EFFECTIVE RESISTANCES AND KIRCHHOFF INDEX OF PRISM GRAPHS

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Abstract. We explicitly compute the effective resistances between any two vertices of a prism graph by using circuit reductions and our earlier findings on a ladder graph. As an application, we derived a closed form formula for the Kirchhoff index of a prism graph. We show as a byproduct that an explicit sum formula involving trigonometric functions hold by comparing our formula for the Kirchhoff index and previously known results in the literature. We also expressed our formulas in terms of certain generalized Fibonacci numbers.

1. Introduction

A prism graph Y_n is a planar graph that looks like a circular ladder with n rungs. Figure 1 illustrates Y_6. Y_n has $2n$ vertices and $3n$ edges. Each of its edges has length 1, so the total length of Y_n is $\ell(Y_n) := 3n$.

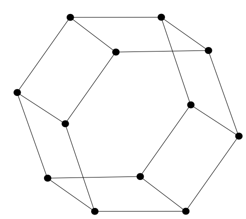

Figure 1. Prism graph Y_6.

We give explicit formulas for the effective resistances between any two vertices of Y_n. We consider Y_n as an electrical network in which we set the resistances along edges as the corresponding edge lengths. If we connect two ladder graphs possibly of different vertex numbers by adding four edges to their end vertices, we obtain a prism graph. We apply circuit reductions to each of those ladder graphs by keeping their four end vertices. This gives us a circuit reduction of the prism graph Y_n. The reduced Y_n will have 8 vertices. Thanks to knowing the resistance values on a ladder graph [3], we can determine the resistance values between the vertices of the reduced Y_n by utilizing the discrete Laplacian and its pseudo inverse of this reduced graph.

Let us define the sequence G_n by the following recurrence relation

$$G_{n+2} = 4G_{n+1} - G_n, \quad \text{if } n \geq 2, \text{ and } G_0 = 0,\ G_1 = 1.$$

We showed that the following equalities hold for Kirchhoff index of Y_n (see Theorem 3.1 and Equation (17) below), where n is a positive integer:

$$Kf(Y_n) = \frac{n(n^2-1)}{6} + \frac{2n^2 G_n^2}{G_{2n} - 2G_n} = \frac{n(n^2-1)}{6} + \frac{n^2}{\sqrt{3}}\Big[\frac{2}{1-(2-\sqrt{3})^n} - 1\Big].$$

Similarly, for any positive integer n, we showed that the following identities of trigonometric sum hold (see Theorem 3.2 and Equation (17) below):

$$\sum_{k=0}^{n-1} \frac{1}{1+2\sin^2(\frac{k\pi}{n})} = \frac{2nG_n^2}{G_{2n}-2G_n} = \frac{n}{\sqrt{3}}\Big[\frac{2}{1-(2-\sqrt{3})^n} - 1\Big].$$

The resistance values on Wheel and Fan graphs (in [1]) and Ladder graphs (in [3] and [4]) are expressed in terms of generalized Fibonacci numbers. Our findings for resistance values on a Prism graph are analogues of those results.

Key words and phrases. Prism graph, effective resistance, Kirchhoff index, circuit reduction.

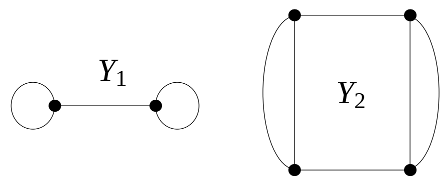

FIGURE 2.

2. Resistances between any pairs of vertices in Y_n

A ladder graph L_n is a planar graph that looks like a ladder with n rungs. It has $2n$ vertices and $3n-2$ edges. Each of its edges has length 1, so the total length of L_n is $\ell(L_n) := 3n-2$. We obtain the prism graph Y_n from L_n by adding two edges connecting the end vertices on the same side.

If we delete an edge that is a rung in Y_n, and then contract two edges that are on each side of that rung, we obtain the prism graph Y_{n-1}. If we apply this process to 3-prism graph Y_3, we obtain the graph on the right in Figure 2. We call it 2-prism graph Y_2. Similarly, if we apply this process to Y_2, we obtain the graph on the left in Figure 2. We call it 1-prism graph Y_1. These graphs are the natural extension of prism graphs to the cases $n = 1, 2$. We see that our formulas for resistance values, Kirchhoff index as well as the spanning tree formulas are also valid for these two cases (see Theorem 3.1 and Equations (10), (16) and (17) below).

For a prism graph Y_n, we label the vertices on the right and left as $\{q_1, q_2, \cdots, q_n\}$ and $\{p_1, p_2, \cdots, p_n\}$, respectively. This is illustrated in Figure 3, where $2 \leq i \leq n$. We want to find the value of $r(p, q)$ for any two vertices p and q of Y_n, where $r(x, y)$ is the resistance function on Y_n. We implement the following strategy to do this:

- Consider Y_n as the union of the ladder graphs L_{n-i+1} and L_{i-1} as illustrated in Figure 4.
- Apply circuit reductions on each of these ladder graphs by keeping the four end vertices as illustrated in Figure 5.
- Use our earlier results on a ladder graph in [3] to find the resistances between the end points of the reduced ladder graphs. Note that certain resistance values are equal to each other because of the symmetries in the ladder graphs.
- So far we obtain the circuit reduction of Y_n by keeping its 8 vertices p_n, q_n, p_1, q_1, p_i, q_i, p_{i-1} and q_{i-1}. This is illustrated in Figure 6. Find the Moore-Penrose inverse L^+ of the discrete Laplacian matrix L of this reduced Y_n. They are 8×8 matrices.
- Use L^+ and Lemma 2.1 to find out the resistances between the 8 vertices of Y_n. Again note that there are symmetries in Y_n and that i is an arbitrary value in $\{2, 3, \ldots, n\}$.

Symmetries in Y_n gives the following identities of resistances:

$$r(p_1, p_i) = r(q_1, q_i) \quad \text{and } r(p_1, q_i) = r(q_1, p_i), \quad \text{for each } i \in \{1, \ldots, n\}. \tag{1}$$

We recall that the resistance values can be expressed in terms of the entries of the pseudo inverse of the discrete Laplacian matrix.

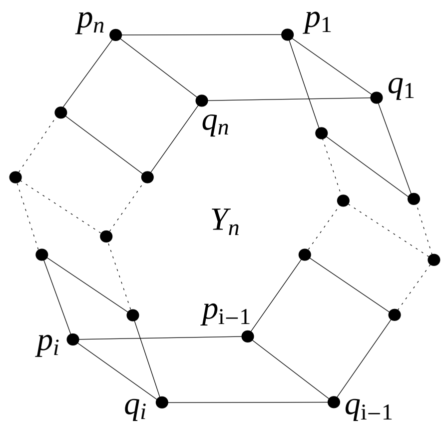

FIGURE 3.

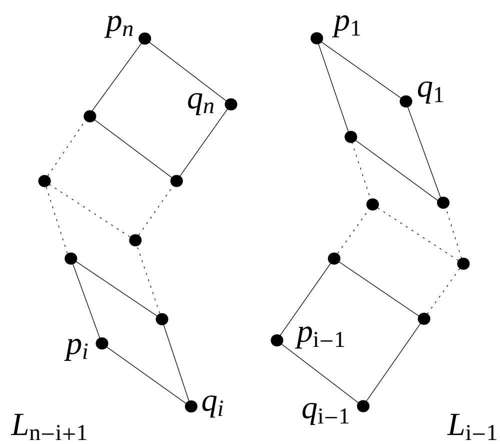

FIGURE 4.

Lemma 2.1. [9], [5, Theorem A] *Suppose G is a graph with the discrete Laplacian* L *and the resistance function $r(x, y)$. For the pseudo inverse* L^+ *of* L, *we have*

$$r(p, q) = l^+_{pp} - 2l^+_{pq} + l^+_{qq}, \quad \textit{for any two vertices } p \textit{ and } q \textit{ of } G.$$

Let $K = 1 + \frac{1}{k} + \frac{1}{m} + \frac{1}{s}$ and $S = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, where k, m, s, a, b and c are the resistance values along the edges given in Figure 6. Considering the ordering of the vertices $V = \{p_1, p_{i-1}, p_i, p_n, q_1, q_{i-1}, q_i, q_n\}$, discrete Laplacian matrix L of the graph (reduced Y_n) given in Figure 6 is as follows:

$$L = \begin{bmatrix} K & -\frac{1}{m} & 0 & -1 & -\frac{1}{s} & -\frac{1}{k} & 0 & 0 \\ -\frac{1}{m} & K & -1 & 0 & -\frac{1}{k} & -\frac{1}{s} & 0 & 0 \\ 0 & -1 & S & -\frac{1}{a} & 0 & 0 & -\frac{1}{b} & -\frac{1}{c} \\ -1 & 0 & -\frac{1}{a} & S & 0 & 0 & -\frac{1}{c} & -\frac{1}{b} \\ -\frac{1}{s} & -\frac{1}{k} & 0 & 0 & K & -\frac{1}{m} & 0 & -1 \\ -\frac{1}{k} & -\frac{1}{s} & 0 & 0 & -\frac{1}{m} & K & -1 & 0 \\ 0 & 0 & -\frac{1}{b} & -\frac{1}{c} & 0 & -1 & S & -\frac{1}{a} \\ 0 & 0 & -\frac{1}{c} & -\frac{1}{b} & -1 & 0 & -\frac{1}{a} & S \end{bmatrix}.$$

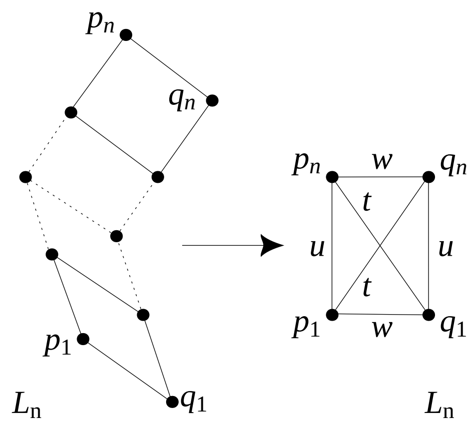

FIGURE 5.

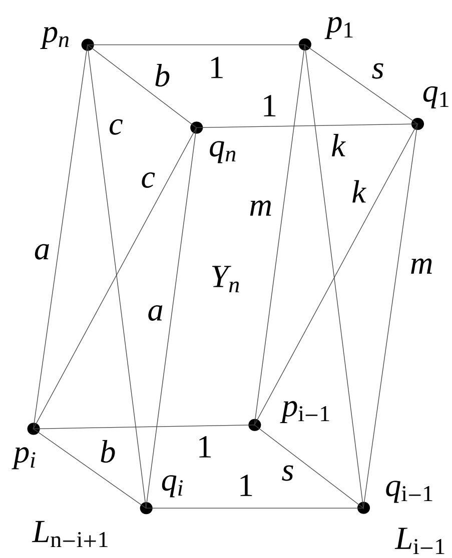

FIGURE 6.

Then we can compute the Moore-Penrose inverse L^+ of L by using [7] with the following formula (see [12, ch 10]):

$$\mathrm{L}^+ = \big(\mathrm{L} - \frac{1}{8}\mathrm{J}\big)^{-1} + \frac{1}{8}\mathrm{J}. \tag{2}$$

where J is of size 8×8 and has all entries 1. Next, we use L^+ and Lemma 2.1 to obtain $r(p_1, p_{i-1})$ and $r(p_1, q_{i-1})$. In this way, we find that

$r(p_1, p_{i-1}) = \frac{P}{Q}$, where $P = m(4bc(ms + k(m + 2s + ms)) + a^2((2 + b)(2 + c)ms + k(2(2 + b)(2 + c)s + m(4 + 2c + 4s + cs + b(2 + c + s)))) + 2a(2c(ms + k(m + 2s + ms)) + b(2(1 + c)ms + k(4(1 + c)s + m(2 + 2c + 2s + cs)))))$ and $Q = 2(c(2m + k(2 + m)) + a((2 + c)m + k(2 + c + m)))(b(2s + m(2 + s)) + a((2 + b)s + m(2 + b + s)))$.

$r(p_1, q_{i-1}) = \frac{R}{S}$, where $R = k(c(4c(2ms + k(m + s + ms)) + b(4(2 + c)ms + k(2(2 + c)s + m(4 + 2c + 4s + cs)))) + a(b(2 + c)(2(2 + c)ms + k((2 + c)s + m(2 + c + 2s))) + c(4(2 + c)ms + k(2(2 + c)s + m(4 + 2c + 4s + cs)))))$ and $S = 2(c(2m + k(2 + m)) + a((2 + c)m + k(2 + c + m)))(c(2s + k(2 + s)) + b((2 + c)s + k(2 + c + s)))$.

We note that these resistance values can be expressed in a more compact form as below:

$$
\begin{aligned}
r(p_1,p_{i-1}) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2+\frac{ac}{a+c}}+\frac{1}{\frac{mk}{m+k}}}+\frac{1}{\frac{1}{2+\frac{ab}{a+b}}+\frac{1}{\frac{ms}{m+s}}}\right),\\
r(p_1,q_{i-1}) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2+\frac{ac}{a+c}}+\frac{1}{\frac{mk}{m+k}}}+\frac{1}{\frac{1}{2+\frac{bc}{b+c}}+\frac{1}{\frac{ks}{k+s}}}\right).
\end{aligned}
\tag{3}
$$

To find the exact values of the resistances in Equation (3), we need the corresponding values from the circuit reduction of L_{n-i+1} and L_{i-1}. Thus, we turn our attention to the circuit reduction of L_n as in Figure 5. Let us apply circuit reductions to a ladder graph L_n by keeping its vertices at its bottom and top so that we obtain a complete graph on 4 vertices. Suppose p_1, q_1 and p_n, q_n are the vertices at the bottom and top of the ladder graph. This is illustrated in Figure 5. Note that by the symmetry in L_n, we have only three distinct edge lengths in the complete graph obtained. Let us consider the ordering of the vertices $\{p_n,\, q_n,\, p_1,\, q_1\}$. Using the notations in Figure 5, the Laplacian matrix M of the reduced graph can be given as follows:

$$
M=\begin{bmatrix}
\frac{1}{t}+\frac{1}{u}+\frac{1}{w} & -\frac{1}{w} & -\frac{1}{u} & -\frac{1}{t}\\
-\frac{1}{w} & \frac{1}{t}+\frac{1}{u}+\frac{1}{w} & -\frac{1}{t} & -\frac{1}{u}\\
-\frac{1}{u} & -\frac{1}{t} & \frac{1}{t}+\frac{1}{u}+\frac{1}{w} & -\frac{1}{w}\\
-\frac{1}{t} & -\frac{1}{u} & -\frac{1}{w} & \frac{1}{t}+\frac{1}{u}+\frac{1}{w}
\end{bmatrix}.
$$

Then we use symmetries in L_n, compute the Moore-Penrose inverse M^+ of M and apply Lemma 2.1 to write

$$
\begin{aligned}
r_{L_n}(p_n,q_n) = r_{L_n}(p_1,q_1) &= \frac{w(wt+u(w+2t))}{2(u+w)(w+t)} = \frac{1}{2}\left(\frac{wt}{w+t}+\frac{uw}{u+w}\right),\\
r_{L_n}(p_n,p_1) = r_{L_n}(q_n,q_1) &= \frac{u(2wt+u(w+t))}{2(u+w)(u+t)} = \frac{1}{2}\left(\frac{uw}{u+w}+\frac{ut}{u+t}\right),\\
r_{L_n}(p_n,q_1) = r_{L_n}(q_n,p_1) &= \frac{t(2uw+(u+w)t)}{2(u+t)(w+t)} = \frac{1}{2}\left(\frac{ut}{u+t}+\frac{wt}{w+t}\right).
\end{aligned}
\tag{4}
$$

Using Equation (4), we derive

$$
\begin{aligned}
A_n &:= \frac{wt}{w+t} = r_{L_n}(p_n,q_n)+r_{L_n}(p_n,q_1)-r_{L_n}(p_n,p_1),\\
B_n &:= \frac{uw}{u+w} = r_{L_n}(p_n,q_n)-r_{L_n}(p_n,q_1)+r_{L_n}(p_n,p_1),\\
C_n &:= \frac{ut}{u+t} = -r_{L_n}(p_n,q_n)+r_{L_n}(p_n,q_1)+r_{L_n}(p_n,p_1).
\end{aligned}
\tag{5}
$$

As particular cases of Equation (5), we have

$$
\frac{ac}{a+c}=C_{n-i+1}, \qquad \frac{ab}{a+b}=B_{n-i+1}, \qquad \frac{bc}{b+c}=A_{n-i+1}, \tag{6}
$$

$$
\frac{mk}{m+k}=C_{i-1}, \qquad \frac{ms}{m+s}=B_{i-1}, \qquad \frac{ks}{k+s}=A_{i-1}, \tag{7}
$$

where we used the notations as in Equation (3) (and so as in Figure 6). In [3], we gave explicit formulas for the resistance values between any two vertices of a ladder graph.

Next, we use [3, Equation 12 at page 959] to write

$$
\begin{aligned}
A_n &= -1-\sqrt{3}+\frac{2\sqrt{3}}{1-(2-\sqrt{3})^n},\\
B_n &= -1-\sqrt{3}+\frac{2\sqrt{3}}{1+(2-\sqrt{3})^n},\\
C_n &= n-1.
\end{aligned} \tag{8}
$$

We use Equations (3), (6) and (7) to derive

$$
\begin{aligned}
r(p_1,p_i) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2+C_{n-i}}+\frac{1}{C_i}}+\frac{1}{\frac{1}{2+B_{n-i}}+\frac{1}{B_i}}\right),\\
r(p_1,q_i) &= \frac{1}{2}\left(\frac{1}{\frac{1}{2+C_{n-i}}+\frac{1}{C_i}}+\frac{1}{\frac{1}{2+A_{n-i}}+\frac{1}{A_i}}\right).
\end{aligned} \tag{9}
$$

where $2 \leq i \leq n$. Next, we use Equation (8) in Equation (10) and then work with [7] to simplify the algebraic expressions as below:

$$
\begin{aligned}
r(p_1,p_i) &= \frac{1+\left(2-\sqrt{3}\right)^n-\left(2-\sqrt{3}\right)^{n-i+1}-\left(2-\sqrt{3}\right)^{i-1}}{2\sqrt{3}\left(1-\left(2-\sqrt{3}\right)^n\right)}+\frac{(n-i+1)(i-1)}{2n},\\
r(p_1,q_i) &= \frac{1+\left(2-\sqrt{3}\right)^n+\left(2-\sqrt{3}\right)^{n-i+1}+\left(2-\sqrt{3}\right)^{i-1}}{2\sqrt{3}\left(1-\left(2-\sqrt{3}\right)^n\right)}+\frac{(n-i+1)(i-1)}{2n}.
\end{aligned} \tag{10}
$$

where $1 \leq i \leq n$.

By using the symmetries of the graph Y_n, we note that for every i and j in $\{1, 2, \dots, n\}$ we have

$$
r(p_1,p_i) = r(p_j,p_k) \quad \text{and} \quad r(p_1,q_i) = r(p_j,q_k), \tag{11}
$$

where $1 \leq k \leq n$ and $k \equiv j+i-1 \mod n$.

Finally, the explicit values of $r(p,q)$ between any two vertices p and q of the prism graph Y_n can be obtained by using Equations (1), (10) and (11).

It follows from Equation (10) that

$$
r(p_1,p_i)+r(p_1,q_i) = \frac{1}{\sqrt{3}}\left(\frac{1+\left(2-\sqrt{3}\right)^n}{1-\left(2-\sqrt{3}\right)^n}\right)+\frac{(n-i+1)(i-1)}{n}. \tag{12}
$$

3. Kirchhoff Index of Y_n

In this section, we obtain an explicit formula for Kirchhoff index of Y_n by using our explicit formulas derived in §2 for the resistances between any pairs of vertices of Y_n. Moreover, we obtain an interesting summation formula by combining our findings and what is known in the literature about Kirchhoff index of Y_n.

Theorem 3.1. *For any positive integer n, we have*

$$
Kf(Y_n) = \frac{n(n^2-1)}{6}+\frac{n^2}{\sqrt{3}}\Big[\frac{2}{1-(2-\sqrt{3})^n}-1\Big].
$$

Proof. With the notation of vertices as in Figure 3 we have

$$\begin{aligned} Kf(Y_n) &= \frac{1}{2}\sum_{p,\, q\in V(Y_n)} r(p,q), \text{ by definition [13].} \\ &= n\sum_{i=1}^{n} r(p_1,p_i)+r(p_1,q_i), \text{ Equations (1) and (11).} \end{aligned}$$

Then the result follows if we use first Equation (12) and do some algebra [7]. □

Alternatively, we can express the Kirchhoff index formula in Theorem 3.1 as follows:

$$Kf(Y_n) = \frac{n(n^2-1)}{6} - \frac{n^2}{\sqrt{3}}\coth\big(\frac{n}{2}\ln(2-\sqrt{3})\big).$$

$Kf(Y_n)$ have rational values. For example, its values for $1 \leq n \leq 10$ are as follows:
1, 11/3, 47/5, 58/3, 655/19, 279/5, 5985/71, 2540/21, 44193/265, 139655/627.

Next, we show that an interesting trigonometric sum identity hold:

Theorem 3.2. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1}\frac{1}{1+2\sin^2(\frac{k\pi}{n})} = \frac{n}{\sqrt{3}}\Big[\frac{2}{1-(2-\sqrt{3})^n}-1\Big].$$

Proof. Prism graph Y_n can be seen as the cartesian product $P_2\Box C_n$, where P_2 is the path graph with 2 vertices and C_n is the cycle graph with n vertices. Moreover, considering the Laplacian eigenvalues of P_2 and C_n we see that the Laplacian eigenvalues of Y_n (see [8], [10] and [11]) are

$$\lambda_{ij} = 4-2\cos\big(\frac{i\pi}{2}\big)-2\cos\big(\frac{2j\pi}{n}\big), \qquad \text{where } i=0,1 \text{ and } j=0,1,\dots,n-1. \tag{13}$$

We recall that [14, pg 644]

$$\sum_{k=1}^{n-1}\frac{1}{\sin^2\big(\frac{k\pi}{n}\big)} = \frac{n^2-1}{3}. \tag{14}$$

Now, we can express the Kirchhoff index via the eigenvalues of the discrete Laplacian matrix of Y_n [13]:

(15)

$$\begin{aligned} Kf(Y_n) &= 2n\sum_{\lambda_{ij}\neq 0}\frac{1}{\lambda_{ij}} \\ &= n\sum_{k=1}^{n-1}\frac{1}{1-\cos\big(\frac{2k\pi}{n}\big)} + n\sum_{k=0}^{n-1}\frac{1}{2-\cos\big(\frac{2k\pi}{n}\big)}, \quad \text{by Equation (13),} \\ &= n\sum_{k=1}^{n-1}\frac{1}{2\sin^2\big(\frac{k\pi}{n}\big)} + n\sum_{k=0}^{n-1}\frac{1}{1+2\sin^2\big(\frac{k\pi}{n}\big)}, \quad \text{using } 1-\cos\big(\frac{2k\pi}{n}\big) = 2\sin^2\big(\frac{k\pi}{n}\big), \\ &= \frac{n(n^2-1)}{6} + n\sum_{k=0}^{n-1}\frac{1}{1+2\sin^2(\frac{k\pi}{n})}, \quad \text{by Equation (14).} \end{aligned}$$

Then the proof is completed by combining Equation (15) and the result in Theorem 3.1. □

4. Recursive Formulations

In this section, we give recursive formulas for the resistance values obtained in §2, the Kirchhoff index of Y_n and the trigonometric formula given in Theorem 3.2. As we did in [3] for Ladder graph, we use the sequence of integers G_n defined by the following recurrence relation

$$G_{n+2} = 4G_{n+1} - G_n, \quad \text{if } n \geq 2, \text{ and } G_0 = 0,\, G_1 = 1.$$

We have

$$G_n = \frac{(2-\sqrt{3})^{-n} - (2-\sqrt{3})^n}{2\sqrt{3}}, \quad \text{for each integer } n \geq 0.$$

The sequence G_n has various well-known properties [15]. For example, it gives the number of spanning trees of L_n [2], and the number of the spanning trees of the prism graph Y_n [6] is given by

$$\frac{n}{2}\Big((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2\Big) = \frac{n}{2}\Big(\frac{G_{2n}}{G_n} - 2\Big). \tag{16}$$

Let $g_n = (2-\sqrt{3})^n = \frac{1}{G_{n+1} - (2-\sqrt{3})G_n}$ for any nonnegative integer n. Then, for any integer $i \in \{1, 2, \ldots, n\}$ we have

$$r(p_1, p_i) = \frac{(n-i+1)(i-1)}{2n} + \frac{G_n^2}{G_{2n} - 2G_n} - \Big(\frac{1}{4\sqrt{3}} + \frac{G_n^2}{2G_{2n} - 4G_n}\Big)(g_{n-i+1} + g_{i-1}),$$

$$r(p_1, q_i) = \frac{(n-i+1)(i-1)}{2n} + \frac{G_n^2}{G_{2n} - 2G_n} + \Big(\frac{1}{4\sqrt{3}} + \frac{G_n^2}{2G_{2n} - 4G_n}\Big)(g_{n-i+1} + g_{i-1}).$$

Similarly, the other resistance values can be expressed in terms of G_n by using the symmetries in Y_n like Equation (1) and Equation (11).

Here are how we can express the results given in Theorem 3.1 and Theorem 3.2 in terms of G_n:

$$Kf(Y_n) = \frac{n(n^2-1)}{6} + \frac{2n^2 G_n^2}{G_{2n} - 2G_n} \qquad \text{and} \qquad \sum_{k=0}^{n-1} \frac{1}{1 + 2\sin^2(\frac{k\pi}{n})} = \frac{2nG_n^2}{G_{2n} - 2G_n}. \tag{17}$$

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