y_0 > 1$ such that $g(y_0) = 0$. Without loss of generality, let y_0 be minimal with this property. In particular, with this choice of y_0 we have $0 < g(x) \le 2$ for all $x \in [1, y_0)$.

We now choose an $x_1 \in (1, y_0)$. As we just said, we have $0 < g(x_1) \le 2$. We may thus find an imaginary α such that

$$g(x_1) = x_1^{\alpha} + x_1^{-\alpha} = 2\cos\left(i\alpha\log x_1\right). \tag{6.20}$$

Similarly to above, starting from this, the two relations (6.15) and (6.16) allow us to compute $g(x)$ for all $x \in [1, y_0]$ of the form $x_{n,m} = x_1^{2^{-n}3^m}$ with $n, m \in \mathbb{N}_{\ge 0}$. More precisely, assuming that we already know $g(x_{n,m})$, by (6.15) and (6.16) we get

$$\begin{aligned} g(x_{n,m}) &= g^2(x_{n+1,m}) - 2,\\ g(x_{n,m+1}) &= g^3(x_{n,m}) - 3g(x_{n,m+1}), \end{aligned} \tag{6.21}$$

as long as $x_{n+1,m}$ respectively $x_{n,m+1}$ is still in the interval $[1, y_0)$. The important point here is that the first equation determines the value $g(x_{n,m+1})$ *uniquely* as the *positive* root of the equation since we know that $g(x) > 0$ for all $x \in [1, y_0)$.

Again, a simple inductive argument — here starting from (6.20) — then shows that, in fact, the precise value of $g(x_{n,m})$ is given by (6.18). Finally, this may be extended to *all* $x_{n,m}$ (that is, without the restriction $x_{n,m} \in [1, y_0)$) by applying (6.21) iteratively. Since also the numbers $x_{n,m} = x_1^{2^{-n}3^m}$ are dense in the positive real numbers, this establishes the claim for Case (2), and thus completes the proof of the lemma. □

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