ALTERNATION ACYCLIC TOURNAMENTS

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ABSTRACT. We define a tournament to be alternation acyclic if it does not contain a cycle in which descents and ascents alternate. Using a result by Athanasiadis on hyperplane arrangements, we show that these tournaments are counted by the median Genocchi numbers. By establishing a bijection with objects defined by Dumont, we show that alternation acyclic tournaments in which at least one ascent begins at each vertex, except for the largest one, are counted by the Genocchi numbers of the first kind. Unexpected consequences of our results include a pair of ordinary generating function formulas for the Genocchi numbers of both kinds and a new very simple model for the normalized median Genocchi numbers.

INTRODUCTION

Genocchi numbers of the first kind are closely related to the Bernoulli and Euler (tangent and secant) numbers, and the first classes of permutations counted by them, introduced by Dumont [9] are *alternating* in one way or another, just like the alternating permutations, counted by the tangent and secant numbers. Whereas the tangent and secant numbers found a geometric interpretation through the work of Purtill [20], Stanley [23] and many people in their wake (using André permutations, first studied by Foata, Schützenberger and Strehl in the 1970-ties [14]), there seems to be far less done in terms of finding geometric interpretations for the various types of Genocchi numbers, studied concurrently with the Genocchi numbers of the first kind. A notable exception is the work of Feigin [12], identifying the Poincaré polynomials of the complete degenerate flag-varieties as q-generalizations of the normalized median Genocchi numbers.

This paper is about the completely accidental discovery of another geometric interpretation of the Genocchi numbers, in the world of hyperplane arrangements. The present author was attempting to attain a better understanding of semiacyclic tournaments, used by Postnikov and Stanley [19], and independently, Shmulik Ravid, to bijectively label the regions created by the Linial arrangement. The subject of this paper is a class of tournaments (we call them *alternation acyclic*), properly containing the class of semiacyclic tournaments. These may be used to bijectively label the regions in a homogeneous variant of the Linial arrangement, which we call the *homogenized Linial arrangement*. The Linial arrangement studied in the literature is a section of our homogenized Linial arrangement (obtained after fixing the values of roughly half of all coordinates at 1). Using the technique of counting points in vector spaces over finite

fields, developed by Athanasiadis [2], we are able to prove that the number of regions created by our homogenized Linial arrangement, and thus the number of alternation acyclic tournaments, is a median Genocchi number. At this time it seems impossible to find a direct combinatorial argument to reproduce this result. On the other hand, using this result it is possible to find a very simple class of objects counted by the median Genocchi numbers, which allow a simple $\mathbb{Z}_2^n$-action, making the known fact transparent, that the median Genocchi number H_{2n+1} is an integer multiple of 2^n. The set of $\mathbb{Z}_2^n$-orbits also has a simple combinatorial representation, perhaps even simpler than the Dellac configurations which are counted by the normalized median Genocchi numbers. Direct counting of alternation acyclic tournaments is not completely impossible: we obtain an explicit combinatorial argument showing that *ascending* alternation acyclic tournaments (in which each numbered element defeats at least one element with a larger number, except for the largest numbered element), are counted by the Genocchi numbers of the first kind. In the general case we obtain recurrences leading to formulas for the ordinary generating functions for the Genocchi numbers of the first and second kinds.

Our paper is structured as follows. After collecting basic facts about Genocchi numbers, hyperplane arrangements in general and the Linial arrangement in particular, we introduce alternation acyclic tournaments in Section 2 and prove their most important properties. In particular, we show that they induce a partial order which we call the right alternating walk order. In Section 3 we show how to encode each alternation acyclic tournament with a pair (π, p), where the permutation π is a linear extension of the alternating walk order, and the parent function p assigns to each element a larger element or the infinity symbol as its parent, thus defining a partial order that is a forest. This representation is not unique, but using them already allows us to show that the regions of a homogenized generalization of the Linial arrangement, introduced in Section 4, are in bijection with our alternation acyclic tournaments. Section 4 also contains the proof of the fact that the number of all alternation acyclic tournaments is a median Genocchi number. We take a closer look at the codes (π, p) in Section 5 and find a way to select unique codes (which we call largest maximal representations) for each alternation acyclic tournament. We also obtain a characterization of all valid codes. Using this characterization we explicitly count ascending alternation acyclic tournaments in Section 6. The key ingredient to obtain this result is a descent-sensitive coding of permutations, using excedant functions, a variant of an idea already present in Dumont's work [9]. Section 7 contains new combinatorial models for the median and normalized median Genocchi numbers. The generating function formulas are derived in Section 8. This paper raises as many questions as it answers: some of these are mentioned in the concluding Section 9.

1. Preliminaries

1.1. **Genocchi numbers.** The *Genocchi numbers G_n of the first kind* are given by the exponential generating function

$$\sum_{n=1}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t + 1}.$$

Their study goes back at least to Seidel [21], who published a triangular table, called *Seidel's triangle*, allowing to compute them recursively. Generalizations and variants of Seidel's triangle include [11, 26]. The first combinatorial models for them were given by Dumont [9]. We will use the following result from his work [9, Corollaire du Théorème 3], which characterizes the signless Genocchi numbers as numbers of *excedant* functions. A function, defined on a set of integers, is *excedant* if it satisfies $f(i) \geq i$ for all i.

Theorem 1.1 (Dumont)**.** *The unsigned Genocchi number $|G_{2n+2}|$ is the number of excedant functions $f : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ satisfying $f(\{1, \ldots, 2n\}) = \{2, 4, \ldots, 2n\}$.*

The following wording (which appears even in the Wikipedia entry for the Genocchi numbers) is easily seen to be equivalent.

Corollary 1.2. *The unsigned Genocchi number $|G_{2n+2}|$ is the number of ordered pairs*

$$((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in \mathbb{Z}^n \times \mathbb{Z}^n$$

such that $1 \leq a_i, b_i \leq i$ hold for all i and the set $\{a_1, b_1, \ldots, a_n, b_n\}$ equals $\{1, \ldots, n\}$.

Indeed, a bijection may be given by associating to each excedant function

$$f : \{1, \ldots, 2n\} \to \{2, 4, \ldots, 2n\}$$

a pair of vectors $((a_1, \ldots, a_n), (b_1, \ldots, b_n))$ given by

$$(a_i, b_i) = \left(n + 1 - \frac{f(2n + 1 - 2i)}{2}, n + 1 - \frac{f(2n + 2 - 2i)}{2} \right) \quad \text{for } i = 1, \ldots, n.$$

Another rephrasing, closer to Dumont's original formulation, in terms of "surjective pistols" may be found in [10] and in [24]. For more information on the Genocchi numbers of the first kind we refer the reader to its entries (A036968 and A001469) in [18].

The *median Genocchi numbers H_{2n-1}*, also called *Genocchi numbers of the second kind*, also appear already in Seidel's triangle. Their study evolved concurrently with the study of the Genocchi numbers of the first kind. For detailed bibliography on them we refer the reader to the above mentioned sources, and their entry (A005439) in [18]. In this paper we will use the following recent result on them, due to Claesson, Kitaev, Ragnarsson and Tenner [8]:

$$H_{2n-1} = \sum_{k=1}^{n} (-1)^{n-k} \cdot (k!)^2 \cdot PS_n^{(k)}. \tag{1.1}$$

Here the numbers $PS_n^{(k)}$ are the *Legendre-Stirling numbers*, see the work of Andrews, Gawronski and Littlejohn [1].

The median Genocchi number H_{2n-1} is known to be an integer multiple of 2^{n-1}, see [4]. The numbers $h_n = H_{2n+1}/2^n$ are the *normalized median Genocchi numbers*. Several combinatorial models of these numbers exists, perhaps the most known are the *Dellac configurations* [7]. Other models may be found in the works of Bigeni [5], Feigin [12, 13] and Han and Zeng [15]. We will present a new combinatorial model for the normalized median Genocchi numbers in Theorem 7.5.

1.2. **Hyperplane arrangements.** A hyperplane arrangement $\mathcal{A}$ is a finite collection of codimension one hyperplanes in a d-dimensional vectorspace over $\mathbb{R}$, which partition the space into regions. The number $r(\mathcal{A})$ of these regions may be found using *Zaslavsky's formula* [25], stating

$$r(\mathcal{A}) = (-1)^d \chi(\mathcal{A}, -1). \tag{1.2}$$

Here $\chi(\mathcal{A}, q)$ is the *characteristic polynomial* of the arrangement, which Zaslavsky expressed in terms of the Möbius function in the intersection lattice of the hyperplanes. Instead of using Zaslavsky's original formulation, we will use the following result of Athanasiadis [2, Theorem 2.2]. In the case when the hyperplanes of $\mathcal{A}$ are defined by equations with integer coefficients, we may consider the hyperplanes defined by the same equations in a vectorspace of the same dimension over a finite field $\mathbb{F}_q$ with q elements, where q is a prime number. If q is sufficiently large, then the number $\chi(\mathcal{A}, q)$ is the number of points in the vector space that do not belong to any hyperplane in the arrangement:

$$\chi(\mathcal{A}, q) = \left| \mathbb{F}_q^d - \bigcup \mathcal{A} \right|. \tag{1.3}$$

1.3. **Semiacyclic tournaments and the Linial arrangement.** This paper is on a class of directed graphs properly containing the class of *semiacyclic tournaments*. A tournament T on the set $\{1, \ldots, n\}$ is a directed graph with no loops nor multiple edges, such that for each pair of vertices $\{i, j\}$ from $\{1, \ldots, n\}$, exactly one of the directed edges $i \to j$ and $j \to i$ belongs to the graph. We consider $\{1, \ldots, n\}$ with the natural order on positive integers. A directed edge $i \to j$ in a cycle is an *ascent* if $i < j$ otherwise it is a *descent*. An *ascending cycle* is a directed cycle in which the number of descents does not exceed the number of ascents. A tournament on $\{1, \ldots, n\}$ is *semiacyclic* if it contains no ascending cycle.

Semiacyclic tournaments arose in the study of the *Linial arrangement* $\mathcal{L}_{n-1}$. This arrangement is the set of hyperplanes

$$x_i - x_j = 1 \quad \text{for } 1 \le i < j \le n$$

in the subspace $V_{n-1} \subset \mathbb{R}^{n-1}$ given by

$$V_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \; : \; x_1 + \cdots + x_n = 0\}.$$

To each region R in $\mathcal{L}_{n-1}$ we may associate a tournament on $\{1, \ldots, n\}$ as follows: for each $i < j$ we set $i \to j$ if $x_i > x_j + 1$ and we set $j \to i$ if $x_i < x_j + 1$. Postnikov and Stanley [19, Proposition 8.5], and independently Shmulik Ravid, observed that, the correspondence above establishes a bijection between the regions of the Linial arrangement $\mathcal{L}_{n-1}$ and the set of semiacyclic tournaments on the set $\{1, \ldots, n\}$.

2. Alternation acyclic tournaments

In this section we define alternation acyclic tournaments and show some of their most basic properties. We define ascents and descents essentially the same way as Postnikov and Stanley do it in [19]. The only minor difference is that we will use the notion of an ascent and a descent on all edges, not only on those contained in a directed cycle.

Definition 2.1. *We call the arrow $i \to j$ an* ascent *if $i < j$ holds, otherwise we call it a* descent. *We will use the notation $i \xrightarrow{a} j$, respectively $i \xrightarrow{d} j$, to denote ascents and descents respectively. A directed cycle $C = (c_0, c_1, \ldots, c_{2k-1})$ is* alternating *if ascents and descents alternate along the cycle, that is, $c_{2j} \xrightarrow{d} c_{2j+1}$ and $c_{2j+1} \xrightarrow{a} c_{2j+2}$ hold for all j (here we identify all indices modulo $2k$). A tournament is* alternation acyclic (or alt-acyclic) *if it contains no alternating cycle.*

Clearly an alternating cycle is also an ascending cycle, hence every semiacyclic tournament is also alternation acyclic. In Section 4 we will state an analogue of [19, Proposition 8.5] for alt-acyclic tournaments, and we will explain how each Linial arrangement is a section of a hyperplane arrangement whose regions are labeled with alt-acyclic tournaments.

A generalization of the notion of a directed cycle is a the notion of a directed closed walk, in which revisiting vertices is allowed. The following, important observation implies that, for tournaments, excluding alternating closed walks vs. alternating cycles makes no difference.

Theorem 2.2. *Suppose a tournament T on $\{1, \ldots, n\}$ contains a closed alternating walk $(c_0, c_1, \ldots, c_{2k-1})$, that is, a closed walk, in which descents and ascents alternate. Then T contains an alternating cycle of length 4.*

Proof. Let $(c_0, c_1, \ldots, c_{2m-1})$ be an alternating closed walk of minimum length in T. It suffices to show that we must have $m = 2$. Indeed, note that a closed alternating walk must have even length, and there is no closed walk $c_0 \to c_1 \to c_0$ in a tournament, so we must have $m \geq 2$. Note also that a closed walk of length 4 must visit 4 distinct vertices, as it can not be the composition of a closed walk of length 3 and one additional edge.

Assume, by way of contradiction that we have $m \geq 3$. As usual, we will identify the indices modulo $2m$. Furthermore, without loss of generality, we will assume that the arrows $c_{2i} \to c_{2i+1}$ are descents and the arrows $c_{2i-1} \to c_{2i}$ are ascents.

It suffices to show that in such a closed alternating walk we must have $c_{2i} < c_{2i+4}$ for all i. Since we assumed $m \geq 3$, this will yield a contradiction of the form $c_0 < c_4 < \cdots < c_0$. We will distinguish two cases:

Case 1: $c_{2i} = c_{2i+3}$ holds. In this case the statement follows from the fact that $c_{2i+3} \to c_{2i+4}$ is an ascent.

Case 2: $c_{2i} \neq c_{2i+3}$ holds. In this case it suffices to show that we must have $c_{2i} < c_{2i+3}$: the statement follows then by transitivity from $c_{2i+3} \xrightarrow{a} c_{2i+4}$. Assume by way of contradiction that $c_{2i} > c_{2i+3}$ holds. Then either we have $c_{2i+3} \xrightarrow{a} c_{2i}$ and $(c_{2i}, c_{2i+1}, c_{2i+2}, c_{2i+3})$ is an alternating 4-cycle, or we have $c_{2i} \xrightarrow{d} c_{2i+3}$ and we may use this descent to replace the subwalk $(c_{2i} \xrightarrow{d} c_{2i+1} \xrightarrow{a} c_{2i+2} \xrightarrow{d} c_{2i+3})$ in the closed walk, thus obtaining a shorter alternating closed walk. In either case, we reach a contradiction with the minimality of m. $\square$

The following consequence of Theorem 2.2 is analogous to a result by Postnikov and Stanley [19, Theorem 8.6] which characterizes semiacyclic tournaments as tournaments containing no ascending cycle of length at most 4.

Corollary 2.3. *A tournament T on $\{1, \ldots, n\}$ is alternation acyclic if and only if it contains no alternating cycle of length 4.*

Another way to characterize alternation acyclic tournaments is to describe them in terms of the right-alternating walk relation.

Definition 2.4. *In a tournament T on $\{1, \ldots, n\}$, there is a* right-alternating walk *from u to v if $u = v$ or there is a directed walk $u = w_0 \xrightarrow{d} w_1 \xrightarrow{a} w_2 \xrightarrow{d} \cdots \xrightarrow{d} w_{2i-1} \xrightarrow{a} w_{2i} = v$ from u to v in which descents and ascents alternate, the first edge being a descent and the last edge being an ascent. We will use the notation $u \leq_{ra} v$ when there is a right-alternating walk from u to v, and we will refer to $\leq_{ra}$ as the* right-alternating walk order *induced by T. We will also use the shorthand notation $u <_{ra} w$ when $u \leq_{ra} v$ and $u \neq v$ hold.*

Proposition 2.5. *A tournament T on $\{1, \ldots, n\}$ is alternation acyclic, if and only the induced right-alternating walk order is a partial order.*

Proof. The relation $\leq_{ra}$ is by definition reflexive and it is obviously transitive, as the concatenation of right-alternating walks yields a right-alternating walk. Hence the relation $\leq_{ra}$ is a partial order if and only if it is antisymmetric. This property is easily seen to be equivalent to the non-existence of a nontrivial closed alternating walk, whose non-existence is equivalent to the non-existence of an alternating 4-cycle by Theorem 2.2. As noted in Corollary 2.3, the non-existence of an alternating 4-cycle is equivalent to the tournament being alt-acyclic. □

Remark 2.6. There is an apparent asymmetry in the definition of the right-alternating walk order. One could analogously define the left-alternating walk order $\leq_{la}$ using alternating walks that begin with an ascent and end with a descent. It is similarly easy to see the analogue of Proposition 2.5 stating that a tournament is alt-acyclic, if and only if $\leq_{la}$ is a partial order. It should be noted that the class of alternation acyclic tournaments is closed under reversing all directed edges and it is also closed under renumbering the vertices such that each $i \in \{1, \ldots, n\}$ is replaced by $n + 1 - i$. Under each of these operations, the role of the partial order $\leq_{ra}$ is taken over by the partial order $\leq_{la}$ and vice versa.

3. Representing alt-acyclic tournaments as biordered forests

A partially ordered set is a *forest* if every element is covered by at most one element. A formula counting linear extensions of a forest is due to Knuth [17]. For a bibliography on generalizations and recent results we refer the reader to [16]. Note that Hivert and Reiner use the dual definition of a forest, in which every element covers at most one element. We follow the definition of Björner and Wachs [6]. In this section we will show that every alternation acyclic tournament may be represented as a tournament induced by a biordered forest, where one of the orders is a linear extension, and the other one is an arbitrary permutation. We will think of the linear extension as a numbering of the elements from 1 to n, and we will encode the second numbering by a word $\pi = \pi(1)\pi(2) \cdots \pi(n)$, where the label of $j \in \{1, \ldots, n\}$ is the position $\pi^{-1}(j)$ of the number j in π

If an element i is covered by an element j in a forest, we will write $j = p(i)$ and say that j is the *parent* of i. We will also use the notation $p(i) = \infty$ when i has no parent, and we will say that i is a *root*. In fact, the Hasse diagram will be a union of trees, and

the roots will be exactly the maximum elements of these trees. Marking the root of each tree defines the partial order. We fix a linear extension of a forest, by numbering its elements in increasing order from 1 to n, where n is the number of the elements. The parent function $p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\}$ must then satisfy $i < p(i)$ for all $i \in \{1, 2 \ldots, n\}$. The converse is also true:

Proposition 3.1. *Given a forest on the set $\{1, 2, \ldots, n\}$, defined by the parent function $p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\}$, the order $1 < 2 < \cdots < n$ is a linear extension of the forest, if and only if the parent function satisfies $i < p(i)$ for all $i \in \{1, 2, \ldots, n\}$.*

Proof. If the numbering represents a linear extension, then the condition $i < p(i)$ is necessary. Conversely, assume the function p satisfies the stated property. If i is less than j in the order of the forest, then for the length ℓ of the shortest path from i to j in the Hasse diagram we have $j = p^\ell(i)$, implying $i < p(i) < p(p(i)) < \cdots < p^\ell(i) = j$. $\quad\square$

From now on we will identify each element of the forest with its label in a fixed linear extension, and we encode the forest with its parent function $p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\}$. Next we give a second labeling of the vertices in terms of the *inverse* of a permutation π of the set $\{1, \ldots, n\}$.

Definition 3.2. *Given a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, we will say that the* labeling induced by the positions in π *is the labeling that associates to each $i \in \{1, 2, \ldots, n\}$ the position $\pi^{-1}(i)$ of i in π.*

Definition 3.3. *We will refer to an ordered triplet of a forest, one of its linear extensions and an arbitrary numbering of its elements as a* biordered forest. *We will encode the forest and its linear extension with the corresponding parent function $p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\}$, satisfying $p(i) > i$ for all i, and the second numbering by the permutation π whose positions induce the second numbering. We will refer to the pair (π, p) as the* code *of the biordered forest.*

The following statement is obvious.

Theorem 3.4. *The correspondence described in Definition 3.3 establishes a bijection between ordered triplets formed by a forest on n elements, a linear extension of this forest, and an arbitrary numbering of its elements and ordered pairs (π, p) formed by a function $p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\}$, satisfying $p(i) > i$ for all i and by a permutation π of $\{1, 2, \ldots, n\}$.*

Remark 3.5. We could also use the language of (P, ω)-partitions, see Stanley's book [22]. This begins with considering a partially ordered set (in our case: a forest) and a bijection $\omega : P \to \{1, 2, \ldots, n\}$. Stanley calls the labeling ω *natural* when ω is a linear extension of P. In such terms, we consider forests with a pair of labelings, one of them natural, the other one arbitrary.

Next we define a tournament induced by a biordered forest.

Definition 3.6. *Let (π, p) be the code of a biordered forest on n elements. We define the tournament $T = T(\pi, p)$ as the tournament induced by the biordered forest to be the tournament whose vertex set is $\{1, 2, \ldots, n\}$ and whose directed edges are defined as follows. For all $u < v$ we set $u \xrightarrow{a} v$ if $p(u) \neq \infty$ and $\pi^{-1}(v) \geq \pi^{-1}(p(u))$ hold, otherwise we set $v \xrightarrow{d} u$.*

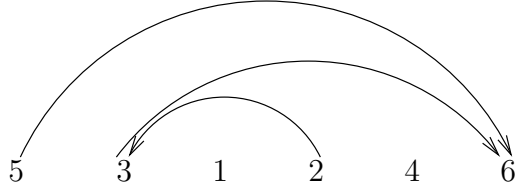We may visualize the pair (π, p) as an arc-diagram, shown in Figure 1.

FIGURE 1. Arc representation of a pair (π, p)

We line up the vertices of the tournament left to right, in the order $\pi(1)$, $\pi(2)$, ..., $\pi(n)$. The permutation π in Figure 1 is 531246. Next, for each i such that $p(i) \neq \infty$, we draw a directed arc from i to $p(i)$. For example, in the picture there is an arc from $\pi(4) = 2$ in position 4 to $\pi(2) = 3$ in position 2, indicating $p(2) = 3$. The number 3 the leftmost number larger than 2 for which $2 \xrightarrow{a} 3$. All numbers larger than 2 that are to the left of 3 defeat 2, and 2 defeats all numbers larger than 2 to the right of 3. Hence we have $5 \xrightarrow{d} 2$, $2 \xrightarrow{a} 3$, $2 \xrightarrow{a} 4$ and $2 \xrightarrow{a} 6$. Similarly we have $p(3) = 6$ and so the only ascent starting at 3 is $3 \xrightarrow{a} 6$. The parent of the numbers $\pi(3) = 1$, $\pi(5) = 4$ and $\pi(6) = 6$ is ∞, no arc begins at these vertices, no ascent starts at these vertices. The parent function p defines a forest with three connected components, the roots of these three trees are 1, 4 and 6, respectively.

The next two statements explain how biordered forests are related to alt-acyclic tournaments.

Theorem 3.7. *Every biordered forest (π, p) induces an alternation acyclic tournament T. Furthermore, the permutation π is a linear extension of the right alternating walk order induced by T.*

Proof. Let (π, p) be the code of the biordered forest on n elements, and let $T = T(\pi, p)$ be the tournament induced by it. First we show that T is alt-acyclic. By Corollary 2.3 it suffices to show that there is no alternating cycle of length 4 in T. Assume, by way of contradiction, that $u_1 \xrightarrow{a} u_2 \xrightarrow{d} u_3 \xrightarrow{a} u_4 \xrightarrow{d} u_1$ holds for some $\{u_1, u_2, u_3, u_4\} \subseteq \{1, 2, \ldots, n\}$. Since we have $u_1 \xrightarrow{a} u_2$ and $u_4 \xrightarrow{d} u_1$ we obtain that u_4 must appear to the left of $p(u_1)$ whereas u_2 can not appear to the left of $p(u_1)$ in π. In other words, we must have
$$\pi^{-1}(u_4) < \pi^{-1}(p(u_1)) \leq \pi^{-1}(u_2).$$
Similarly $u_2 \xrightarrow{d} u_3$ and $u_3 \xrightarrow{a} u_4$ imply that we must have
$$\pi^{-1}(u_2) < \pi^{-1}(p(u_3)) \leq \pi^{-1}(u_4).$$
This is a contradiction, as u_2 and u_4 can not mutually precede each other in π.

To show the second part of the statement it suffices to prove that $\pi^{-1}(u) < \pi^{-1}(v)$ holds whenever v covers u in the right alternating walk order. For an arbitrary pair (u, v), satisfying $u \leq_{ra} v$, the statement $\pi^{-1}(u) \leq \pi^{-1}(v)$ follows then by considering a saturated chain from u to v. If v covers u in the right alternating walk order, then there is a w such that $u \xrightarrow{d} w \xrightarrow{a} v$ holds. By the definition of $T(\pi, p)$, u must be

to the left of $p(w)$, whereas v can not be to the left of $p(w)$ in π. In other words, $\pi^{-1}(u) < \pi^{-1}(p(w)) \leq \pi^{-1}(v)$ must hold, and u is to the left of v in π. $\qquad \square$

Our next result is the converse of Theorem 3.7.

Theorem 3.8. *Let T be an alternation acyclic tournament on $\{1, 2, \ldots, n\}$, and let π be any linear extension of the right alternating walk order. Then there is a unique parent function $p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\}$ such that the tournament induced by (π, p) is T.*

Proof. Clearly, for each $u \in \{1, \ldots, n\}$, the only way to define $p(u)$ is to set $p(u)$ equal to the leftmost v in π such that $u \xrightarrow{a} v$ holds, if such a v exists, and to set $p(u) = \infty$ when no ascent begins at u. We only need to verify that the tournament $T(\pi, p)$ induced by the biordered forest with code (π, p) is the same as the tournament T we started with. Consider a pair (u, v) of vertices satisfying $u < v$. If $p(u) = \infty$ or v is to the left of $p(u) \neq \infty$ then, by the definition of p, we must have $v \xrightarrow{d} u$ in T and also in $T(\pi, p)$. Also by definition, $u \xrightarrow{a} v$ holds in both tournaments, if $v = p(u)$. We are left to consider the case when v is to the right of $p(u)$ in π. In $T(\pi, p)$ we must have $u \xrightarrow{a} v$, the only remaining question is, could we have $v \xrightarrow{d} u$ in the tournament T, for such a vertex v? The answer is no, since $v \xrightarrow{d} u \xrightarrow{a} p(u)$ would imply $v <_{ra} p(u)$ in contradiction with π being a linear extension of the partial order $\leq_{ra}$. $\qquad \square$

Remark 3.9. For any alt-acyclic tournament T, the element 1 is always incomparable to the other elements of $\{1, \ldots, n\}$ in the right alternating walk order, hence the partial order $\leq_{ra}$ has always at least two linear extensions. This makes the use of biordered forests to directly count alt-acyclic tournaments difficult. We will see two different ways to overcome this difficulty in Sections 4 and 5.

4. Counting alternation acyclic tournaments using hyperplane arrangements

In this section we introduce a hyperplane arrangement whose regions are in bijection with the alternation acyclic tournaments. Using a result of Athanasiadis [2], we will be able to count them.

Definition 4.1. *Consider the vector space $\mathbb{R}^{2n-1}$ with coordinate functions $x_1, \ldots, x_n$, $y_1, \ldots, y_{n-1}$. We define the* homogenized Linial arrangement $\mathcal{H}_{2n-2}$ *as the set of hyperplanes*

$$x_i - x_j = y_j \quad 1 \leq i < j \leq n \tag{4.1}$$

in the subspace $U_{2n-2} \subset \mathbb{R}^{2n-1}$ given by

$$U_{2n-2} = \{(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) \in \mathbb{R}^{2n-1} \; : \; x_1 + \cdots + x_n = 0\}.$$

Remark 4.2. Restricting our arrangement in $\mathbb{R}^{2n-1}$ to the set U_{2n-2} does not change the number of regions, because of the following observation: given a point

$$(x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) \in \mathbb{R}^{2n-1},$$

all points of the line

$$\{(x_1 + t, \ldots, x_n + t, y_1, \ldots, y_{n-1}) \; : \; t \in \mathbb{R}\}$$

belong to the same region of $\mathcal{H}_{2n-2}$, considered as a hyperplane arrangement in $\mathbb{R}^{2n-2}$, since $(x_i + t) - (x_j + t) = x_i - x_j$ holds for all $1 \leq i < j \leq n$. There is exactly one choice of t on this line for which the sum of the x-coordinates is zero. Intersecting our picture with U_{2n-2} allows us to get rid of an inessential dimension. It also makes our definition more compatible with the usual definition of the Linial arrangement, due to the following observation. Intersecting $\mathcal{H}_{2n-2}$ with all hyperplanes of the form $y_j = 1$ yields a hyperplane arrangement, which, after discarding the redundant y-coordinates, is exactly the Linial arrangement $\mathcal{L}_{n-1}$.

Next we associate to each region R of the homogenized Linial arrangement $\mathcal{H}_{2n-2}$ a tournament $T(R)$ on $\{1, \dots, n\}$ as follows. For each $i < j$, set $i \to j$ if the points of the region satisfy $x_i - y_i > x_j$, and set $j \to i$ if $x_i - y_i < x_j$ holds for all points in the region. The correspondence $R \mapsto T(R)$ is clearly well-defined and injective.

Theorem 4.3. *The correspondence $R \mapsto T(R)$ establishes a bijection between all regions of the homogenized Linial arrangement $\mathcal{H}_{2n-2}$ and all alternation acyclic tournaments on the set $\{1, \dots, n\}$*

Proof. First we show that every tournament associated to a region is alt-acyclic. Assume, by way of contradiction, that there is a region R, such that the tournament $T(R)$ is not alt-acyclic. By Corollary 2.3 this implies the existence of an alternating 4-cycle $i_1 \xrightarrow{d} i_2 \xrightarrow{a} i_3 \xrightarrow{d} i_4 \xrightarrow{a} i_1$. By the definition of $T(R)$, all points of the region R satisfy

$$x_{i_1} > x_{i_2} - y_{i_2} > x_{i_3} \quad \text{because of } i_1 \xrightarrow{d} i_2 \xrightarrow{a} i_3, \text{ and}$$

$$x_{i_3} > x_{i_4} - y_{i_4} > x_{i_1} \quad \text{because of } i_3 \xrightarrow{d} i_4 \xrightarrow{a} i_1.$$

We obtain the contradiction $x_{i_1} > x_{i_1}$.

Next we show that every alt-acyclic tournament T on $\{1, \dots, n\}$ is of the form $T(R)$ for some region R. Consider an alt-acyclic tournament T. By Theorem 3.8, the tournament T is induced by a biordered forest with code (π, p). Let us set

$$x_i = \frac{n+1}{2} - \pi^{-1}(i) \quad \text{for } i = 1, 2, \dots, n$$

and let us set

$$y_i := \pi^{-1}(p(i)) - \pi^{-1}(i) - 1/2 \quad \text{for } i = 1, \dots, n-1.$$

Observe first that we have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \left(\frac{n+1}{2} - \pi^{-1}(i) \right) = \frac{n(n+1)}{2} - \sum_{i=1}^{n} i = 0,$$

hence the point $(x_1, \dots, x_n, y_1, \dots, y_{n-1})$ belongs to the vector space U_{2n-2}. Observe next, that for each $i < j$ the difference $x_i - x_j = \pi^{-1}(j) - \pi^{-1}(i)$ is the difference between the positions of j and i. This integer is strictly more than $y_i = \pi^{-1}(p(i)) - \pi^{-1}(i) - 1/2$ exactly when $j = p(i)$ or j is to the right of $p(i)$ in π. Therefore $T(R)$ is exactly the tournament induced by the biordered forest whose code is (π, p). $\square$

Now we are ready to prove one of the main results of or paper.

Theorem 4.4. *The number of alternation acyclic tournaments on the set $\{1, \dots, n\}$ is the median Genocchi number H_{2n-1}.*

Proof. By Theorem 4.3 the statement is equivalent to showing that the number of re-
gions in the homogenized Linial arrangement $\mathcal{H}_{2n-2}$ is H_{2n-1}. We will find this number
using Zaslavsky's formula (1.2), where we compute the characteristic polynomial using
Athanasiadis' result (1.3). To simplify our calculations, instead of applying (1.3) to the
hyperplane arrangement $\mathcal{H}_{2n-2}$ directly, we will count the regions of the hyperplane ar-
rangement $\widehat{\mathcal{H}}_{2n}$, given by the equations (4.1) in $\mathbb{R}^{2n}$ with coordinate functions $x_1,\ldots,x_n$,
$y_1,\ldots,y_n$. In other words, rather than removing one inessential dimension by restricting
to the subspace U_{2n-2} (keep in mind Remark 4.2 pointing out that this restriction does
not change the number of regions), we add an additional inessential dimension y_n that is
not involved in the equations defining the hyperplanes. The proof of Theorem 4.3 may
be applied to show that the number of regions is the same as the number of alt-acyclic
tournaments on $\{1,\ldots,n\}$ (with the remark that the value of y_n may be chosen in an
arbitrary fashion). Let us now consider the hyperplane arrangement $\widehat{\mathcal{H}}_{2n}$ as the subset
of $\mathbb{F}_q^{2n}$ for some very large prime q. Let us introduce the shorthand notation $\chi(n,k,q)$
for

$$\left| \left\{ (x_1,\ldots,x_n,y_1,\ldots,y_n) \in \mathbb{F}_q^{2n} - \bigcup \widetilde{\mathcal{H}}_{2n} \; : \; |\{x_1-y_1,\ldots,x_n-y_n\}| = k \right\} \right|$$

that is, the number of those points in $\mathbb{F}_q^{2n}-\bigcup \widetilde{\mathcal{H}}_{2n}$, for which the set $\{x_1-y_1,\ldots,x_n-y_n\}$
has k elements. By (1.3), we must have

$$\chi(\widetilde{\mathcal{H}}_{2n},q) = \sum_{k=1}^{n} \chi(n,k,q). \tag{4.2}$$

We claim that the numbers $\chi(n,k,q)$ satisfy the recurrence

$$\chi(n,k,q) = (q-k)\cdot k\cdot \chi(n-1,k,q) + (q-k+1)^2\cdot \chi(n-1,k-1,q) \quad \text{for } n\geq 2. \tag{4.3}$$

Indeed, let us first select the values of $x_1,\ldots,x_{n-1}$ and $y_1,\ldots,y_{n-1}$ in such a way that
they satisfy those equations from (4.1) which do not involve x_n or y_n. This amounts to
selecting a point in $\mathbb{F}_q^{2n-2}-\bigcup \widetilde{\mathcal{H}}_{2n-2}$ in such a way that the set $\{x_1-y_1,\ldots,x_{n-1}-y_{n-1}\}$
must have k or $k-1$ elements. Depending on the choice between k and $k-1$, this
selection may be performed in $\chi(n-1,k,q)$ or $\chi(n-1,k-1,q)$ ways, respectively. In
the case when $\{x_1-y_1,\ldots,x_{n-1}-y_{n-1}\}$ has k elements, there are $q-k$ ways to select
the value of x_n from the complement of the set $\{x_1-y_1,\ldots,x_{n-1}-y_{n-1}\}$. Once this
selection is made, we may select y_n in k ways, making sure that x_n-y_n belongs to the
set $\{x_1-y_1,\ldots,x_{n-1}-y_{n-1}\}$. Similarly, in the case when $\{x_1-y_1,\ldots,x_{n-1}-y_{n-1}\}$ has
$k-1$ elements, there are $q-k+1$ ways to select the value of x_n, and also $q-k+1$ ways
to select the value of y_n afterward. Both x_n and x_n-y_n must belong to the complement
of $\{x_1-y_1,\ldots,x_{n-1}-y_{n-1}\}$ in this case. Using the initial condition

$$\chi(1,k,q) = \delta_{1,k}q^2 \tag{4.4}$$

(where $\delta_{1,k}$ is the Kronecker delta function), the polynomials $\chi(n,k,q)$ may be com-
puted. Since, for each n, the ambient space is $2n$ dimensional, the number of regions
of $\widetilde{\mathcal{H}}_{2n}$ is equal to

$$(-1)^{2n}\chi\left(\widetilde{\mathcal{H}}_{2n},-1\right) = \chi\left(\widetilde{\mathcal{H}}_{2n},-1\right) = \sum_{k=1}^{n} \chi(n,k,-1).$$

Introducing $r(n, k) := \chi(n, k - 1)$, the initial condition (4.4) yields $r(1, 1) = 1$ and the recurrence (4.3) may be rewritten as

$$r(n, k) = -(k + 1) \cdot k \cdot r(n - 1, k) + k^2 \cdot r(n - 1, k - 1). \tag{4.5}$$

Introducing

$$PS_n^{(k)} = \frac{(-1)^{n-k} \cdot r(n, k)}{(k!)^2},$$

the initial condition $r(1, 1) = 1$ may be transcribed as $PS_1^{(1)} = 1$, and the recurrence (4.5) may be rewritten as

$$PS_n^{(k)} = k(k + 1) \cdot PS_{n-1}^{(k)} + PS_{n-1}^{(k-1)}. \tag{4.6}$$

Equation (4.6) is a recurrence relation satisfied by the Legendre-Stirling numbers, shown by Andrews, Gawronski and Littlejohn [1, Theorem 5.3], and the initial conditions also match. We obtain that $r(n, k) = (-1)^{n-k} \cdot (k!)^2 \cdot PS_n^{(k)}$, and that

$$r\left(\mathcal{H}_{2n-2}\right) = r\left(\widehat{\mathcal{H}}_{2n}\right) = \sum_{k=1}^{n} (-1)^{n-k} \cdot (k!)^2 \cdot PS_n^{(k)}.$$

It was shown in [8] (see Equation (1.1)) that the above sum equals the median Genocchi number H_{2n-1}.

$\square$

5. DIRECT COUNTING USING THE LARGEST MAXIMUM ORDER

By Theorem 3.8, given an alternation acyclic tournament T, after fixing a linear extension π of the partial order $\leq_{ra}$, there is a unique parent function p such that the biordered forest encoded by (π, p) induces T. In this section we fix one such linear extension for each alternation acyclic tournament and describe how to recognize the valid pairs (π, p). This will allow us to directly count alternation acyclic tournaments of various kinds.

Definition 5.1. *For an alternation acyclic tournament T on $\{1, \ldots, n\}$, we define the largest maximal order to be the permutation $\lambda = \lambda(1) \cdots \lambda(n)$, in which for each k, the vertex $\lambda(k)$ is the largest maximal element in the poset obtained by restricting the partial order $\leq_{ra}$ to the set $\{\lambda(1), \ldots, \lambda(k)\}$. We call the unique pair (λ, p) inducing T the largest maximal representation of T.*

Note that the largest maximal order is necessarily a linear extension of the partial order $\leq_{ra}$. For example, the largest maximal order for the tournament induced by the pair (π, p) shown in Figure 1 is 125346, and the largest maximal representation is shown in Figure 2. It is easy to verify that this diagram induces the same tournament, the fact that this is the largest maximal representation will be easily verifiable using Theorem 5.3 below.

Remark 5.2. Consider the largest maximal representation shown in Figure 3. Here $\pi(4) = 4$ is the largest maximal element of the set $\{1, 2, 5, 4\}$ because we have $5 \xrightarrow{d} 3 \xrightarrow{a} 4$ and so $5 <_{ra} 4$ holds. We only discarded the vertex 3 from the set of elements to be considered as a maximal element, but we can not correctly interpret the restriction of
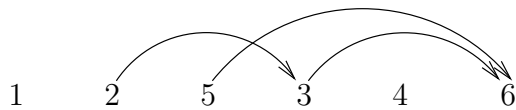
FIGURE 2. Largest maximal representation of the tournament induced in Figure 1

the partial order to the subset $\{1, 2, 5, 4\}$ without considering the relation of 3 to 4 and 5 in the entire tournament.
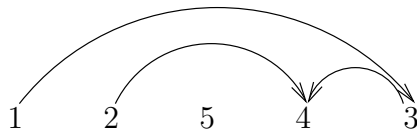

FIGURE 3. Largest maximal representation illustrating the importance of "discarded" elements

The next theorem completely characterizes the largest maximal representations. Recall that $i \in \{1, \ldots, n\}$ is a *descent* of the permutation π of $\{1, \ldots, n\}$ if $\pi(i) > \pi(i+1)$ holds.

Theorem 5.3. *Given a permutation λ of $\{1, \ldots, n\}$ and a parent function*

$$p : \{1, 2, \ldots, n\} \to \{2, \ldots, n\} \cup \{\infty\},$$

the pair (λ, p) is the largest maximal representation of the tournament induced by (λ, p) if and only if for each descent i of λ, the vertex $\lambda(i+1)$ belongs to the range of p.

Proof. Assume first that (λ, p) is a largest maximal representation and that i is a descent of λ. By definition $\lambda(i)$ is a maximal element in the subset $\{\lambda(1), \ldots, \lambda(i)\}$, ordered by $\leq_{ra}$, but it is not a maximal element in the subset $\{\lambda(1), \ldots, \lambda(i), \lambda(i+1)\}$, since $\lambda(i+1)$ is the largest maximal element in the latter set, and it is smaller than $\lambda(i)$. Hence $\lambda(i+1)$ must cover $\lambda(i)$ in the restriction of $\leq_{ra}$ to $\{\lambda(1), \ldots, \lambda(i+1)\}$. Since, for any, $k > i$, the relation $\lambda(k) <_{ra} \lambda(i+1)$ can not hold, the relation $\lambda(i) <_{ra} \lambda(i+1)$ is also a cover relation in the entire set $\{1, \ldots, n\}$. Thus there is a $j \in \{1, \ldots, n\}$ such that $\lambda(i) \xrightarrow{d} j \xrightarrow{a} \lambda(i+1)$ holds. Since $\lambda(i)$ is immediately to the left of $\lambda(i+1)$, we must have $p(j) = \lambda(i+1)$.

Assume next that an alt-acyclic tournament is induced by a code (π, p), in which for each descent i of π, the element $\pi(i+1)$ belongs to the range of p. We will show by induction on k that for each $k \in \{1, \ldots, n\}$ the vertex $\pi(k)$ is the largest maximal element of the set $\{\pi(1), \ldots, \pi(k)\}$. For $k = 1$ we must have $\pi(1) = 1$ since setting $1 = \pi(i+1)$ for some $i \geq 1$ would make i a descent and 1 is never in the range of the parent function p. Assume now that the statement holds for some k and consider the set $\{\pi(1), \ldots, \pi(k+1)\}$. Since, by Theorem 3.7, the permutation π is a linear extension of the partial order $\leq_{ra}$, the element $\pi(k+1)$ is a maximal element of the set $\{\pi(1), \ldots, \pi(k+1)\}$ ordered by $\leq_{ra}$, we only need to show that it is the largest maximal element. There is nothing to prove when $\pi(k) < \pi(k+1)$ holds: adding $\pi(k+1)$ to the set $\{\pi(1), \ldots, \pi(k)\}$ can only decrease the list of the maximal elements and, by our

induction hypothesis, $\pi(k)$ was the largest element on this list before we added $\pi(k+1)$. We are left to consider the case when $\pi(k) > \pi(k+1)$ holds, that is, k is a descent. By our assumption there is a $j < \pi(k+1)$ satisfying $\pi(k+1) = p(j)$. Consider any $i \leq k$ for which $\pi(i) > \pi(k+1)$ holds. This element is to the left of $\pi(k+1) = p(j)$ and it is larger than j. Hence we have $\pi(i) \overset{d}{\to} j \overset{a}{\to} \pi(k+1)$, implying $\pi(i) <_{ra} \pi(k+1)$. We obtained that no element of $\{\pi(1), \ldots, \pi(k+1)\}$ that is larger than $\pi(k+1)$ can be a maximal element in this set, with respect to $\leq_{ra}$. Therefore $\pi(k+1)$ is the largest maximal element. $\square$

Theorem 5.3 allows us to count alt-acyclic tournaments in a recursive fashion, by using the following *reduction* operation.

Definition 5.4. *Given the largest maximal representation* (λ, p) *of an alternation acyclic tournament* T *on* $\{1, \ldots, n\}$ *for some* $n \geq 2$*, we define its* reduction *to the set* $\{1, \ldots, n-1\}$ *to be the alternation acyclic tournament* T' *with largest maximal representation* (λ', p') *where*

$$\lambda'(i) = \begin{cases} \lambda(i) & \text{if } i < \lambda^{-1}(n); \\ \lambda(i+1) & \text{if } i \geq \lambda^{-1}(n); \end{cases} \quad \text{and} \quad p'(i) = \begin{cases} p(i) & \text{if } p(i) \neq n; \\ \infty & \text{if } p(i) = n. \end{cases}$$

In other words, the permutation $\lambda'(1) \cdots \lambda'(n-1)$ *is obtained from* $\lambda(1) \cdots \lambda(n)$ *by deleting the letter* n*, and the parent function* p' *is obtained from* p *by changing all values* $p(i) = n$ *to* $p'(i) = \infty$*.*

Proposition 5.5. *If* (λ, p) *is the largest maximal representation of an alternation acyclic tournament* T *on* $\{1, \ldots, n\}$ *then the pair* (λ', p') *given in Definition 5.4 is the largest maximal representation of an alternation acyclic tournament on* $\{1, \ldots, n-1\}$*.*

Proof. Clearly λ' is a permutation of $\{1, \ldots, n-1\}$, and the function p' maps $\{1, \ldots, n-1\}$ into $\{2, \ldots, n-1\} \cup \{\infty\}$ in such a way that $p'(i) > i$ holds for all i. We only need to verify that for every descent i of λ', the element $\lambda'(i+1)$ is in the range of p'. This is most easily verified by visualizing the reduction operation in terms of the arc representations. In such terms, the reduction operation removes the letter n, and redirects all arrows ending in n to point to ∞. If a letter in λ' is less than the letter immediately preceding it, the same remains true even after inserting the letter n. (Note that $\lambda^{-1}(n)$ is a descent unless $\lambda^{-1}(n) = n$. Finally the range of p' is obtained from the range of p by removing n from it (if it was present). $\square$

Definition 5.6. *We say that an alternation acyclic tournament has* type (n, i, j) *if it is a tournament on* $\{1, \ldots, n\}$*, and the parent function* p *in its largest maximal representation* (λ, p) *satisfies* $|p^{-1}(\infty)| = i$ *and* $|p(\{1, \ldots, n\})| = j+1$*. We will denote the number of alternation acyclic tournaments of type* (n, i, j) *with* $A(n, i, j)$*.*

Note that $p(n) = \infty$ always holds, so $A(n, i, j) > 0$ can only hold when $i \geq 1$. Similarly, $j \geq 0$ must hold.

Theorem 5.7. *The numbers* $A(n, i, j)$ *satisfy the initial condition* $A(1, i, j) = \delta_{i,1} \cdot \delta_{0,j}$ *(where* $\delta_{i,1} \cdot \delta_{0,j}$ *is a product of Kronecker deltas), and the recurrence relation*

$$A(n, i, j) = \sum_{k=i}^{n-1} \binom{k}{i-1} \cdot j \cdot A(n-1, k, j-1) + (j+1) \cdot A(n-1, i-1, j) \quad \text{for } n \geq 2.$$

$$
n=2 \quad
\begin{array}{c|cc}
1 & 1 & \\
0 & 0 & 1 \\
\hline
\end{array}
\begin{array}{c}
j \\ \diagup \\ i
\end{array}
\begin{array}{cc}
1 & 2
\end{array}
\qquad\qquad
n=3 \quad
\begin{array}{c|ccc}
2 & 1 & & \\
1 & 1 & 4 & \\
0 & 0 & 0 & 1 \\
\hline
\end{array}
\begin{array}{c}
j \\ \diagup \\ i
\end{array}
\begin{array}{ccc}
1 & 2 & 3
\end{array}
$$

$$
n=4 \quad
\begin{array}{c|cccc}
3 & 1 & & & \\
2 & 5 & 11 & & \\
1 & 1 & 5 & 11 & \\
0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\begin{array}{c}
j \\ \diagup \\ i
\end{array}
\begin{array}{cccc}
1 & 2 & 3 & 4
\end{array}
\qquad
n=5 \quad
\begin{array}{c|ccccc}
4 & 1 & & & & \\
3 & 16 & 26 & & & \\
2 & 17 & 58 & 66 & & \\
1 & 1 & 6 & 16 & 26 & \\
0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\begin{array}{c}
j \\ \diagup \\ i
\end{array}
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5
\end{array}
$$

TABLE 1. The values of $A(n,i,j)/j!$ for $2 \le n \le 5$

Proof. Suppose we have an alternation acyclic tournament T of type (n,i,j), and consider its reduction T'. We claim that the type of T' must be either $(n-1,k,j-1)$ for some $k \ge i$ or $(n-1,i-1,j)$. Indeed, if n is in the range of p then the range of p' has one less element, and $p'^{-1}(\infty) = p^{-1}(\infty) - \{n\} \cup p^{-1}(n)$ properly contains $p^{-1}(\infty) - \{n\}$. If n is not in the range of p then $p^{-1}(n) = \emptyset$, $p'^{-1}(\infty) = p^{-1}(\infty) - \{n\}$ has exactly one less element than $p^{-1}(\infty)$, and the range of p' equals the range of p.

We claim that any alt-acyclic tournament T' of type $(n-1,k,j-1)$ (where $k \ge i$) is the reduction of exactly $\binom{k}{i-1} \cdot j$ alt-acyclic tournaments of type (n,i,j). Indeed, unless n is inserted as the last letter of λ, it creates a descent, so it must be inserted right before vertex that is in the range of p'. There are j ways to perform this insertion. Furthermore, we must take a $(k-i+1)$-element subset of $p'^{-1}(\infty)$ and reassign them to have n as their parent. A similar, but simpler reasoning shows that for any alt-acyclic tournament T' of type $(n-1,i-1,j)$ there are exactly $(j+1)$ alt-acyclic tournaments of type (n,i,j) whose reduction is T. $\square$

An immediate consequence of Theorem 5.7 is the following.

Corollary 5.8. *The numbers $A(n,i,j)/j!$ are integers, they are given by the initial condition $A(1,i,j)/j! = \delta_{i,1} \cdot \delta_{0,j}$, and the recurrence relation*

$$
\frac{A(n,i,j)}{j!} = \sum_{k=i}^{n-1} \binom{k}{i-1} \cdot \frac{A(n-1,k,j-1)}{(j-1)!} + (j+1) \cdot \frac{A(n-1,i-1,j)}{j!} \quad \text{for } n \ge 2.
$$

We computed the numbers $A(n,i,j)/j!$ using Maple and the formula given in Corollary 5.8 for $n \le 5$. These are given in Table 5. A generating function formula for the numbers $A(n,i,j)/j!$ will be given in Section 8. Inspecting the tables we can make several observations, some of which are easy to show.

Proposition 5.9. $A(n,i,j) = 0$ *holds for $i + j > n$.*

Indeed, for the largest maximal representation (λ, p), to have $j + 1$ elements in the range of p, we need at least j elements of $\{1, \ldots, n\}$ to have a parent different from ∞.

Proposition 5.10. $A(n, i, 0) = \delta_{i,n}$ *where* $\delta_{i,n}$ *is the Kronecker delta.*

Indeed, when the range of p is $\{\infty\}$ then all elements have ∞ as their parent. It is only a little harder to show that in the main diagonal of each table we have the *Eulerian* numbers.

Proposition 5.11. *The number* $A(n, n - j, j)/j!$ *is the number of permutations of* $\{1, \ldots, n\}$ *having exactly j descents.*

Proof. Because of Proposition 5.9, when we set $i = n - j$ the recurrence given in Corollary 5.8, only the term associated to $k = n - j$ will have a positive contribution. By $\binom{n-j}{n-j-1} = n - j$ we get

$$\frac{A(n, n - j, j)}{j!} = (n - j) \cdot \frac{A(n - 1, k, j - 1)}{(j - 1)!} + (j + 1) \cdot \frac{A(n - 1, n - j - 1, j)}{j!}$$

for $n \geq 2$. This is exactly the recurrence for the Eulerian numbers, and the initial conditions match. $\qquad\square$

It may be a little harder to notice that the numbers in the first column multiplied by the factorial of the row index add up to the Genocchi numbers of the first kind, that is,

$$|G_{2n}| = \sum_{j=0}^{n-1} A(n, 1, j). \tag{5.1}$$

We will devote Section 6 to proving Equation (5.1).

6. Counting ascending alternation acyclic tournaments

Definition 6.1. *We call an alternation acyclic tournament T on $\{1, \ldots, n\}$ ascending if every $i < n$ is the tail of an ascent, that is, for each $i < n$ there is a $j > i$ such that $i \to j$.*

Lemma 6.2. *An alternating acyclic tournament T on $\{1, \ldots, n\}$ is ascending if and only if it has type $(n, 1, j)$ for some j.*

Indeed, for any biordered forest inducing T, if (π, p) is the code of the biordered forest, $p(i) = \infty$ holds if and only if i is not the tail of any ascent. An alt-acyclic tournament is ascending if and only if n is the only element of $\{1, \ldots, n\}$ whose parent is ∞.

Because of Lemma 6.2, Equation (5.1) may be rephrased as follows.

Theorem 6.3. *The number of ascending alternation acyclic tournaments on $\{1, \ldots, n\}$ is the unsigned Genocchi number of the first kind $|G_{2n}|$.*

We will prove Theorem 6.3 by showing that the largest maximal representations of such tournaments are in bijection with a set whose cardinality is known to be a Genocchi number of the first kind. The key ingredient in establishing our bijection is the following result.

Theorem 6.4. *There is a bijection between the set of all permutations π of $\{1, \ldots, n\}$ and the set of excedant functions $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that, for each π, a number $k \in \{1, \ldots, n\}$ does not belong to the set $\{f(1), \ldots, f(n)\}$ if and only if $\pi(i + 1) = k$ for some descent i of π.*

Proof. We will describe our bijection using the process of inserting the numbers $1, 2, \ldots, n$ into the permutation π in decreasing order. In order to reduce the number of cases, we place before the first number $\pi(1)$ the number $\pi(0) := 0$ and after the last number $\pi(n)$ the number $\pi(n+1) = n+1$. Thus every number is inserted between two numbers. For example, for $n = 6$ and the permutation $\pi(1)\cdots\pi(6) = 615342$ we have the insertion process

$$07 \to 067 \to 0657 \to 06547 \to 065347 \to 0653427 \to 06153427.$$

The number $f(i)$ is computed in step $n + 1 - i$ when we insert i into the permutation between the numbers u and v, using the following rule:

$$f(i) := \begin{cases} v & \text{if } u > v; \\ \overleftarrow{u} & \text{if } 0 < u < v; \\ i & \text{if } u = 0 \end{cases} \tag{6.1}$$

Here $\overleftarrow{u}$ is the leftmost number w in the current word such that the consecutive subword $w \cdots u$ is *decreasing*, that is, each number in it is smaller than the immediately preceding number. (We have $\overleftarrow{u} = u$ exactly when u is immediately preceded by a smaller number.) In our example we have $(f(1), \ldots, f(6)) = (5, 4, 4, 6, 6, 6)$. In the third step, when we inserted 4 between 5 and 7, we set $f(4) = \overleftarrow{5} = 6$, in the fifth step, when we inserted 2 between 4 and 7, we set $f(2) = \overleftarrow{4} = 4$. In the last step, when we inserted 1 between 6 and 5, we set $f(1) = 5$.

The operation $\pi \mapsto f$ is well-defined. The numbers $f(i)$ clearly satisfy $i \leq f(i) \leq n$. Since the number of all words $\pi(1)\cdots\pi(n)$ is the same as the number of all excedant functions $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$, to show that we defined a bijection, it suffices to show that our assignment is injective: there is at most one way to reconstruct a permutation from an excedant function f.

We always have $f(n) = n$ and the first step is to insert n between 0 and $n+1$, the last line of the definition (6.1) is applicable. Assume, by induction, that there is only one way to reconstruct the insertion of n, $n - 1$, $\ldots$, $i + 1$, based on the knowledge of $f(n)$, $f(n-1)$, $\ldots$, $f(i+1)$. Consider $f(i)$ satisfying $i \leq f(i) \leq n$, and let us show that there is only one way to insert i that yields the given value of $f(i)$. Only the last line of the definition (6.1) allows setting $f(i) = i$, the value of $f(i)$ is greater than i on the other two lines. Thus, in the case when $f(i) = i$, we must insert i right after 0 as the first new number in our permutation. From now on we may assume that $f(i) = w$ for some $w > i$. Let w' be the immediate predecessor of w in our current word. We distinguish two cases depending on how w' and w compare. If $w' > w$ then i can not be inserted anywhere after w, since the only way to obtain $f(i) = w$ would be to insert i between some u and v satisfying $u < v$ and $w = \overleftarrow{u}$. This is impossible: if $w \cdots u$ is a decreasing subword, then so is $w'w \cdots u$ and so $\overleftarrow{u}$ is either w' or an even earlier number. Thus i must be inserted somewhere before w, and the only way to get $f(i) = w$ when w is a number to the right of the place of insertion is to insert i right before w. We are left with the case when $w' < w$. If we insert i anywhere before w, we can not get $f(i) = w$, only w' or a number to the left of it. We must therefore insert i after w in such a way that the second line of (6.1) can be used and it yields $f(i) = w$. We must find a u such that the v succeeding u is larger than u and the subword $w \cdots u$ is decreasing. In

other words, we must take the rightmost u such that $w \cdots u$ is a decreasing consecutive subword.

We are left to show that the set $\{f(1), \ldots, f(n)\}$ contains all numbers between 1 and n except those values that are immediately preceded by a larger number in the permutation $\pi(1) \cdots \pi(n)$. We prove the following generalization of this statement by induction: at step $n + 1 - i$ of the insertion process, the set $\{f(i), \ldots, f(n)\}$ contains all elements of the set $\{i, \ldots, n\}$ except those numbers, which are immediately preceded by a larger number in the current permutation of $n, n - 1, \ldots, i$. At the first step n is inserted and it is preceded by a smaller number. We set $f(n) = n$. Assume the statement is true up to step $n - i$ and consider the insertion of i. If i is inserted right after 0, the current set of numbers immediately preceded by a smaller number does not change, and $f(i) = i$ is added to the set $\{f(i + 1), \ldots, f(n)\}$. In all other cases i is inserted right after a larger number and $i \notin \{f(i), \ldots, f(n)\}$. If i is inserted between u and v satisfying $u > v$, then v which was hitherto immediately preceded by a larger number, it is now immediately preceded by the smaller number i. The set of numbers immediately preceded by a smaller numbers gains i as a new element and loses v as an element, no other change occurs. This change is properly reflected in setting $f(i) = v$. Finally, if $u < v$ holds, then the only change to the set of numbers immediately preceded by a larger number is the addition of i to this set. This is properly handled, if we select $f(i)$ to be a number that is already present in the set $\{f(i + 1), \ldots, f(n)\}$. Selecting $f(i) = \overleftarrow{u}$ fits the bill, as $\overleftarrow{u}$ can not be immediately preceded by a larger number. $\qquad\square$

Definition 6.5. *We call the excedant function* $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *associated to the permutation* π *by the algorithm described in the proof of Theorem 6.4 the* descent-sensitive code *of the permutation* π.

Proof of Theorem 6.3: We count ascending alternation acyclic tournaments by counting the codes (λ, p) of their largest maximal representations. Clearly, for all alt-acyclic tournaments $p(n) = \infty$ must hold, and the tournament is ascending exactly when $p(i) \neq \infty$ holds for all $i < n$. Thus the restriction of p to $\{1, \ldots, n - 1\}$ is a function $\widetilde{p} : \{1, \ldots, n - 1\} \to \{2, \ldots, n\}$, and this restriction $\widetilde{p}$ completely determines p. Let us replace each permutation λ by its descent-sensitive code $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Note that we must have $\lambda(1) = 1$ for the largest maximal order of each alt-acyclic tournament and this equality is equivalent to $f(1) = 1$. Hence the function f is completely defined by its restriction $\widetilde{f}$ to the set $\{2, \ldots, n\}$, which sends this set into itself. The description given in Theorem 5.3 may be restated as follows: the pair of functions (f, p) comes from a largest maximal code (λ, p) if and only if we have

$$\{p(1), \ldots, p(n - 1)\} \cup \{f(2), \ldots, f(n)\} \supseteq \{2, \ldots, n\}. \tag{6.2}$$

Let us associate to each pair of functions

$$(\widetilde{p}, \widetilde{f}) : \{1, \ldots, n - 1\} \times \{2, \ldots, n\} \to \{2, \ldots, n\} \times \{2, \ldots, n\}$$

a function $\widehat{f} : \{1, \ldots, 2n - 2\} \to \{2, 4, \ldots, 2n - 2\}$ as follows:

$$\widehat{f}(i) = \begin{cases} 2\widetilde{f}\left(\frac{i}{2} + 1\right) - 2 & \text{if } i \text{ is even;} \\ 2\widetilde{p}\left(\frac{i+1}{2}\right) - 2 & \text{if } i \text{ is odd.} \end{cases} \tag{6.3}$$

The assignment $(\widetilde{p}, \widetilde{f}) \mapsto \widehat{f}$ is a bijection between the set of all functions

$$(\widetilde{p}, \widetilde{f}) : \{1, \ldots, n-1\} \times \{2, \ldots, n\} \to \{2, \ldots, n\} \times \{2, \ldots, n\}$$

and the set of all functions $\widehat{f} : \{1, \ldots, 2n-2\} \to \{2, 4, \ldots, 2n-2\}$; the inverse is given by the formulas

$$\widetilde{p}(j) = \frac{\widehat{f}(2j-1)}{2} + 1 \quad \text{and} \quad \widetilde{f}(j) = \frac{\widehat{f}(2j-2)}{2} + 1.$$

Clearly $\widehat{f}$ takes only even values. The condition $\widetilde{p}(i) \geq i + 1$ for $1 \leq i \leq n - 1$ is equivalent to $\widehat{f}(2j - 1) \geq 2j$ for $1 \leq j \leq n - 1$. The condition $\widetilde{f}(i) \geq i$ for $2 \leq i \leq n$ is equivalent to $\widehat{f}(2j) \geq 2j$ for $1 \leq j \leq n - 1$. Finally condition (6.2) is equivalent to

$$\widehat{f}(\{1, \ldots, 2n-2\}) = \{2, 4, \ldots, 2n-2\}.$$

We obtain that the number of valid pairs $(\widetilde{p}, \widetilde{f})$ is the same as the number of excedant functions counted in Dumont's Theorem 1.1. $\qquad \square$

7. New combinatorial models for the Genocchi numbers

Adjusting the proof of Theorem 6.3 allows us to state a variant of Dumont's theorem for the median Genocchi numbers. If we drop the requirement of the alt-acyclic tournament being ascending, we may have $p(i) = \infty$ for some $i < n$. Let us define $\widehat{f} : \{1, \ldots, 2n-1\} \to \{1, \ldots, 2n-1\}$ by setting $\widehat{f}(i) = 2n - 1$ when i is odd and $p((i+1)/2) = \infty$, and let us keep the rest of the definition given in (6.3) unchanged. Since the rest of our reasoning remains virtually unchanged and the number of vertices whose parent is ∞ is the second coordinate of the type, we obtain the following result.

Theorem 7.1. *For each $i \in \{1, \ldots, n\}$, the sum $\sum_{j=0}^{n} A(n, i, j)$ equals the number of excedant functions $f : \{1, 2, \ldots, 2n-1\} \to \{1, 2, \ldots, 2n-1\}$ satisfying the following conditions:*

 (1) *$f(2k) \leq 2n - 2$ holds for $k = 1, \ldots, n - 1$;*
 (2) *$f(\{1, 2, \ldots, 2n-1\}) = \{2, 4, \ldots, 2n-2\} \cup \{2n-1\}$;*
 (3) *$|f^{-1}(\{2n-1\})| = i$.*

Taking into account Theorem 4.4, we obtain the following result on the median Genocchi numbers.

Corollary 7.2. *The median Genocchi number H_{2n-1} is the number of excedant functions $f : \{1, 2, \ldots, 2n-1\} \to \{1, 2, \ldots, 2n-1\}$ satisfying $f(2k) \leq 2n - 2$ for $k = 1, \ldots, n - 1$ and*

$$f(\{1, 2, \ldots, 2n-1\}) = \{2, 4, \ldots, 2n-2\} \cup \{2n-1\}.$$

Theorem 7.1 may be rephrased in a way that is analogous to Corollary 1.2. Proving this statement directly is easier than to derive it from Theorem 7.1.

Proposition 7.3. *For each $i \in \{1, \ldots, n\}$, the sum $\sum_{j=0}^{n} A(n, i, j)$ is the number of ordered pairs*

$$((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1})) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$$

satisfying the following conditions:

(1) $0 \le a_k \le k$ *and* $1 \le b_k \le k$ *hold for all* $k \in \{1, \ldots, n-1\}$;
(2) *the set* $\{a_1, b_1, \ldots, a_{n-1}, b_{n-1}\}$ *contains* $\{1, \ldots, n-1\}$;
(3) $|\{k \in \{1, \ldots, n-1\} : a_k = 0\}| = i$.

Proof. Consider the largest maximal representation (λ, p) of an alternation acyclic tournament and let us replace λ with its descent-sensitive code $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$. As noted in the proof of Theorem 6.3 we have $f(1) = 1$ and $p(n) = \infty$, hence the pair (λ, p) is completely determined by the restriction of f to $\{2, \ldots, n\}$ and the restriction of p to $\{1, \ldots, n-1\}$. Let us define the vectors $(a_1, \ldots, a_{n-1})$ and $(b_1, \ldots, b_{n-1})$ by setting

$$a_k = \begin{cases} n + 1 - p(n-k) & \text{if } p(n-k) \ne \infty, \\ 0 & \text{if } p(n-k) = \infty \end{cases}$$

and $b_k = n + 1 - f(n + 1 - k)$ for $k = 1, \ldots, n - 1$. The condition $p(n - k) \in \{n - k + 1, \ldots, n\} \cup \{\infty\}$ is equivalent to $0 \le a_k \le k$ and the condition $n+1-k \le f(n+1-k) \le n$ is equivalent to $1 \le b_k \le k$. Condition (6.2) is equivalent to Condition (2) in our statement. Finally, $|\{k \in \{1, \ldots, n-1\} : a_k = 0\}|$ is clearly the number of elements sent into ∞ by p. $\qquad\square$

In analogy to Corollary 7.2, Theorem 4.4 and Proposition 7.3 have the following consequence.

Corollary 7.4. *The median Genocchi number* H_{2n-1} *is the total number of all ordered pairs*

$$((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1})) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$$

such that $0 \le a_i \le i$ *and* $1 \le b_i \le i$ *hold for all* i *and the set* $\{a_1, b_1, \ldots, a_{n-1}, b_{n-1}\}$ *contains* $\{1, \ldots, n-1\}$.

Corollary 7.4 makes the divisibility of H_{2n-1} by 2^{n-1} especially transparent. Furthermore, it inspires the following, very simple model for the normalized median Genocchi numbers.

Theorem 7.5. *The normalized median Genocchi number* h_n *is the number of sequences* $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_n, v_n\}$ *subject to the following conditions:*
    (1) *the set* $\{u_k, v_k\}$ *is a (one- or two-element) subset of* $\{1, \ldots, k\}$;
    (2) *the set* $\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$ *equals* $\{1, \ldots, n\}$.

Proof. By Corollary 7.4, the median Genocchi number is the number of pairs of vectors $((a_1, \ldots, a_n), (b_1, \ldots, b_n))$ such that $0 \le a_k \le k$ and $1 \le b_k \le k$ hold for all k and the set $\{a_1, b_1, \ldots, a_n, b_n\}$ contains $\{1, \ldots, n\}$. Let us first define a $\mathbb{Z}_2^n$-action of the set of all such vectors. We define the involution ϕ_k for $k \in \{1, \ldots, n\}$ as follows. The map ϕ_k sends

$$((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \quad \text{into}$$
$$((a_1, \ldots, a_{k-1}, a'_k, a_{k+1}, \ldots, a_n), (b_1, \ldots, b_{k-1}, b'_k, b_{k+1}, \ldots, b_n))$$

where

$$(a'_k, b'_k) = \begin{cases} (b_k, a_k) & \text{if } a_k \ne b_k \text{ and } a_k \ne 0; \\ (0, b_k) & \text{if } a_k = b_k; \\ (b_k, b_k) & \text{if } a_k = 0. \end{cases}$$

In other words, the map ϕ_k changes only the k-th coordinates of $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, it swaps a_k and b_k if $\{a_k, b_k\}$ is a two element subset of $\{1, \ldots, k\}$ and it swaps the pair (b_k, b_k) with the pair $(0, b_k)$. Note that in this second case, we have

$$\{a_k, b_k\} \cap \{1, \ldots, k\} = \{b_k\}$$

for $a_k = 0$, as well as for $a_k = b_k$. The action of the involutions ϕ_k is free, as they act on different coordinates. An orbit representative for this action is the sequence of sets

$$\{a_1, b_1\} \cap \{1\}, \{a_2, b_2\} \cap \{1, 2\}, \ldots, \{a_n, b_n\} \cap \{1, \ldots, n\}.$$

In the case when $a_k \neq 0$ we may set $u_k = a_k$ and $v_k = b_k$, and in the case when $a_k = 0$, we may set $u_k = b_k$ and $v_k = b_k$. This orbit representative is valid if and only if the set $\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}$ equals $\{1, \ldots, n\}$. $\qquad\square$

8. Generating function formulas

In this section we prove a generating function formula for the numbers $A(n, i, j)$ introduced in Section 5. Considering Theorems 4.4 and 6.3 we will obtain some formulas for the generating functions of the Genocchi numbers of both kinds which seem hard to see directly.

We begin with introducing the generating function

$$\alpha(x, y, t) = \sum_{n=0}^{\infty} \sum_{i=1}^{n} \sum_{j=0}^{n-1} \frac{A(n, i, j)}{j!} x^i y^j t^n.$$

Let us denote the coefficient of t^n in $\alpha(x, y, t)$ by $\alpha_n(x, y)$.

Corollary 5.8 maybe rewritten as

$$\alpha_1(x, y) = x \quad \text{and} \tag{8.1}$$

$$\alpha_{n+1}(x, y) = xy(\alpha_n(x+1, y) - \alpha_n(x, y)) + x\alpha_n(x, y) + xy\frac{\partial}{\partial y}\alpha_n(x, y) \quad \text{for } n \geq 1. \tag{8.2}$$

Using these equations we may write a linear differential equation for $\alpha(x, y, t)$ in the variable y, which we may attempt to solve by the method known as "variation of parameters". We will not follow through with this process, as it may lead to questions of convergence which we can avoid by "cheating" and simply announcing that we want to introduce the formal power series

$$\beta_n(x, y) = \alpha_n(x, y) \cdot e^{-y}.$$

For these, equations (8.1) and (8.2) may be rewritten as

$$\beta_1(x, y) = xe^{-y} \quad \text{and} \tag{8.3}$$

$$\beta_{n+1}(x, y) = xy\beta_n(x+1, y) + x\beta_n(x, y) + xy\frac{\partial}{\partial y}\beta_n(x, y) \quad \text{for } n \geq 1. \tag{8.4}$$

Let us define the polynomial $\beta_{n,k}(x)$ as the coefficient of y^k in $\beta_n(x, y)$. Equations (8.3) and (8.4) may be transformed into

$$\beta_{1,k}(x) = x \cdot \frac{(-1)^k}{k!} \quad \text{for } k \geq 0 \text{ and} \tag{8.5}$$

$$\beta_{n+1,k}(x) = x(\beta_{n,k-1}(x+1) + (k+1)\beta_{n,k}(x)) \quad \text{for } n \geq 1 \text{ and } k \geq 0. \tag{8.6}$$

Note that (8.6) also holds for $k = 0$, once we set $\beta_{n,-1}(x) = 0$ for all n. Let us set finally $\gamma_{n,k}(x) = \beta_{n,k}(x - k)$. Equations (8.3) and (8.4) may be transformed into

$$\gamma_{1,k}(x, y) = (x - k) \cdot \frac{(-1)^k}{k!} \quad \text{for } k \geq 0 \text{ and} \tag{8.7}$$

$$\gamma_{n+1,k}(x) = (x - k)(\gamma_{n,k-1}(x) + (k + 1)\gamma_{n,k}(x)) \quad \text{for } n \geq 1 \text{ and } k \geq 0. \tag{8.8}$$

Again we set $\gamma_{n,-1}(x) = 0$ for all n. This is an array of polynomials that is easy to compute after introducing

$$\gamma_k(x, t) = \sum_{n=1}^{\infty} \gamma_{n,k}(x)t^n.$$

For $k = 0$, Equation (8.7) and repeated use of Equation (8.8) yields $\gamma_{n,0} = x^n$ for $n \geq 1$. Hence we have

$$\gamma_0(x, t) = \frac{xt}{1 - xt}. \tag{8.9}$$

For $k \geq 1$, Equation (8.8) implies the recurrence

$$\gamma_k(x, t) = \frac{(x - k)t}{1 - (x - k)(k + 1)t} \cdot \left(\frac{(-1)^k}{k!} + \gamma_{k-1}(x, t) \right). \tag{8.10}$$

Using Equations (8.9) and (8.10), an easy induction on k implies

$$\gamma_k(x, t) = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{(k - i)!} \prod_{\ell=0}^{i} \frac{(x - k + \ell)t}{1 - (x - k + \ell)(k + 1 - \ell)t}. \tag{8.11}$$

Next we introduce

$$\widetilde{\beta}_k(x, t) = \sum_{n=0}^{\infty} \beta_{n,k}(x)t^n.$$

The definition of $\gamma_{n,k}(x)$ implies $\beta_{n,k}(x) = \gamma_{n,k}(x + k)$ and $\widetilde{\beta}_k(x, t) = \gamma_k(x + k, t)$. Hence Equation (8.11) may be rewritten as

$$\widetilde{\beta}_k(x, t) = \sum_{i=0}^{k} \frac{(-1)^{k-i}}{(k - i)!} \prod_{\ell=0}^{i} \frac{(x + \ell)t}{1 - (x + \ell)(k + 1 - \ell)t}. \tag{8.12}$$

Finally, as an immediate consequence of the definitions we have

$$\alpha(x, y, t) = \sum_{k=0}^{\infty} \widetilde{\beta}_k(x, t) \cdot y^k \cdot e^y.$$

Combining the last equation with Equation (8.12) we obtain the formula

$$\alpha(x, y, t) = \sum_{j=0}^{\infty} y^j \sum_{k=0}^{j} \frac{1}{(j - k)!} \sum_{i=0}^{k} \frac{(-1)^{k-i}}{(k - i)!} \prod_{\ell=0}^{i} \frac{(x + \ell)t}{1 - (x + \ell)(k + 1 - \ell)t}$$

The last formula may be made look slightly more elegant by observing that

$$\prod_{\ell=0}^{i}(k + 1 - \ell) = (k + 1)!/(k - i)!$$

Therefore we obtain the following result.

Theorem 8.1. *The generating function $\alpha(x,y,t) = \sum_{n,i,j} A(n,i,j)x^i y^j t^n/j!$ is given by*

$$\alpha(x,y,t) = \sum_{j=0}^{\infty} \frac{y^j}{j!} \sum_{k=0}^{j} \binom{j}{k} \frac{1}{k+1} \sum_{i=0}^{k} (-1)^{k-i} \prod_{\ell=0}^{i} \frac{(x+\ell)(k+1-\ell)t}{1-(x+\ell)(k+1-\ell)t}.$$

By Theorem 4.4 the generating function of the median Genocchi numbers H_{2n-1} is obtained by substituting $x = 1$ and replacing each y^j with $j!$ in $\alpha(x,y,t)$. Theorem 8.1 thus has the following consequence.

Corollary 8.2. *The median Genocchi numbers satisfy*

$$\sum_{n=1}^{\infty} H_{2n-1}t^n = \sum_{j=0}^{\infty}\sum_{k=0}^{j} \binom{j}{k} \frac{1}{k+1} \sum_{i=0}^{k} (-1)^{k-i} \prod_{\ell=0}^{i} \frac{(1+\ell)(k+1-\ell)t}{1-(1+\ell)(k+1-\ell)t}.$$

By Theorem 6.3, the generating function of the Genocchi numbers of the first kind is obtained by replacing each y^j by $j!$ and then taking the coefficient of x in in $\alpha(x,y,t)$. To use Theorem 8.1, observe that all powers of x occur in the products of the form

$$\prod_{\ell=0}^{i} \frac{(x+\ell)(k+1-\ell)t}{1-(x+\ell)(k+1-\ell)t}.$$

Here, for $\ell = 0$, the factor

$$\frac{x(k+1)t}{1-x(k+1)t} = x(k+1)t + x^2(k+1)^2 t^2 \cdots$$

has no constant term, and the coefficient of x is $(k+1)t$. We can take out this factor, simplify by (k+1), and only the constant terms of the remaining factors contribute to the coefficient of x. Theorem 8.1 thus has the following consequence.

Corollary 8.3. *The Genocchi numbers of the first kind satisfy*

$$\sum_{n=1}^{\infty} |G_{2n}|t^n = t \cdot \sum_{j=0}^{\infty}\sum_{k=0}^{j} \binom{j}{k} \sum_{i=0}^{k} (-1)^{k-i} \prod_{\ell=1}^{i} \frac{\ell(k+1-\ell)t}{1-\ell(k+1-\ell)t}.$$

9. Concluding remarks

Dumont's first permutation models for the Genocchi numbers were created by finding a class of excedant functions first [9, Corollaire du Théorème 3], and then establishing a bijection between excedant functions and permutations [9, Section 5]. This bijection is very different from, although similar in spirit to our Theorem 6.4. Using the bijection presented in Theorem 6.4, new classes of permutations counted by Genocchi numbers of the first kind may be introduced, however these classes will be very similar if not identical to the examples obtained by Dumont, after combining his bijection with Foata's fundamental transformation [14] which transforms counting excedances into counting descents. Dumont's bijection between permutations and excendant functions makes identifying excedances easy, whereas our bijection is poised on identifying descents. More interesting results could be hoped for by finding new permutation models for median Genocchi numbers using Corollaries 7.2 and 7.4. The curiosity of all results presented in this paper is that objects counted by Genocchi numbers of the first kind

are presented as subsets of objects counted by median Genocchi numbers: it is usually the other way around in the literature.

It seems somewhat difficult to find a bijection between the combinatorial model presented for the normalized median Genocchi numbers in Theorem 7.5 and the Dellac configurations [7] or Feigin's combinatorial model [12, 13]. At the very least, a more direct proof of Theorem 7.5 would be desirable.

This paper arose in a search for generalizations of semiacyclic tournaments that appear in the work of Postnikov and Stanley [19]. In particular, we have found a hyperplane arrangement, whose regions are counted by the median Genocchi numbers, known to be multiples of powers of 2. Semiacyclic tournaments count regions in the Linial arrangement, which is a section of the arrangement we presented in this paper. The number of semiacyclic tournaments on n vertices is known to be

$$2^{-n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}$$

see [19, Theorem 8.1]. It is hard to miss in the above formula that the sum after the factor 2^{-n} is obviously an integer, but not obviously a multiple of 2^n. No combinatorial proof of this divisibility is known, perhaps the q-counting of the regions of the Linial arrangement by Athanasiadis [3] comes closest. Perhaps the q-counting of the regions of our homogenized Linial arrangement, combined with a better understanding how the Linial arrangement appears as a section of our arrangement could help find some additional explanations how divisibility by a power of 2 appears in both settings.

Acknowledgments

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