Passing through a stack k times

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We consider the number of passes a permutation needs to take through a stack if we only pop the appropriate output values and start over with the remaining entries in their original order. We define a permutation π to be k-pass sortable if π is sortable using k passes through the stack. Permutations that are 1-pass sortable are simply the stack sortable permutations as defined by Knuth. We define the permutation class of 2-pass sortable permutations in terms of their basis. We also show all k-pass sortable classes have finite bases by giving bounds on the length of a basis element of the permutation class for any positive integer k. Finally, we define the notion of tier of a permutation π to be the minimum number of passes *after* the first pass required to sort π. We then give a bijection between the class of permutations of tier t and a collection of integer sequences studied by Parker [16]. This gives an exact enumeration of tier t permutations of a given length and thus an exact enumeration for the class of $(t+1)$-pass sortable permutations. Finally, we give a new derivation for the generating function in [16] and an explicit formula for the coefficients.

1. Introduction

We begin with the notion of permutation (or pattern) containment.

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Definition 1.1. *A permutation $\pi = \pi_1\pi_2\ldots\pi_n \in S_n$ is said to* contain *a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ if there exist indices $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_k \le n$ such that $\pi_{\alpha_i} < \pi_{\alpha_j}$ if and only if $\sigma_i < \sigma_j$. Otherwise, we say π* avoids *σ.*

Example 1.2. The permutation $\pi = 4127356$ contains 231 since the $4, 7, 3$ appear in the same relative order as $2, 3, 1$. However, π avoids 321 since there is no decreasing subsequence of length three in π.

A stack is a last-in first-out sorting device that utilizes push and pop operations. In Volume 1 of *The Art of Computer Programming* [11], Knuth showed that the permutation π can be sorted (meaning that by applying push and pop operations to the sequence of entries $\pi_1, \ldots, \pi_n$ one can output the sequence $1, \ldots, n$) if and only if π avoids the permutation 231. Shortly thereafter Tarjan [22], Even and Itai [9], Pratt [17], and Knuth himself in *Volume 3* [12] studied sorting machines made up of multiple stacks in series or in parallel.

Classifying the permutations that are sortable by such a machine is one of the key areas of interest in this field. To better do so, we will use the following definitions.

Definition 1.3. *A* permutation class *is a downset of permutations under the containment order. Every permutation class can be specified by the set of minimal permutations which are* not *in the class called its* basis*. For a set B of permutations, we denote by $\mathrm{Av}(B)$ the class of permutations which do not contain any element of B.*

For example, Knuth's result says that the stack-sortable permutations are precisely $\mathrm{Av}(231)$, that is the basis for the stack sortable permutations is $\{231\}$. Given any naturally defined sorting machine, the set of sortable permutations forms a class. This is often because a subpermutation of a sortable permutation can be sorted by ignoring the operations corresponding to absent entries. [1]

Given that the class of permutations sortable by a single stack have a basis of only one element, namely 231, expecting that the sortable permutations for a network made up of more than one stack would also have a finite basis seems reasonable. However, this is not the case for machines made up $k \ge 2$ stacks in series or in parallel. That these machines must have infinite bases was shown by Murphy [15] and Tarjan [22], respectively. Moreover, the exact enumeration question is unknown; see Albert, Atkinson, and Linton [1] for the best known bounds. For a general overview of stack sorting, we refer the reader to Bóna [5].

In part because of the difficulties noted above, numerous researchers have considered weaker machines. Atkinson, Murphy, and Ruškuc [2] considered sorting with two *increasing* stacks in series, i.e., two stacks whose entries must be in increasing order when read from top to bottom[2]. They characterized the permutations this machine can sort with an infinite list of forbidden patterns, and also found the enumeration of these permutations. Interestingly, these permutations are in bijection with the 1342-avoiding permutations previously counted by Bóna [4]. Similarly, the third author [20] studied a machine where the first stack must have entries in decreasing order when read from top to bottom. This permutation class of sortable permutations was shown to be $\mathrm{Av}(3241, 3142)$, known to be enumerated by the Schröder numbers by Kremer [13, 14] and later an explicit bijection was given by Schroeder and the third author [18]. A different version, sorting with a stack of depth 2 followed by a standard stack (of infinite depth), was studied by Elder [8]. He characterized the sortable permutations with a finite list of forbidden patterns, but did not enumerate these permutations.

[1] An exception is West's notion of 2-stack-sortability [23], which is due to restrictions on how the machine can use its two stacks. Namely this machine prioritizes keeping large entries from being placed above small entries. Because of this limitation, this machine can sort 35241, but not its subpermutation 3241.

[2] Even without this restriction, the final stack must be increasing if the sorting is to be successful.

We apply a sorting algorithm on a stack whereby the entries the permutation are pushed into the stack in the usual way. We remove or pop entries from the stack only if they are the next needed entry for the output (the next entry of the identity permutation). That is, we will allow larger entries to be placed above smaller entries, but we will not allow entries to be pushed to the output prematurely. In particular, this means that if a permutation contains the pattern 231, then there will be entries left in the stack after all legal moves have been made. In this case, the algorithm is repeated on the stack working from the bottom of the stack to the top. That is, the remaining entries are returned to the input to be read in the same order they were the first time.

In some respects this is similar to West's notion of 2-stack-sortability [23] where entries were run through a single stack twice. However, unlike West's algorithm that prioritizes keeping the stack in increasing order, we will prioritize only outputting the proper entries. The main advantage is that our sortable permutations will form a permutation class, as removing entries from a permutation will not impede its ability to be sorted by this algorithm.

Definition 1.4. *We will call each repetition of the algorithm used to sort a permutation a pass. Further, the tier of a permutation σ will refer to the minimum number of times we need to restart the sorting; that is, the tier is one less than the minimal number of passes necessary to sort σ. We use $t(\sigma)$ to denote the tier of the permutation σ.*

Example 1.5. The permutation 231 has tier $t(231) = 1$, all other elements of S_3 have tier 0.

We translate the original stack sortable requirement to the following theorem.

Theorem 1.6. *(Knuth) A permutation π has tier 0, that is π can be sorted via single pass through the stack, if and only if π avoids the pattern 231.*

We note that another way to think of our sorting machine when applied to a permutation with tier t is as a network of $t + 1$ input-restricted deques in series with a special output condition. Namely, entries of our permutation may only enter the top of the deque and then either exit the top of the deque to go to the output (immediately passing through the other deques if one is so inclined), or exit the bottom of the deque (and enter the top of the next deque) when no more entries are available to enter the deque and the top entry of the deque is not the next entry to be output.

2. SEPARABLE PAIRS AND PERMUTATION CLASSES

In order to investigate the tiers of permutations more generally, we first give an explicit condition on permutations that describes their tier.

Definition 2.1. *Let $\sigma \in S_m$ and let $i \in \{1, 2, ..., n - 1\}$. We say that the integers $(i + i, i)$ are a separated pair in a permutation $\sigma \in S_n$ if there is a subsequence in σ of the form $(i + 1, k, i)$ where $k > i + 1$.*

Equivalently one could say that $(i + 1, i)$ are a separated pair in σ if they occur as part of a 231 pattern where $i + 1$ is the middle valued number and i is the smallest number in the pattern. We may also say that the element k *separates* $i + 1$ and i. The avoidance of a separated pair in a permutation σ is in fact equivalent to σ avoiding the pattern 231. In particular, Proposition 2.3 is equivalent to Lemma 2 (after taking the complement of the inverse of the permutations involved) of a paper by Claesson [6] . Claesson was studying what was known then as *generalized pattern* avoidance, introduced by Babson and Steingrímsson [3]. Now such patterns are known by the less misleading term, *bivincular*. For a thorough study of such pattern avoidance we refer the reader to the survey [21] by Steingrímsson and book [10] by Kitaev.

Example 2.2. The permutation $\pi = 5412736$ contains two 231 patterns, namely 573 and 473. However, π contains only one separated pair, namely $(4, 3)$.

Proposition 2.3. *A permutation π has tier $t(\pi) > 0$, that is π cannot be sorted in a single pass through the stack, if and only if π contains a separated pair.*

Proof. We simply prove that containing a 231 pattern is equivalent to having a separated pair. If the permutation π contains a separated pair then obviously it contains a 231 pattern. Conversely, assume π contains a 231 pattern which we denote (i, j, k) with $k < i < j$. If $k = i - 1$ we are done. Otherwise consider the position of $i - 1$ relative to j. If $i - 1$ is on the same side of j as k then $(i, i - 1)$ form a separated pair in π. Otherwise, $i - 1$ is on the same side of j as i in which case we have another 231 pattern in π, $(i - 1, j, k)$. We continue inductively and see that in any case π must contain a separated pair. □

More generally we see that the number of separated pairs in a permutation characterizes its tier.

Theorem 2.4. *The tier of a permutation under this sorting algorithm is exactly the number of separated pairs in the permutation.*

Proof. Assume the permutation π has t separated pairs. We note that if $t = 0$, then we know π is sortable from Proposition 2.3 and hence the tier of π is zero.

Consider a pass through the stacks where i is pushed to the output, but $i + 1$ is not. The sorting algorithm will need to be restarted in this scenario if and only if $i + 1$ lies in the stack below another number, say j which must necessarily be larger than $i + 1$ since the algorithm has reached i. Hence $(i + 1, i)$ was a separated pair in π. Moreover, $(i + 1, i)$ must be the smallest separated pair in π for i to be pushed to the output in this pass. Restarting then continues with the remaining permutation values larger than i. □

Example 2.5. The permutation $\pi = 356124$ has two separated pairs, $(3, 2), (5, 4)$ and thus has tier 2. We show the sorting of π using three passes through a stack in Figure 1.

The notion of separated pairs also allows one to more easily study the possible tiers of permutations that require more passes through the stacks. For example, the permutation $\pi = 4637251$ contains the four separated pairs $(2, 1), (3, 2), (4, 3)$, and $(6, 5)$ hence π is a tier 4 permutation. Before continuing to investigate separated pairs, we note that the number of separated pairs are preserved in permutation classes. This in turn shows that the k-pass sortable permutations form a permutation class for any value of k.

Proposition 2.6. *If σ and τ are two permutations and σ is contained in τ then τ has at least as many separated pairs as σ. Equivalently the number of separated pairs in σ is less than or equal to the number in any permutation that contains it.*

Proof. The proof is simply a generalization of the argument in the proof or Proposition 2.3. Assume $\sigma = \sigma_1\sigma_2...\sigma_k$ is contained in $\tau = \tau_1\tau_2...\tau_n$. Let $\tau_\sigma = \tau_{\sigma_1}\tau_{\sigma_2}, ...\tau_{\sigma_k}$ be the subsequence of τ corresponding to the pattern σ. We argue that every separated pair in σ forces a distinct separated pair in τ.

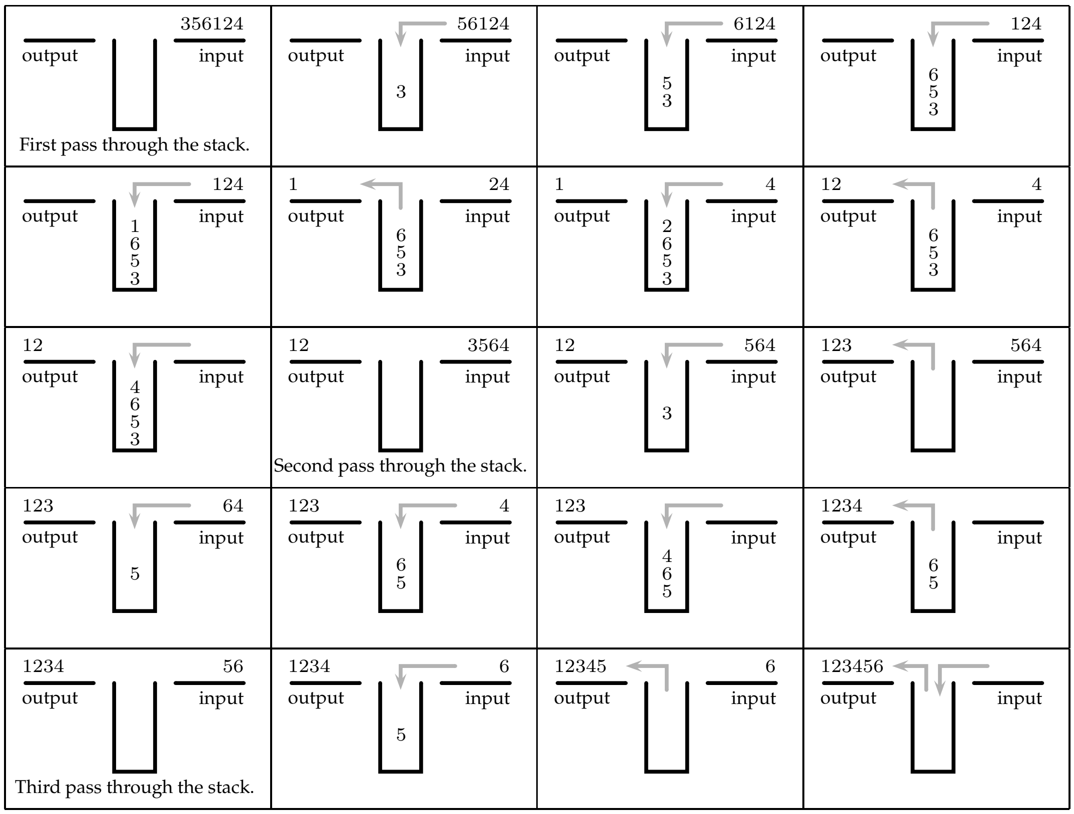

Figure 1: Sorting the permutation 356124 with $k = 3$ passes through a stack.

Assume (σ_i, σ_k) is a separated pair, i.e. $\sigma_i = \sigma_k + 1$ and there is a larger entry σ_j which separates them. We argue that there is a separated pair in τ say $(m+1, m)$ with $\tau_{\sigma_k} \leq m < \tau_{\sigma_i}$ using the same argument as in Proposition 2.3.

Since σ_i and σ_k were consecutive integers in σ, there are no integers in the subsequence τ_σ between τ_{σ_k} and τ_{σ_i}. Hence for every separated pair in σ there is a distinct separated pair in τ and thus the tier of τ is at least that of σ. $\square$

Corollary 2.7. *Given a nonnegative integer t, the permutations of tier less than or equal to t form a permutation class. That is, the k-pass sortable permutations form a permutation class for any positive integer k.*

The proof is simply to note that the permutation of tier at most t are those that avoid all permutations of tier $(t+1)$. This yields another way to see that the k-pass sortable permutations form a permutation class for any positive integer k.

2.1 Explicit Basis for Maximum Tier 1 Permutations

We now classify permutations that have maximum tier $t = 1$. That is, if we allow ourselves a maximum of one re-use of the stack to complete the sorting process, which permutations are sortable?

Theorem 2.8. *A permutation π is 2-pass sortable, i.e. $t(\pi) \leq 1$, if and only if π avoids*

$$24153, 24513, 24531, 34251, 35241, 42513, 42531, 45231, 261453, 231564, 523164.$$

Proof. First one can check that each of the listed permutations has exactly two separated pairs. Moreover, the removal of any entry in any of these permutations also removes at least one separated pair. Hence these permutations are all minimal basis elements for the class of 2-pass sortable permutations.

Next we note that any basis permutation cannot have more than two separated pairs, for if we remove the smallest entry of the smallest separated pair, then the permutation has exactly one less separated pair and is still not 2-pass sortable.

Hence every basis element has exactly two separated pairs. For a permutation σ to meet the minimal length requirement, one of the separated pairs in σ must be $(2, 1)$.

In one case, the basis element σ has two separated pairs of the form $(2, 1)$ and $(3, 2)$. As $3, 2, 1$ must appear in descending order, the basis elements in this instance are 34251 and 35241.

Otherwise, suppose the two separated pairs are $(2, 1)$ and $(b + 1, b)$ where $b > 2$. There are then six possible relative orders of these elements in a basis element σ. For clarity, we consider each of these cases separately.

Case 1: The permutation σ contains the subsequence $(b + 1)b21$. However, then σ contains $(b + 1)kb21$ where $k > b + 1$. Thus removing b and rescaling means our entry $b + 1$ is part of a separated pair with another entry. Thus the subsequence $(b + 1)b21$ does not lead to basis elements.

Case 2: The permutation σ contains the subsequence $(b + 1)2b1$. Then we need only separate the pair $(b + 1, b)$ and thus we have either 45231 or 42531.

Case 3: The permutation σ contains the subsequence $(b + 1)21b$. Then we need to separate both pairs, possibly with a single entry between the 2 and the 1 if it is large enough. As in Case 1, we do not get a minimal basis element if a larger entry appears immediately following $b + 1$ as it makes the b redundant. Hence, the permutations obtained here are only $42513, 523164$.

Case 4: The permutation σ contains the subsequence $2(b + 1)b1$. We just need to separate the pair $(b + 1, b)$, and the only possibility is 24531.

Case 5: The permutation σ contains the subsequence $2(b + 1)1b$. Here we need only separate the pair $(b + 1, b)$. Hence we have 24513 or 24153.

Case 6: The permutation σ contains the subsequence $21(b + 1)b$. In this final case, we need to separate both pairs individually. Thus the basis elements obtained are $231564, 251463, 261453$. However, 251463 is not minimal as it contains 24153. ☐

We have included the avoidance numbers for this basis in Column 2 of Table 2 in Section 4, and the data in Table 1 in Section 3 gives the number of permutations of exact tier t.

3. THE MAXIMUM TIER OF PERMUTATION OF A GIVEN LENGTH

Continuing the use of separated pairs, we may now show that in fact there is a finite basis for each class by bounding the length of potential basis elements. Given a nonnegative integer t, let P_t denote the set of all permutations of tier at most t and let B_t be the basis for this set. That is $\sigma \in P_t$ if and only if $\sigma \in \mathrm{Av}(B_t)$.

Proposition 3.1. *Given any $\sigma \in B_t$, the length of σ is at most $3(t+1)$.*

Proof. First note that if the tier of σ is greater than $t+1$ then σ contains a permutation of tier exactly $t+1$ which can replace σ in the basis. Thus the tier of σ is $t+1$. If the length of $\sigma \in B_t$ is greater than $3(t+1)$ then some number occurring in σ is not part of a separated pair, nor necessary to separate a pair. Hence there is a shorter permutation which is contained in σ of tier $(t+1)$ which can replace σ in the basis. ☐

From Theorem 1.6, we know B_0 has one length 3 basis element. And by Theorem 2.8, we see the basis B_1 consists of eleven basis elements; eight of length 5 and three of length 6. The question of the shortest possible element in a given basis is much more subtle. Equivalently one can ask, what is the maximal possible tier of a permutation of length n?

Notation 3.2. *Let $\tau(n)$ represent the maximum tier of any permutation of length n, and as before let $t(\sigma)$ represent the tier of σ.*

Example 3.3. From Theorems 1.6 and 2.8, we have that $\tau(1) = \tau(2) = 0$, $\tau(3) = \tau(4) = 1$, and $\tau(5) = 2$. From Table 1 in Section 4, we see $\tau(6) = 3$, $\tau(7) = \tau(8) = 4$, $\tau(9) = 5$ and $\tau(10) = 6$.

We will prove an exact formula for $\tau(n)$ later, however we include proofs of a few lower bounds on $\tau(n)$ to demonstrate some specific constructions.

Lemma 3.4. *For all positive integers n, we have $\tau(n) \geq \lfloor \frac{n-1}{2} \rfloor$.*

Proof. One can use n to separate as many distinct pairs as possible. In particular, we create the separated pairs $(n-1, n-2), (n-3, n-4), \ldots, (n - 2 * \lfloor \frac{n-1}{2} \rfloor + 1, n - 2 * \lfloor \frac{n-1}{2} \rfloor)$. Such a permutation must have at least $\lfloor \frac{n-1}{2} \rfloor$ separated pairs and the bound follows. ☐

Example 3.5. If $n = 7$, use 7 to separate $(6, 5), (4, 3), (2, 1)$ as in 6427531.

For odd lengths we can do better.

Lemma 3.6. *If n is odd with $n = 2k + 1$, then $\tau(n) = k + \tau(k)$.*

Proof. To prove this lemma, we first give a construction for a particular permutation of high tier giving a lower bound for $\tau(n)$, then argue it is optimal.

We create the largest set of consecutive separated pairs we can have, $(k+1, k), (k, k-1), (k-1, k-2), \ldots, (2, 1)$ and use the numbers from $k+2$ up to $n = 2k+1$ to separate them. Since all of these numbers are larger than all of the pairs we are separating, they can be used interchangeably. In particular, we arrange these separators in an optimal $\tau(\frac{n-1}{2}) = \tau(k)$ pattern. Thus the number of separated pairs is at least $k + \tau(k)$ and hence $\tau(n) \geq \frac{n-1}{2} + \tau(\frac{n-1}{2})$.

Further, we note that among the elements $1, 2, 3, \ldots, k+2$, we can form at most k separated pairs in a permutation of length $n = 2k+1$. This is because there are only $k+1$ possible pairs of consecutive elements, but if all $k+2$ were placed in descending order (as needed to form all $k+1$ separated pairs), then there would $k+1$ gaps to fill with larger entries, but only $k-1$ large elements remaining. Hence we can optimize our construction by putting only the first $k+1$ elements in descending order as $k+2$ is the most useful as a "separator". Now when considering arranging the remaining k large entries, we see that the above construction is optimal as the smaller elements cannot be used to separate larger elements. Hence $\tau(n) \geq \frac{n-1}{2} + \tau(\frac{n-1}{2})$ as well.

To prove $\tau(n)$ is equal to this bound, we consider the effects of moving some of the entries from $1, 2, \ldots, k+1$ or moving the entries from $k+2$ to n. The construction above produces the maximal number of separated pairs among the first $k+1$ elements. If we rearrange these elements in our permutation, the only possible new separated pair we can construct is $(k+2, k+1)$ since all of the lower values are already separated pairs, and these values are too small to separate any of the larger pairs. However in order to create a new separated pair $(k+2, k+1)$ we need to "un-separate" at least one other pair. Thus the net change in tier of such a rearrangement is at most zero. Similarly, since the values $k+2$ up to n are already in an optimal $\tau(\frac{n-1}{n})$ configuration, we cannot create any extra separated pairs among these larger elements. Thus the construction is produces permutations of maximal tier.

□

To illustrate the above we consider a few examples shown below.

Example 3.7. If $n = 7$ we begin with the sequence 4321 then place $5, 6,$ and 7 between each pair to separate them. Since we can do this any way we please, we use the optimal $\tau(3)$ pattern (i.e. 231) for these elements to yield 4637251 which has optimal tier 4.

Example 3.8. If $n = 9$, we begin with 54321 and use $6, 7, 8, 9$ to separate the consecutive pairs in an optimal length 4 pattern such as 2314 to yield 574836291 which has tier $4 + 1 = 5$ and $\tau(9) = 5$.

We also note that increasing the length by one increases the maximal tier by at most one.

Lemma 3.9. *For any positive integer n we have $\tau(n+1) \leq \tau(n) + 1$.*

Proof. Let σ be in S_{n+1} with tier $\tau(n+1)$. If we remove the number 1 from σ and reduce all of the remaining numbers by 1 to create a permutation $\rho \in S_n$. By removing 1 from σ we have at most removed one separated pair, hence $t(\rho) \geq t(\sigma) - 1$ or $\tau(n+1) = t(\sigma) \leq t(\tau) + 1 \leq \tau(n) + 1$. □

Also, since we can find a length $n+1$ permutation containing a given length n permutation, we can use the argument in Lemma 3.6 to get a lower bound for the case when n is even as well.

Corollary 3.10. *If $n = 2k$ then $\tau(n) \geq k - 1 + \tau(k-1)$.*

However, while we still can have only $\frac{n-2}{2} = \lfloor \frac{n-1}{2} \rfloor$ separated pairs from our smallest $\frac{n+2}{2}$ entries, we can duplicate the above construction and have $\tau(\frac{n}{2})$ separated pairs amongst our largest $\frac{n}{2}$ entries. Hence we have the following lemma.

Lemma 3.11. *If $n = 2k$ then $\tau(n) \geq k - 1 + \tau(k)$.*

Combining Lemmas 3.6, 3.11, we get the following result.

Theorem 3.12. *The maximum tier of a permutation of length n satisfies the recurrence*

$$\tau(n) = \left\lfloor \frac{n-1}{2} \right\rfloor + \tau\left(\left\lfloor \frac{n}{2} \right\rfloor\right).$$

Moreover

$$\tau(n) = \sum_{j\geq 1} \left\lfloor \frac{n-2^{j-1}}{2^j} \right\rfloor = n - 1 - \lfloor \log_2(n) \rfloor. \tag{1}$$

Proof. The recursion formula for $\tau(n)$ generalizes lemmas 3.6, 3.11. The first equality follows from iterating the recursive formula. For example,

$$\tau(n) = \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \right\rfloor + \tau\left(\left\lfloor \frac{\lfloor \frac{n}{2} \rfloor}{2} \right\rfloor\right).$$

Let $n = \sum_{i=0}^{k} c_i 2^i$ and note that in general

$$\left\lfloor \frac{\lfloor \frac{n}{2^i} \rfloor}{2} \right\rfloor = \left\lfloor \frac{c_i + c_{i+1}2 + c_{i+2}2^2 + \cdots + c_k 2^{k-i}}{2} \right\rfloor = c_{i+1} + c_{i+2}2 + c_{i+3}2^2 + \cdots + c_k 2^{k-i-1} = \left\lfloor \frac{n}{2^{i+1}} \right\rfloor$$

and we have

$$\left\lfloor \frac{\lfloor \frac{n}{2^i} \rfloor - 1}{2} \right\rfloor = \left\lfloor \frac{n-2^i}{2^{i+1}} \right\rfloor \text{ since } \left\lfloor \frac{n}{2^i} \right\rfloor - 1 = \left\lfloor \frac{n-2^i}{2^i} \right\rfloor$$

and the first equality in (1) follows.

To show the second equality in formula (1), we evaluate the individual terms in the sum

$$\begin{aligned}\left\lfloor \frac{n-2^{j-1}}{2^j} \right\rfloor &= \left\lfloor \frac{c_0}{2^j} + \frac{c_1}{2^{j-1}} + \cdots + \frac{c_{j-1}}{2} + c_j + c_{j+1}2 + \cdots c_k 2^{k-j} - \frac{1}{2} \right\rfloor \\ &= c_j + c_{j+1}2 + \cdots + c_k 2^{k-j} + \left\lfloor \frac{c_0}{2^j} + \frac{c_1}{2^{j-1}} + \cdots + \frac{c_{j-1}}{2} - \frac{1}{2} \right\rfloor.\end{aligned}$$

Finally we note that

$$\begin{aligned}\left\lfloor \frac{c_0}{2^j} + \frac{c_1}{2^{j-1}} + \cdots + \frac{c_{j-1}}{2} - \frac{1}{2} \right\rfloor &= \begin{cases} 0 & c_{j-1} = 1 \\ -1 & c_{j-1} = 0 \end{cases} \\ &= c_{j-1} - 1.\end{aligned}$$

Thus $\left\lfloor \frac{n-2^{j-1}}{2^j} \right\rfloor = \left[\sum_{i=j}^{k} c_i 2^{i-j} \right] + (c_{j-1} - 1)$. Summing on j yields

$$\begin{aligned}\tau(n) = \sum_{j\geq 1} \left\lfloor \frac{n-2^{j-1}}{2^j} \right\rfloor &= \sum_{j=1}^{k} c_j \sum_{i=0}^{j-1} 2^i - \sum_{j=0}^{k} (c_j - 1) \\ &= \sum_{j=0} c_j (2^j - 1) - \sum_{j=0}^{k} (c_j - 1) = n - (k+1) \\ &= n - 1 - \lfloor \log_2(n) \rfloor.\end{aligned}$$

□

From Theorem 3.12 we get a curious result for lengths of the form $n = 2^k - 1$.

Corollary 3.13. *Let $n = 2^k - 1$ for some integer k, then every permutation of tier $\tau(n)$ is constructed with the method given in Lemma 3.6. Moreover there is exactly one such permutation.*

Proof. Given the formula in Theorem 3.12, if $n = 2^k - 1$ then $\tau(n+1) = \tau(n)$. Assume that σ is a permutation of tier $\tau(n)$ and note $\frac{n-1}{2} = 2^{k-1} - 1$ which we will call m for convenience. Assume σ is not of the form $(m+1)a_2(m)a_4(m-1)a_6 \dots (2)a_{n-1}(1)$ where $a_2a_4 \dots a_{n-1}$ has tier $\tau(2^{k-1} - 1)$ and assume that j is the first element that is not part of a separated pair $(j+1, j)$. If $j < m+1$ then create a new permutation $\hat{\sigma}$ by first moving all entries less than $j+1$ to the right (if necessary) so that there is only one separator for each separated pair, then increase all values in σ larger than j by one, and add a $j+1$ in the first position. We have that $(j+1, j)$ is now a separated pair in $\hat{\sigma}$ without affecting any other separated pairs, thus $t(\hat{\sigma}) = t(\sigma) + 1 = \tau(n) + 1$ contradicting the result in Theorem 3.12. Thus every maximal tier permutation of length $2^k - 1$ has the form above, and thus the number of such permutations is the same as the number of length $2^{k-1} - 1$ with maximal tier. Repeating the argument, eventually we get that the number of such permutations is the same as the number of length one permutations of maximal tier and the result follows. □

Example 3.14. For length $n = 2^4 - 1 = 15$ the maximal tier is 11 and the unique permutation of this length and tier is $\pi =$ 8 12 7 14 6 11 5 15 4 10 3 13 2 9 1.

We can also consider how to find the tier of a permutation obtained by combining two permutations via a specific kind of concatenation to get a new permutation.

Definition 3.15. *An* interval *of a permutation π is a consecutive subsequence of π that contains consecutive values.*

Example 3.16. The permutation $\sigma = 685712943$ contains maximal intervals $6857, 12, 9, 43$.

Definition 3.17. *A permutation π is said to be* plus-decomposable *if π is the concatenation of two non-empty intervals ω and τ' where the values of ω are less than those of τ'. Further, if we rescale the entries of τ' by subtracting the length of ω from each entry of τ' to get a permutation τ, we denote $\pi = \omega \oplus \tau$. If a permutation is not plus-decomposable, we say the permutation is* plus-indecomposable.

Example 3.18. The permutation $\pi = 43126758$ is plus-decomposable and can be written as $\pi = 4312 \oplus 231 \oplus 1$. The permutation $\sigma = 685712943$ is plus-indecomposable.

Proposition 3.19. *If a permutation π is plus-decomposable, say $\pi = \sigma \oplus \tau$, then the tier of π is the sum of the tiers of σ and τ, i.e. $t(\pi) = t(\sigma) + t(\tau)$.*

Proof. Consider the process of sorting π. As every entry of the σ portion of π must be pushed to the output before any entry of τ', we cannot start sorting τ' until the last pass needed to sort σ has commenced. This final pass where part of σ is still being sorted is the $(t(\sigma) + 1)$st pass.

During the $(t(\sigma) + 1)$st pass, every remaining entry of the σ portion can be output before τ' is considered and thus this pass can also be used as the first pass for τ'. Then τ' requires $t(\tau)$ more passes to be sorted. Hence π is $[t(\sigma) + t(\tau) + 1]-$pass sortable and thus $t(\pi) = t(\sigma) + t(\tau)$. □

Definition 3.20. *A permutation π is said to be* minus-decomposable *if π is the concatenation of two non-empty intervals ω' and τ where the values of ω' are greater than those of τ. As before, if we rescale the entries of ω' by subtracting the length of τ from each entry of ω' to get a permutation ω, we denote $\pi = \omega \ominus \tau$.*

	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6
n = 1	1						
n = 2	2						
n = 3	5	1					
n = 4	14	10					
n = 5	42	70	8				
n = 6	132	424	160	4			
n = 7	429	2382	1978	250	1		
n = 8	1430	12804	19508	6276	302		
n = 9	4862	66946	168608	106492	15674	298	
n = 10	16796	343772	1337684	1445208	451948	33148	244

Table 1: Number of permutations of length n and exact tier t

Example 3.21. The permutation $\pi = 67584312$ is minus-decomposable and can be written as $\pi = 2314 \ominus 1 \ominus 1 \ominus 12$.

Proposition 3.22. *If a permutation π is minus-decomposable, say $\pi = \sigma \ominus \tau$, and σ is k-pass sortable and τ is m-pass sortable, then π is $(k+m)$-pass sortable, i.e. $t(\pi) = t(\sigma) + t(\tau) + 1$ if the last entry of σ is not 1, otherwise π is $(k+m-1)$ sortable and $t(\pi) = t(\sigma) + t(\tau)$.*

Proof. Again, consider the process of sorting π. Every entry of τ must be pushed to the output before any entry of σ', we cannot start sorting σ' until after the last pass needed to sort τ has been completed. If the last entry of σ' is the smallest entry of σ' then the last pass to sort τ is also the first pass to sort σ', hence π is $(k+m)-$pass sortable and $t(\pi) = t(\sigma) + t(\tau)$. Otherwise another pass is required to begin sorting σ' and hence π is $[k+m+1]-$pass sortable and thus $t(\pi) = t(\sigma) + t(\tau) + 1$. □

Note that if a permutation is minus-indecomposable and length greater than one, then it cannot have its smallest value in the last position. This leads to the following corollary:

Corollary 3.23. *Let π be a permutation and assume $\pi = \sigma_1 \ominus \sigma_2 \ominus \cdots \ominus \sigma_p$ where each σ_i is minus-indecomposable, and let r denote the number of the σ_i with $i < p$ of length one, then $t(\pi) = p - r - 1 + \sum_{i=1}^{p} t(\sigma_i)$.*

4. EXPLICIT GENERATING FUNCTION FOR K-PASS PERMUTATIONS OF LENGTH N

A simple program was written in SAGE [7] to compute the tier of all permutations up to length 10. The data for the number of permutations of a given length and exact tier is given in Table 1. The numbers given in this triangular form appear in the OEIS A122890 and A158830 [19]. The sequences found in the OEIS were created by manipulating generating series for iterated functions, however one version (A122890) does have an equivalent combinatorial interpretation found in Parker's thesis [16]. We modify Parker's description slightly to align the data properly (in particular effecting a row-reversal which gives OEIS A158830).

Parker's original description of OEIS A122890

Assume n, t are positive integers and let $W(n,t)$ be the number of sequences $a_1a_2\ldots a_n$ of length n, such that each $1 \le a_i \le i$ and there are exactly t indices i such that $a_i \le a_{i+1}$.

Example 4.1. We have $W(4,2) = 10$ as it counts the sequences:

$$1121, 1131, 1132, 1211, 1212, 1213, 1214, 1221, 1231, 1232.$$

For convenience, we reindex the sequence and consider the complementary condition on the indices. That is, assume n, t are non-negative integers with $n > 0$, and let $T(n,t)$ be the number of sequences $a_na_{n-1}\ldots a_1$ of length n where $1 \le a_{n-i+1} \le i$ for each i, and there are exactly t values of i such that $a_{i+1} > a_i$. We say the sequence has a *descent* at i if $a_{i+1} > a_i$ and in this language $T(n,t)$ is the number of length n sequences with entries bounded by the index such that there are exactly t descents. That is, $T(n,t) = W(n, n-1-t)$. Note that the reindexing has no effect on the problem and counting descents instead of non-descents reverses the rows of the data in OEIS A122890. We will refer to these reindexed sequences as *Parker sequences*.

Example 4.2. We have $T(4,1) = W(4,2) = 10$ and counts the sequences given in Example 4.1.

Example 4.3. $T(5,2) = 8$ and counts the sequences:

$$11321, 12121, 12131, 12132, 12141, 12142, 12143, 12321.$$

We now prove a bijection between permutations of length n and tier t and the Parker sequences counted by $T(n,t)$.

Theorem 4.4. *The number of permutations of length n and tier t is $T(n,t)$.*

Proof. We first define a bijection between the set of integer sequences defined above and the set of permutations. We then show a permutation has $(i+1, i)$ as a separated pair if and only if the associated sequence has a descent at index $i+1$.

Let $P = \{a_na_{n-1}\ldots a_1 \mid 1 \le a_{n-i+1} \le i, \forall i, n\}$, that is P is the set of all Parker sequences, and let S be the set of all permutations. Let $f : P \to S$ be defined as $f(a_na_{n-1}\ldots a_1) = \pi_n\pi_{n-1}\ldots\pi_1$ where the element 1 is placed in π_{a_1}. Then for each i from 2 to n, place the element i in the a_ith position of the remaining positions in π, counting from the right.

The map is a bijection as the process is invertible. That is, given a permutation $\pi = \pi_n\pi_{n-1}\ldots\pi_1$, create the sequence $f^{-1}(\pi) = a_na_{n-1}\ldots a_1$ by letting a_i be the relative position (from right to left) of the element i among the elements greater or equal to i in the permutation π. The bounds on the entries must obey Parker's restriction as there are $n-i+1$ numbers greater or equal to i in π.

We now prove that there is a descent at i for the sequence a if and only if $(i+1, i)$ is a separated pair in $f(a)$. Assume there is a descent at index i, i.e. $a_{i+1} > a_i$ since a_{i+1} appears before a_i in a. This would imply that $i+1$ is placed to the left of i in the permutation $f(a)$. Since the inequality is strict there must also be at least one unoccupied position between these two elements after we place $i+1$ in the permutation. The only elements remaining are larger than $i+1$. Hence at least one of these larger elements must separate $i+1$ and i in $f(a)$.

Conversely, assume there is a separated pair in π, say $(i+1, i)$. Consider the associated sequence $a = f^{-1}(\pi)$. Then $a_{i+1} \ge a_i$. since $i+1$ must be placed to the left i. Further, since there must be a larger element separating $i+1$ and i in π, we have $a_{i+1} > a_i$. Hence there is a descent in a at index i.

Since the number of descents a Parker sequence has is the same as the number of separated pairs the associated permutation has, we have the number of permutations of length n and tier t is $T(n,t)$. $\square$

Example 4.5. As an example we compute $f(12133)$ for the bijection given in the proof of Theorem 4.4. Begin with the last number in the sequence 3 and place the 1 into the third position from the right in the permutation so we have $**1**$. Then we consider the next element in the sequence from the right, since it is also 3, we place the element 2 in the third remaining position from the right and we have $*21**$. The next element in the sequence is 1 hence we place the element 3 in the first remaining position from the right, i.e. $*21*3$. The we place the element 4 in the second remaining position from the right as the next entry is 2 to get $421*3$. Finally 5 must be placed in the only remaining position (which is the first from the right) to get 42153.

Corollary 4.6. *The number of permutations of length n sortable by a stack with (at most) k passes, or $Av(B_{k-1})$, is* $\displaystyle\sum_{j=0}^{k-1} T(n,j)$.

4.1 Explicit formula

The sequences considered in the OEIS were originally constructed by manipulating generating functions. For the sake of completeness we include the constructions here. We also add a new construction of the generating function that gives an explicit formula for $T(n,t)$.

OEIS 158830 Construction

Let $\hat{C}(x) = xC(x)$ where $C(x)$ is the generating function for the Catalan numbers or equivalently the avoidance numbers of any permutation of length 3. Let the numbers in the n-th row be the coefficients of the n-th iterate of $\hat{C}(x)$. The multiply the generating function represented by the t-th column by $(1-x)^t$. The resulting entry in row n and column t is $T(n,t)$.

OEIS 122890 Construction, Parker [16]

Let $a_1(x) = x$, $a_2(x) = x + x^2$ and for all $n \geq 2$, let $a_n(x)$ be the $n-1$ iterate of $x + x^2$. Write the coefficients of $a_n(x)$ as the entries in row n and multiply the generating function represented by column j by $(1-x)^j$. $T(n,t)$ is the entry in position $(n, n-t)$.

Alternate Construction

To develop a recurrence for the number of permutations of length n and tier t, we introduce a set of functions f_k from permutations of length n to length $n+1$. In particular, given a permutation α of length n, let $f_k(\alpha)$ be the permutation obtained by increasing every number in α by one, then inserting a one into the $(k+1)$-st position. For example $f_2(3412) = 45123$ and $f_3(3412) = 45213$.

Assume $\beta = f_k(\sigma)$ for some permutation σ of length n and some integer $0 \leq k \leq n$. We note β has the same number of separated pairs $(i+1,i)$ with $i \geq 2$ as σ. Thus the $t(\beta)$ is either $t(\sigma)$ or $t(\sigma)+1$ depending on whether $(2,1)$ is a separated pair in β. We note that $(2,1)$ is a separated pair in β if and only if k is at least two larger than the position of the 1 in σ. That is, if the 1 in σ occurs in the j-th position and $k \geq j+2$, we will have $(2,1)$ as a separated pair in $\beta = f_k(\sigma)$.

Let $P(n,t,k)$ be the number of permutations of length n, tier t (where $n > 0, t \geq 0$), and have the 1 in

	t = 0	t ≤ 1	t ≤ 2	t ≤ 3	t ≤ 4	t ≤ 5	t ≤ 6
n = 1	1	1	1	1	1	1	1
n = 2	2	2	2	2	2	2	2
n = 3	5	6	6	6	6	6	6
n = 4	14	24	24	24	24	24	24
n = 5	42	112	120	120	120	120	120
n = 6	132	556	716	720	720	720	720
n = 7	429	2811	4789	5039	5040	5040	5040
n = 8	1430	14234	33742	40018	40320	40320	40320
n = 9	4862	71808	240416	346908	362582	362880	362880
n = 10	16796	360568	1698252	3143460	3595408	3628556	3628800

Table 2: Number of permutations of length n and tier at most t

the k-th position (thus $1 \leq k \leq n$). Clearly we have $T(n,t) = \displaystyle\sum_{k=1}^{n} P(n,t,k)$.

Example 4.7. For example for length $n = 3$ we have $P(3,0,1) = 2$ (for the permutations $123, 132$), $P(3,0,2) = 2$ (for $213, 312$), $P(3,0,3) = 1$ (for 321), $P(3,1,1) = 1$ (for 231), and otherwise $P(3,t,k) = 0$.

Theorem 4.8. *For all integers $n > 0$, $t \geq 0$, $1 \leq k \leq n+1$ we have*

$$P(n+1,t,k) = \sum_{j \geq k-1} P(n,t,j) + \sum_{j \leq k-2} P(n,t-1,j).$$

Proof. First note that every permutation β of length $n+1$ with a 1 in the k-th position arises exactly once from applying one of the f_k operators to a permutation σ of length n. Consider $\beta = f_k(\sigma)$. If we apply f_k to a permutation then the tier is either fixed or increases by one. The tier is fixed exactly when $(2,1)$ is not a new separated pair in the permutation of length $n+1$. That is, the tier increases when the 1 of β is appears at least two slots to the right of the 1 of σ so that there is larger element to separate the pair $(2,1)$ of β. $\square$

Example 4.9. Consider the permutation 24153, since the 1 occurs in the third position,

$$t(f_k(24153)) = \begin{cases} t(24153) = 2 & \text{if } k \leq 4 \\ t(24153) + 1 = 3 & \text{if } k \geq 5 \end{cases}$$

The data in Table 2 gives the number of elements in each permutation class at a given length and up to a given tier. We also note that we were able to compute the number of basis elements in B_3. There are 4 of length 6, 116 of length 7, 67 of length 8 and 12 of length 9 (note the maximal length of a basis element would be 12).

Now, we are ready to find an explicit formula for the generating function $T_t(x) = \sum_{n \geq 0} T(n,t)x^n$. In order to do that, we define $P_{n,t}(v) = \sum_{k=1}^{n} P(n,t,k)v^{k-1}$. By multiplying the recurrence relation in the statement of Theorem 4.8 by v^{k-1}, we have

$$\sum_{k=1}^{n+1} P(n+1,t,k)v^{k-1} = \sum_{k=1}^{n+1} v^{k-1} \sum_{j \geq k-1} P(n,t,j) + \sum_{k=1}^{n+1} v^{k-1} \sum_{j \leq k-2} P(n,t-1,j),$$

which, by exchanging the order of the sums, implies

$$\sum_{k=1}^{n+1} P(n+1,t,k)v^{k-1} = P_{n,t}(1) + \sum_{k=1}^{n} \frac{v(1-v^k)}{1-v} P(n,t,k) + \sum_{k=1}^{n} \frac{v^{k+1}-v^{n+1}}{1-v} P(n,t-1,k).$$

Thus, by definitions of $P_{n,t}(v)$, we obtain

$$P_{n+1,t}(v) = P_{n,t}(1) + \frac{v}{1-v}(P_{n,t}(1) - vP_{n,t}(v)) + \frac{v^2}{1-v}(P_{n,t-1}(v) - v^{n-1}P_{n,t-1}(1)). \tag{2}$$

Define $P_n(w,v) = \sum_{t\geq 0} P_{n,t}(v)w^t$. Then, by multiplying (2) by w^t and summing over $t \geq 1$, we obtain

$$P_{n+1}(w,v) = P_n(w,1) + \frac{v}{1-v}(P_n(w,1) - vP_n(w,v)) + \frac{v^2 w}{1-v}(P_n(w,v) - v^{n-1}P_n(w,1)) \tag{3}$$

with $P_1(v,w) = 1$. Now, we define $P(z;w,v) = \sum_{n\geq 1} P_n(w,v)z^n$ to be the generating function for $P_n(w,v)$. By multiplying (3) by z^n and summing over $n \geq 1$, we have

$$P(z;w,v) - z = zP(z;w,1) + \frac{vz}{1-v}(P(z;w,1) - vP(z;w,v)) + \frac{vwz}{1-v}(vP(z;w,v) - P(vz;w,1)),$$

which is equivalent to

$$\left(1 + \frac{v^2 z(1-w)}{1-v}\right) P(z;w,v) = z + \frac{z}{1-v}P(z;w,1) - \frac{vwz}{1-v}P(vz;w,1).$$

To solve this functional equation, we apply the kernel method and take

$$v = C(z(1-w)) = \frac{1-\sqrt{1-4z(1-w)}}{2z(1-w)},$$

which cancels $P(z;w,v)$, where $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. This gives

$$P(z;w,1) = C(z(1-w)) - 1 + wC(z(1-w))P(zC(z(1-w));w,1). \tag{4}$$

Define $\rho_0(z) = C(z(1-w))$ and $\rho_j(z) = C\left(z(1-w)\prod_{i=0}^{j-1}\rho_i(z)\right)$ for all $j \geq 1$. Then, by assuming $0 < |w| < 1$, $|z| < 1$ and iterating (4), we have

$$\begin{aligned}
P(z;w,1) &= \rho_0(z) - 1 + w\rho_0(z)P(z\rho_0(z);w,1) \\
&= \rho_0(z) - 1 + w\rho_0(z)(\rho_0(z\rho_0(z)) - 1) + w^2\rho_0(z)\rho_0(z\rho_0(z))P(z\rho_0(z)\rho_0(z\rho_0(z));w,1) \\
&= \rho_0(z) - 1 + w\rho_0(z)(\rho_1(z) - 1) + w^2\rho_0(z)\rho_1(z)P(z\rho_0(z)\rho_1(z);w,1) \\
&= \cdots,
\end{aligned}$$

which leads to the following result.

Theorem 4.10. *The generating function*

$$P(z;w,1) = \sum_{n\geq 1}\sum_{t\geq 0}\sum_{k=1}^{n} P(n,t,k)w^t z^k = \sum_{n\geq 1}\sum_{t\geq 0} T(n,t)w^t z^n$$

is given by

$$T(z,w)=\sum_{j\geq 0}(\rho_j(z)-1)w^j\prod_{i=0}^{j-1}\rho_i(z).$$

Define $\psi_j(z)=\sqrt{2\psi_{j-1}(z)-1}$ with $\psi_1(z)=\sqrt{1-4z(1-w)}$ and $\psi_0=1-2z(1-w)$. By induction on $j\geq 0$, we obtain

$$\rho_j(z)=\frac{1-\psi_{j+1}(z)}{1-\psi_j(z)}.$$

Thus, $\prod_{i=0}^{j-1}\rho_i(z)=\frac{1-\psi_j(z)}{2z(1-w)}$ for all $j\geq 0$. Hence, by Theorem 4.10, we have the following formula.

Theorem 4.11. *The generating function $P(z;w,1)=T(z,w)$ is given by*

$$T(z,w)=\sum_{j\geq 0}\frac{\psi_j(z)-\psi_{j+1}(z)}{2z(1-w)}w^j,$$

where $\psi_j(z)=\sqrt{2\psi_{j-1}(z)-1}$ with $\psi_1(z)=\sqrt{1-4z(1-w)}$ and $\psi_0=1-2z(1-w)$.

In order to find the generating function for $T_t(z)=\sum_{n\geq 1}T(n,t)z^n$, we have to find the coefficient of w^t in $T(z,w)$. Thus, by Theorem 4.11, we have

$$T_t(z)=\sum_{j=0}^{t}[w^{t-j}]\left(\frac{\psi_j(z)-\psi_{j+1}(z)}{2z(1-w)}\right). \tag{5}$$

For example, for $t=0$, we have

$$T_0(z)=[w^0]\left(\frac{\psi_0(z)-\psi_1(z)}{2z(1-w)}\right)=\frac{1-2z-\sqrt{1-4z}}{2z}=C(z)-1,$$

as expected. For $t=1$, we have

$$\begin{aligned}
T_1(z)&=[w^1]\left(\frac{\psi_0(z)-\psi_1(z)}{2z(1-w)}\right)+[w^0]\left(\frac{\psi_1(z)-\psi_2(z)}{2z(1-w)}\right)\\
&=[w^1]\left(\frac{1-\sqrt{1-4z(1-w)}}{2z(1-w)}-1\right)+[w^0]\left(\frac{\sqrt{1-4z(1-w)}-\sqrt{2\sqrt{1-4z(1-w)}-1}}{2z(1-w)}\right)\\
&=-\frac{1}{\sqrt{1-4z}}+\frac{1-\sqrt{1-4z}}{2z}+\left(\frac{\sqrt{1-4z}-\sqrt{2\sqrt{1-4z}-1}}{2z}\right)\\
&=\frac{1-\sqrt{2\sqrt{1-4z}-1}}{2z}-\frac{1}{\sqrt{1-4z}}\\
&=z^3+10z^4+70z^5+424z^6+2382z^7+12804z^8+66946z^9+343772z^{10}+\cdots.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
T_2(z)&=\frac{1-\sqrt{2\sqrt{2\sqrt{1-4z}-1}-1}}{2z}+\frac{z}{\sqrt{1-4z}^3}-\frac{1}{\sqrt{1-4z}\sqrt{2\sqrt{1-4z}-1}}\\
&=8z^5+160z^6+1978z^7+19508z^8+168608z^9+1337684z^{10}+10003422z^{11}+\cdots,
\end{aligned}$$

Note that $\psi_j^k(z)=\sum_{i\geq 0} I^k\binom{k/2}{i}(-2)^i\psi_{j-1}^i(z)$ for all $j\geq 1$ and $\psi_1^k=\sum_{i\geq 0}\binom{k/2}{i}(-4)^iz^i(1-w)^i$, where $I^2=-1$. Thus, for all $j\geq 2$,

$$\psi_j(z)=\sum_{i_1,\ldots,i_j\geq 0} I^{1+3i_j+\cdots+3i_3+2i_2+2i_1}2^{i_j+\cdots+i_2+2i_1}\binom{1/2}{i_j}\binom{i_j/2}{i_{j-1}}\cdots\binom{i_2/2}{i_1}z^{i_1}(1-w)^{i_1}.$$

Hence, for all $s\geq 0$ and $j\geq 2$,

$$[w^s]\frac{\psi_j(z)}{2z(1-w)}=\sum_{i_1,\ldots,i_j\geq 0} I^{1+3i_j+\cdots+3i_3+2i_2+2i_1+2s}2^{i_j+\cdots+i_2+2i_1-1}\binom{1/2}{i_j}\binom{i_1-1}{s}z^{i_1-1}\prod_{k=2}^{j}\binom{i_k/2}{i_{k-1}}$$

and

$$[w^s]\frac{\psi_1(z)}{2z(1-w)}=\sum_{i_1\geq 0} I^{2i_1+2s}2^{2i_1-1}\binom{1/2}{i_1}\binom{i_1-1}{s}z^{i_1-1}.$$

Hence, by (5), we can write an explicit formula for the generating function $T_t(z)$ in terms of multi sums.

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