On Level-1 Consensus Ensuring Stable Social Choice

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Abstract

Level-1 consensus is a property of a preference profile. Intuitively, it means that there exists some preference relation such that, when ordering the other preference-relations by increasing distance from it, the closer preferences are more frequent in the profile. This is a desirable property, since it enhances the stability of the social choice by guaranteeing that there exists a Condorcet winner and it is elected by all scoring rules.

In this paper, we present an algorithm for checking whether a given preference-profile exhibits level-1 consensus. We apply this algorithm to a large number of preference-profiles, both real and randomly-generated, and find that level-1 consensus is very improbable. We back this empirical findings by a simple theoretical proof that, under the impartial culture assumption, the probability of level-1 consensus approaches zero when the number of individuals approaches infinity.

Motivated by these observations, we show that the level-1 consensus property can be weakened retaining the stability implications. The weaker level-1 consensus is considerably more probable, both empirically and theoretically. In fact, under the impartial culture assumption, the probability converges to a positive number when the number of individuals approaches infinity.

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I. INTRODUCTION

Recently, Mahajne et al. [9] have proposed the concept of level-1 consensus of a preference profile showing that it considerably enhances the stability of social choice. In particular, if a preference profile exhibits level-1 consensus around a given preference relation $\succ_0$ with respect to the inversion metric, then:[1]

- There exists a Condorcet winner;
- The Condorcet winner is chosen by all the scoring rules;
- With an odd number of individuals, the majority relation is transitive and equals $\succ_0$.

The current study focuses on two questions:

1. How can a preference profile be tested for level-1 consensus?
2. How likely is it that level-1 consensus exists?

Questions of the former type have been recently studied with respect to various domain restrictions. For example, Escoffier et al. [5] provide an efficient way to check whether a preference profile is *single-peaked*, Bredereck et al. [3] provide an efficient way to check whether a preference profile is *single-crossing*, and Barberà and Moreno [1] ask whether the satisfaction of their proposed *top monotonicity* condition (a sufficient condition for an extension of the median-voter theorem to hold) is easy to check.

Questions of the latter type have been studied in the social choice literature with respect to various domain restrictions that guarantee social stability, e.g., the existence of Condorcet winners under the majority rule [6, 14].

Our answer to the first question is an efficient algorithm for determining whether a preference profile exhibits level-1 consensus. In case of such consensus, we further identify the preference relations around which it occurs.

Our answer to the second question is that level-1 consensus is highly improbable. We applied our algorithm on a recently-released dataset of 315 real-world preference-profiles from

[1] In fact, Mahajne et al. [9] define a family of conditions called level-r consensus, where r is an integer between 1 and $K!/2$ and K is the number of alternatives. But for the sake of simplicity, in the present paper we focus on level-1 consensus which is the strongest condition in this family.

Note that recently Poliakov [13] proved that level-r consensus is equivalent to level-1 consensus whenever $r \leq (K-1)!$.

various sources [12] and found that none of them exhibits level-1-consensus. Moreover, experiments performed on thousands of profiles generated randomly according to Mallows' phi model [10] revealed that, for a wide range of parameter settings, profiles exhibiting level-1 consensus were rare. To support these findings, we prove that under the standard probabilistic setting of equally-probable preference relations, the probability of level-1 consensus goes to zero when the number of individuals is sufficiently large.

Motivated by these results, we established that the level-1 consensus property can be weakened, enhancing the stability of social choice. Applying the modified algorithm to the above mentioned dataset, it has been found that 39 out of 315 profiles exhibit level-1 consensus. The weaker level-1 consensus property is also much more probable in the settings of randomly-generated profiles we tested. In particular, under the impartial assumption the probability of the weaker level-1 consensus property converges to a positive number for any number of individuals.

II. DEFINITIONS

Let $A = \{a_1, \ldots, a_K\}$ be a set of $K \geq 3$ alternatives and let $N = \{1, \ldots, n\}$ be a set of individuals. Also, let $\mathcal{P}$ be the subset of complete, transitive and antisymmetric binary relations on A. We will refer to the elements of $\mathcal{P}$ as *preference relations* or simply as *preferences.* A *preference profile* or simply a *profile* is a list $\pi = (\succ_1, \ldots, \succ_n)$ of preference relations on A such that for each $i \in N$, $\succ_i$ is the preference relation of individual i. We denote by $\mathcal{P}^n$ the set of all possible preference profiles.

Let $\pi = (\succ_1, \ldots, \succ_n)$ be a preference profile. For each preference $\succ \in \mathcal{P}$, let $\mu_\pi(\succ) := |\{i \in N : \succ_i = \succ\}|$ = the number of individuals whose preference is $\succ$, which in this study is referred to as the *frequency* of $\succ$.

Definition 1. The *inversion distance* between two preferences $\succ, \succ'$, denoted $d(\succ, \succ')$, is the number of pairs of alternatives that are ranked differently by the two preferences, i.e, the number of sets $\{a, b\} \subseteq A$ such that $a \succ b$ and $b \succ' a$ or vice-versa.

It is known that the inversion-distance is a metric on $\mathcal{P}$ [7]. It is clear from the definition that the inversion distance can vary between 0 and $\binom{K}{2}$, the number of subsets of two alternatives.

For example, if there are three alternatives and $a_1 \succ a_3 \succ a_2$ and $a_2 \succ' a_3 \succ' a_1$, then $d(\succ, \succ') = 3$ since all three pairs of alternatives are ranked differently by $\succ$ and $\succ'$.

Definition 2. Let $\succ$ and $\succ'$ be two different preference relations on A, and let $\succ_0 \in P$ be a preference relation on A. We say that $\succ$ *is closer than* $\succ'$ *to* $\succ_0$ if $d(\succ, \succ_0) < d(\succ', \succ_0)$.

Definition 3. Let $\succ_0 \in \mathcal{P}$. A preference profile $\pi \in \mathcal{P}^n$ exhibits *consensus of level*-1 *around* $\succ_0$ if the following two conditions hold:

1. For all pairs of preference relations $\succ, \succ' \in \mathcal{P}$, $\succ \geq_{\succ_0} \succ' \Rightarrow \mu_\pi(\succ) \geq \mu_\pi(\succ')$.

2. There is at least one pair $\succ, \succ' \in \mathcal{P}$, such that $\succ >_{\succ_0} \succ'$ and $\mu_\pi(\succ) > \mu_\pi(\succ')$.

III. OPERATIVE TEST FOR LEVEL-1 CONSENSUS

Given a preference profile $\pi \in \mathcal{P}^n$, we would like to check whether there exists some preference relation $\succ_0$ such that π exhibits level-1 consensus around it. Our operative test relies on the observation that the two conditions in Definition 3 are equivalent to the following:

(Condition 1) For all $\succ', \succ$, if $\mu_\pi(\succ') > \mu_\pi(\succ)$, then $d(\succ', \succ_0) < d(\succ, \succ_0)$.

(Condition 2) There exists a pair $\succ', \succ$ such that $\mu_\pi(\succ') > \mu_\pi(\succ)$.

Our operative test proceeds in several steps.

First, we calculate the frequency $\mu_\pi(\succ)$ of each of the preferences $\succ \in \pi$. Let n' be the number of distinct preferences in π. Note that $n' \leq n$ and also $n' \leq K!$, since with K alternatives there are at most $K!$ possible preferences. Now Condition 2 is easily checked: it is satisfied if-and-only-if (a) there exists a pair of preferences in π with different frequencies, or (b) $n' < K!$ (since this implies that there exists a preference not in π with frequency 0).

If Condition 2 is satisfied, it only remains to check whether Condition 1 is satisfied as well.

We order the preferences in descending order of $\mu_\pi(\succ)$, and rename them $\succ_1, \succ_2, \ldots, \succ_{n'}$, such that $\mu_\pi(\succ_1) \geq \mu_\pi(\succ_2) \geq \cdots \geq \mu_\pi(\succ_{n'})$. This enables us to identify the candidates for level-1-consensus. Since $d(\succ_0, \succ_0) = 0$, Condition 1 immediately implies that each candidate $\succ_0$ must be a preference with maximal frequency. So, the list of candidates are $\succ_1, \succ_2, \ldots, \succ_h$ such that $h \leq n'$ is the largest index for which $\mu_\pi(\succ_1) = \mu_\pi(\succ_2) = \cdots = \mu_\pi(\succ_h)$.

Now we can directly check Condition 1. This condition should be checked separately for each candidate preference $\succ_0$. Given a candidate, we can calculate its inversion-distance from each preference $\succ_i \in \pi$, $d(\succ_i, \succ_0)$. Now, we represent the profile π relative to $\succ_0$ in the form of a scatter-plot, which will lead to a straight-forward assessment of the profile's consensus status. Our scatter-plot is a plot whose x-axis denotes the distance $d(\succ_i, \succ_0)$ and whose y-axis denotes the frequency $\mu_\pi(\succ_i)$. Notice that there may be several different preference relations $\succ_i$ with the same frequency, $\mu_\pi(\succ_i) = m$. Therefore, for each integer value on the y-axis of the scatter-plot, m, we may have several corresponding values on the x-axis, which can be represented by a horizontal segment whose maximum and minimum borders are given by $\max_{i:\mu_\pi(\succ_i)=m} \ d(\succ_i, \succ_0)$ and $\min_{i:\mu_\pi(\succ_i)=m} \ d(\succ_i, \succ_0)$, respectively.

Condition 1 above requires that, for every two frequencies $m_1 > m_2$, *all* preferences with frequency m_1 are closer to $\succ_0$ than *all* preferences with frequency m_2: $\max_{i:\mu_\pi(\succ_i)=m_1} \ d(\succ_i, \succ_0) < \min_{i:\mu_\pi(\succ_i)=m_2} \ d(\succ_i, \succ_0)$. Graphically (see Figure 1), this means that when we scan the scatter plot from top to bottom, we must see non-overlapping intervals ordered strictly from left to right.

Three examples are shown in Figure 1. The left example is positive: there are 5 non-overlapping intervals (two of which consist of a single point), and when they are scanned from top to bottom, they are ordered strictly from left to right. Therefore Condition 1 holds. The middle and right examples are negative: the second and third intervals from the top overlap. For instance, in the middle example the overlap is in a single point, $x = 2$. This point corresponds to two distinct preferences with different frequencies (5 and 4), which are both found at distance 2 from the candidate preference; these preferences violate Condition 1.

The process of 'scanning the scatter plot from top to bottom' can be formalized as follows. Order the list of preferences by a double order criterion: the primary criterion is descending frequency (as before), and the secondary criterion is ascending distance. So the preferences are partitioned to equivalence-classes by their frequency: the classes are ordered by descending frequency, and within each equivalence-class, the preferences are ordered by ascending distance from $\succ_0$. Preferences with both the same frequency and the same

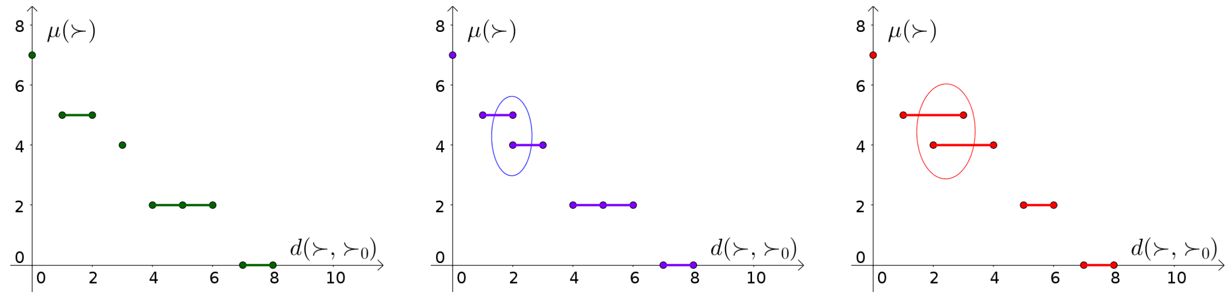

FIG. 1. Scatter-plot representation of profiles relative to a candidate preference enables to graphically check whether level-1 consensus around that candidate is satisfied. The left plot satisfies Condition 1 since the horizontal segments are decreasing and non-overlapping. The middle and right plots violate Condition 1 since there are overlapping segments (circled).

distance are ordered arbitrarily. Under this ordering, the following lemma holds:

Lemma 1. *If condition 1 is violated for* any *pair of preference relations in* π*, then it is violated for an adjacent pair* $\succ_i, \succ_{i+1}$ *for some* i.

The lemma is easy to understand based on the graphical criterion outlined above. A formal proof is provided in Appendix A.

Lemma 1 implies that in order to ensure that Condition 1 is satisfied for all preferences in π, it is sufficient to scan the ordered list of preferences from $\succ_1$ to $\succ_{n'}$, and check if there is some i such that $\mu_\pi(\succ_i) > \mu_\pi(\succ_{i+1})$, yet $d(\succ_i, \succ_0) \geq d(\succ_{i+1}, \succ_0)$.

If Condition 1 holds for all preferences in π, it remains to check that it holds for preferences not in π, i.e, preferences with zero frequency. Let $D = d(\succ_{n'}, \succ_0)$ be the largest distance of a preference $\succ_{n'}$ in π relative to $\succ_0$. Condition 1 implies that, if $\mu_\pi(\succ) = 0$, then $d(\succ, \succ_0) > D$. Therefore, we have to check that the distances between $\succ_0$ and preferences not in π are all larger than D. Equivalently, we can ensure that all preferences with distance at most D are in π. This can be checked by calculating the *number* of possible preferences with distance at most D and verifying that it is equal to the total number n of preferences in π. Since this number involves all possible preferences, it does not depend on the candidate $\succ_0$. Therefore, we can calculate this number assuming w.l.o.g that $\succ_0$ is the identity permutation $1, \ldots, K$. Then, n is the number of permutations on K elements with at most D *inversions* (out-of-order elements). This can be written as:

$$\sum_{j=0}^{D} T(K, j)$$

where $T(k, j)$ is the number of permutations on K elements with *exactly* j inversions.[2]

The number $T(K, j)$ can be calculated using the following recurrence relation[3]:

- $T(K, 0) = 1$: there is exactly one permutation with zero inversions — the identity permutation.
- $T(0, j) = 0$: there are no permutations with 0 elements.
- $T(K, j) = \sum_{i=0}^{\min(K-1,j)} T(K-1, j-i)$: For any permutation of $1, \ldots, K$ with j inversions, let i be the number of elements that come after element K in that permutation. Since K is larger than all other elements, there are exactly i inversions involving K. Therefore, if we remove K, we get a permutation of $1, \ldots, K-1$ with exactly $j - i$ inversions. By summing the counts of these permutations for all possible values of i (namely, $i \geq 0, i \leq j, i \leq K-1$) we get $T(K, j)$.

To summarize, we formally present our algorithm using two procedures.

Procedure Check_Consensus(π):

1. Calculate the frequency $\mu_\pi(\succ)$ of each preference $\succ \in \pi$.
2. Define K = number of alternatives, n' = number of distinct profiles in π.
3. If all frequencies are equal and $n' = K!$, return "no consensus; Condition 2 violated".
4. Order the preferences by descending frequency: $\mu_\pi(\succ_1) \geq \mu_\pi(\succ_2) \geq \cdots \geq \mu_\pi(\succ_{n'})$.
5. Set $M := \mu_\pi(\succ_1)$ to be the maximum frequency of a preference relation.
6. For $j = 1, 2, \ldots n'$ while $\mu_\pi(\succ_j) = M$:
   If Condition_1_Is_Satisfied(π, $\succ_j$), then return $\succ_j$.
7. Return "no consensus; Condition 1 is violated for all candidates".

[2] $T(K, d)$ is also called the *Mahonian number*; see OEIS sequence A008302, https://oeis.org/A008302.

[3] Explained by Vineel Kumar Reddy Kovvuri in http://stackoverflow.com/a/25747326/827927

Procedure Condition_1_Is_Satisfied(π, $\succ_0$):

1. Within every group of preferences with the same frequency, order the preferences by ascending inversion-distance from $\succ_0$.

2. For $i = 1, 2, \ldots, n' - 1$:
   If $\mu_\pi(\succ_i) > \mu_\pi(\succ_{i+1})$ and $d(\succ_i, \succ_0) \geq d(\succ_{i+1}, \succ_0)$, return False.

3. Set $D := d(\succ_{n'}, \succ_0)$. If $\sum_{j=0}^{D} T(K, j) = n$, return True. Else, return False.

We now analyze the run-time of the above algorithm.

In procedure *Condition_1_Is_Satisfied*, we have to calculate the distance between $\succ_0$ and each of the other $n'-1$ preferences. Calculating the inversion distance between a given pair of preferences can be done by a recently-developed algorithm [4] with runtime of $O(K\sqrt{\log K})$. We then have to order the n' distinct preferences and then scan them from top to bottom. Ordering n' items can be done in time $O(n' \log n')$. The value of n' is at most the maximum of n (the number of voters) and $K!$ (the number of possible preferences). So the run-time is simultaneously bounded by $O(K!K \log K)$ and $O(n \cdot K\sqrt{\log K} + n \cdot \log n)$; the former bound is relevant when the number of alternatives is small relative to the number of voters ($K! \leq n$) and the latter when it is large ($K! > n$).

As will be explained below, the probability of having two preferences with exactly the same frequency is low, so in most cases we will have to run the procedure *Condition_1_Is_Satisfied* only once. However, in the improbable case in which there are many preferences with the same frequency, it would have to run at most n' times. Therefore, the worst-case run-time is $O(n' \cdot (n'K\sqrt{\log K} + n' \log n')) = O(n'^2 K\sqrt{\log K} + n'^2 \log n')$.

IV. PROBABILITY OF LEVEL-1 CONSENSUS

Equipped with a procedure for checking level-1 consensus, we set out to check how likely is this property in various settings. We conducted several simulation experiments.

In the first experiment we used the PrefLib database [12], an online database of real-world preference-profiles collected from various sources. This database includes 315 full preference profiles, with different numbers of alternatives and voters; see Table I for statistics. For each preference profile, we used the algorithm described in the previous section to check whether there exists a 1-level consensus. The results were striking: none of the 315 profiles exhibited a level-1 consensus.

In the second experiment we used preference-profiles that were generated according to *Mallows' phi model* [10], which was claimed to favor level-1 consensus [9]. Mallows' model assumes that there is a "correct" preference $\succ_*$, and the actual preferences of the voters are noisy variants of it. The probability of a preference $\succ$ depends on its inversion distance from the correct preference: $d(\succ, \succ_*)$. The strength of this dependence is determined by a parameter $\phi \in (0, 1]$, where lower ϕ means higher dependence; when $\phi \to 0$ the preferences of all voters are identical and equal to $\succ_*$, while when $\phi = 1$ the preference of each voter is selected uniformly at random from the $K!$ possible orderings on K items. In general, the probability of each preference-relation $\succ$ is given by [8]:

$$\text{Prob}\left[\, \succ \quad \mid \; \phi, \succ_* \,\right] = \frac{1}{Z} \cdot \phi^{d(\succ, \succ_*)}$$

where Z is a normalization factor.

We considered all 6 combinations of $K \in \{3, 4, 5\}$ alternatives and $n \in \{100, 1000\}$ voters, where ϕ varied between 0 and 1. For each combination of K, n, ϕ we ran 1000 experiments and calculated (a) the percentage of profiles that exhibit level-1 consensus, (b) the percentage of profiles that are single-peaked,[4] and (c) the percentage of profiles that exhibit a weakened variant of level-1 consensus, which is presented in the next section. The results are shown in Figure 2. As can be expected, consensus always exists when $\phi = 0$, since in this case there is a deterministic consensus on the "true" preference. Additionally, when there are 100 voters and 3 alternatives and ϕ is small, a small positive percentage of the profiles exhibit a level-1 consensus (top left plot). In all other cases, the percentage of level-1 consensus profiles drops to 0 when $\phi \geq 0.05$.

Why is level-1 consensus so rare? Intuitively, the reason is that it requires preferences to have exactly the same frequency in the population. Condition 1 implies that if $d(\succ_i, \succ_0) = d(\succ_j, \succ_0)$ then $\mu_\pi(\succ_i) = \mu_\pi(\succ_j)$. For every $K \geq 3$ and for every candidate $\succ_0$, there exist at least two preferences with the same distance from the candidate, $d(\succ_i, \succ_0) = d(\succ_j, \succ_0)$. Hence, a necessary condition for level-1 consensus is that there exist at least two preferences

[4] This calculation was done for the sake of comparison. It was implemented using Nicholas Mattei's PrefLib tools, which are freely available at GitHub: https://github.com/nmattei/PrefLib-Tools .

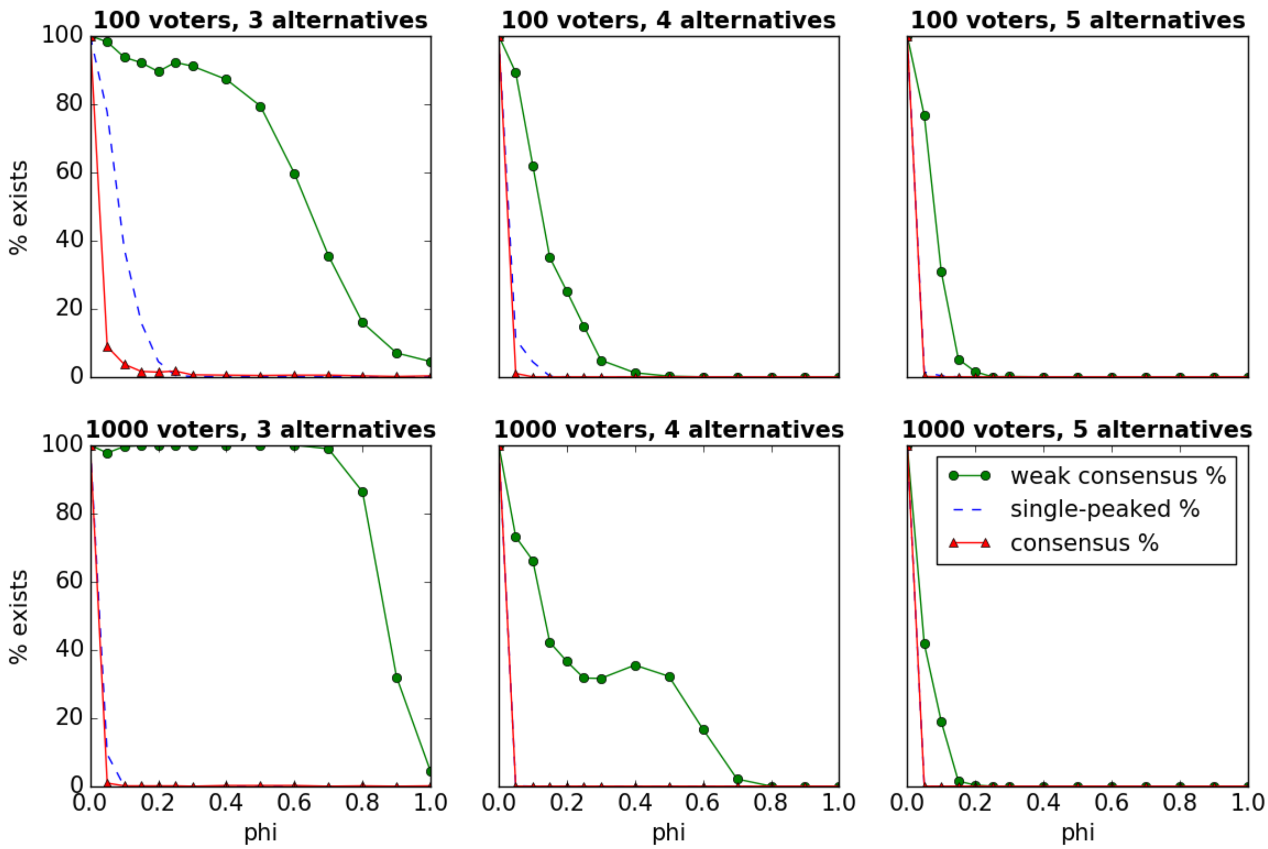

FIG. 2. Percentage of profiles, from a set of profiles selected at random according to Mallows' phi model, which exhibit level-1-consensus or single-peakedness.

with exactly the same frequency. The probability of this event goes to 0 as the number of voters goes to ∞.

Below we present an approximate asymptotic upper bound on the probability of level-1 consensus for the case $\phi = 1$. This is the case of impartial culture, in which all $K!$ preferences are equally probable. Let $\succ_0$ be the fixed preference $K \succ_0 \cdots \succ_0 1$. Recall that $T(K,d)$ is the number of distinct preferences whose inversion-distance from $\succ_0$ is exactly d. Since d varies between 0 and $\binom{K}{2}$:

$$\sum_{d=0}^{\binom{K}{2}} T(K,d) = K!$$

For each d, it is required that all $T(K,d)$ profiles have exactly the same frequency. Let $P(n,K,d)$ be the probability that, with n voters and K alternatives, all profiles with distance

d from $\succ_0$ will have the same frequency. Then, the probability of having level-1 consensus is at most:

$$P(n, K) = \prod_{d=0}^{\binom{K}{2}} P(n, K, d)$$

Under the impartial culture assumption, the frequency of a profile is a random variable distributed like $\textsc{Binom}[n,\ 1/K!]$. Its mean value is $\mu = n/K!$ and its standard deviation is $\sigma = \sqrt{(n/K!) \cdot (1 - 1/K!)}$. We make the following simplifying assumptions:

1. The number of voters n is sufficiently large so that the binomial variable can be approximated by a normal random variable distributed like $\textsc{Normal}[\mu, \sigma]$.

2. The frequencies of different profiles are independent random variables (in fact, the frequencies are dependent. The sum of all frequencies must equal n, so when the frequency of one preference is higher, it is more likely that the frequency of another preference will be lower. However, we are interested in the probability that two variables are equal, which should not be much affected by this dependence).

Under these assumptions, $P(n, K, d)$ can be estimated as the probability that $T(K, d)$ i.i.d. normal random variables will have the same value x, when x is integrated between $-\infty$ and ∞:[5]

$$\begin{aligned} P(n, K, d) &\approx \int_{-\infty}^{\infty} (2\pi\sigma^2)^{-T(K,d)/2} \exp\left(-\frac{T(K, d) \cdot (x - \mu)^2}{2\sigma^2}\right) dx \\ &= (2\pi\sigma^2)^{-T(K,d)/2} \sqrt{\frac{2\pi\sigma^2}{-T(K, d)}} \\ &= \frac{1}{\sqrt{T(K, d)\, (2\pi\sigma^2)^{T(K,d)-1}}} \end{aligned}$$

And their product is:

$$\begin{aligned} P(n, K) &\approx \frac{1}{\prod_{d=0}^{\binom{K}{2}} \sqrt{T(K, d)\, (2\pi\sigma^2)^{T(K,d)-1}}} \\ &= \frac{1}{\sqrt{\prod_{d=0}^{\binom{K}{2}} T(K, d)} \cdot \sqrt{(2\pi\sigma^2)^{\sum_{d=0}^{\binom{K}{2}} (T(K,d)-1)}}} \end{aligned}$$

[5] We are grateful to Hernan J. Gonzalez for his help in solving this problem.

This expression depends on n only through σ. Hence, to get an asymptotic approximation for how $P(n, K)$ changes as a function of n, we remove the factors that do not depend on n:

$$P(n,K) < \frac{1}{\sqrt{(2\sigma^2)^{\sum_{d=0}^{\binom{K}{2}}(T(K,d)-1)}}}$$
$$= \frac{1}{\sqrt{(2\sigma^2)^{K!-\binom{K}{2}-1}}}$$

Now we substitute $\sigma^2 = (n/K!) \cdot (1 - 1/K!) > (n/K!) \cdot (1/2)$ and get:

$$P(n,K) < \frac{1}{\sqrt{(n/K!)^{K!-\binom{K}{2}-1}}}$$

Recall that $P(n, k)$ is the probability of level-1 consensus around a *fixed* preference; the probability of level-1 consensus around *any* preference is, by the union bound, at most this probability times the number of possible preferences, i.e,

$$K! \cdot P(n,K) < \frac{K!}{\sqrt{(n/K!)^{K!-\binom{K}{2}-1}}}$$

For example, with $K = 3$ alternatives we get an upper bound of $6/\sqrt{(n/6)^2} = O(1/n)$; with $K = 4$ alternatives the upper bound is $24/\sqrt{(n/24)^{17}} = O(1/n^{8.5})$. In any case, the upper bound goes to 0 when $n \to \infty$, and the rate of convergence to 0 becomes much faster when K is larger.

V. A WEAKER BUT MORE PROBABLE VARIANT OF LEVEL-1 CONSENSUS

Motivated by the low probability for a level-1 consensus, we suggest below a weakened variant of this property, where Condition 1 is replaced with:

(Condition 1') For all $\succ', \succ$, if $\mu_\pi(\succ') > \mu_\pi(\succ)$, then $d(\succ', \succ_0) \leq d(\succ, \succ_0)$.

It will be shown below that changing $d(\succ', \succ_0) < d(\succ, \succ_0)$ to $d(\succ', \succ_0) \leq d(\succ, \succ_0)$ significantly increases the probability that the condition is satisfied, while keeping the desirable stability properties of the original condition. Moreover, these stability properties hold even without Condition 2.[6]

[6] Our proofs below closely follow the proofs of [9]. Their proofs are stated for level-r consensus for general r, and indeed Condition 1' can also be adapted to general r, but for the sake of simplicity we prefer to focus on the case $r = 1$.

Code	Description	# profiles	# alternatives	# voters
ED-00004 1–100: Netflix Prize Data [2]	Rankings of movies by consumers.	100	3	100–1000
ED-00004 101-200: Netflix Prize Data [2]	Rankings of movies by consumers.	100	4	100–1000
ED-00006: Skate Data	Ranking of skaters by judges in competitions.	20	10–25	8–10
ED-00009: AGH Course Selection	Ranking of courses by university students.	2	7–9	≈ 150
ED-00011: Web Search	Ranking of search-phrases by search-engines.	3	100–250	5
ED-00012: T shirt	Ranking of T-shirt designs by researchers.	1	10	30
ED-00014: Sushi Data	Ranking of sushi kinds by consumers.	1	10	5000
ED-00014: Sushi Data	Ranking of sushi kinds by consumers.	1	10	5000
ED-00015: Clean Web Search	Ranking of search-phrases by search-engines.	79	10–250	4
ED-00024: Mechanical Turk Dots [11]	Ranking of dot-sets by Amazon-Turk workers.	4	4	≈ 800
ED-00025: Mechanical Turk Puzzle [11]	Ranking of puzzles by Amazon-Turk workers.	4	4	≈ 800
ED-00032: Education Surveys	Ranking of issues by informatics students.	1	6	15

TABLE I. Summary of PrefLib [12] data-sets used in our experiments.

Given a preference-profile π, define the *majority relation* M_π as follows: $aM_\pi b$ iff, in a vote between a and b, a beats b by a weak majority. I.e, the number of preferences in π by which $a \succ b$ is at least as large as the number of preferences in π by which $b \succ a$.

Lemma 2. *Let $\pi \in \mathcal{P}^n$ be a preference-profile and $\succ_0 \in \mathcal{P}$ a preference-relation such that Condition 1' is satisfied. Then, for any two alternatives a, b, if $a \succ_0 b$ then $aM_\pi b$.*

Moreover, if n is odd then the opposite is also true: if $aM_\pi b$ then $a \succ_0 b$.

Proof. $\implies$: Suppose that $a \succ_0 b$. Partition $\mathcal{P}$, the set of $K!$ possible preferences, to two subsets:

- The subset $C(a > b)$ containing the $K!/2$ preferences for which $a \succ b$;
- The subset $C(b > a)$ containing the $K!/2$ preferences for which $b \succ a$.

Let $w^{ab} : C(b > a) \to C(a > b)$ be the bijection that takes a preference in $C(b > a)$ and switches a with b in the ranking. Then, since $a \succ_0 b$, for every preference $\succ \in C(b > a)$ it holds that $d(w^{ab}(\succ), \succ_0) < d(\succ, \succ_0)$ (see proof in Appendix B).

By Condition 1', this implies that $\mu_\pi(w^{ab}(\succ)) \geq \mu_\pi(\succ)$. So for every preference $\succ$ by which b is preferred to a, there is a preference $w^{ab}(\succ)$ by which a is preferred to b, which is at least as frequent. Therefore, a beats b by a weak majority: $aM_\pi b$.

$\Longleftarrow$: When n is odd, every majority is strict. Hence, if $aM_\pi b$ then it is not true that $bM_\pi a$. Hence, by the first part of the lemma, it is not true that $b \succ_0 a$. Since $\succ_0$ is a complete ranking, $a \succ_0 b$. □

We denote by $Best(\succ_0)$ the alternative ranked first according to $\succ_0$.

A *Condorcet winner* of π is an alternative a that beats all other alternative by a weak majority, i.e, for any other alternative b, $aM_\pi b$.

Theorem 1. *Let $\pi \in \mathcal{P}^n$ be a preference-profile and $\succ_0 \in \mathcal{P}$ a preference-relation such that Condition 1' is satisfied. Then $Best(\succ_0)$ is a Condorcet winner of π.*

Moreover, if n is odd then $\succ_0$ coincides with the majority relation M_π, and $\succ_0$ is the unique preference in $\mathcal{P}$ for which Condition 1' is satisfied.

Proof. Let $a_1 := Best(\succ_0)$. So for every $b \neq a$, $a_1 \succ_0 b$. By Lemma 2, this implies that $a_1 M_\pi b$. Hence, a_1 is a Condorcet winner of π.

When n is odd, Lemma 2 implies that $a \succ_0 b$ iff $aM_\pi b$, so $\succ_0 \equiv M_\pi$. This is true for any preference in $\mathcal{P}$ for which Condition 1' holds, so any such preference coincides with $\succ_0$. □

A *scoring rule* is a rule characterized by a vector S of K scores, $S_1 \geq \cdots \geq S_K$. Given a profile π, for each preference $\succ \in \pi$, the rule assigns score S_1 to the alternative ranked first by $\succ$, S_2 to the alternative ranked second by $\succ$, and so on. The rule then sums the scores assigned to each alternative by all preferences in π, and selects the alternative/s that received the highest total score.

Lemma 3. *Let $\pi \in \mathcal{P}^n$ be a preference-profile and $\succ_0 \in \mathcal{P}$ a preference-relation such that Condition 1' is satisfied. Then, for any two alternatives a, b and any scoring-rule S, if $a \succ_0 b$ then the score of a is at least as large as the score of b.*

Proof. Partition $\mathcal{P}$ to two halves, $C(a > b)$ and $C(b > a)$, and define the bijection w^{ab} between them, as in the proof of Theorem 1.

For every scoring-rule S and preference $\succ \in \mathcal{P}$, define $\Delta_S(\succ)$ as the difference between the score of a in $\succ$ and the score of b in $\succ$. So:

- For every preference $\succ \in C(a > b)$, $\Delta_S(\succ)$ is weakly-positive.
- For every preference $\succ \in C(b > a)$, $\Delta_S(\succ)$ is weakly-negative.
- For every preference $\succ \in \mathcal{P}$, $\Delta_S(\succ) = -\Delta_S(w^{ab}(\succ))$.

Given the scoring-rule S and the profile π, define $\Delta_S(\pi)$ as the difference between the total score of a in π and the total score of b in π. Then, by definition:

$$
\begin{aligned}
\Delta_S(\pi) &= \sum_{\succ \in \mathcal{P}} \mu_\pi(\succ) \cdot \Delta_S(\succ) \\
&= \sum_{\succ \in C(a>b)} \mu_\pi(\succ) \cdot \Delta_S(\succ) + \sum_{\succ \in C(b>a)} \mu_\pi(\succ) \cdot \Delta_S(\succ) \\
&= \sum_{\succ \in C(a>b)} \mu_\pi(\succ) \cdot \Delta_S(\succ) + \mu_\pi(w^{ab}(\succ)) \cdot \Delta_S(w^{ab}(\succ)) \\
&= \sum_{\succ \in C(a>b)} \mu_\pi(\succ) \cdot \Delta_S(\succ) - \mu_\pi(w^{ab}(\succ)) \cdot \Delta_S(\succ) \qquad \text{(Since } \Delta_S(w^{ab}(\succ)) = -\Delta_S(\succ)\text{)} \\
&= \sum_{\succ \in C(a>b)} \Delta_S(\succ) \cdot [\mu_\pi(\succ) - \mu_\pi(w^{ab}(\succ))]
\end{aligned}
$$

Since $a \succ_0 b$, for every preference $\succ \in C(a > b)$, the lemma in Appendix B implies that $d(w^{ab}(\succ), \succ_0) > d(\succ, \succ_0)$. Hence, by Condition 1', $\mu_\pi(\succ) \geq \mu_\pi(w^{ab}(\succ)$. Hence, all terms in the last sum are weakly-positive. Hence, $\Delta_S(\pi) \geq 0$ and the lemma is proved. □

Theorem 2. *Let $\pi \in \mathcal{P}^n$ be a preference-profile and $\succ_0 \in \mathcal{P}$ a preference-relation such that Condition 1' is satisfied. Then every scoring-rule assigns a highest total score to Best$(\succ_0)$.*

Proof. Follows directly from Lemma 3. □

The procedure for checking Condition 1' is very similar to the one for checking Condition 1 in Section III. There are two differences: the inequality that causes the procedure to fail is $d(\succ_i, \succ_0) > d(\succ_{i+1}, \succ_0)$ (instead of $d(\succ_i, \succ_0) \geq d(\succ_{i+1}, \succ_0)$), and in the last step we have to check that no preference outside π has distance less than D (instead of less-than-or-equal-to D):

Procedure Condition_1'_Is_Satisfied(π, $\succ_0$):

1. Order preferences with the same frequency by ascending inversion-distance from $\succ_0$.

2. For $i = 1, 2, \ldots, n' - 1$:
   If $\mu_\pi(\succ_i) \; > \; \mu_\pi(\succ_{i+1})$ and $d(\succ_i, \succ_0) > d(\succ_{i+1}, \succ_0)$, return False.

3. Set $D := d(\succ_{n'}, \succ_0)$. Set n^* to the number of profiles in π whose distance to $\succ_0$ is at most $D - 1$. If $\sum_{j=0}^{D-1} T(K, j) = n^*$, return True. Else, return False.

We applied the modified procedure to the same experimental settings described in Section IV and estimated the probability of having the weakened level-1-consensus. Out of the 315 PrefLib profiles, **39** exhibit weak level-1 consensus. All 39 profiles are from the dataset labeled "ED-00004 1–100", where all profiles have 3 alternatives. This means that 39% of all profiles with 3 alternatives exhibited the weak level-1 consensus (in contrast to **0** which exhibit level-1 consensus).

The results of the experiments on random profiles are shown in Figure 2; it is evident that in all settings, including the most difficult setting of impartial culture ($\phi = 1$), weak level-1-consensus is substantially more probable than level-1-consensus.

Finally we present a theoretical calculation showing that, even in the impartial culture setting, and even when $n \to \infty$, the probability of weak-level-1-consensus approaches a positive constant (in contrast to the probability of level-1-consensus, which approaches 0).

As explained above, the probability that two or more preferences will have exactly the same frequency goes to 0 when $n \to \infty$, so for simplicity we neglect this possibility and assume that each preference relation has a different frequency. Under this assumption, there always exists a unique preference with maximum frequency; let's call it $\succ_0$. This is the only candidate for satisfying Condition 1'. Below we calculate the probability that Condition 1' holds for this preference.

For every $i \geq 1$, define:

$$F_i \; := \; \{ \; \mu_\pi(\succ) \mid d(\succ, \succ_0) = i \; \}$$

so F_i contains the frequencies of all preferences whose distance from $\succ_0$ is exactly i. Note that F_i is non-empty only when $i \leq \binom{K}{2}$, since $\binom{K}{2}$ is the maximum possible inversion-distance between two preferences on K alternatives.

Condition 1' is equivalent to the requirement that each integer in F_i is larger than each integer in F_j, for every $i < j$. Note that condition 1' does not put any restriction on the frequencies within F_i.

Let $F := \cup_i F_i$ = the set of frequencies of the $K! - 1$ preferences different than $\succ_0$. The total number of different orders on F is $|F|!$. The total number of orders that satisfy Condition 1' is $|F_1|! \cdot |F_2|! \cdots |F_{\binom{K}{2}}|!$. Since all preferences are equally likely, we can assume that all $|F|!$ orders are equally likely. Therefore, the probability that the order of frequencies satisfies Condition 1' is at least:

$$\frac{|F_1||F_2| \cdots |F_{\binom{K}{2}}|}{|F|!} \tag{1}$$

which is a positive constant that does not approach 0 even when $n \to \infty$.

As an illustration, we calculate the probability of Condition 1' for $K = 3$ alternatives. In this case we have $|F_1| = 2$ and $|F_2| = 2$ and $|F_3| = 1$ and $|F| = 2+2+1 = 5$. Therefore, the probability that Condition 1' is satisfied is at least $2! \cdot 2! \cdot 1!/5! = 1/30 \approx 0.033$. Indeed, in our experiments with $\phi = 1$, the fraction of profiles with weak level-1-consensus was 0.043 for 1000 voters (and at most 0.06 for less than 1000 voters). This is slightly higher than the lower bound of 0.033, which can be explained by the fact that, when n is finite, there is a positive probability for having two profiles with the same frequency.

When $K > 3$, the probability of Condition 1' in impartial culture is much lower. For example, for $K = 4$ the lower bound is only about 10^{-12}. Indeed, we found no profiles that exhibit weak level-1 consensus in our experiments with $\phi = 1$ and $K \geq 4$. However, the probability is still positive and does not approach 0.

VI. REPRODUCIBILITY OF EXPERIMENTS

Our experiments can be reproduced by re-running the code, which is freely available through the following GitHub fork: https://github.com/erelsgl/PrefLib-Tools

VII. CONCLUSION

We presented a practical procedure for checking whether a preference profile exhibits a level-1 consensus. Realizing that this property is highly improbable, we have slightly weakened the condition for a level-1 consensus, proving that it preserves the desirable stability of the social choice. Furthermore, the modified condition is considerably more likely to be satisfied. This was demonstrated synthetically for the impartial culture setting and over a database of real-world preference profiles.

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Appendix A: Proof of Lemma 1

This section provides a formal proof to the following lemma used in subsection III.

Lemma 1. Suppose that the preference relations in a profile π are ordered by two criteria: first by frequency $\mu_\pi(\succ_i)$, then by distance $d(\succ_i, \succ_0)$, where $\succ_0$ is a fixed preference. In this ordering, if Condition 1 is violated for *any* pair of preference relations in π, then it is violated for an adjacent pair $\succ_i, \succ_{i+1}$ for some i.

Proof. Suppose that there exist indices $i < j$ such that Condition 1 is violated for the pair $\succ_i, \succ_j$, i.e, $\mu_\pi(\succ_i) > \mu_\pi(\succ_j)$ but $d(\succ_i, \succ_0) \geq d(\succ_j, \succ_0)$. We now prove the lemma by induction on the difference of indices, $j - i$.

Base: If $j - i = 1$, then $\succ_i$ and $\succ_j$ are already adjacent, so we are done.

Step: Suppose that $j - i > 1$. We prove that there exists a pair with a smaller difference that violates condition 1. We consider several cases.

Case #1: i is not the largest index in its equivalence class. i.e, there exists $i' > i$ such that $\mu_\pi(\succ_{i'}) = \mu_\pi(\succ_i)$. Then, by the secondary ordering criterion, $d(\succ_{i'}, \succ_0) \geq d(\succ_i, \succ_0)$, condition 1 is violated for the pair $\succ_{i'}$ and $\succ_j$.

Case #2: j is not the smallest index in its equivalence class. i.e, there exists $j' < j$ such that $\mu_\pi(\succ_{j'}) = \mu_\pi(\succ_j)$. Then, by the secondary ordering criterion, $d(\succ_j, \succ_0) \geq d(\succ_{j'}, \succ_0)$, condition 1 is violated for the pair $\succ_i$ and $\succ_{j'}$.

Otherwise, i is the largest index in its equivalence class, j is the smallest index in its equivalence class, but still $i + 1 < j$. This means that the equivalence classes of i and j are not adjacent, i.e, $\mu_\pi(\succ_i) > \mu_\pi(\succ_{i+1}) > \mu_\pi(\succ_j)$. Now there are two remaining cases:

Case #3: $d(\succ_i, \succ_0) \geq d(\succ_{i+1}, \succ_0)$, in which case condition 1 is violated for the adjacent pair $\succ_i$ and $\succ_{i+1}$ and we are done.

Case #4: $d(\succ_{i+1}, \succ_0) > d(\succ_i, \succ_0)$. This implies $d(\succ_{i+1}, \succ_0) > d(\succ_j, \succ_0)$, so condition 1 is violated for the pair $\succ_{i+1}$ and $\succ_j$ and we are done. □

Appendix B: Proof related to Lemma 2

This section provides a formal proof to an intuitive claim made during the proof of Lemma 2. Let a, b be two fixed alternatives. Let $C(a > b)$ be the set of preferences by which $a \succ b$ and $C(b > a)$ the set of preferences by which $b \succ a$. Let $w^{ab} : C(b > a) \to C(a > b)$ be a function that takes a preference-relation and creates a new preference-relation by switching the position of a and b in the ranking.

Lemma 4. *If $a \succ_0 b$, then for every preference $\succ \in C(b > a)$:*

$$d(\succ, \succ_0) \ > \ d(w^{ab}(\succ), \succ_0)$$

Proof. Define $D(\succ, \succ_0)$ as the set of pairs-of-alternatives $\{i, j\}$ that are ranked differently in $\succ$ and in $\succ_0$. By definition, the inversion distance is the cardinality of this set:

$$d(\succ, \succ_0) = |D(\succ, \succ_0)|$$

so it is sufficient to show that there are more pairs in $D(\succ, \succ_0)$ than in $D(w^{ab}(\succ), \succ_0)$. To show this, we consider all possible pairs-of-alternatives; for each pair, we calculate its contribution to the difference in cardinalities $|D(\succ, \succ_0)| - |D(w^{ab}(\succ), \succ_0)|$, and show that the net difference is positive.

- The pair $\{a, b\}$ is in $D(\succ, \succ_0)$ but not in $D(w^{ab}(\succ), \succ_0)$, so this pair contributes $+1$ to the difference.

- Any pair that contains neither a nor b is not affected by the switch. I.e, each pair $\{c, e\}$ where $c, e \neq a, b$ is in $D(\succ, \succ_0)$ if-and-only-if it is in $D(w^{ab}(\succ), \succ_0)$, so it contributes 0 to the difference.

- Let c be an alternative that is ranked by $\succ$ above b or below a, i.e, either $c \succ b \succ a$ or $b \succ a \succ c$. Then, the order between c to a and b is not affected by the switch, so

$\{c, a\}$ is in $D(\succ, \succ_0)$ if-and-only-if it is in $D(w^{ab}(\succ), \succ_0)$, and similarly for $\{c, b\}$, so it contributes 0 to the difference.

- Let c be an alternative that is ranked by $\succ$ between a and b, i.e, $b \succ c \succ a$. Then, the switch w^{ab} changes the direction of both the pair $\{c, a\}$ and the pair $\{c, b\}$. We have to calculate how much these two pairs contribute to the difference.

  Suppose that the pair $\{c, a\}$ contributes -1 to the difference. This means that this pair is in $D(w^{ab}(\succ), \succ_0)$ but not in $D(\succ, \succ_0)$. This means that $\{c, a\}$ is ranked by $\succ$ the same as by $\succ_0$. By assumption $c \succ a$, so also $c \succ_0 a$.

  By assumption $a \succ_0 b$, so by transitivity, $c \succ_0 b$. But $b \succ c$. This means that the pair $\{c, b\}$ is in $D(\succ, \succ_0)$ but not in $D(w^{ab}(\succ), \succ_0)$, so it contributes $+1$ to the difference.

  Similarly, if the pair $\{c, b\}$ contributes -1 to the difference, then the pair $\{c, a\}$ contributes $+1$.

  Therefore, the net contribution of the two pairs $\{c, a\}$ and $\{c, b\}$ is at least 0.

We proved that the contribution of each pair is at least 0, and the contribution of the pair $\{a, b\}$ is $+1$. Therefore, the difference $d(\succ, \succ_0) - d(w^{ab}(\succ), \succ_0)$ is positive and the lemma is proved. $\square$

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