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THE COHOMOLOGY GROUPS OF REAL TORIC VARIETIES ASSOCIATED TO WEYL CHAMBERS OF TYPE C AND D

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Abstract. Given a root system, the Weyl chambers in the co-weight lattice give rise to a real toric variety, called the real toric variety associated to the Weyl chambers. We compute the integral cohomology groups of real toric varieties associated to the Weyl chambers of type C_n and D_n, completing the computation for all classical types.

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1. Introduction

For a root system of rank n, its Weyl chambers with the co-weight lattice give rise to an n-dimensional non-singular complete fan. By the fundamental theorem of toric geometry, the fan corresponds to a smooth compact toric variety, which is called the *toric variety associated to the Weyl chambers*. The real locus of the toric variety forms a real variety called the *real toric variety associated to the Weyl chambers*. For each irreducible root system of type R, we denote the associated toric variety by X_R and the real toric variety by $X^{\mathbb{R}}_R$. In this paper, we are interested in the topology of $X^{\mathbb{R}}_R$.

In general, since a real toric variety $X^{\mathbb{R}}$ is the fixed point set of the involution in a toric variety X, the rth mod 2-cohomology group of $X^{\mathbb{R}}$ is isomorphic to the $2r$th mod 2-cohomology group of X, which is completely determined by the number of cones of the corresponding fan. Therefore, the mod 2-Betti numbers of $X^{\mathbb{R}}_R$ have been known as corollaries of study of X_R such as [9, 12, 6] and [1]. However, not so much is known about their rational and integral cohomology groups. Here, we review the known results in this direction. For the classical types of A_n and B_n, the rth $\mathbb{Q}$-Betti numbers of $X^{\mathbb{R}}_{A_n}$ and $X^{\mathbb{R}}_{B_n}$ are computed in [7, 4]:

$$\beta^r(X^{\mathbb{R}}_{A_n};\mathbb{Q}) = \binom{n+1}{2r} a_{2r}, \quad \text{and}$$

$$\beta^r(X^{\mathbb{R}}_{B_n};\mathbb{Q}) = \binom{n}{2r} b_{2r} + \binom{n}{2r-1} b_{2r-1},$$

where $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sec x + \tan x$ and $\sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \frac{1}{\cos x - \sin x}$. Remark that a_n is known as the nth Euler zigzag number (see A000111 of [11]), and b_n is known as the nth generalized Euler number or

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Springer number (see A001586 of [11]). See Table 1. (We set $a_k = b_k = \binom{n}{k} = 0$ for a negative integer $k < 0$ as a convention.)

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
a_n	1	1	1	2	5	16	61	272	1385	7936	$\cdots$
b_n	1	1	3	11	57	361	2763	24611	250737	2873041	$\cdots$

TABLE 1. The list of a_n and b_n for small n

For the exceptional types of $R = G_2$, F_4, and E_6, the non-zero $\mathbb{Q}$-Betti numbers of $X_R^{\mathbb{R}}$ have been computed in [3]. See Table 2.

Types R	$\beta^0(X_R^{\mathbb{R}})$	$\beta^1(X_R^{\mathbb{R}})$	$\beta^2(X_R^{\mathbb{R}})$	$\beta^3(X_R^{\mathbb{R}})$
G_2	1	9		
F_4	1	57	264	
E_6	1	36	1323	4392

TABLE 2. The list of non-zero Betti numbers of $X_{G_2}^{\mathbb{R}}$, $X_{F_4}^{\mathbb{R}}$ and $X_{E_6}^{\mathbb{R}}$

It should be noted that the real toric varieties in the above cases have only 2-torsion in the cohomology groups. Hence, by the universal coefficient theorem, one can compute their integral cohomology groups.

In this paper, we compute the rational Betti number of real toric varieties associated to the Weyl chambers of classical type C and D. Furthermore, we show that they have only 2-torsion in the cohomology group. By the universal coefficient theorem, our results completely determine the integral cohomology groups of $X_{C_n}^{\mathbb{R}}$ and $X_{D_n}^{\mathbb{R}}$, thus completing the computation of the integral cohomology groups of the real toric varieties associated to the Weyl chambers of all classical types.

Theorem 1.1. *Let $s_m = 2^m - 1$ and $t_m = (m-2)2^{m-1} + 1$. The rth $\mathbb{Q}$-Betti numbers of $X_{C_n}^{\mathbb{R}}$ $(n \geq 3)$ and $X_{D_n}^{\mathbb{R}}$ $(n \geq 4)$ are as follows.*

$$\beta^r(X_{C_n}^{\mathbb{R}};\mathbb{Q}) = \binom{n}{2r-2} 2^{2r-2} s_{n-2r+2} a_{2r-2} + \binom{n}{2r}(2b_{2r} - 2^{2r} a_{2r}), \qquad \text{and}$$
$$\beta^r(X_{D_n}^{\mathbb{R}};\mathbb{Q}) = \binom{n}{2r-4} 2^{2r-4} t_{n-2r+4} a_{2r-4} + \binom{n}{2r}(2b_{2r} - 2^{2r} a_{2r}),$$

where $a_k = b_k = \binom{n}{k} = 0$ for a negative integer $k < 0$. Furthermore, $X_{C_n}^{\mathbb{R}}$ and $X_{D_n}^{\mathbb{R}}$ have only 2-torsion in the cohomology groups.

Tables 3 and 4 are the lists of non-zero $\mathbb{Q}$-Betti numbers of $X_{C_n}^{\mathbb{R}}$ and $X_{D_n}^{\mathbb{R}}$ for $n \leq 11$. The Euler characteristic of X is denote by $\chi(X)$.

n	$\beta^0(X_{C_n}^{\mathbb{R}})$	$\beta^1(X_{C_n}^{\mathbb{R}})$	$\beta^2(X_{C_n}^{\mathbb{R}})$	$\beta^3(X_{C_n}^{\mathbb{R}})$	$\beta^4(X_{C_n}^{\mathbb{R}})$	$\beta^5(X_{C_n}^{\mathbb{R}})$	$\beta^6(X_{C_n}^{\mathbb{R}})$	$\chi(X_{C_n}^{\mathbb{R}})$
3	1	13	12					0
4	1	27	106					80
5	1	51	450	400				0
6	1	93	1410	5222				-3904
7	1	169	3794	30954	27328			0
8	1	311	9436	129416	474850			354560
9	1	583	22572	448728	3617778	3191040		0
10	1	1113	53040	1399020	18908730	69295142		-51733504
11	1	2157	123640	4102164	80153898	645241762	569068544	0

TABLE 3. The list of non-zero Betti numbers of $X_{C_n}^{\mathbb{R}}$ for $n \leq 11$

Only two exceptional cases remain to be open.

Question 1.2. Compute the integral cohomology groups of $X_{E_7}^{\mathbb{R}}$ and $X_{E_8}^{\mathbb{R}}$.

n	$\beta^0(X^{\mathbb{R}}_{D_n})$	$\beta^1(X^{\mathbb{R}}_{D_n})$	$\beta^2(X^{\mathbb{R}}_{D_n})$	$\beta^3(X^{\mathbb{R}}_{D_n})$	$\beta^4(X^{\mathbb{R}}_{D_n})$	$\beta^5(X^{\mathbb{R}}_{D_n})$	$\beta^6(X^{\mathbb{R}}_{D_n})$	$\chi(X^{\mathbb{R}}_{D_n})$
4	1	12	51	24				16
5	1	20	219	200				0
6	1	30	639	2642	1200			-832
7	1	42	1511	15470	14000			0
8	1	56	3149	59864	242114	109312		76032
9	1	72	6077	182472	1816146	1639680		0
10	1	90	11237	479040	8778330	35366822	15955200	-11101184
11	1	110	20437	1143824	32715210	324103714	292512000	0

TABLE 4. The list of non-zero Betti numbers of $X^{\mathbb{R}}_{D_n}$ for $n \leq 11$

2. Real toric varieties associated to Weyl chambers

Let X be a smooth compact toric variety and $X^{\mathbb{R}}$ its real toric variety. By the fundamental theorem of toric geometry, X is assigned by a non-singular complete fan Σ. Assume that Σ has m rays, and we put $[m] = \{1, 2, \ldots, m\}$ the set of rays of Σ. We obtain a (star-shaped) simplicial complex K on $[m]$, whose simplices correspond to the sets of rays spanning cones of Σ. In addition, the primitive integral vectors of the rays define a map λ from $[m]$ to $\mathbb{Z}^n$; $\lambda(v)$ is the direction of the ray $v \in [m]$. Note that Σ is completely determined by a pair of K and λ, and, therefore, so is X.

Similarly, the real toric variety $X^{\mathbb{R}}$ is also determined by a pair of K and the composition $\lambda^{\mathbb{R}} \colon [m] \xrightarrow{\lambda} \mathbb{Z}^n \xrightarrow{\text{mod } 2} \mathbb{Z}_2^n$, where $\mathbb{Z}_2$ is the field with two elements. Throughout the paper, we identify the power set $2^{[m]}$ of $[m]$ with $\mathbb{Z}_2^m$ in the following way: for $S \subset [m]$, the corresponding element $S \in \mathbb{Z}_2^m$ is $\sum_{i \in S} \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith standard vector of $\mathbb{Z}_2^m$. Then, we may regard $\lambda^{\mathbb{R}}$ as the linear map $\Lambda \colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$, which is called the *characteristic matrix* of $X^{\mathbb{R}}$. We make note of the fact that $X^{\mathbb{R}}$ does not depend on the choice of a basis of $\mathbb{Z}_2^n$. Therefore, topological invariants of $X^{\mathbb{R}}$ should be computed in terms of K and Λ. Our method in this paper is based on the following cohomology formulae of real toric varieties.

Theorem 2.1 ([13, 5, 2])**.** *Let k be a positive integer. Then, as graded modules,*

$$H^*(X^{\mathbb{R}}; \mathbb{Q}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda)} \widetilde{H}^{*-1}(K_S; \mathbb{Q}),$$

$$H^*(X^{\mathbb{R}}; \mathbb{Z}_{2k+1}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda)} \widetilde{H}^{*-1}(K_S; \mathbb{Z}_{2k+1}), \qquad \text{and}$$

$$H^*(X^{\mathbb{R}}; \mathbb{Z}_{2^k+1}) \cong \bigoplus_{S \in \mathrm{Row}(\Lambda)} \widetilde{H}^{*-1}(K_S; \mathbb{Z}_{2^k}),$$

where $\mathrm{Row}(\Lambda)$ is the subspace of $\mathbb{Z}_2^m$ spanned by the rows of Λ, and K_S is the induced subcomplex of K on the vertices indicated by $S \in 2^{[m]} = \mathbb{Z}_2^m$.

In the rest of this section, we review the construction of the non-singular complete fan associated to Weyl chambers (see, for example, [12]). Let V be a finite dimensional real Euclidean space, and $\Phi_R \subset V$ an irreducible root system of type R. Take a fixed set of simple roots $\Delta_R = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi_R$. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ be the set of fundamental co-weights, that is, $(\omega_i, \alpha_j) = \delta_{ij}$ with respect to the inner product in the ambient space V. Then, we can assume $V = \mathbb{R}\langle \Omega \rangle$. Consider the co-weight lattice $\mathbb{Z}\langle \Omega \rangle = \{v \in V \mid (v, \alpha) \in \mathbb{Z} \text{ for any } \alpha \in \Phi_R\}$. Denote by $W_R = \langle s_1, \ldots, s_n \rangle$ the Weyl group generated by the simple reflections acting on V. Then, W_R decomposes V into several chambers, giving a non-singular complete fan in $\mathbb{R}\langle \Omega \rangle$ with the lattice structure $\mathbb{Z}\langle \Omega \rangle \subset \mathbb{R}\langle \Omega \rangle$. Let $V_R = W_R \cdot \Omega = \{v_1, \ldots, v_m\}$ be the set of rays spanning the chambers, and $K_R \subset 2^{V_R}$ the corresponding simplicial complex called the *Coxeter complex* of type R [8, §1.15], whose maximal simplices are the chambers. Then the corresponding characteristic matrix is obtained as $\Lambda_R = (v_1, v_2, \ldots, v_m)$, where the columns are the mod 2 coordinates of the rays with respect to any basis of $\mathbb{Z}\langle \Omega \rangle$. Observe that the maximal cones in the fan correspond to sets of simple roots; for each set of simple roots Δ in Φ_R the corresponding cone is $C_\Delta = \{v \in V \mid (v, \alpha) > 0 \text{ for any } \alpha \in \Delta\}$. This observation is useful to compute K_R and Λ_R explicitly.

We fix some notations which are used later;

- $[n] = \{1, 2, \dots, n\}$ and $[\pm n] = \{\pm 1, \pm 2, \dots, \pm n\}$.
- For $I \subset [\pm n]$, $I^+ = \{i \in [n] \mid i \in I\}$.
- For $I \subset [\pm n]$, $I^- = \{i \in [n] \mid -i \in I\}$.
- For $I \subset [\pm n]$, $I^\pm = I^+ \cup I^-$.

We also recall the following facts for the proof of the main theorem.

Lemma 2.2 (Section 3 of [14]). *The cohomology groups of $\mathcal{C}_{2r}^{odd}$ and $\mathcal{B}_{2r}^{odd}$ are*

$$\widetilde{H}^*(\mathcal{C}_{2r}^{odd};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{b_n} & (* = r-1) \\ 0 & (* \neq r-1), \end{cases} \qquad \widetilde{H}^*(\mathcal{B}_{2r}^{odd};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{a_n} & (* = r-1) \\ 0 & (* \neq r-1), \end{cases}$$

where $\mathcal{C}_{2r}^{odd}$ is the poset complex of the odd rank-selected lattice of faces of the cross-polytope over $2r$ points, and $\mathcal{B}_{2r}^{odd}$ is the poset complex of the odd rank-selected Boolean algebra on $2r$ points.

3. Type C_n

In this section, since $\Phi_{C_n} = \Phi_{B_n}$ for $n \leq 2$, we assume that $n \geq 3$. The root system Φ_{C_n} of type C_n consists of $2n^2$ roots

$$\pm 2\varepsilon_i \ (1 \leq i \leq n) \quad \text{and} \quad \pm \varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq n),$$

where ε_i is the ith standard vector of $\mathbb{R}^n = V$. Then, the co-weight lattice $\mathbb{Z}\langle\Omega\rangle$ is

$$\mathbb{Z}\langle\Omega\rangle = \left\{ \frac{1}{2}(\ell_1\varepsilon_1 + \cdots + \ell_n\varepsilon_n) \ \middle|\ \ell_i \in \mathbb{Z}, \text{ and } \ell_i \equiv \ell_j \pmod 2 \text{ for all } i, j \right\}.$$

Choose a basis $\{\epsilon_i \mid 1 \leq i \leq n\}$ of $\mathbb{Z}\langle\Omega\rangle$ defined by $\epsilon_i := \varepsilon_i$ for $i = 1, \dots, n-1$ and $\epsilon_n := \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$.

Any set of simple roots of type C_n is written as

$$\Delta = \{\mu_1\varepsilon_{\sigma(1)} - \mu_2\varepsilon_{\sigma(2)}, \mu_2\varepsilon_{\sigma(2)} - \mu_3\varepsilon_{\sigma(3)}, \dots, \mu_{n-1}\varepsilon_{\sigma(n-1)} - \mu_n\varepsilon_{\sigma(n)}, 2\mu_n\varepsilon_{\sigma(n)}\},$$

where $\mu_j = \pm 1$ and $\sigma\colon [n] \to [n]$ is a permutation. Note that every line containing a ray in V_{C_n} is the intersection of $n-1$ hyperplanes normal to $\Delta \setminus \{\alpha\}$ for $\alpha \in \Delta$, and the direction of the ray is determined by α. For $\alpha \in \Delta$, there exists a unique primitive integral vector $\beta_{\alpha,\Delta} \in \mathbb{Z}\langle\Omega\rangle$ such that $(\beta_{\alpha,\Delta}, \alpha') = 0$ for all $\alpha' \in \Delta \setminus \{\alpha\}$ and $(\beta_{\alpha,\Delta}, \alpha) > 0$. More precisely, $\beta_{\alpha,\Delta} = \sum_{k=1}^{i} \mu_k\varepsilon_{\sigma(k)}$ for $\alpha = \alpha_i^\sigma$, where $\alpha_i^{\mu,\sigma} := \mu_i\varepsilon_{\sigma(i)} - \mu_{i+1}\varepsilon_{\sigma(i+1)}$ for $i = 1, \dots, n-1$ and $\alpha^{\mu,\sigma_n} := 2\mu_n\varepsilon_{\sigma(n)}$. We label the ray corresponding to $\beta_{\alpha,\Delta}$ by the subset $I = \{\mu_1\sigma(1), \dots, \mu_i\sigma(i)\}$ of $[\pm n]$. We note that the label subset I satisfies

$$I^+ \cap I^- = \emptyset. \tag{1}$$

Conversely, for each $I \subset [\pm n]$ satisfying (1), one can find α and Δ such that $\alpha \in \Delta$ and $\beta_{\alpha,\Delta} = \beta_I := \sum_{k\in I} \operatorname{sign}(k)\varepsilon_{|k|}$. Therefore, we have an identification of V_{C_n} as

$$V_{C_n} := \{I \subset [\pm n] \mid I^+ \cap I^- = \emptyset\}.$$

Under this identification, a maximal cone C_Δ corresponds to a sequence of nested sets $I_1 \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n$.

Now, let us consider the characteristic map λ_{C_n}. For a vertex $I \in V_{C_n}$, $\lambda_{C_n}(I)$ is the primitive vector in $\mathbb{Z}\langle\Omega\rangle$ in the direction of β_I. If neither n nor $-n$ is contained in I,

$$\beta_I = \sum_{k\in I^+} \varepsilon_k - \sum_{k\in I^-} \varepsilon_k = \sum_{k\in I^+} \epsilon_k - \sum_{k\in I^-} \epsilon_k.$$

Otherwise,

$$\beta_I = \begin{cases} \left(\displaystyle\sum_{k\in I^+\setminus\{n\}} \epsilon_k - \sum_{k\in I^-} \epsilon_k \right) + 2\epsilon_n - \epsilon_1 - \cdots - \epsilon_{n-1}, & \text{if } n \in I; \\ \left(\displaystyle\sum_{k\in I^+} \epsilon_k - \sum_{k\in I^-\setminus\{n\}} \epsilon_k \right) - 2\epsilon_n + \epsilon_1 + \cdots + \epsilon_{n-1}, & \text{if } -n \in I. \end{cases}$$

Since β_I is primitive in $\mathbb{Z}\langle\Omega\rangle$ if and only if $|I| \le n-1$, we see

$$\lambda_{C_n}(I) = \begin{cases} \beta_I, & \text{if } |I| \le n-1; \\ \frac{1}{2}\beta_I, & \text{if } |I| = n. \end{cases}$$

It follows

$$\lambda^{\mathbb{R}}_{C_n}(I) = \begin{cases} \sum_{k \in I^{\pm}} \mathbf{e}_k, & \text{if } |I| \le n-1 \text{ and } n \notin I^{\pm}; \\ \sum_{k \notin I^{\pm}} \mathbf{e}_k, & \text{if } |I| \le n-1 \text{ and } n \in I^{\pm}; \\ \mathbf{e}_n + \sum_{k \in I^-} \mathbf{e}_k, & \text{if } |I| = n \text{ and } n \in I; \\ \mathbf{e}_n + \sum_{k \in I^+} \mathbf{e}_k, & \text{if } |I| = n \text{ and } -n \in I, \end{cases} \tag{2}$$

where $\mathbf{e}_i$ is the ith standard vector of $\mathbb{Z}_2^n$. Let Λ_{C_n} be the characteristic matrix of $X^{\mathbb{R}}_{C_n}$ determined by $\lambda^{\mathbb{R}}_{C_n}$, that is, Λ_{C_n} is an $n \times (3^n - 1)$ matrix with elements in $\mathbb{Z}_2$ whose columns are indexed by $I \in V_{C_n}$. Its column indexed by I is denoted by $\lambda^{\mathbb{R}}_{C_n}(I)$.

In order to apply Theorem 2.1 to compute the cohomology of $X^{\mathbb{R}}_{C_n}$, we consider $(K_{C_n})_S$ for all $S \in \operatorname{Row}(\Lambda_{C_n})$. For this, it is convenient to identify $\operatorname{Row}(\Lambda_{C_n})$ with the power set of $[n]$ in the following way; we associate to $\{i_1, \ldots, i_r\} \subset [n]$ the sum of $i_1, \ldots, i_r$th rows of Λ_{C_n}. We note the rows of Λ_{C_n} are symmetric except for the nth row, hence we deal with the case for $n \in S$ and $n \notin S$ separately.

For each $S \subset [n]$, the vertex set $V(S)$ of $(K_{C_n})_S$ is separated into two parts $V_1(S) = \{I \in V(S) \mid |I| < n\}$ and $V_2(S) = \{I \in V(S) \mid |I| = n\}$. More precisely,

$$V_1(S) = \left\{ I \in V_{C_n} \mid |I^{\pm} \cap (S \cup \{n\})| \text{ is odd and } |I| < n \right\},$$

and $V_2(S)$ is as follows.

- When $|S|$ is odd and $n \notin S$, $V_2(S) = \{I \in V_{C_n} \mid |I^- \cap (S \cup \{n\})| \text{ is odd and } |I| = n\}$.
- When $|S|$ is even and $n \notin S$, $V_2(S) = \{I \in V_{C_n} \mid |I^- \cap S| \text{ is odd and } |I| = n\}$.
- When $|S|$ is odd and $n \in S$, $V_2(S) = \{I \in V_{C_n} \mid |I^- \cap S| \text{ is even and } |I| = n\}$.
- When $|S|$ is even and $n \in S$, $V_2(S) = \{I \in V_{C_n} \mid |I^- \cap (S \setminus \{n\})| \text{ is even and } |I| = n\}$.

Moreover, $(K_{C_n})_S$ can be regarded as the poset complex of the poset on $V(S) = V_1(S) \cup V_2(S)$ ordered by inclusion.

Proposition 3.1. *We have*

$$(K_{C_n})_{\emptyset} \simeq \emptyset \quad \text{and} \quad (K_{C_n})_{\{n\}} \simeq \bigvee^{2^n - 1} S^0.$$

Proof. The former statement is obvious from the definition. Note that, by (2), $\lambda^{\mathbb{R}}_{C_n}(I)$ has the $\mathbf{e}_n$-term if and only if I has n elements. Therefore, $(K_{C_n})_{\{n\}}$ consists of 2^n distinct points, which proves the latter statement. □

Lemma 3.2. *Let $n \ge 3$. For a nonempty subset $S \subset [n-1]$, $(K_{C_n})_S$ is isomorphic to $(K_{C_n})_{S \cup \{n\}}$.*

Proof. Put $T = S \cup \{n\}$. We will construct an explicit isomorphism between $(K_{C_n})_S$ and $(K_{C_n})_T$. For each $I \in V_{C_n}$ and each $x \in [n]$, let I^{-x} be the set obtained from I by reversing the sign of the elements in $I \cap \{x, -x\}$, that is,

$$I^{-x} = \{-i \mid i \in I \cap \{x, -x\}\} \cup \{i \mid i \in I \setminus I \cap \{x, -x\}\}.$$

Note that if $I \cap \{x, -x\} = \emptyset$ then $I^{-x} = I$. For any $I_1, I_2 \subset V_{C_n}$ and $x \in [n]$, it is trivial to see that $I_1 \subset I_2$ if and only if $I_1^{-x} \subset I_2^{-x}$. We define a simplicial involution φ of K_{C_n} by

$$\varphi(I) := \begin{cases} I^{-n}, & \text{if } |S| \text{ is odd;} \\ I^{-1}, & \text{if } |S| \text{ is even.} \end{cases}$$

We show it restricts to a bijection from $V(S)$ to $V(T)$, which proves the assertion. For this, it is sufficient to show that $\varphi(I) \in V(T)$ for each $I \in V(S)$. We observe that $|I| = |\varphi(I)|$ and $\varphi(I)^{\pm} = I^{\pm}$. Therefore, if $I \in V_1(S)$, then $\varphi(I) \in V_1(T)$. Suppose that $|I| = n$ and $n \in I$. Then $|I^- \cap S|$ is odd. If $|S|$ is odd, then $-n \in \varphi(I)$ and $|I^+ \cap S|$ is even, and therefore, so is $|\varphi(I)^+ \cap S|$ since $\varphi(I)^+ = I^+$. If

$|S|$ is even, then $n \in \varphi(I)$ and $|I^- \cap S|$ is odd, and therefore, $|\varphi(I)^- \cap S|$ differs from $|I^- \cap S|$ by 1, so it is even. The case where $|I| = n$ and $-n \in I$ is similar. Thus, for $I \in V_2(S)$, $\varphi(I)$ is in $V_2(T)$. ☐

Lemma 3.3. *Let $n \geq 3$. For any subset $S \subset [n-1]$ of odd cardinality and for any $a \in [n-1] \setminus S$, $(K_{C_n})_S$ is isomorphic to $(K_{C_n})_{S \cup \{a\}}$.*

Proof. Put $T = S \cup \{a\}$, and $|S| = 2r - 1$ with $r \geq 1$. For $I \in V_{C_n}$, we let $\psi(I)$ be the subset of $[\pm n]$ obtained from I by switching $\pm a$ with $\pm n$. It is easy to see that ψ defines an involution of K_{C_n}. Furthermore, one can check that ψ maps $V(S)$ to $V(T)$, inducing an isomorphism between $(K_{C_n})_S$ and $(K_{C_n})_T$. ☐

By Lemma 3.2, we may assume $n \notin S$. Also by Lemma 3.3, we see $(K_{C_n})_S$ depends only on $\lfloor \frac{|S|+1}{2} \rfloor$ when $n \notin S$. From now on, we assume that $S = \{1, 2, \dots, 2r-1\}$ with $r \geq 1$ and $2r - 1 < n$. We invoke the famous "Quillen's Theorem A" to find a simpler complex which is homotopy equivalent to $(K_{C_n})_S$.

In what follows, we do not distinguish a poset from its order complex by abuse of notation. Let $K^r_{C_n}$ be a poset on $U \cup W$ ordered by inclusion, where $U = \{I \in V_1(S) \mid I^\pm \subset S \cup \{n\}\}$ and $W = V_2(S)$. Note that $K^r_{C_n}$ is a subposet of $(K_{C_n})_S$.

Theorem 3.4 (Proposition 1.6 of [10])**.** *Let X be a subposet of a poset Y. If $X_{\leq y}$ is contractible for all $y \in Y$, then the inclusion $X \subset Y$ is a homotopy equivalence.*

Lemma 3.5. *We have a homotopy equivalence $(K_{C_n})_S \simeq K^r_{C_n}$.*

Proof. By Theorem 3.4, we only have to check that $(K^r_{C_n})_{\leq I}$ is contractible for each vertex I of $(K_{C_n})_S$. If $|I| = n$, then $(K^r_{C_n})_{\leq I}$ is a cone whose apex is I. For $|I| < n$, let J be the maximum subset of I satisfying $J^\pm \subset (S \cup \{n\})$. Then, J is the unique maximum of $(K^r_{C_n})_{\leq I}$, and $(K^r_{C_n})_{\leq I}$ is a cone whose apex is I. ☐

Now we investigate the cohomology of $K^r_{C_n}$. For simplicity, we just put $K = K^r_{C_n}$.

Observe that $K_U \cong \mathcal{C}^{odd}_{2r}$. In particular, by Lemma 2.2,

$$\widetilde{H}^*(K_U) \cong \widetilde{H}^*(\mathcal{C}^{odd}_{2r}) \cong \begin{cases} \mathbb{Z}^{b_{2r}} & (* = r-1) \\ 0 & (* \neq r-1). \end{cases} \tag{3}$$

Since K is obtained by attaching r-dimensional simplices to K_U, $\widetilde{H}^*(K) \cong \widetilde{H}^*(K_U) \cong 0$ for $* < r-1$ and for $* > r$. We compute $\widetilde{H}^r(K)$ and $\widetilde{H}^{r-1}(K)$ in the following. Let $\mathcal{L} = \{J = (\pm 1, \pm 2, \dots, \pm(2r-1), \pm n) \mid |J^-| \text{ is odd }\}$. Then, $|\mathcal{L}| = 2^{2r-1}$. For $J \in \mathcal{L}$, we see $K_{<J} \simeq \mathcal{B}^{odd}_{2r}$ and $K_{>J}$ is 2^{n-2r} points. Therefore, for $V_J = \{I \in V \mid I \subset J \text{ or } I \supset J\}$, we have

$$K_{V_J} \simeq K_{<J} \star K_{>J} \simeq \mathcal{B}^{odd}_{2r} \star \bigvee^{2^{n-2r}-1} S^0,$$

where $X \star Y$ denotes the join of two topological space X and Y, and by Lemma 2.2,

$$\widetilde{H}^*(K_{V_J}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{(2^{n-2r}-1)a_{2r}} & (* = r) \\ 0 & (* \neq r). \end{cases}$$

Observe that $K = \bigcup_{J \in \mathcal{L}} K_{V_J}$. Since $|J \cap J'| \leq 2r-2$ for any two distinct elements J and J' of $\mathcal{L}$, the dimension of $K_{V_J} \cap \bigcup_{J' \in \mathcal{L} \setminus \{J\}} K_{V_{J'}}$ is less than $r-1$. By the Mayer-Vietoris sequence, we have

$$\widetilde{H}^r(K) = \widetilde{H}^r\left(\bigcup_{J \in \mathcal{L}} K_{V_J}\right) \cong \bigoplus_{J \in \mathcal{L}} \widetilde{H}^r(K_{V_J}) \cong \mathbb{Z}^{2^{2r-1}(2^{n-2r}-1)a_{2r}}.$$

We now show inductively on the cardinality of $\mathcal{L}' \subset \mathcal{L}$ that

$$\widetilde{H}^{r-1}\left(\bigcup_{J \in \mathcal{L}'} K_{V_J} \cup K_U\right) \cong \mathbb{Z}^{b_{2r} - |\mathcal{L}'| a_{2r}}.$$

The case when $|\mathcal{L}'| = 0$ follows from (3). The cases when $|\mathcal{L}'| > 0$ are proved by the Mayer-Vietoris sequence

$$\begin{aligned}
0 \leftarrow & \widetilde{H}^r\left(\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U\right) \oplus \widetilde{H}^r\left(K_{V_{J'}}\right) \overset{\cong}{\leftarrow} \widetilde{H}^r\left(\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U \cup K_{V_{J'}}\right) \\
\overset{0}{\leftarrow} & \widetilde{H}^{r-1}\left((\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U) \cap K_{V_{J'}}\right) \leftarrow \widetilde{H}^{r-1}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U\right) \oplus \widetilde{H}^{r-1}\left(K_{V_{J'}}\right) \\
\leftarrow & \widetilde{H}^{r-1}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U \cup K_{V_{J'}}\right) \leftarrow \widetilde{H}^{r-2}\left((\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U) \cap K_{V_{J'}}\right) = 0,
\end{aligned}$$

where $J' \notin \mathcal{L}'$ since $\widetilde{H}^*\left(\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U \cap K_{V_{J'}}\right) = \widetilde{H}^*(K_{<J'}) \cong \begin{cases} \mathbb{Z}^{a_{2r}} & (*=r-1) \\ 0 & (*\neq r-1) \end{cases}$ and $\widetilde{H}^{r-1}\left(K_{V_{J'}}\right) \cong 0$. The first isomorphism follows from the fact that $\dim(K_U) = r-1$ and the previous computation.

We conclude that, by Lemma 3.5,

$$\widetilde{H}^*((K_{C_n})_S) \cong \widetilde{H}^*(K) \cong \begin{cases} \mathbb{Z}^{(2^{n-2r}-1)2^{2r-1}a_{2r}} & (*=r) \\ \mathbb{Z}^{b_{2r}-2^{2r-1}a_{2r}} & (*=r-1) \\ 0 & (\text{otherwise}). \end{cases}$$

Combining this with Proposition 3.1 and Theorem 2.1, we obtain the type C_n part of Theorem 1.1.

4. Type D_n

The (real) toric variety associated to the Weyl chamber of type D_n can be obtained similarly to the Type C_n case. Since $\Phi_{D_n} = \Phi_{A_n}$ for $n \leq 3$, we consider the case when $n \geq 4$. The root system Φ_{D_n} of type D_n consists of $2n(n-1)$ roots

$$\pm\varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq n),$$

where ε_i is the ith standard vector of $\mathbb{R}^n = V$. Then, the co-weight lattice $\mathbb{Z}\langle\Omega\rangle$ is

$$\Lambda = \left\{ \frac{1}{2}(\ell_1\varepsilon_1 + \cdots + \ell_n\varepsilon_n) \ \middle|\ \ell_i \in \mathbb{Z}, \text{ and } \ell_i \equiv \ell_j \pmod 2 \text{ for all } i,j \right\}.$$

Choose a basis $\{\epsilon_i \mid 1 \leq i \leq n\}$ of $\mathbb{Z}\langle\Omega\rangle$ defined by $\epsilon_i := \varepsilon_i$ for $i = 1,\dots,n-1$ and $\epsilon_n := \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$. Any set of simple roots of type D_n is written as

$$\Delta = \{\mu_1\varepsilon_{\sigma(1)} - \mu_2\varepsilon_{\sigma(2)}, \mu_2\varepsilon_{\sigma(2)} - \mu_3\varepsilon_{\sigma(3)}, \dots, \mu_{n-1}\varepsilon_{\sigma(n-1)} - \mu_n\varepsilon_{\sigma(n)}, \mu_{n-1}\varepsilon_{\sigma(n-1)} + \mu_n\varepsilon_{\sigma(n)}\},$$

where $\mu_j = \pm 1$ and $\sigma\colon [n] \to [n]$ is a permutation. Put $\alpha_i^{\mu,\sigma} := \mu_i\varepsilon_{\sigma(i)} - \mu_{i+1}\varepsilon_{\sigma(i+1)}$ for $i = 1,\dots,n-1$ and $\alpha_n^{\mu,\sigma} := \mu_{n-1}\varepsilon_{\sigma(n-1)} + \mu_n\varepsilon_{\sigma(n)}$. For $\alpha \in \Delta$, there exists a unique primitive integral vector $\beta_{\alpha,\Delta}$ such that $(\beta_{\alpha,\Delta}, \alpha') = 0$ for all $\alpha' \in \Delta \setminus \{\alpha\}$ and $(\beta_{\alpha,\Delta}, \alpha) > 0$. We label the ray corresponding to $\beta_{\alpha,\Delta}$ by the subset I of $[\pm n]$ satisfying (1) as we did for the case of type C_n. Then, we have

α	$\beta_{\alpha,\Delta}$	I
$\alpha_i^{\mu,\sigma}$ for $1 \leq i \leq n-2$	$\sum_{k=1}^{i} \mu_k\varepsilon_{\sigma(k)}$	$\{\mu_1\sigma(1),\dots,\mu_i\sigma(i)\}$
$\alpha_{n-1}^{\mu,\sigma}$	$\left(\sum_{k=1}^{n-1} \mu_k\varepsilon_{\sigma(k)}\right) - \mu_n\varepsilon_{\sigma(n)}$	$\{\mu_1\sigma(1),\dots,\mu_{n-1}\sigma(n-1), -\mu_n\sigma(n)\}$
$\alpha_n^{\mu,\sigma}$	$\sum_{k=1}^{n} \mu_k\varepsilon_{\sigma(k)}$	$\{\mu_1\sigma(1),\dots,\mu_n\sigma(n)\}$

Conversely, for each $I \subset [\pm n]$ satisfying both (1) and $|I| \neq n-1$, one can find α and Δ such that $\alpha \in \Delta$ and $\beta_{\alpha,\Delta} = \beta_I$. Therefore, we have an identification of V_{D_n} as

$$V_{D_n} = \{I \subset [\pm n] \mid I^+ \cap I^- = \emptyset \text{ and } |I| \neq n-1\}.$$

Under this identification, a maximal cone C_Δ corresponds to n such subsets $I_1 \subsetneq \cdots \subsetneq I_{n-2} \subsetneq I_{n-1} \cap I_n$. This gives the complete combinatorial structure of K_{D_n}.

We note that $V_{D_n} \subset V_{C_n}$. By similar computation to the case of Type C_n, one can see that the characteristic map λ_{D_n} is given exactly same as (2), that is, $\lambda_{D_n}(I) = \lambda_{C_n}(I)$ for $I \in V_{D_n} \subset V_{C_n}$. Hence, the characteristic matrix Λ_{D_n} is an $n \times (3^n - 1 - n \cdot 2^{n-1})$ matrix with elements in $\mathbb{Z}_2$ obtained from Λ_{C_n} by restricting columns to those corresponding to V_{D_n}.

From now on, let us consider the topology of $X^{\mathbb{R}}_{D_n}$ for $n \geq 4$. We identify again $\mathrm{Row}(\Lambda_{D_n})$ with the power set of $[n]$ as we did in the case of Type C_n. In order to compute the cohomology of $X^{\mathbb{R}}_{D_n}$ it is enough to consider $(K_{D_n})_S$ for all $S \subset [n]$ by Theorem 2.1. We note the rows of Λ_{D_n} are symmetric except for the nth row, hence we deal with the case for $n \in S$ and $n \notin S$ separately.

For $m \geq 0$, put

$$t_m = (m-2)2^{m-1} + 1.$$

Proposition 4.1. *For $n \geq 4$,*

$$(K_{D_n})_\emptyset \simeq \emptyset \quad \text{and} \quad (K_{D_n})_{\{n\}} \simeq \bigvee^{t_n} S^1.$$

Proof. Since the former statement is obvious from the definition, let us prove the latter one. Note that $(K^{\mathbb{R}}_{D_n})_{\{n\}} = K_W$, where $W = \{I \in V(K^{\mathbb{R}}_{D^n}) \mid |I| = n\}$. Recall that I_1 and I_2 in W are connected if and only if $|I_1 \cap I_2| = n-1$. Hence, each vertex has n edges, and there is no 2-face in K_W. Therefore, K_W is homotopy equivalent to $\bigvee^{t_n} S^1$ as desired. □

By the same argument as in Lemmas 3.2 and 3.3, we have the following two lemmas.

Lemma 4.2. *For a positive integer $n \geq 4$ and for a nonempty subset $S \subset [n-1]$, $(K_{D_n})_S$ is homotopy equivalent to $(K_{D_n})_{S \cup \{n\}}$.*

Lemma 4.3. *For a positive integer $n \geq 4$ and for odd subset $S \subset [n-1]$ for any $a \in [n-1] \setminus S$, $(K_{D_n})_S$ is homotopy equivalent to $(K_{D_n})_{S \cup \{a\}}$.*

By Lemmas 4.2 and 4.3, we may assume that $S = \{1, 2, \ldots, 2r-1\}$ with $r \geq 1$ and $2r - 1 < n$.

We note that each maximal simplex of $(K_{D_n})_S$ contains exactly one pair of elements I_1 and I_2 such that $|I_1 \cap I_2| = n - 1$. Let $V(S)$ be the vertex set of $(K_{D_n})_S$, and $V' = \{J \in [\pm n] \mid J = I_1 \cap I_2$ for some $I_1, I_2 \in V(S)$, and $|J| = n-1\}$. Similarly to Lemma 3.5, we have the following lemma.

Lemma 4.4. *We have a homotopy equivalence $(K_{D_n})_S \simeq K^r_{D_n}$, where $K^r_{D_n}$ is the poset on $U \cup W \cup V'$ ordered by inclusion, where*

$$U = \{I \in V(S) \mid I^\pm \subset S \cup \{n\} \text{ and } |I| < n-1\}, \quad \text{and} \quad W = \{I \in V(S) \mid |I| = n\}.$$

Proof. We adjoin new vertices labelled with $I_1 \cap I_2$ on the edge between I_1 and I_2, then take the subdivision of the simplex. Then, $(K_{D_n})_S$ is homeomorphic to the poset complex of the vertex set $V(S) \cup V'$, where the poset structure is given by the inclusion of vertex labels. Then, this poset complex is indeed homotopy equivalent to $K^r_{D_n}$ by Theorem 3.4. □

Now we investigate the cohomology of $K^r_{D_n}$. We put $K = K^r_{D_n}$ for simplicity. We first consider the case when $n = 2r$ and $2r+1$, where $t_{n-2r} = 0$. In this case, K is exactly same as in the type C_n case. Thus, we have

$$\widetilde{H}^*(K) \cong \begin{cases} \mathbb{Z}^{b_{2r} - 2^{2r-1} a_{2r}} & (* = r-1) \\ 0 & (\text{otherwise}). \end{cases}$$

Now we proceed to the case when $n > 2r + 1$. One can see $K_U \simeq \mathcal{C}^{odd}_{2r}$. Since K is obtained by attaching $r+1$-dimensional simplices to K_U, by Lemma 2.2 we have $\widetilde{H}^*(K) \cong \widetilde{H}^*(K_U) \cong 0$ for $* < r-1$ and for $* > r+1$. Let $\mathcal{L} = \{J = (\pm 1, \pm 2, \ldots, \pm(2r-1), \pm n) \mid |J^-| \text{ is odd }\}$. Then, $|\mathcal{L}| = 2^{2r-1}$. For $J \in \mathcal{L}$, we see $K_{<J} \simeq \mathcal{B}^{odd}_{2r}$. Therefore, for $V_J = \{I \in V \mid I \subset J \text{ or } I \supset J\}$, we have $K_{V_J} \simeq K_{<J} \star K_{>J} \simeq \mathcal{B}^{odd}_{2r} \star K_{>J}$ for $J \in \mathcal{L}$, and $K = \bigcup_{J \in \mathcal{L}} K_{V_J}$. Furthermore, $K_{>J}$ has $(n-2r)2^{n-1-2r}$ vertices in V' and 2^{n-2r} vertices in W. Since each vertex in V' is adjacent to exactly two edges, the first Betti number of $K_{>J}$ is $2(n-2r)2^{n-1-2r} - ((n-2r)2^{n-1-2r} + 2^{n-2r}) + 1 = t_{n-2r}$. In summary, we have $K_{>J} \simeq \bigvee^{t_{n-2r}} S^1$ and

$$\widetilde{H}^*(K_{V_J}) \cong \begin{cases} \mathbb{Z}^{t_{n-2r} a_{2r}} & (* = r+1) \\ 0 & (\text{otherwise}). \end{cases}$$

We now show inductively on the cardinality of $\mathcal{L}' \subset \mathcal{L}$ that

$$\widetilde{H}^*\left(\bigcup_{J\in\mathcal{L}'} K_{V_J} \cup K_U\right) \cong \begin{cases} \mathbb{Z}^{|\mathcal{L}'|t_{n-2r}a_{2r}} & (*=r+1) \\ \mathbb{Z}^{b_{2r}-|\mathcal{L}'|a_{2r}} & (*=r-1) \\ 0 & (\text{otherwise}). \end{cases}$$

This follows from the Mayer-Vietoris sequence

$$\begin{aligned} 0=\widetilde{H}^r\left((\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U)\cap K_{V_{J'}}\right) &\leftarrow \widetilde{H}^{r+1}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U\right)\oplus \widetilde{H}^{r+1}\left(K_{V_{J'}}\right) \leftarrow \widetilde{H}^{r+1}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U \cup K_{V_{J'}}\right) \\ \leftarrow \widetilde{H}^r\left((\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U)\cap K_{V_{J'}}\right) &\leftarrow \widetilde{H}^{r}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U\right)\oplus \widetilde{H}^{r}\left(K_{V_{J'}}\right) \leftarrow \widetilde{H}^{r}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U \cup K_{V_{J'}}\right) \\ \leftarrow \widetilde{H}^{r-1}\left((\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U)\cap K_{V_{J'}}\right) &\leftarrow \widetilde{H}^{r-1}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U\right)\oplus \widetilde{H}^{r-1}\left(K_{V_{J'}}\right) \leftarrow \widetilde{H}^{r-1}\left(\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U \cup K_{V_{J'}}\right) \\ \leftarrow \widetilde{H}^{r-2}\left((\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U)\cap K_{V_{J'}}\right) &= 0. \end{aligned}$$

where $J' \notin \mathcal{L}'$ since $\widetilde{H}^*\left((\bigcup_{J\in\mathcal{L}'} K_{V_J}\cup K_U)\cap K_{V_{J'}}\right) = \widetilde{H}^*(K_{<J'}) \cong \begin{cases} \mathbb{Z}^{a_{2r}} & (*=r-1) \\ 0 & (*\neq r-1) \end{cases}$ and the terms in the second row are all trivial.

By Lemma 4.4, we conclude that

$$\widetilde{H}^*((K_{D_n})_S) \cong \widetilde{H}^*(K) \cong \begin{cases} \mathbb{Z}^{2^{2r-1}t_{n-2r}a_{2r}} & (*=r+1) \\ \mathbb{Z}^{b_{2r}-2^{2r-1}a_{2r}} & (*=r-1) \\ 0 & (\text{otherwise}). \end{cases}$$

Combining this with Proposition 4.1 and Theorem 2.1, we obtain the type D_n part of Theorem 1.1.

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